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Boundary driven Markov gas: duality and scaling limits

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Abstract. Inspired by the recent work of Bertini and Posta [5], who introduced the boundary driven Brownian gas on $[0, 1]$, we study boundary driven systems of independent particles in a general setting, including particles jumping on finite graphs and diffusion processes on bounded domains in $\mathbb{R}^d$. We prove duality with a dual process that is absorbed at the boundaries, thereby creating a general framework that unifies dualities for boundary driven systems in the discrete and continuum setting. We use duality first to show that from any initial condition the systems evolve to the unique invariant measure, which is a Poisson point process with intensity the solution of a Dirichlet problem. Second, we show how the boundary driven Brownian gas arises as the diffusive scaling limit of a system of independent random walks coupled to reservoirs with properly rescaled intensity.

Keywords. Boundary driven systems, Brownian gas, duality, orthogonal polynomials, point processes, scaling limits.

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1 Introduction

With great admiration, we dedicate this article to Errico Presutti on the occasion of his 80th birthday.

1.1 Background and motivation

Boundary driven systems are important in the study of non-equilibrium steady states [4, 12, 24]. In the context of interacting particle systems on finite graphs, boundary driving means that one adds "external reservoirs" from which particles can enter and leave the system. Only some vertices of the graph are in contact with the external reservoirs. We call these "boundary sites" and all the remaining vertices "bulk sites". Particles can then enter and leave the system only through the boundary sites. This mechanism is usually modeled via birth and death processes. Birth and death rates are chosen in a manner adapted to the system, i.e. in such a way to mimic the jump rates in the bulk. This implies that the stationary measure of each reservoir is a marginal of the stationary measure of the entire system. The simplest setting is a one-dimensional finite chain, with two reservoirs, one at the left end and the other at the right end of the chain. The usual objects of study in this context are the stationary distribution of such non-equilibrium systems and its macroscopic properties (e.g. the density profile, the current and their large deviations).

In the *discrete setting* of finite graphs, boundary driven systems of independent particles (and more generally zero-range processes) have a special status, because the non-equilibrium steady states are inhomogeneous product measures, in the case of independent particles product of Poisson distributions. For one dimensional systems, the parameters of these product measures then interpolate linearly between the densities λ_L and λ_R of the left reservoir and right reservoir (see e.g. Section 4.2.3 of [7]). For a class of particle systems (including independent particles), one has the property of *duality* [11], which allows to express the n-point time-dependent correlation functions in terms of the evolution of n (dual) particles. In the discrete setting, these dual particles evolve on a larger system, where absorbing extra sites have been added, representing the action of reservoirs of the original system. Duality has been an essential tool to study detailed properties of different boundary driven systems such as the so-called KMP model (see [21]) and the Exclusion process (see e.g. [13], [15], [17], [18]). See [7] for an account of dualities in the discrete boundary driven setting. Given the broad applicability of duality there is the need to extend it to continuum systems.

In [14] the authors started the study of self-duality beyond the discrete setting, i.e., self-duality of general independent Markov processes evolving as point configurations, which is the analogue of particle configurations in the discrete setting. There, self-duality turned out to be a general property of the evolution of the n-th factorial moment measure, which can be expressed via the evolution of n (dual) particles. The authors of [14] considered the setting of closed systems with a conserved number of particles. The goal of our paper will be to initiate the analysis of duality for *boundary-driven systems in the continuum*, starting from the case of independent particles. We believe that the framework we build here can be used as well for boundary driven interacting particle systems in the continuum, but we leave this for future research.

To achieve our goal, a proper definition of the action of reservoirs in the continuum has to be considered. In the interval $[0, 1]$, the naive idea would be to study a system of independent Brownian motions that are absorbed at the boundaries 0 and 1, with additional creation of particles at 0 and 1. However, as was noticed in [5], this approach does not work, because in the continuum particles put at the boundary would immediately leave via that same boundary. Therefore, the problem of modeling reservoirs in the continuum is more involved than in the discrete setting. In [5] the *boundary-driven Brownian gas* on $[0, 1]$ has been defined as the sum of two independent processes: one process modeling the evolution of the particles initially present in the system and moving as independent Brownian motions absorbed at 0 and at 1; and another Poisson point process adding particles on $(0, 1)$ with well-chosen intensity. The creation of particles no longer takes place at the boundaries only, rather particles are created everywhere in $(0, 1)$ with an intensity that guarantees the prescribed densities of the reservoirs. The authors in [5] then proceed by proving that this process is Markov.

One of the main aims in this paper is to establish in the setting of the boundary driven Brownian gas, the kind of duality results proved in [7, 15] for discrete boundary driven systems. To do this, we use the set-up introduced in [14] for closed systems in the continuum and extend it to the boundary driven Brownian gas. In particular we show that the time-dependent n-th factorial moment measures of this system can be written in terms of n dual Brownians, absorbed at the boundaries. Next, a second aim is to generalize this duality to the abstract setting of general boundary driven systems of independent particles in the continuum. For this we will need to generalize the construction of Bertini and Posta [5] first to systems of independent diffusion processes evolving on regular domain $\mathfrak{D} \subset \mathbb{R}^d$ and second to systems of general independent Markov processes which are allowed to jump and which thus can leave $\mathfrak{D}$ without hitting its boundary. As a by-product of such general construction and our duality relations two results will follow. We shall prove that in the discrete setting of a one-dimensional chain, modelling the reservoirs as: i) birth and death processes at the boundaries or ii) by a Poissonian addition of particles everywhere, are indeed equivalent processes. Furthermore the boundary driven Brownian gas (in the continuum) arises as the diffusive scaling limit of the model with birth and death processes (in the discrete) when the intensities are also scaled with the system size.

1.2 Duality results for independent random walks

For the reader's convenience we recall the standard dualities of independent particles in the discrete setting, both in the case of closed and open systems.

1.2.1 Closed systems

Let us consider a system of simple independent random walks, namely the Markov process $\{\eta_t,\ t \geq 0\}$ with $\eta_t = \{\eta_t(x)\}_{x\in\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ where

$$\eta_t(x) := \text{number of particles at } x \text{ at time } \ t \geq 0$$

whose generator acts on bounded and local functions $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ as

$$\begin{aligned}(Lf)(\eta) = \sum_{\|x-y\|=1} \frac{1}{2}\, [\eta(x)(f(\eta + \delta_x - \delta_y) - f(\eta)) \\ + \eta(y)(f(\eta + \delta_y - \delta_x) - f(\eta))] \,.\end{aligned} \tag{1.1}$$

Here the sum is restricted to nearest neighbour sites and $\eta+\delta_x-\delta_y$ denotes the configuration where a particle has been moved from x to y in the configuration η. We then have that $\{\eta_t,\ t \geq 0\}$ is self-dual with self-duality function given by

$$D^{\mathrm{cl}}(\xi, \eta) = \prod_{x\in\mathbb{Z}^d} d(\xi(x), \eta(x)) \tag{1.2}$$

for $\xi,\ \eta \in \mathbb{N}^{\mathbb{Z}^d}$ with single-site self-duality function given by

$$d(k, n) = (n)_k \mathbf{1}_{\{k\leq n\}} := \frac{n!}{(n-k)!} \mathbf{1}_{\{k\leq n\}}. \tag{1.3}$$

If we denote by $\mathbb{E}^{\mathrm{IRW}}_\eta$ the expectation w.r.t. the law of the process evolving according to the generator given in (1.1) and starting from $\eta \in \mathbb{N}^{\mathbb{Z}^d}$, the self-duality relation is then expressed in the following way: for any $\eta,\ \xi \in \mathbb{N}^{\mathbb{Z}^d}$ and for any $t \geq 0$,

$$\mathbb{E}^{\mathrm{IRW}}_\xi [D^{\mathrm{cl}}(\xi_t, \eta)] = \mathbb{E}^{\mathrm{IRW}}_\eta [D^{\mathrm{cl}}(\xi, \eta_t)]. \tag{1.4}$$

The self-duality functions given in (1.2), that we refer to as *classical self-dualities* are products of falling factorial polynomials and they have been used to prove the hydrodynamic limit (see [11]). More recently they have been generalized to the context of systems of independent particles evolving in the continuum (e.g., $\mathbb{R}^d$, and more generally in Borel spaces) in [14].

1.2.2 Open systems

Let us further consider a system of simple independent random walks on a finite chain $V_N := \{1, \dots, N\}$ where the boundary points $\{1, N\}$ are in contact with reservoirs with intensity parameters $\lambda_L, \lambda_R \in (0, \infty)$. Namely, we consider the Markov process $\{\zeta_t,\ t \geq 0\}$ with state space $\mathbb{N}^{V_N}$ and whose generator acts on functions $f : \mathbb{N}^{V_N} \to \mathbb{R}$ as

$$L_{\text{res}} f(\zeta) = L_{\text{bulk}} f(\zeta) + L_{\text{left}} f(\zeta) + L_{\text{right}} f(\zeta), \tag{1.5}$$

where L_{bulk} denotes the generator of continuous-time symmetric independent random walkers jumping with rate $\frac{1}{2}$ over the edges $(i, i+1)$, $i \in \{1, \dots, N-1\}$ and where $L_{\text{left}}, L_{\text{right}}$ denote the boundary generators, modelling the contact with the reservoirs, which are given by

$$L_{\text{left}} f(\zeta) = \zeta(1)[f(\zeta - \delta_1) - f(\zeta)] + \lambda_L[f(\zeta + \delta_1) - f(\zeta)]$$

and

$$L_{\text{right}} f(\zeta) = \zeta(N)[f(\zeta - \delta_N) - f(\zeta)] + \lambda_R[f(\zeta + \delta_N) - f(\zeta)].$$

These generators describe the exit and entrance of particles via the reservoirs at left and right boundaries of the chain. Each particle can leave the system through the right or left end at rate 1, and at rate λ_L (resp. λ_R) particles enter the system at the left (resp. right) end. In the following we shall call the process ζ_t the *"reservoir process with parameters λ_L, λ_R"*.

In [7] the authors proved that the reservoir process with parameters λ_L, λ_R is dual to a system of independent random walkers on the lattice $\{0, \dots, N+1\}$ with absorbing boundaries. In the dual process the absorbing sites 0 and $N+1$ replace the reservoirs of the original process. With abuse of notation we shall use, for the dual process, the name $\{\xi_t,\ t \geq 0\}$ as in the previous paragraph, although now, in the boundary-driven context, the dual has absorbing boundary sites. The duality function D^{λ_L, λ_R} can be written as

$$D^{\lambda_L, \lambda_R}(\xi, \zeta) = \lambda_L^{\xi(0)} \lambda_R^{\xi(N+1)} D^{\text{cl}}(\xi, \zeta), \tag{1.6}$$

where $\xi \in \mathbb{N}^{\{0,\dots,N+1\}}$, $\zeta \in \mathbb{N}^{\{1,\dots,N\}}$ and $D^{\rm cl}(\cdot,\cdot)$ is given in (1.2) but now the product is over V_N and not over $\mathbb{Z}^d$, namely

$$D^{\rm cl}(\xi,\zeta) = \prod_{i=1}^{N} d(\xi(i),\zeta(i))$$

with $d(k,n) = \frac{n!}{(n-k)!}\mathbf{1}_{k\le n}$. Let us denote by $\mathbb{E}^{\rm res}_\zeta$ the expectation in the reservoir process with parameters λ_L, λ_R starting from $\zeta \in \mathbb{N}^{\{1,\dots,N\}}$. Moreover we denote by $\mathbb{E}^{\rm abs}_\xi$ the expectation in the dual process starting from an initial configuration $\xi \in \mathbb{N}^{\{0,\dots,N+1\}}$. Then we have the following duality result: for any $\zeta \in \mathbb{N}^{V_N}$, $\xi \in \mathbb{N}^{\{0,\dots,N+1\}}$ and $t \ge 0$

$$\mathbb{E}^{\rm res}_\zeta\left[D^{\lambda_L,\lambda_R}(\xi,\zeta_t)\right] = \mathbb{E}^{\rm abs}_\xi\left[D^{\lambda_L,\lambda_R}(\xi_t,\zeta)\right] \tag{1.7}$$

or equivalently

$$\mathbb{E}^{\rm res}_\zeta\left[\lambda_L^{\xi(0)}\lambda_R^{\xi(N+1)}D^{\rm cl}(\xi,\zeta_t)\right] = \mathbb{E}^{\rm abs}_\xi\left[\lambda_L^{\xi_t(0)}\lambda_R^{\xi_t(N+1)}D^{\rm cl}(\xi_t,\zeta)\right]. \tag{1.8}$$

The main aim of this paper is to extend the above duality result to general systems of boundary-driven independent particles. The random walk dynamics of each particle will be replaced by a generic Markov process. As a consequence we shall consider boundary driven systems of independent particles evolving not necessarily on the lattice, rather on generic regular domains $\mathfrak{D} \subset \mathbb{R}^d$, $d \ge 1$.

1.3 Outline

The rest of our paper is organized as follows. In Section 2 we introduce basic notations. As a preliminary step, in Section 3 we present duality results for closed systems of independent particles in the continuum. First we recall a self-duality result from [14]. Second, we prove a duality result, where the dual system is deterministic and follows the backward Kolmogorov equation associated to the single particle; we then use this duality result to provide a simple proof of Doob's theorem. Section 4 contains the main result of this paper regarding *boundary driven systems*. We start by recalling the definition of the boundary driven Brownian gas on $[0,1]$, introduced in [5]. We then generalize this construction to general independent diffusion processes moving on regular domains $\mathfrak{D} \subset \mathbb{R}^d$ and finally to general independent Markov processes which can make jumps and thus can exit $\mathfrak{D}$ without hitting its boundary. For those systems we formulate, with increasing generality, the duality results in Theorems 4.1, 4.2 and 4.6, and in particular we use Theorem 4.2 to characterize the unique invariant measure of the systems. In Section 5, we use the duality result to show that the boundary-driven Brownian gas introduced in [5] is the scaling limit of the reservoir process of independent random walks with generator (1.5). Namely, we prove that the latter equals in distribution the 'boundary driven random walk gas' and that, when the parameters are scaled as λ_L/N, λ_R/N, it converges on the diffusive scale to the boundary driven Brownian gas with parameters λ_L, λ_R. Finally, in Section 6, orthogonal dualities are treated, extending to the continuum results from [15].

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2 Setting and notations

We will work in the context of independent particles moving in a state space E, which is assumed to be a Polish space, equipped with its Borel σ-algebra $\mathscr{E}$. In the relevant examples, $E = \mathbb{R}^d$, or E is a closed subset of $\mathbb{R}^d$ with regular boundary, or in the discrete setting $E = \mathbb{Z}^d$ or a finite graph. However, for the general duality results which we state here, there is no need to restrict to the finite dimensional setting.

2.1 Labeled independent particles

A single particle is moving as a Markov process $\{X_t : t \geq 0\}$ on E. A finite number of (labeled) independent particles is the process $\mathscr{X}_t = (X_t(1), \ldots, X_t(\mathbf{N}))$ arising from joining $\mathbf{N} \in \mathbb{N}$ independent copies of $\{X_t : t \geq 0\}$, possibly starting from different initial locations $X_0(i) = x_i \in E$. We denote by $\mathbb{E}_{x_1,\ldots,x_{\mathbf{N}}}$ the expectation of $\{\mathscr{X}_t, t \geq 0\}$ starting from $(x_1, \ldots, x_{\mathbf{N}})$, by S_t the semigroup of the Markov process $\{X_t : t \geq 0\}$, defined via $S_t f(x) = \mathbb{E}_x f(X_t)$, and by $S_t^{\otimes \mathbf{N}}$ the associated semigroup of $\mathbf{N}$ independent copies of $\{X_t : t \geq 0\}$. By independence we have

$$S_t^{\otimes \mathbf{N}} \prod_{i=1}^{\mathbf{N}} f_i(x_i) = \prod_{i=1}^{\mathbf{N}} \mathbb{E}_{x_i} \left[f_i(X_t(i)) \right] = \prod_{i=1}^{\mathbf{N}} S_t f_i(x_i).$$

We denote by S_t^* the dual semigroup working on measures μ (on $(E, \mathscr{E})$), defined via

$$\int f dS_t^* \mu = \int S_t f d\mu. \tag{2.1}$$

We remind the reader that we call a σ-finite measure m on E *reversible* if

$$\int_E S_t f \, g \, \mathrm{d}m = \int_E f \, S_t g \, \mathrm{d}m$$

for any $f, g \in L^2(E, m)$ and $t > 0$. Moreover, we say that the Markov process $\{X_t,\, t \geq 0\}$ is *strongly reversible* if there exists a reversible σ-finite measure m such that the transition probability measure is absolutely continuous w.r.t. m, i.e., there exists a transition density

$$\mathfrak{p}_t : E \times E \to [0, \infty)$$

such that, for all $t > 0$,

$$S_t f(x) = \int f(y)\mathfrak{p}_t(x, y)m(dy) = \int f(y)\mathfrak{p}_t(y, x)m(dy), \tag{2.2}$$

where the symmetry $\mathfrak{p}_t(x, y) = \mathfrak{p}_t(y, x)$ follows from the assumed reversibility of m. Relevant examples to keep in mind are i) Brownian motion, where m is the Lebesgue measure; ii) symmetric random walk, where m is the counting measure; iii) the Ornstein Uhlenbeck process, where m is the Gaussian measure.

2.2 Point configurations

It is convenient for our purposes to describe the motion of independent particles modulo permutation, i.e. via configurations. More precisely, the initial configuration associated to $\mathbf{N}$ labeled particle positions $(x_1, \ldots, x_{\mathbf{N}}) \in E^{\mathbf{N}}$ is defined as

$$\eta = \sum_{i=1}^{\mathbf{N}} \delta_{x_i}, \tag{2.3}$$

which is viewed as a point configuration on E. The configuration at time t is then defined as

$$\eta_t = \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)}, \tag{2.4}$$

where $X_0(i) = x_i$. Notice that by the fact that the independent particles are indistinguishable, $\{\eta_t, t \geq 0\}$ is a Markov process on the space of point configurations with total mass $\mathbf{N}$. More generally, if we have a point configuration on E, with potentially infinitely many particles, i.e., $\eta = \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$, where we now also allow $\mathbf{N} = \infty$, then we define the configuration at time $t > 0$ as in (2.4). In case we work with infinitely many particles, we have to assume that the initial configuration is such that no explosions take place, i.e., such that at any time $t > 0$, the configuration $\eta_t = \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)}$ is a well-defined point configuration. In this paper, however, in order to avoid technicalities, we will restrict to systems with finitely many particles. We denote by $\mathbb{E}_\eta$ the expectation in the configuration process $\{\eta_t, t \geq 0\}$.

For a configuration η we define its associated *n-th factorial measure* by

$$\eta^{(n)} := \sum_{1\leq i_1,\ldots,i_n\leq \mathbf{N}}^{\neq} \delta_{(x_{i_1},\ldots,x_{i_n})}, \tag{2.5}$$

where the superscript $\neq$ means that the summation is over n mutually distinct indexes $i_1,\ldots,i_n$ taken from $\{1,\ldots,\mathbf{N}\}$, with $\mathbf{N}=\eta(E)$. The measure $\eta^{(n)}$ is a point-measure on E^n. Intuitively speaking, $\eta^{(n)}$ corresponds to un-normalized sampling of n different particles out of the configuration η and takes the name *factorial* from the following identity: for any $B\in\mathscr{E}$

$$\eta^{(n)}(B^n) = (\eta(B))_n$$

with $(m)_n := m(m-1)\cdots(m-n+1)$ denoting the n-th falling factorial.

An important object of study is the expectation $\mathbb{E}[\eta^{(n)}]$ that is called the *n-th factorial moment measure.* Here $\mathbb{E}$ refers to the average w.r.t. the randomness of the distribution of points in η. We have that $\mathbb{E}[\eta^{(n)}]$ is a measure on E^n, and, in particular,

$$\mathbb{E}[\eta^{(n)}(B^n)] = \mathbb{E}[(\eta(B))_n]$$

provides the n-th factorial moment of the number of points of $B\in\mathscr{E}$. An important special case is when the points in η are distributed according to a Poisson point process with intensity measure λ: it is well know (see, e.g., [23, (4.11)]) that in this case one has

$$\mathbb{E}[\eta^{(n)}] = \lambda^{\otimes n} \tag{2.6}$$

which is a particular instance of the Mecke's equation.

In the next sections we will study, by duality, the expectation of the n-th factorial measure of the configuration at time t, i.e. $\mathbb{E}_\eta[\eta_t^{(n)}]$, which will be called *the n-th factorial moment measure at time t.*

3 General duality results for independent particles

In this section we review some known duality results for closed (i.e., without reservoirs) systems of independent particles: namely self-duality and duality w.r.t. deterministic systems.

3.1 Intertwining and self-duality

We now recall an intertwining and a self-duality result for independent particles taken from [14]. As already mentioned, in order to avoid technicalities, the results below are stated for finitely many particles. However, whenever the infinitely many particle limit is well-defined, by passing to this limit, the result extends immediately to the infinite case.

The following result (originally stated in [14, Theorem 3.1 and 3.2]) states that the expectation of the n-th factorial moment measure at time t can be expressed in terms of n independent evolutions.

Theorem 3.1. *Let η be a finite point configuration as defined in* (2.3). *Assume that the particles evolve independently according to the Markov process $\{X_t : t \geq 0\}$.*

a) (Intertwining) The following identity holds

$$\mathbb{E}_\eta[\eta_t^{(n)}] = (S_t^{\otimes n})^* \eta^{(n)}, \tag{3.1}$$

where S_t^* is the dual semigroup defined in (2.1).

b) (Self-duality) If $\{X_t : t \geq 0\}$ is strongly reversible with reversible measure m then one can express the density of n-th factorial moment measure $\mathbb{E}_\eta[\eta_t^{(n)}]$ w.r.t. $m^{\otimes n}$ via

$$\frac{\mathrm{d}\mathbb{E}_\eta[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(z_1, \dots, z_n) = \int \prod_{i=1}^n \mathfrak{p}_t(z_i, y_i)\eta^{(n)}(\mathrm{d}(y_1, \dots, y_n)), \tag{3.2}$$

where $\mathfrak{p}_t(\cdot, \cdot)$ is the transition density defined in (2.2).

We provide here a proof of the above result, which relies on generating functions. This generating function approach is well suited to study boundary driven systems in Section 4 below.

Proof. We start from the following identity from [23, Lemma 4.11], for a general finite random point configuration. Let $u : E \to (0, 1)$ then

$$\exp\left(\int \log(1 - u(z))\eta(\mathrm{d}z)\right) =$$
$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int u^{\otimes n}(z_1, \dots, z_n)\eta^{(n)}(\mathrm{d}(z_1, \dots, z_n)). \tag{3.3}$$

We can now use this identity to prove (3.1). Let us adopt the abbreviation $u_t(z) = S_t u(z) = \mathbb{E}_z[u(X_t)]$. Using the independence of the processes $X_t(i)$, $i \in \{1, \dots, N\}$, we compute

$$\mathbb{E}_\eta\left[\exp\left(\int \log(1 - u(z))\eta_t(\mathrm{d}z)\right)\right]$$

$$
\begin{aligned}
&=\mathbb{E}_{x_1,\dots,x_{\mathbf{N}}}\left[\prod_i(1-u(X_t(i)))\right]\\
&=\prod_i \mathbb{E}_{x_i}\left[1-u(X_t(i))\right]=\prod_i(1-u_t(x_i))=\exp\left(\int\log(1-u_t(z))\eta(\mathrm{d}z)\right)\\
&=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int u_t^{\otimes n}(z_1,\dots,z_n)\eta^{(n)}(\mathrm{d}(z_1,\dots,z_n))\\
&=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\dots,z_n)(S_t^{\otimes n})^*\eta^{(n)}(\mathrm{d}(z_1,\dots,z_n)), \qquad (3.4)
\end{aligned}
$$

where we used (3.3) in the fourth identity. On the other hand, using (3.3) once more, we have

$$
\begin{aligned}
&\mathbb{E}_\eta\left[\exp\left(\int\log(1-u(z))\eta_t(\mathrm{d}z)\right)\right]\\
&= 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\dots,z_n)\mathbb{E}_\eta[\eta_t^{(n)}](\mathrm{d}(z_1,\dots,z_n)) \qquad (3.5)
\end{aligned}
$$

and therefore, from (3.5) and (3.4) we conclude

$$
\begin{aligned}
&1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\dots,z_n)\mathbb{E}_\eta[\eta_t^{(n)}](\mathrm{d}(z_1,\dots,z_n))\\
&= 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\dots,z_n)(S_t^{\otimes n})^*\eta^{(n)}(\mathrm{d}(z_1,\dots,z_n)). \qquad (3.6)
\end{aligned}
$$

Because this holds for all u, identifying term by term in the above series and using a standard density argument for symmetric functions (linear combinations of functions of the form $u(z_1)u(z_2)\dots u(z_n)$ are dense in the set of symmetric functions), we obtain (3.1).

If in addition we assume strong reversibility, we then have, for any $f : E^n \to \mathbb{R}$ bounded,

$$
\begin{aligned}
&\int f(z_1,\dots,z_n)\mathbb{E}_\eta[\eta_t^{(n)}](\mathrm{d}(z_1,\dots,z_n))\\
&=\int f(z_1,\dots,z_n)(S_t^{\otimes n})^*\eta^{(n)}(\mathrm{d}(z_1,\dots,z_n))\\
&=\int (S_t^{\otimes n}f)(z_1,\dots,z_n)\eta^{(n)}(\mathrm{d}(z_1,\dots,z_n))\\
&=\int\left(\int f(y_1,\dots,y_n)\prod_{i=1}^n\mathfrak{p}_t(z_i,y_i)m^{\otimes n}(\mathrm{d}(y_1,\dots,y_n))\right)\eta^{(n)}(\mathrm{d}(z_1,\dots,z_n))
\end{aligned}
$$

$$= \int \left(\int f(z_1, \dots, z_n) \prod_{i=1}^{n} \mathfrak{p}_t(z_i, y_i) \eta^{(n)}(\mathrm{d}(y_1, \dots, y_n)) \right) m^{\otimes n}(\mathrm{d}(z_1, \dots, z_n)),$$

from which (3.2) follows. □

Remark 3.2. As noticed in [14, Remark 2.3(iii)], for the system of independent random walks on $\mathbb{Z}^d$ with generator given in (1.1), (3.2) is equivalent to the classic self-duality relation (1.4). Indeed for a singleton $(z_1, \dots, z_n)$, $z_i \in \mathbb{Z}^d$, we have the relation (see [14, Lemma 2.1])

$$\eta^{(n)}(\{(z_1, \dots, z_n)\}) = D^{\mathrm{cl}}(\delta_{z_1} + \ldots + \delta_{z_n}, \eta) \tag{3.7}$$

with D^{cl} defined in (1.2) and thus

$$\mathbb{E}_\eta[\eta_t^{(n)}](\{(z_1, \dots, z_n)\}) = \mathbb{E}_\eta^{\mathrm{IRW}}[D^{\mathrm{cl}}(\delta_{z_1} + \ldots + \delta_{z_n}, \eta_t)]$$

and

$$\int \prod_{i=1}^{n} \mathfrak{p}_t(z_i, y_i)\, \eta^{(n)}(\mathrm{d}(y_1, \dots, y_n)) = \mathbb{E}^{\mathrm{IRW}}_{\delta_{z_1} + \ldots + \delta_{z_n}}[D^{\mathrm{cl}}(\xi_t, \eta)],$$

where ξ_t denotes the configuration of independent random walks at time t starting from $\xi_0 = \delta_{z_1} + \ldots + \delta_{z_n}$.

3.2 Duality w.r.t. the associated deterministic system

The so-called "associated deterministic system" is a dynamical system on functions $f : E \to \mathbb{R}$ which follows the flow of the Kolmogorov backwards equation of the Markov process $\{X_t,\, t \geq 0\}$. More precisely for $f : E \to \mathbb{R}$ we define $f_t(x) = S_t f(x) = \mathbb{E}_x[f(X_t)]$. This flow f_t is the solution of the system of ODE given by

$$\frac{df_t(x)}{dt} = \mathscr{L} f_t(x), \tag{3.8}$$

with $\mathscr{L}$ being the Markov generator associated to $\{S_t, t \geq 0\}$. Notice that, by the Markov semigroup property, $f_t > 0$ when $f > 0$. For $f : E \to (0, \infty)$ and $\mathscr{X} = (x_1, \dots, x_{\mathbf{N}})$, we define the function

$$\mathscr{D}(f, \mathscr{X}) = \prod_{i=1}^{\mathbf{N}} f(x_i) \tag{3.9}$$

or, alternatively, in terms point configurations we set

$$D(f, \eta) := e^{\int \log f \mathrm{d}\eta}.$$

Duality between the configuration process and the deterministic system is then formulated as follows.

Theorem 3.3. *The process* $\{\mathscr{X}_t : t \geq 0\} = \{(X_t(1), \ldots, X_t(\mathbf{N})),\ t \geq 0\}$ *is dual to the deterministic evolution on functions* $f : E \to (0, \infty)$ *defined via* $f_t(x) = S_t f(x) = \mathbb{E}_x[f(X_t)]$*, with duality function* (3.9)*, i.e.,*

$$\mathbb{E}_{\mathscr{X}}\left[\mathcal{D}(f, \mathscr{X}_t)\right] = \mathcal{D}(f_t, \mathscr{X}), \tag{3.10}$$

or, equivalently, in terms of the point configuration process

$$\mathbb{E}_{\eta}\left[D(f, \eta_t)\right] = D(f_t, \eta). \tag{3.11}$$

Proof. The proof is straightforward, indeed by the independence of the particles and by the definition of f_t, we have

$$\mathbb{E}_{\mathscr{X}}\left[\prod_{i=1}^{\mathbf{N}} f(X_t(i))\right] = \prod_{i=1}^{\mathbf{N}} \mathbb{E}_{\mathscr{X}}\left[f(X_t(i))\right] = \prod_{i=1}^{\mathbf{N}} f_t(x_i).$$

□

3.2.1 Doob's theorem

Let us now consider the connection between the duality result of Section 3.2 with the time evolution of Poisson point processes. It is well-known that independent Markovian particle evolutions preserve Poisson processes: we refer to this result as Doob's theorem but it can also be viewed as a consequence of the random displacement theorem (see, e.g., [23]).

We briefly recall the definition of a Poisson point process. For a σ-finite measure μ on $(E, \mathscr{E})$ the Poisson point process with intensity measure μ is defined as the random point configuration $\eta = \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$, defined on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ such that

1. For every $\omega \in \Omega$, the map $\mathscr{E} \ni A \to \eta(\omega, A)$ is a $\mathbb{N}$-valued measure on the σ-algebra $\mathscr{E}$.

2. For $A_1, \ldots A_n \in \mathscr{E}$, n disjoint measurable subsets of E, $\{\eta(A_i), i = 1, \ldots, n\}$ are independent Poisson random variables with parameter $m_i = \int_{A_i} \mu(\mathrm{d}z)$.

See [23] for background on Poisson point processes. We denote by $\mathscr{P}_\mu$ the law of the Poisson point process with intensity measure μ. We recall the reader that a Poisson point process is uniquely characterized by its Laplace functional, i.e., by

$$\int \left(e^{\int f(z)\eta(\mathrm{d}z)}\right) \mathscr{P}_\mu(\mathrm{d}\eta) = e^{\int (e^{f(z)}-1)\mu(\mathrm{d}z)} \tag{3.12}$$

for all f for which the integral $\int (e^{f(z)} - 1)\mu(\mathrm{d}z)$ is finite.

Often we will consider the setting in which the measure μ is absolutely continuous with respect to a σ-finite *reference* measure m on $(E, \mathscr{E})$, with density $\rho : E \to [0, +\infty)$, i.e. $\mu(dz) = \rho(z)m(dz)$. In such a setting, with a small abuse of notation, we will denote by $\mathscr{P}_\rho$ the law of the Poisson point processes with intensity measure $\rho(z)m(dz)$. We denote by $\mathbb{E}_{\mathscr{P}_\rho}$ the expectation of the process of independent particles moving according to the Markovian dynamics corresponding to the semigroup S_t whose associated point configuration is initially distributed as $\mathscr{P}_\rho$.

The following result then proves Doob's theorem via the duality (3.10).

Theorem 3.4. *Let* $\mathbf{N}$ *be a random variable on* $\mathbb{N}$ *and let* $\{\mathscr{X}_t, t \geq 0\} = \{(X_t(1), \ldots, X_t(\mathbf{N})), t \geq 0\}$ *be a system of independent particles initialized at time zero from a Poisson point process with intensity measure* μ*. Then the distribution of the* $\mathbf{N}$ *particles at time* $t \geq 0$*, namely the random point configuration* $\sum_i \delta_{X_t(i)}$*, is a Poisson point process with intensity measure* $\mu_t = S_t^* \mu$*, where* S_t^* *denotes the dual semigroup introduced in* (2.1).

In particular, if $\mu(dz) = \rho(z)m(\mathrm{d}z)$*, where* m *is a reversible measure of the Markov process* $\{X_t : t \geq 0\}$*, then* $\mu_t(dz) = \rho_t(z)m(\mathrm{d}z)$ *with*

$$\rho_t(z) = \mathbb{E}_z[\rho(X_t)] = S_t\rho(z).$$

If m *is a stationary measure of* $\{X_t : t \geq 0\}$*,* $\rho \in L^2(m)$ *and* $\mu(dz) = \rho(z)m(\mathrm{d}z)$ *then* $\mu_t(dz) = \rho_t(z)m(\mathrm{d}z)$ *with*

$$\rho_t = S_t^\dagger \rho,$$

and $S_t^\dagger$ *denotes the adjoint semigroup of* S_t *in* $L^2(m)$.

Proof. Using (3.11) and (3.12), we obtain

$$\begin{aligned}
\int \mathbb{E}_\eta \left(e^{\int \log f(z) \eta_t(\mathrm{d}z)} \right) \mathscr{P}_\mu(\mathrm{d}\eta) &= \int \mathbb{E}\left(D(f, \eta_t)\right) \mathscr{P}_\mu(\mathrm{d}\eta) \quad (3.13)\\
= \quad & \int e^{\int \log f_t(z) \eta(\mathrm{d}z)} \mathscr{P}_\mu(\mathrm{d}\eta) = e^{\int (S_t f - 1) d\mu} \\
= \quad & e^{\int (S_t(f-1)) d\mu} = e^{\int (f-1) d S_t^* \mu}.
\end{aligned}$$

From this we infer that η_t is again a Poisson point process with intensity $\mu_t = S_t^* \mu$. In particular, if $\mu(dz) = \rho(z)m(dz)$ and m is reversible then $S_t^* \mu(dz) = (S_t \rho)(z) m(dz)$. More generally, if $\mu(dz) = \rho(z)m(dz)$ and m is stationary then $S_t^* \mu(dz) = (S_t^\dagger \rho)(z) m(dz)$. ☐

Corollary 3.5. *In the setting of Theorem 3.4, the Poisson point processes with intensity measure* $c \cdot m(dz)$ *parametrized by a constant* $c > 0$ *are reversible for* $\{\mathscr{X}_t, t \geq 0\}$*. More generally, if* m *is a stationary measure of* $\{X_t : t \geq 0\}$*, the Poisson point process with intensity* $\rho(z)m(dz)$ *is stationary if and only if*

$$S_t^\dagger \rho = \rho.$$

Proof. When c is positive constant and $\mu = cm$ we have, using (3.11) and (3.12),

$$\begin{aligned}&\mathbb{E}_{\mathscr{P}_\mu}\left(\mathbb{E}_\eta\left[e^{\int \log f \mathrm{d}\eta_t}\right] e^{\int \log g \mathrm{d}\eta}\right) = \mathbb{E}_{\mathscr{P}_\mu}\left(e^{\int \log S_t f \mathrm{d}\eta} e^{\int \log g \mathrm{d}\eta}\right)\\ =&\mathbb{E}_{\mathscr{P}_\mu}\left(e^{\int \log((S_t f) g) \mathrm{d}\eta}\right) = e^{c\int ((S_t f)g-1)\mathrm{d}m}\end{aligned}$$

and using the self-adjointness of S_t we obtain

$$e^{c\int ((S_t f)g-1)\mathrm{d}m} = e^{c\int ((S_t g)f-1)\mathrm{d}m} = \mathbb{E}_{\mathscr{P}_\mu}\left(\mathbb{E}_\eta\left[e^{\int \log g \mathrm{d}\eta_t}\right] e^{\int \log f \mathrm{d}\eta}\right)$$

which implies reversibility of $\mathscr{P}_\mu$.

The second statement follows immediately via Theorem 3.4. □

4 Duality for boundary driven systems of independent particles

In this section we will present a duality result for boundary driven systems of independent particles which generalizes the duality relation (1.7), valid only in the discrete setting, that has been previously obtained in [7], [15]. In Section 4.1, we recall the definition of the *boundary driven Brownian gas* recently introduced in [5] and we state a duality result in this context (see Theorem 4.1 below). In Section 4.2 we consider more general systems of independent diffusion processes on regular domains $\mathfrak{D} \subset \mathbb{R}^d$. We prove first an intertwining result and secondly a duality result under an extra assumption on the transition probabilities of a single particle. In Section 4.3 we introduce a further generalization of the construction of Bertini and Posta in [5], namely boundary driven Markov processes with jumps, which can exit the domain without hitting its boundary.

4.1 The boundary driven Brownian gas on $[0,1]$: definition and duality

Let $E = [0,1]$ and denote by $\{W_t, t \geq 0\}$ a standard Brownian motion absorbed upon hitting 0 or 1. Let us denote by τ_0, τ_1 the hitting times of 0, resp. 1, of $\{W_t, t \geq 0\}$. We denote by $\mathbb{P}_x^{\text{abs}}$ and by S_t respectively the distribution of the trajectories of $\{W_t, t \geq 0\}$ starting from $x \in [0,1]$ and the semigroup of the process. It is well known that the transition probability $p_t(\cdot,\cdot) : [0,1] \times \mathscr{B}([0,1]) \to [0,1]$ of the absorbed Brownian motion satisfies

$$p_t(x, \mathrm{d}y) = \mathfrak{p}_t(x,y)\,\mathrm{d}y \qquad \forall x, y \in (0,1) \tag{4.1}$$

with $\mathfrak{p}_t(x,y)=\mathfrak{p}_t(y,x)$ a symmetric function referred as transition density (see, e.g., [6, p. 122] for an explicit formula of $\mathfrak{p}_t(x,y)$). With a slight abuse of notation we denote by $\mathfrak{p}_t(x,0)$ (respectively $\mathfrak{p}_t(x,1)$) the probability, starting from $x\in[0,1]$, of being absorbed at 0 (resp. at 1) by the time $t\geq 0$. We then have, for any $x\in(0,1)$,

$$\int_0^1 \mathfrak{p}_t(x,y)\,\mathrm{d}y+\mathfrak{p}_t(x,0)+\mathfrak{p}_t(x,1)=1 \tag{4.2}$$

and for any $f:[0,1]\to\mathbb{R}$ bounded

$$S_t f(x)=\int_0^1 \mathfrak{p}_t(x,y)f(y)\,\mathrm{d}y+f(0)\mathfrak{p}_t(x,0)+f(1)\mathfrak{p}_t(x,1).$$

For $\xi:=\sum_{i=1}^{\mathbf{N}}\delta_{x_i}$, $x_i\in(0,1)$ and $\mathbf{N}\in\mathbb{N}$, we then consider the point configuration (on $[0,1]$) valued Markov process given by

$$\begin{cases}\xi_t:=\sum_{i=1}^{\mathbf{N}}\delta_{W_t(i)}\ ,\\ \xi_0=\xi\end{cases}$$

where $\{W_t(i)\}_{t\geq 0}$ are independent copies of $\{W_t\}_{t\geq 0}$ such that $W_0(i)=x_i$ for any $i\in[\mathbf{N}]$. The transition function $P_t(\xi,\cdot)$ of the process $\{\xi_t,t\geq 0\}$ is then given by the image of $\otimes_{i=1}^{\mathbf{N}}p_t(x_i,\cdot)$ under the mapping $(x_i)_{i=1}^{\mathbf{N}}\to\sum_{i=1}^{\mathbf{N}}\delta_{x_i}$. For $\boldsymbol{x}=(x_1,\ldots,x_{\mathbf{N}})\in(0,1)^{\mathbf{N}}$, we denote by $\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{x}}$ the expectation in the process $\{\xi_t,t\geq 0\}$ starting from $\xi_0=\sum_{i=0}^{N}\delta_{x_i}$. Finally, let Θ_t be a Poisson point configuration on $(0,1)$ with time dependent intensity $\lambda_t(dx)$ given by

$$\lambda_t(\mathrm{d}x)=\lambda(t,x)\,\mathrm{d}x \tag{4.3}$$

and

$$\begin{aligned}\lambda(t,x)&=\lambda_L\mathbb{P}^{\mathrm{abs}}_x(\tau_0\leq t)+\lambda_R\mathbb{P}^{\mathrm{abs}}_x(\tau_1\leq t)\\&=\lambda_L\mathfrak{p}_t(x,0)+\lambda_R\mathfrak{p}_t(x,1)\end{aligned} \tag{4.4}$$

for some $\lambda=(\lambda_L,\lambda_R)\in\mathbb{R}^2_+$. Moreover $\{\xi_t,\,t\geq 0\}$ and $\{\Theta_t,\,t\geq 0\}$ are independent. The process $\{\Theta_t,\,t\geq 0\}$, by adding particles in the bulk $(0,1)$, models in turn the effect of the reservoirs at 0 and 1 (cf. [5, (2.1), (2.2)]). The *boundary driven Brownian gas* is then defined, for any $t>0$, by

$$\eta_t=\xi_t\big|_{(0,1)}+\Theta_t \tag{4.5}$$

viewed as a point configuration on $(0,1)$ and such that $\eta_0=\xi_0\big|_{(0,1)}$. Here $\xi_t\big|_{(0,1)}$ denotes the restriction of the point configuration ξ_t to $(0,1)$.

The motivation for this definition can be found in [5]. In Section 5.2 below we will show how the boundary driven Brownian gas arises as a scaling limit of the reservoir process on a chain $\{1,\dots,N\}$ defined in (1.5).

Let us recall that $\eta^{(n)}$ denotes the n-th factorial measure corresponding to the initial configuration $\eta_0=\eta$ made of $\mathbf{N}$ particles, and $\eta_t^{(n)}$ denotes the n-th factorial measure corresponding to η_t, i.e. the configuration at time t with $\mathbf{N}_t$ particles. We denote by $\mathbb{E}_\eta^\lambda$ the expectation in the process defined via (4.5) initialized from η. We will use the following abbbreviations: $\boldsymbol{x}=(x_1,\dots,x_n)$, for $I=\{i_1,\dots,i_k\}\subset\{1,\dots,n\}$ we put $\boldsymbol{x}_I=(x_{i_1},\dots,x_{i_k})$ and we write $[k]$ for $\{1,\dots,k\}$. We shall also use the following shorthand for the transition density in the rest of the paper

$$\mathfrak{p}_t^{(n)}(\boldsymbol{x},\boldsymbol{y})=\prod_{i=1}^n\mathfrak{p}_t(x_i,y_i). \tag{4.6}$$

For the boundary driven Brownian gas the following duality result holds, where the dual process is a system of independent absorbed Brownian motions.

Theorem 4.1. *For the boundary driven Brownian gas $\{\eta_t,t\ge 0\}$, the n-th factorial moment measure at time $t>0$ is absolutely continuous w.r.t. $m^{\otimes n}$, with m denoting the Lebesgue measure on $(0,1)$ with the following density:*

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}_\eta^\lambda[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(\boldsymbol{z}) \quad &= \quad \sum_{I\subset[n]}\mathbb{E}_{\boldsymbol{z}_I}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,1\})=|I|\}}\right]\cdot\\ &\cdot\int_{(0,1)^{n-|I|}}\mathfrak{p}_t^{(n-|I|)}(\boldsymbol{z}_{[n]\setminus I},\boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}).\end{aligned} \tag{4.7}$$

This result can be read as a duality relation in the spirit of (3.2): in order to know the n-th order factorial moment measure at time $t>0$, one has to follow (not more than) n dual particles. However, due to the presence of reservoirs, we have factors

$$\mathbb{E}_{\boldsymbol{z}_I}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\xi_t(\{0,1\})=|I|}\right]$$

which can be considered as corresponding to $|I|$ "absorbed" dual particles. This result has to be compared with the discrete setting, namely (1.8), where an analogous term multiplying the product of falling factorial polynomials appears in the duality function and the process with reservoirs is dual to an absorbing process with two extra sites associated to the reservoirs (see [7], [15] and [13]).

In the next subsection we state and prove a more general version of Theorem 4.1, which applies to independent diffusions on regular domains $\mathfrak{D} \subset \mathbb{R}^d$ and includes also an intertwining result. (4.7) for the boundary driven Brownian gas on $(0,1)$ will then follow as a particular case of Theorem 4.2.

4.2 Boundary driven diffusion processes: definition and duality

Let $\mathfrak{D}$ be a regular domain of $\mathbb{R}^d$, where by regular domain we mean an open, simply connected and bounded subset $\mathfrak{D} \subset \mathbb{R}^d$ such that its boundary $\partial\mathfrak{D}$ is Lipschitz. Let $\{Y_t, t \geq 0\}$ be the diffusion process on $\mathbb{R}^d$ with generator

$$\mathscr{L} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} \tag{4.8}$$

with regular coefficient functions $a_{i,j}$, b_j and with $a = (a_{i,j})$ symmetric, non-degenerate and positive definite. We then denote by $\{X_t, t \geq 0\}$ the Markov process on $\bar{\mathfrak{D}} = \mathfrak{D} \cup \partial\mathfrak{D}$ with semigroup $\{S_t, t \geq 0\}$, evolving as $\{Y_t, t \geq 0\}$ on $\mathfrak{D}$ and absorbed upon hitting $\partial\mathfrak{D}$. More specifically the regularity assumptions on the coefficients are that $\frac{\partial^2 a_{i,j}}{\partial_{x_i}\partial_{x_\ell}}$ and $\frac{\partial b_i}{\partial_{x_j}}$ are locally uniformly Hölder continuous on $\mathfrak{D} \cup \partial\mathfrak{D}$ (see, e.g., [20]). Denote by $\mathbb{P}_x^{\text{abs}}$ (resp. $\mathbb{E}_x^{\text{abs}}$) the distribution (resp. the expectation) of the trajectories of $\{X_t, t \geq 0\}$ starting from $x \in \mathfrak{D}$. We then assume that

$$\mathbb{P}_x^{\text{abs}}\left(\tau_{\partial\mathfrak{D}} < \infty\right) = 1, \quad \forall x \in \mathfrak{D}, \tag{4.9}$$

where $\tau_{\partial\mathfrak{D}}$ denotes the hitting time of $\partial\mathfrak{D}$.

For $\xi := \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$, $x_i \in \mathfrak{D}$, we consider the point configuration (on $\bar{\mathfrak{D}}$) valued Markov process given by

$$\begin{cases} \xi_t := \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)} \ , \\ \xi_0 = \xi \ , \end{cases}$$

where $\{X_t(i)\}_{t\geq 0}$ are independent copies of $\{X_t\}_{t\geq 0}$ such that $X_0(i) = x_i$ for any $i \in [\mathbf{N}]$. For $\boldsymbol{x} = (x_1, \ldots, x_{\mathbf{N}}) \in \mathfrak{D}^{\mathbf{N}}$, we denote by $\mathbb{E}_{\boldsymbol{x}}^{\text{abs}}$ the expectation in the process $\{\xi_t, t \geq 0\}$ starting from $\xi_0 = \sum_{i=0}^{N} \delta_{x_i}$.

Let μ be a finite measure on $\mathfrak{D}$ and let $\lambda : \partial\mathfrak{D} \to \mathbb{R}_+$ be a bounded measurable function modelling the reservoir intensity at any $x \in \partial\mathfrak{D}$. If λ satisfies the just mentioned assumptions it is said to be regular. Finally we define Θ_t the Poisson point process on $\mathfrak{D}$ with time dependent intensity

$\lambda_t(\mathrm{d}x)$ given by

$$\lambda_t(\mathrm{d}x) = \left(\int_{\partial\mathfrak{D}} \lambda(y)\, \mathbb{P}_x(\tau_{\partial\mathfrak{D}} \le t, X_{\tau_{\partial\mathfrak{D}}} \in \mathrm{d}y) \right) \mu(\mathrm{d}x). \tag{4.10}$$

Notice that the above definition of reservoir density is a generalization of (4.4). The *boundary driven diffusion gas in the domain $\mathfrak{D}$ with reservoir intensity λ and a priori measure μ*, denoted by $\{\eta_t, t \ge 0\}$, is then given, for any $t \ge 0$, by

$$\begin{cases} \eta_t = \xi_t\big|_{\mathfrak{D}} + \Theta_t \ , \\ \eta_0 = \xi_0 = \sum_{i=1}^N \delta_{x_i}, \ x_i \in \mathfrak{D} \end{cases} \tag{4.11}$$

viewed as a point configuration on $\mathfrak{D}$, where $\xi_t\big|_{\mathfrak{D}}$ is the restriction of ξ_t to $\mathfrak{D}$.

We denote by $\mathbb{E}_\eta^\lambda$ the expectation in the process defined via (4.11) initialized from η. Following the strategy of [5], it can be shown that $\{\eta_t,\, t \ge 0\}$ is a Markov process when the transition probability $p_t(x, \mathrm{d}y)$ of the process $\{X_t, t \ge 0\}$ satisfies

$$p_t(x, \mathrm{d}y) m(\mathrm{d}x) = p_t(y, \mathrm{d}x) m(\mathrm{d}y) \qquad \text{on} \qquad \mathfrak{D} \times \mathfrak{D} \tag{4.12}$$

for a finite measure m and for the reservoir intensity in (4.10) we choose $\mu = m$. We refer to Theorem A in the Appendix for further details.

We are now ready to state the main results of this section, namely a general intertwining relation for the factorial moment measure at time $t > 0$ of the boundary driven diffusion processes on a d-dimensional regular domain $\mathfrak{D}$ and a duality result, under an extra symmetry assumption on the transition probability of $\{X_t, t \ge 0\}$ (see (4.14) below), generalizing Theorem 4.1.

Theorem 4.2. *Let $\{\eta_t, t \ge 0\}$ be the boundary driven diffusion gas defined in* (4.11)*. Then for all $n \in \mathbb{N}$ and $t \ge 0$, it holds:*

a) for all bounded, measurable and permutation invariant $f : \mathfrak{D}^n \to \mathbb{R}$

$$\begin{aligned} &\mathbb{E}_\eta^\lambda \left[\int_{\mathfrak{D}^n} f(\boldsymbol{z}) \eta_t^{(n)}(\mathrm{d}\boldsymbol{z}) \right] \\ &= \sum_{k=0}^n \binom{n}{k} \int_{\mathfrak{D}^n} f(\boldsymbol{z}) \lambda_t^{\otimes k}(\mathrm{d}\boldsymbol{z}_{[k]}) \otimes (S_t^{\otimes n-k})^* \eta^{(n-k)}(\mathrm{d}\boldsymbol{z}_{[n]\setminus[k]}); \end{aligned} \tag{4.13}$$

b) assume further that the transition probability of $\{X_t, t \ge 0\}$ satisfies

$$p_t(x, \mathrm{d}y) = \mathfrak{p}_t(x, y) m(\mathrm{d}y) \tag{4.14}$$

for a symmetric function $\mathfrak{p}_t(x,y)$ and a finite measure m on $\mathfrak{D}$. Then, choosing $\mu = m$, the following holds

$$\frac{\mathrm{d}\mathbb{E}^{\lambda}_{\eta}[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(\boldsymbol{z}) = \sum_{I\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_I}\left[e^{\int_{\partial\mathfrak{D}} \log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=|I|\}}\right] \cdot \int_{\mathfrak{D}^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(\boldsymbol{z}_{[n]\setminus I},\boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}) \tag{4.15}$$

Remarks 4.3.

i) Notice that, if $\lambda(x)\in\{\lambda_L,\lambda_R\}$ for any $x\in\partial\mathfrak{D}$ and it is regular (as defined above), setting $\partial\mathfrak{D}_L=\{x\in\partial\mathfrak{D}:\ \lambda(x)=\lambda_L\}$, we then have

$$\frac{\mathrm{d}\mathbb{E}^{\lambda}_{\eta}[\eta^{(n)}]}{\mathrm{d}m^{\otimes n}}(\boldsymbol{z}) = \sum_{I\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_I}\left[\lambda_L^{\xi_t(\partial\mathfrak{D}_L)}\lambda_R^{\xi_t(\partial\mathfrak{D}\setminus\partial\mathfrak{D}_L)}\mathbf{1}_{\{\xi_t(\partial E)=|I|\}}\right] \times \int_{\mathfrak{D}^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(\boldsymbol{z}_{[n]\setminus I},\boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}), \tag{4.16}$$

which is the multidimensional analogue of (4.7) when there are two possible values for the reservoir intensity.

ii) The multidimensional Brownian motion satisfies (4.14) with m given by the Lebesgue measure (see, e.g. [3, Theorem 4.4]): thus the multidimensional boundary driven Brownian gas satisfies (4.15).

iii) In one dimension, all diffusion processes satisfy (4.14) (see, e.g. [6, pag.13]). In particular, consider the diffusion process on $\mathbb{R}$ with generator

$$\mathscr{L}f(y) = \frac{1}{2}\,\sigma^2(y)\,\frac{\mathrm{d}^2 f}{\mathrm{d}y^2}(y) + b(y)\,\frac{\mathrm{d}f}{\mathrm{d}y}(y), \tag{4.17}$$

where the drift b and the diffusivity σ are continuous functions and with $\sigma^2(x)\geq\delta>0$ for each $x\in(0,1)$. Then (4.14) holds with m given by

$$m(\mathrm{d}x) = \frac{1}{\sigma^2(x)}\ \exp\left(2\int_{x_0}^{x}\frac{b(y)}{\sigma^2(y)}\,\mathrm{d}y\right)\mathrm{d}x\,, \tag{4.18}$$

for an arbitrary $x_0\in(0,1)$ (see, e.g. [6, pag.17]).

iv) We refer to [20] for conditions on the coefficients $a_{i,j}$ and b_i in (4.8) ensuring that (4.14) holds.

Proof. Coherently with what we have done in Section 3.1, we provide a proof relying on generating functions which uses the identity (3.3). Let $u : \mathfrak{D} \to \mathbb{R}$ bounded and measurable. By the independence of Θ_t and ξ_t we have

$$
\begin{aligned}
&\mathbb{E}_\eta^\lambda\left[\exp\left(\int_{\mathfrak{D}} \log(1-u(z))\eta_t(\mathrm{d}z)\right)\right]\\
=&\mathbb{E}\left[\exp\left(\int_{\mathfrak{D}} \log(1-u(z))\Theta_t(\mathrm{d}z)\right)\right]\mathbb{E}_\eta^{\mathrm{abs}}\left[\exp\left(\int_{\mathfrak{D}} \log(1-u(z))\xi_t\big|_{\mathfrak{D}}(\mathrm{d}z)\right)\right],
\end{aligned}
$$

where $\mathbb{E}$ denotes the expectation in the Poisson point process Θ_t. Notice in particular that

$$
\mathbb{E}_\eta\left[\exp\left(\textstyle\int_{\mathfrak{D}} \log(1-u(z))\xi_t\big|_{\mathfrak{D}}(\mathrm{d}z)\right)\right] = \mathbb{E}_\eta\left[\exp\left(\textstyle\int_{\mathfrak{D}} \log(1-u(z))\xi_t(\mathrm{d}z)\right)\right]
$$

and that for $\{\xi_t, t \geq 0\}$ Theorem 3.1 applies.

Using (3.3) combined with (2.6) and (3.1), we obtain

$$
\begin{aligned}
&\mathbb{E}_\eta^\lambda\left[\exp\left(\int \log(1-u(z))\eta_t(\mathrm{d}z)\right)\right]\\
=&\left(1+\sum_{n=1}^\infty \frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\dots,z_n)\lambda_t^{\otimes n}(\mathrm{d}(z_1\dots z_n))\right)\\
&\quad\times\left(1+\sum_{n=1}^\infty \frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\dots,z_n)(S_t^{\otimes n})^*\eta^{(n)}(\mathrm{d}(z_1,\dots,z_n))\right)\\
=&1+\sum_{k,l}\frac{(-1)^{k+l}}{k!\,l!}\int u^{\otimes(k+l)}(z_1,\dots,z_{k+l})\lambda_t^{\otimes k}(\mathrm{d}(z_1,\dots,z_k))\\
&\qquad\otimes(S_t^{\otimes l})^*\eta^{(l)}(\mathrm{d}(z_{k+1},\dots,z_{k+l}))\,.
\end{aligned}
\tag{4.19}
$$

On the other hand we have

$$
\begin{aligned}
&\mathbb{E}_\eta^\lambda\left[\exp\left(\int \log(1-u(z))\eta_t(\mathrm{d}z)\right)\right]\\
=&1+\sum_{n=1}^\infty \frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\dots,z_n)\mathbb{E}_\eta^\lambda[\eta_t^{(n)}](\mathrm{d}(z_1,\dots,z_n)).
\end{aligned}
$$

Then, via identification of the terms with n-fold tensor product of u in the last expression in (4.19) and the right hand side of the above identity, we obtain the following equality for all $n \in \mathbb{N}$:

$$
\begin{aligned}
&\int u^{\otimes n}(z_1,\dots,z_n)\mathbb{E}^{\lambda}_{\eta}[\eta^{(n)}_t](\mathrm{d}(z_1,\dots,z_n))\\
&=\sum_{k=0}^{n}\binom{n}{k}\int u^{\otimes n}(z_1,\dots,z_n)\lambda_t^{\otimes k}(\mathrm{d}(z_1,\dots,z_k))\\
&\qquad\qquad\qquad\qquad \otimes (S_t^{\otimes(n-k)})^{*}\eta^{(n-k)}(\mathrm{d}(z_{k+1},\dots,z_n))\,.
\end{aligned}
$$

Via the above mentioned density argument of linear combinations of $u^{\otimes n}$ this implies (4.13).

Recalling the definition of $\lambda_t(\mathrm{d}z)$ we have that

$$
\begin{aligned}
&\lambda_t^{\otimes k}(\mathrm{d}(z_1,\dots,z_k))\\
&=\left(\prod_{i=1}^{k}\int_{\partial\mathfrak{D}}\lambda(u)\mathbb{P}^{\mathrm{abs}}_{z_i}(\tau_{\partial\mathfrak{D}}\le t, X_{\tau_{\partial\mathfrak{D}}}\in \mathrm{d}u)\right)\mu^{\otimes k}(\mathrm{d}(z_1,\dots,z_k))\\
&=\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_{[k]}}\left[e^{\int_{\partial\mathfrak{D}}\log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}}\right]\mu^{\otimes k}(\mathrm{d}(z_1,\dots,z_k)).
\end{aligned}
$$

If now we assume that (4.14) holds and choosing $\mu=m$, we obtain, integrating a bounded and permutation invariant function $f_n:\mathfrak{D}^n\to\mathbb{R}$

$$
\begin{aligned}
&\mathbb{E}^{\lambda}_{\eta}\left[\int_{\mathfrak{D}^n} f_n(z_1,\dots,z_n)\eta^{(n)}_t(\mathrm{d}(z_1,\dots,z_n))\right]\\
&=\sum_{k=0}^{n}\binom{n}{k}\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_{[k]}}\left[e^{\int_{\partial\mathfrak{D}}\log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}}\right]\\
&\qquad\qquad\times\int_{\mathfrak{D}^n} f_n(\boldsymbol{z})m^{\otimes k}(\mathrm{d}\boldsymbol{z}_{[k]})\otimes(S_t^{\otimes(n-k)})^{*}\eta^{(n-k)}(\mathrm{d}\boldsymbol{z}_{[n]\setminus[k]})\\
&=\sum_{k=0}^{n}\binom{n}{k}\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_{[k]}}\left[e^{\int_{\partial\mathfrak{D}}\log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}}\right]\\
&\quad\times\int_{\mathfrak{D}^n}\left(\int_{\mathfrak{D}^{n-k}} f_n(z_1,\dots,z_k,y_1,\dots,y_{n-k})\,\mathfrak{p}_t^{(n-k)}(\boldsymbol{z}_{[n]\setminus[k]},\boldsymbol{y})m^{\otimes n-k}(\mathrm{d}\boldsymbol{y})\right)\\
&\qquad\qquad\qquad\times m^{\otimes k}(\mathrm{d}\boldsymbol{z}_{[k]})\otimes\eta^{(n-k)}(\mathrm{d}\boldsymbol{z}_{[n]\setminus[k]}),
\end{aligned}
$$

where we recall that $\mathfrak{p}_t^{(r)}((v_1,\dots,v_r),(u_1,\dots,u_r))=\prod_i^r\mathfrak{p}_t(v_i,u_i)$. Exchanging the integrals and using the the symmetry of the functions $\mathfrak{p}_t(\cdot,\cdot)$ leads to

$$
\begin{aligned}
&\mathbb{E}^{\lambda}_{\eta}\left[\int_{\mathfrak{D}^n} f_n(z_1,\dots,z_n)\eta^{(n)}_t(\mathrm{d}(z_1,\dots,z_n))\right]\\
&=\sum_{k=0}^{n}\binom{n}{k}\int_{\mathfrak{D}^n} m^{\otimes k}(\mathrm{d}\boldsymbol{z}_{[k]})\otimes m^{\otimes n-k}(\mathrm{d}\boldsymbol{y})\,f_n(z_1,\dots,z_k,y_1,\dots,y_{n-k})
\end{aligned}
$$

$$\times\left(\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_{[k]}}\left[e^{\int_{\partial\mathfrak{D}}\log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}}\right]\right.$$
$$\left.\cdot\int_{\mathfrak{D}^{n-k}}\mathfrak{p}_t^{(n-k)}(\boldsymbol{y},\boldsymbol{z}_{[n]\setminus[k]})\,\eta^{(n-k)}(\mathrm{d}\boldsymbol{z}_{[n]\setminus[k]})\right)$$

which, upon renaming the variables, can be rewritten as

$$\mathbb{E}^{\lambda}_{\eta}\left[\int_{\mathfrak{D}^n} f_n\mathrm{d}\eta_t^{(n)}\right]=\int_{\mathfrak{D}^n} f_n(\boldsymbol{z})\left(\sum_{k=0}^{n}\binom{n}{k}\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_{[k]}}\left[e^{\int_{\partial\mathfrak{D}}\log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}}\right]\right.$$
$$\left.\times\int_{\mathfrak{D}^{n-k}}\mathfrak{p}_t^{(n-k)}(\boldsymbol{z}_{[n]\setminus[k]},\boldsymbol{y})\,\eta^{(n-k)}(\mathrm{d}\boldsymbol{y})\right)m^{\otimes n}(\mathrm{d}\boldsymbol{z}).$$

By taking the symmetrization of the above expression in brackets in the right hand side we obtain (4.15) and the proof is concluded . □

We conclude this section by looking at the evolution of a Poisson distributed particle cloud and by using duality to show the existence and the uniqueness of the stationary distribution for the system of boundary driven independent particles. Let ν be a finite measure on $\mathfrak{D}$ and denote by $\mathscr{P}_\nu$ the Poisson point process with intensity ν.

Theorem 4.4. *Let $\{\eta_t,\, t\geq 0\}$ be the boundary driven diffusion gas in the domain $\mathfrak{D}$ defined in* (4.11) *and let $\mu(dx)$ be the finite measure on $\mathfrak{D}$ appearing in* (4.10).

i) If η_0 is distributed according to $\mathscr{P}_\nu$, then η_t is the restriction to $\mathfrak{D}$ of the Poisson process on $\bar{\mathfrak{D}}$ with intensity

$$S_t^*\nu+\lambda_t, \tag{4.20}$$

with λ_t defined in (4.10).

ii) Assume further (4.12) *and take $\mu=m$. Then, the unique stationary measure for the boundary driven diffusion process is given by the distribution of a Poisson point process with intensity*

$$\lambda_\infty(\mathrm{d}x)=h(x)m(\mathrm{d}x),$$

where

$$h(x)=\int_{\partial\mathfrak{D}}\lambda(u)\mathbb{P}_x^{\mathrm{abs}}(X_{\tau_{\partial\mathfrak{D}}}\in\mathrm{d}u).$$

Moreover, for any initial configuration η, the distribution of η_t converges weakly as $t\to\infty$ to the distribution of the Poisson point process with intensity $\lambda_\infty(\mathrm{d}x)=h(x)m(\mathrm{d}x)$.

Remark 4.5. Notice that, when λ is a continuous function, $h : \mathfrak{D} \to \mathbb{R}$ given in Theorem 4.4(ii) is the solution of the following Dirichlet problem

$$\begin{cases} \mathscr{L} h = 0 & \text{in } \mathfrak{D} \\ h = \lambda & \text{on } \partial\mathfrak{D} \end{cases} \tag{4.21}$$

where $\mathscr{L}$ is the generator given in (4.17). In particular, for the one-dimensional boundary driven Brownian gas, $\mathscr{L} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and

$$h(x) = \lambda_L(1-x) + \lambda_R x.$$

Proof. By Doob's theorem (Theorem 3.4), we know that the evolution of $\mathscr{P}_\nu$ under independent copies of absorbed particles $\{X_t, t \geq 0\}$ is equal to a Poisson point process with intensity $S_t^* \nu$. Therefore (4.20) follows from the fact that the independent sum of two Poisson point processes is a Poisson point process with intensity measure the sum of the intensity measures. Further, notice that for every finite measure μ on $\mathfrak{D}$, we have

$$(S_t^{\otimes n})^* \mu^{\otimes n} \longrightarrow 0$$

as $t \to \infty$ because eventually all the mass from μ will be absorbed at the boundary $\partial\mathfrak{D}$. Therefore, by taking the limit $t \to \infty$ in (4.13), only the term $k = n$ survives and thus, the n-th factorial moment measures converge to $\lambda_\infty^{\otimes n}(\mathrm{d}\boldsymbol{x}) = \left(\prod_{i=1}^n h(x_i)\right) \mu^{\otimes n}(\mathrm{d}\boldsymbol{x})$ with

$$h(x) = \int_{\partial\mathfrak{D}} \lambda(u) \mathbb{P}_x^{\mathrm{abs}}(X_{\tau_{\partial\mathfrak{D}}} \in \mathrm{d}u).$$

This shows that the limiting distribution of η_t is indeed Poisson with intensity measure λ_∞. Since (4.12) implies that $\{\eta_t, t \geq 0\}$ is Markov, we conclude that the distribution of the Poisson point process with intensity λ_∞ is the unique stationary measure. □

4.3 Boundary driven Markov gas

In this section we provide another extension of the construction of Bertini and Posta [5] for systems of particles that can make jumps and thus, they do not necessarily hit the boundary when exiting a regular domain. Therefore, instead of associating a reservoir parameter function λ to the boundary of the domain only, we need to associate it rather to the complement of the domain. We therefore consider particles that evolve on a regular domain and are absorbed upon hitting a point in the complement of this domain. The examples that we have in mind are jump Markov processes (see, e.g., [22, Eq. 4]) with generator given by

$$\mathscr{L} f(x) = \int_{\mathbb{R}^d} a(x-y)\,[f(y) - f(x)]\;\mathrm{d}y\,, \tag{4.22}$$

with $a(-x) = a(x)$ and $f : \mathbb{R}^d \to \mathbb{R}$ a Borel measurable function with compact support (a system of particles evolving accordingly to (4.22) is then called *free Kawasaki dynamics*) and standard rotationally symmetric α stable processes (see, e.g., [10]) with generator given by

$$\mathscr{L} = \Delta^{\alpha/2} \tag{4.23}$$

for $\alpha \in (0, 2)$.

Let $\{Y_t, t \geq 0\}$ be a strong Markov process on $\mathbb{R}^d$. Let $\mathfrak{D}$ be a regular domain of $\mathbb{R}^d$ (see Section 4.2) and define $\mathfrak{D}^{\mathrm{ext}} := \mathbb{R}^d \setminus \mathfrak{D}$.

Let $\{X_t, t \geq 0\}$ be the Markov process on $\mathbb{R}^d$ with semigroup $\{S_t, t \geq 0\}$ which evolves as $\{Y_t, t \geq 0\}$ on $\mathfrak{D}$ and is absorbed upon hitting $\mathfrak{D}^{\mathrm{ext}}$. We denote by $\mathbb{P}_x^{\mathrm{abs}}$ the distribution of the trajectories of $\{X_t, t \geq 0\}$ starting from $x \in \mathfrak{D}$, by $\tau_{\mathfrak{D}^{\mathrm{ext}}}$ the hitting time of the set $\mathfrak{D}^{\mathrm{ext}}$. We assume $\mathbb{P}_x^{\mathrm{abs}}(\tau_{\mathfrak{D}^{\mathrm{ext}}} < \infty) = 1$.

Let now $\lambda : \mathfrak{D}^{\mathrm{ext}} \to \mathbb{R}_+$ be a bounded measurable function giving the reservoir intensity at any $x \in \mathfrak{D}^{\mathrm{ext}}$ and let $\mu(dx)$ be a finite measure on $\mathfrak{D}$. We then define the point configuration (on $\mathbb{R}^d$) valued process $\{\xi_t, t \geq 0\}$ arising from independent copies of the absorbed Markov process $\{X_t, t \geq 0\}$ starting from $\xi_0 = \sum_i \delta_{x_i}$, $x_i \in \mathfrak{D}$. I.e., for $\xi := \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$, $x_i \in \mathfrak{D}$, we define

$$\begin{cases} \xi_t := \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)} \ , \\ \xi_0 = \xi \end{cases}$$

where $\{X_t(i)\}_{t\geq 0}$ are independent copies of $\{X_t\}_{t\geq 0}$ such that $X_0(i) = x_i$ for any $i \in [\mathbf{N}]$. For $\boldsymbol{x} = (x_1, \ldots, x_{\mathbf{N}}) \in \mathfrak{D}^{\mathbf{N}}$, we denote by $\mathbb{E}_{\boldsymbol{x}}^{\mathrm{abs}}$ the expectation in the process $\{\xi_t, t \geq 0\}$ starting from $\xi_0 = \sum_{i=1}^{N} \delta_{x_i}$. Finally we define $\{\Theta_t,\, t \geq 0\}$, a Poisson point process on $\mathfrak{D}$ independent of $\{\xi_t,\, t \geq 0\}$ and with time dependent intensity $\lambda_t(\mathrm{d}x)$ given by

$$\lambda_t(\mathrm{d}x) = \left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(y)\, \mathbb{P}_x^{\mathrm{abs}}(\tau_{\mathfrak{D}^{\mathrm{ext}}} \leq t, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}y) \right) \mu(\mathrm{d}x), \tag{4.24}$$

which is supposed to be finite. The *boundary driven Markov gas in the domain $\mathfrak{D}$ with reservoir intensity* λ, denoted by $\{\eta_t, t \geq 0\}$, is then given, for any $t \geq 0$, by

$$\eta_t = \xi_t\big|_{\mathfrak{D}} + \Theta_t, \tag{4.25}$$

viewed as a point configuration on $\mathfrak{D}$. We denote by $\mathbb{E}_\eta^\lambda$ the expectation in the process defined via (4.25) initialized from η.

Also in this context, $\{\eta_t,\, t \geq 0\}$ is a Markov process when the transition probability $p_t(x, \mathrm{d}y)$ of the process $\{X_t, t \geq 0\}$ satisfies

$$p_t(x, \mathrm{d}y)m(\mathrm{d}x) = p_t(y, \mathrm{d}x)m(\mathrm{d}y) \qquad \text{on} \qquad \mathfrak{D} \times \mathfrak{D} \tag{4.26}$$

for a finite measure m and for the reservoir intensity in (4.10) we choose $\mu = m$ (see Theorem A in the appendix).

We then have the following result generalizing Theorem 4.2. We omit the proof being a straightforward adaptation of the proof of Theorem 4.2.

Theorem 4.6. *Let $\{\eta_t, t \geq 0\}$ be boundary driven Markov gas defined in (4.25). Then for all $n \in \mathbb{N}$ and $t \geq 0$, it holds:*

a) for all bounded, measurable and permutation invariant $f : \mathfrak{D}^n \to \mathbb{R}$

$$\mathbb{E}^\lambda_\eta \left[\int_{\mathfrak{D}^n} f(\boldsymbol{z}) \eta_t^{(n)}(\mathrm{d}\boldsymbol{z}) \right] = \sum_{k=0}^{n} \binom{n}{k} \int_{\mathfrak{D}^n} f(\boldsymbol{z}) \lambda_t^{\otimes k}(\mathrm{d}\boldsymbol{z}_{[k]}) \cdot \otimes (S_t^{\otimes n-k})^* \eta^{(n-k)}(\mathrm{d}\boldsymbol{z}_{[n]\setminus[k]}); \tag{4.27}$$

b) assume further that

$$p_t(x, \mathrm{d}y) = \mathfrak{p}_t(x, y)m(\mathrm{d}y) \tag{4.28}$$

for a symmetric function $\mathfrak{p}_t(x, y)$ and a finite measure m on $\mathfrak{D}$. Then, choosing $\mu = m$, the following holds

$$\frac{\mathrm{d}\mathbb{E}^\lambda_\eta[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(\boldsymbol{z}) = \sum_{I \subset [n]} \mathbb{E}^{\text{abs}}_{\boldsymbol{z}_I} \left[e^{\int_{\mathfrak{D}^{\text{ext}}} \log(\lambda) \mathrm{d}\xi_t} \mathbf{1}_{\{\xi_t(\mathfrak{D}^{\text{ext}}) = |I|\}} \right] \cdot \int_{\mathfrak{D}^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(\boldsymbol{z}_{[n]\setminus I}, \boldsymbol{y}) \eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y})$$

Remarks 4.7 (Examples).

i) The process $\{Y_t, t \geq 0\}$ with generator given in (4.22) is reversible with respect to the Lebesgue measure (see, e.g., [22, Remark 2.7]) but (4.28) is not satisfied since each particle has a positive probability to stay in the initial position during any time interval $[0, t]$.

ii) A spherically symmetric α-stable processes on $\mathbb{R}^d$ with generator given in (4.23) is strongly reversible w.r. to the Lebesgue measure (see, e.g. [10, Eq. 4.4]) and (4.28) is fulfilled.

5 From the discrete to the continuum

In this section we consider the discrete analogue of the boundary driven Brownian gas. Here by "discrete" we mean that the space on which the particles evolve is a lattice and the independent Brownians are replaced by independent random walks. Our first aim will be to show that such a process is equal (in distribution) to the *reservoirs process* $\{\zeta_t,\, t \geq 0\}$ defined via the generator in (1.5). We will then show how the boundary driven Brownian gas arises as a scaling limit of $\{\zeta_t,\, t \geq 0\}$.

5.1 On the equivalence of two definitions of boundary driven independent random walks

We consider the boundary driven Markov gas as explained in Section 4.3, where the process $\{Y_t, t \geq 0\}$ is chosen to be the rate $\frac{1}{2}$ symmetric nearest neighbor random walk jumping on the integers and domain $\mathfrak{D} = V_N = \{1, \ldots, N\}$ with boundary $\{0, N+1\}$. The restriction to the nearest neighbor case is for simplicity only. The generalization to independent walkers with generic jump rates $c(x, y)$, $x, y \in \mathbb{Z}$, absorbed upon leaving V_N is straightforward and so is the extension to more general graphs. Let $\widetilde{V}_N := \{0, \ldots, N+1\} = V_N \cup \{0, N+1\}$ and $\{X_t, t \geq 0\}$ be the process evolving as $\{Y_t, t \geq 0\}$ on V_N and absorbed when hitting 0 or $N+1$. Notice that in this context of nearest neighbor random walks, $\mathfrak{D}^{\mathrm{ext}}$ reduces to $\{0, N+1\}$. We start the process from an initial configuration $\eta \in \mathbb{N}^{V_N}$, viewed as a point configuration on V_N, i.e. $\eta = \sum_i \delta_{x_i}$, where $x_i \in V_N$ are the initial positions of the particles. We define its time evolution as follows:

$$\eta_t = \xi_t\big|_{V_N} + \Theta_t. \tag{5.1}$$

Here ξ_t is the point configuration on $\widetilde{V}_N$ at time t arising from $\xi_0 = \eta$ when all the particles in η evolve as independent copies of the process X_t defined above. For $\boldsymbol{z} = (z_1, \ldots, z_n) \in V_N^n$, we denote by $\mathbb{E}_{\boldsymbol{z}}^{\mathrm{abs}}$ the expectation in the process $\{\xi_t, t \geq 0\}$ started from $\sum_{i=1}^n \delta_{z_i}$. Further, for $\lambda = (\lambda_L, \lambda_R) \in \mathbb{R}_+^2$, Θ_t is a Poisson point process on V_N with intensity defined by

$$\lambda_t(\mathrm{d}x) = \big(\lambda_L \mathbb{P}_x^{\mathrm{RW}}(X_t = 0) + \lambda_R \mathbb{P}_x^{\mathrm{RW}}(X_t = N+1)\big)\, m(\mathrm{d}x) \tag{5.2}$$

with $m(\mathrm{d}x)$ denoting the counting measure and $\mathbb{P}_x^{\mathrm{RW}}$ the path-space measure of the absorbed random walk $\{X_t, t \geq 0\}$. Thus, $\{\Theta_t(\{x\}), x \in V_N\}$ are independent random variables which are Poisson distributed with parameter $\lambda_t(x)$. The process defined in (5.1) is the discrete analogue of the process defined in (4.5) and from Theorem (4.6) applied to this context we have that

$$\frac{\mathrm{d}\mathbb{E}^\lambda_\eta[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(\boldsymbol{z}) = \sum_{I\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_I}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,N+1\})=|I|\}}\right] \cdot \int_{V_N^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(\boldsymbol{z}_{[n]\setminus I},\boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}), \quad (5.3)$$

where $\mathfrak{p}_t^{(n)}(\boldsymbol{z},\boldsymbol{y}) = \prod_{i=1}^n \mathbb{P}^{\mathrm{RW}}_{z_i}(X_t = y)$ and $\mathbb{E}^\lambda_\eta$ denotes the expectation in the process $\{\eta_t, t\geq 0\}$ starting from η.

Let us now compare the process η_t with the process ζ_t, the reservoir process with parameters λ_L, λ_R introduced in Section 1.2.

Theorem 5.1. *Let $\eta \in \mathbb{N}^{V_N}$. Then $\{\zeta_t, t\geq 0\}$, denoting the reservoir process with parameters λ_L,λ_R and generator given in* (1.5) *started from η, and $\{\eta_t, t\geq 0\}$, denoting the boundary driven Markov gas defined in* (5.1) *started from η, are equal in distribution.*

Notice that in the statement of the Theorem we are implicitly identifying the point configuration η_t with the vector $(\eta_t(\{x\}))_{x\in V_N}$ of occupation variables.

Proof. In order to prove the result we will make use of the duality relations (1.7) and (5.3).

Indeed, it suffices to show that for all $\xi = \sum_{i=1}^n \delta_{z_i}$, $z_i \in \mathbb{N}^{V_N}$ one has for all η and $t \geq 0$

$$\mathbb{E}^{\mathrm{res}}_\eta\left[\prod_{x=1}^N d(\xi(\{x\}),\zeta_t(x))\right] = \mathbb{E}^\lambda_\eta\left[\prod_{x=1}^N d(\xi(\{x\}),\eta_t(\{x\}))\right]. \quad (5.4)$$

By (1.6) and (1.7) we have

$$\mathbb{E}^{\mathrm{res}}_\eta\left[\prod_{x=1}^N d(\xi(\{x\}),\zeta_t(x))\right] = \mathbb{E}^{\mathrm{abs}}_\xi\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\prod_{x=1}^N d\left(\xi_t(\{x\}),\zeta(x)\right)\right].$$

where we remind that ξ_t is the point configuration on $\tilde{V}_N$ at time t arising from $\xi_0 = \eta$ when all the particles in η evolve as independent random walkers on $\tilde{V}_N$ absorbed at $\{0, N+1\}$. On the other hand, by (3.7), we have

$$\mathbb{E}^\lambda_\eta\left[\prod_{x=1}^N d(\xi(\{x\}),\eta_t(\{x\}))\right] = \mathbb{E}^\lambda_\eta[\eta_t^{(n)}(\{(z_1,\dots,z_n)\})]$$

and by (5.3)

$$\mathbb{E}^{\lambda}_{\eta}\left[\eta^{(n)}_t(\{z_1,\ldots,z_n\})\right] = \sum_{I\subset[n]}\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_I}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,N+1\})=|I|\}}\right] \cdot\int_{V_N^{n-|I|}}\mathfrak{p}^{(n-|I|)}_t(\boldsymbol{z}_{[n]\setminus I},\boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}). \quad (5.5)$$

It thus remains to show that

$$\sum_{I\subset[n]}\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_I}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,N+1\})=|I|\}}\right] \cdot\int_{V_N^{n-|I|}}\mathfrak{p}^{(n-|I|)}_t(\boldsymbol{z}_{[n]\setminus I},\boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}) = \mathbb{E}^{\mathrm{abs}}_{\xi}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\prod_{x\in V_N}d\left(\xi_t(\{x\}),\eta(\{x\})\right)\right].$$

Notice that, for any $I\subset[n]$, by (3.7), we have

$$\int_{V_N^{n-|I|}}\mathfrak{p}^{(n-|I|)}_t(\boldsymbol{z}_{[n]\setminus I},\boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}) = \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_{[n]\setminus I}}\left[\prod_{x=1}^{N}d(\xi_t(\{x\}),\eta(\{x\}))\right]. \quad (5.6)$$

We thus have,

$$\begin{aligned}&\mathbb{E}^{\lambda}_{\eta}\left[\eta^{(n)}_t(\{z_1,\ldots,z_n\})\right]\\ &=\sum_{I\subset[n]}\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_I}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,N+1\})=|I|\}}\right] \cdot\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_{[n]\setminus I}}\left[\prod_{x=1}^{N}d(\xi_t(\{x\}),\eta(\{x\}))\right]\\ &=\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\prod_{x=1}^{N}d(\xi_t(\{x\}),\eta(\{x\}))\right],\end{aligned}$$

where we used (5.5) and (5.6) in the first equality and the independence of particles in the second equality. □

5.2 Scaling limit

In this section we show how the process of independent random walkers with reservoirs λ_L and λ_R, when appropriately rescaled in space and time, and with rescaling of the reservoirs intensities, converges to the boundary driven Brownian gas. We start with the following lemma.

Lemma 5.2. *Consider $\{\Theta^{(N)}\}_{N\geq 1}$ a sequence of Poisson point processes on $(0,1)$ with the intensity measures*

$$\lambda^{(N)}(\mathrm{d}x) = \left(\frac{1}{N}\sum_{i=1}^{N} a_N(\tfrac{i}{N})\delta_{i/N}\right)(\mathrm{d}x) \tag{5.7}$$

with $a_N : \{\frac{1}{N}, \dots, \frac{N-1}{N}, 1\} \to \mathbb{R}_+$. Assume furthermore that whenever $i/N \to x \in [0,1]$ then also

$$a_N(\tfrac{i}{N}) \to \alpha(x), \tag{5.8}$$

where $\alpha : [0,1] \to \mathbb{R}$ is a smooth function. Then as $N \to \infty$, $\Theta^{(N)}$ converges to the Poisson point process Θ with intensity $\alpha(x)\mathrm{d}x$.

Proof. Because sequences of Poisson point processes converge when the sequences of their intensity measures converge, it suffices to prove that (5.7) converges weakly to $\alpha(x)\mathrm{d}x$ as $N \to \infty$. Let $f : [0,1] \to \mathbb{R}$ be continuous, then

$$\int f(x)\lambda^{(N)}(\mathrm{d}x) = \frac{1}{N}\sum_{i=1}^{N} f(\tfrac{i}{N})a_N(\tfrac{i}{N}).$$

By the condition on $a_N(\frac{i}{N})$, this sum converges to the Riemann integral $\int_0^1 f(x)\alpha(x)dx$. □

We then have the following result.

Theorem 5.3. *Consider the reservoir process $\{\zeta_{N,t}, t \geq 0\}$ on the chain $\{1, \dots, N\}$, with reservoirs parameters $\frac{\lambda_L}{N}, \frac{\lambda_R}{N}$ and generator given in* (1.5). *Define $\eta_{N,t}(\mathrm{d}x)$ the point configuration on $[0,1]$ via*

$$\mathscr{Z}_{N,t}(\mathrm{d}x) = \left(\sum_{i=1}^{N} \zeta_{N,tN^2}(i)\delta_{i/N}\right)(\mathrm{d}x). \tag{5.9}$$

Assume that at time $t = 0$,

$$\mathscr{Z}_{N,0} = \sum_{i=1}^{\mathbf{N}} \delta_{x_i^{(N)}/N}, \tag{5.10}$$

where $x_i^{(N)}/N \to x_i \in (0,1)$ for all $i = 1, \dots, \mathbf{N}$.

Then as $N \to \infty$ the process $\{\mathscr{Z}_{N,t}(\mathrm{d}x), t \geq 0\}$ converges (in the sense of convergence of finite dimensional distributions) to the boundary driven Brownian gas with parameters λ_L, λ_R, started at the configuration $\sum_{i=1}^{\mathbf{N}} \delta_{x_i}$.

Proof. As a consequence of Theorem 5.1, the reservoir process $\zeta_{N,t}$ equals (in distribution) the boundary driven Markov gas $\eta_{N,t}$ obtained as a sum of the configuration arising from letting the particles initially in the system evolve according to independent random walkers absorbed at 0 and $N+1$, and adding an independent Poisson point process on V_N with intensity

$$\lambda_t(i) = \tfrac{\lambda_L}{N}\, \mathbb{P}_i^{\mathrm{RW}}(X_t = 0) + \tfrac{\lambda_R}{N}\, \mathbb{P}_i^{\mathrm{RW}}(X_t = N+1), \tag{5.11}$$

where $\{X_t, t \geq 0\}$ denotes the random walk on V_N absorbed at the boundary $\{0, N+1\}$. Therefore after diffusive rescaling of space and time, the intensity of the Poisson point process on $(0,1)$ modelling the reservoirs effect becomes

$$\lambda_t^{(N)}(\mathrm{d}x) = \sum_{i=1}^{N} \big(\tfrac{\lambda_L}{N}\, \mathbb{P}_i^{\mathrm{RW}}(X_{tN^2} = 0) + \tfrac{\lambda_R}{N}\, \mathbb{P}_i^{\mathrm{RW}}(X_{tN^2} = N+1) \big)\, \delta_{i/N}(dx). \tag{5.12}$$

Because the absorbed random walk X_{tN^2}/N converges weakly, as $N \to \infty$, to the Brownian motion on $[0,1]$ absorbed at the boundaries, we can apply Lemma 5.2 with

$$a_N(\tfrac{i}{N}) = \lambda_L\, \mathbb{P}_i^{\mathrm{RW}}(X_{tN^2} = 0) + \lambda_R\, \mathbb{P}_i^{\mathrm{RW}}(X_{tN^2} = N+1)$$

which converges, in the sense given by (5.8), to

$$\alpha(x) = \lambda_L\, \mathbb{P}_x^{\mathrm{abs}}(\tau_L \leq t) + \lambda_R\, \mathbb{P}_x^{\mathrm{abs}}(\tau_R \leq t) \tag{5.13}$$

where τ_L, τ_R are the hitting times of 0, resp. 1, and $\mathbb{P}_x^{\mathrm{abs}}$ the path space measure of Brownian motion started from x and absorbed whenever hitting $0, 1$. Therefore, the Poisson point processes (5.12) converge to the Poisson point processes with intensity (5.13).

We know now that the absorbed random walk X_{tN^2}/N weakly converges to the absorbed Brownian motion. In the same way, a system of independent absorbed random walkers, weakly converge to a system of independent absorbed Brownian motions. As a consequence, the point configuration corresponding to the time evolution of the independent walkers initially in the system converge to the point configuration corresponding to the evolution of independent absorbed Brownian motions.

Since the evolution of the particles initially in the system and the added Poisson process are independent, in the scaling limit, we obtain the sum of the evolution of the particles initially in the system and an independent Poisson point process with intensity (5.13), which is the boundary driven Brownian gas with reservoir parameters λ_L, λ_R. □

6 Orthogonal dualities

Duality results with orthogonal polynomials duality functions are well known in the discrete context for independent random walkers, both for systems without reservoirs (closed systems) [16] and for systems with reservoirs (open systems) [15]. In this context the orthogonality property of the duality function has proven to be crucial in several applications involving, in particular, the study of the fluctuation fields [1, 2, 9], and the study of the speed of relaxation to equilibrium [8]. A generalisation of these duality results to the continuum case has been obtained in [15] only for the closed system case. In order to have a complete analogy with the duality theory for independent random walks on a finite chain with reservoirs, we now investigate orthogonal dualities for the boundary driven Brownian gas.

6.1 Known orthogonal dualities

6.1.1 Closed discrete systems

Orthogonal self-duality functions are well known for the system of simple symmetric independent random walkers on $\mathbb{Z}^d$ described in Section 1.2 (see [16], [25]). More precisely, for any $\theta > 0$, the factorized functions given by

$$D_\theta^{\mathrm{or}}(\xi,\eta) = \prod_{x\in\mathbb{Z}^d} C_{\xi(x)}(\eta(x),\theta), \tag{6.1}$$

where $C_k(n,\theta)$ are the Charlier polynomials defined as

$$C_k(n,\theta) = \sum_{\ell=0}^{k} \binom{k}{\ell} (-\theta)^{k-\ell} (n)_\ell \tag{6.2}$$

($(n)_\ell$ denotes the ℓ-th falling factorial) are self-duality functions for the Markov process $\{\eta_t,\, t\geq 0\}$ with generator given in (1.1). The dualities in (6.1) satisfy the following orthogonality relation w.r.t. the measure $\mu_\theta^{\mathrm{rev}} = \otimes_{x\in\mathbb{Z}^d}\mathrm{Poisson}(\theta)$ which is reversible for $\{\eta_t,\, t\geq 0\}$: for any $\xi,\, \xi' \in \mathbb{N}^{\mathbb{Z}^d}$

$$\int D_\theta^{\mathrm{or}}(\xi,\eta) D_\theta^{\mathrm{or}}(\xi',\eta)\mathrm{d}\mu_\theta^{\mathrm{rev}}(\eta) = \mathbf{1}_{\{\xi=\xi'\}}\frac{\xi!}{\theta^{|\xi|}},$$

where $\xi! := \prod_{x\in\mathbb{Z}^d}\xi(x)!$ and $|\xi| = \sum_{x\in\mathbb{Z}^d}\xi(x)$.

Notice that the relation between orthogonal and classical dualities is given by (see [15, Remark 4.2])

$$D_\theta^{\mathrm{or}}(\xi,\eta) = \sum_{\xi'\leq\xi} (-\theta)^{|\xi|-|\xi'|}\binom{\xi}{\xi'} D^{\mathrm{cl}}(\xi',\eta) = \sum_{I\subset[n]} (-\theta)^{n-|I|} D^{\mathrm{cl}}\left(\sum_{i\in I}\delta_{y_i},\eta\right), \tag{6.3}$$

where $\xi = \sum_{i=1}^n y_i$ and where $\xi' \le \xi$ means that $\xi'(x) \le \xi(x)$ for any $x \in \mathbb{Z}^d$ and $\binom{\xi}{\xi'} := \prod_{x\in\mathbb{Z}^d} \binom{\xi(x)}{\xi'(x)}$.

6.1.2 Open discrete systems

Let us now reconsider the reservoir process with parameters λ_L, λ_R defined in Section 1.2. In [15] the authors proved that the following functions, for $\theta > 0$,

$$D^{\text{or}}_{\text{res},\theta}(\xi,\zeta) = (\lambda_L - \theta)^{\xi(0)} D^{\text{or}}_\theta(\xi,\zeta)(\lambda_R - \theta)^{\xi(N+1)} \tag{6.4}$$

with

$$D^{\text{or}}_\theta(\xi,\zeta) = \prod_{x\in\widetilde{V}_N} C_{\xi(x)}(\zeta(x),\theta)$$

are duality functions between $\{\zeta_t, t \ge 0\}$ the Markov process on $V_N = \{1,\dots,N\}$ with generator given in (1.5) and $\{\xi_t,\, t \ge 0\}$ the system of random walkers on $\widetilde{V}_N = \{0,\dots,N+1\}$ absorbed at $\{0, N+1\}$. Notice that the orthogonality relation is w.r.t. $\mu_\theta = \otimes_{x\in V_N}\text{Poisson}(\theta)$ which is not stationary for the reservoir process with general parameters λ_L, λ_R, but it is reversible for the reservoir process with parameters $\lambda_L = \lambda_R = \theta$, the last case referred as the reservoir process in *equilibrium*.

6.1.3 Closed systems in the continuum

Generalizations of orthogonal self-dualities for systems considered in Section 3.1, namely closed systems of independent Markov processes on general Polish spaces E, has been recently studied in [14]. More precisely, let $\eta_t = \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)}$ with $\{X_t(i),\, t \ge 0\}$ independent copies of a Markov process on E started from x_i, strongly reversible w.r.t. to a measure m. Then, the measure defined for any $t \ge 0$, $n \in \mathbb{N}$ and $\theta > 0$ as

$$\eta_t^{(n),\theta}(\mathrm{d}\boldsymbol{z}) := \sum_{I\subset[n]} (-\theta)^{n-|I|}\, \eta_t^{(|I|)}(\mathrm{d}\boldsymbol{z}_I)\, m^{\otimes^{n-|I|}}(\mathrm{d}\boldsymbol{z}_{[n]\setminus I}) \tag{6.5}$$

satisfies the following duality relation (see [14, Corollary 4.2])

$$\frac{\mathrm{d}\mathbb{E}^\lambda_\eta[\eta_t^{(n),\theta}]}{\mathrm{d}m^{\otimes n}}(z_1,\dots,z_n) = \int \prod_{i=1}^n \mathfrak{p}_t(z_i,y_i)\eta^{(n),\theta}(\mathrm{d}(y_1,\dots,y_n))$$

and generalizes the orthogonal self-dualities given in (6.1) in the following sense:

i) let

$$\mathbf{1}_{\boldsymbol{B}}(z_1,\dots,z_n) := \left(\mathbf{1}_{B_1}^{\otimes d_1} \otimes \cdots \otimes \mathbf{1}_{B_K}^{\otimes d_K}\right)(z_1,\dots,z_n)$$

for $\boldsymbol{B} = \{B_1,\dots,B_K\}$ a family of mutually disjoint sets in E and $\{d_1,\dots,d_K\}$ such that $\sum_{i=1}^K d_i = n$, then

$$\begin{aligned}&\int \mathbf{1}_{\boldsymbol{B}}(z_1,\dots,z_n)\,\eta^{(n),\theta}(d(z_1,\dots,z_n))\\ &= \prod_{\ell=1}^{K}(-\theta m(B_\ell))^{d_\ell}\; C_{d_\ell}(\eta(B_\ell);\theta m(B_\ell))\end{aligned} \tag{6.6}$$

with $C_k(n,x)$ being the Charlier polynomials defined above.

ii) If we denote by $\mathscr{P}_{\theta m}$ the distribution of a Poisson point process with intensity measure θm, then, the following orthogonal relation holds

$$\begin{aligned}&\mathbb{E}_{\mathscr{P}_{\theta m}}\left[\left(\int f_n \mathrm{d}\zeta^{(n),\theta}\right)\left(\int g_{n'} \mathrm{d}\zeta^{(n'),\theta}\right)\right]\\ =&\mathbf{1}_{\{n=n'\}}\cdot n!\int f_n\, g_n\,\mathrm{d}(\theta m)^{\otimes n}\end{aligned} \tag{6.7}$$

for $\zeta \sim \mathscr{P}_{\theta m}$ and $f_n : E^n \to \mathbb{R}$, $g_{n'} : E^{n'} \to \mathbb{R}$ bounded and permutation invariant functions.

We refer to [23] for a proof of the two above facts.

The aim of the next section is to generalize the orthogonal dualities for the reservoir system given in (6.4) in the context of the boundary driven Brownian gas on $(0,1)$.

6.2 Orthogonal dualities for the boundary driven Brownian gas

Let us now consider the boundary driven Brownian gas on $(0,1)$ with parameters λ_L and λ_R

$$\eta_t = \xi_t + \Theta_t$$

defined in Section 4.1. We have previously proved that the factorial measure $\eta_t^{(n)}$ is the right object to study in order to have a duality result for boundary driven system. Inspired by the relation highlighted in the previous subsection between classical and orthogonal dualities we now study for any $n \in \mathbb{N}$ and $\theta > 0$

$$\eta_t^{(n),\theta}(\mathrm{d}\boldsymbol{z}) := \sum_{I\subset[n]} (-\theta)^{n-|I|}\, \eta_t^{(|I|)}(\mathrm{d}\boldsymbol{z}_I)\, m^{\otimes^{n-|I|}}(\mathrm{d}\boldsymbol{z}_{[n]\setminus I}), \tag{6.8}$$

viewed as a measure on $(0,1)^n$. Here $m(\mathrm{d}z)$ is the Lebesgue measure on $(0,1)$ and the orthogonality properties (6.6) and (6.7) hold for (6.8) for, respectively, $\boldsymbol{B} = \{B_1,\dots,B_K\}$ a family of mutually disjoint sets in $(0,1)$ with $\{d_1,\dots,d_K\}$ such $\sum_{i=1}^K d_i = n$, and bounded and permutation invariant functions $f_n : (0,1)^n \to \mathbb{R}$, $g_{n'} : (0,1)^{n'} \to \mathbb{R}$.

Notice that the orthogonality relations holds true w.r.t. the intensity measure of the Poisson point process whose distribution is reversible for the boundary driven Brownian gas *in equilibrium*, namely with $\lambda_L = \lambda_R = \theta$.

Moreover, since we will integrate the above defined measure $\eta^{(n),\theta}$ against $\mathfrak{p}_t^{(n)}(\cdot,\cdot) : [0,1]^n \times [0,1]^n \to \mathbb{R}$, i.e. a function defined on $\{0,1\}$ as well, we extend $\eta^{(n),\theta}$ in the following way: we define $\bar{m}(dz) = m(dz) + \delta_0(dz) + \delta_1(dz)$ and we denote

$$\eta_t^{[n],\theta}(\mathrm{d}\boldsymbol{z}) := \sum_{I\subset[n]} (-\theta)^{n-|I|}\, \eta_t^{(|I|)}(\mathrm{d}\boldsymbol{z}_I)\, \bar{m}^{\otimes^{n-|I|}}(\mathrm{d}\boldsymbol{z}_{[n]\setminus I}) \tag{6.9}$$

whenever integrated against functions being non zero also at the boundary $[0,1]$. Notice that $\int_{[0,1]} \mathfrak{p}_t(x,y)\bar{m}(\mathrm{d}y) = 1$ for any $x \in [0,1]$ and that we used the brackets $[\cdot]$ in the upper index of $\eta_t^{[n],\theta}$ to emphasize the difference with $\eta_t^{(n),\theta}$.

We then have the following theorem, providing orthogonal dualities between the boundary driven Brownian gas and the system of independent Brownian motions on $[0,1]$ absorbed at the boundaries.

Theorem 6.1. *For the boundary driven Brownian gas, the expectation of the measure given in* (6.8) *at time $t \geq 0$ is absolutely continuous w.r.t. $m^{\otimes n}$ with the following density:*

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}_\eta^\lambda[\eta_t^{(n),\theta}]}{\mathrm{d}m^{\otimes n}}(\boldsymbol{z}) \quad = \quad & \sum_{J\subset[n]} \mathbb{E}_{\boldsymbol{z}_J}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\ & \cdot \int_{E^{n-|J|}} \mathfrak{p}_t^{(n-|J|)}(\boldsymbol{z}_{[n]\setminus J},\boldsymbol{y})\eta^{[n-|J|],\theta}(\mathrm{d}\boldsymbol{y}).\end{aligned}$$

Proof. Using (6.8) and (4.7) we have

$$\mathbb{E}_\eta^\lambda[\eta_t^{(n),\theta}](\mathrm{d}\boldsymbol{z}) = \mathbb{E}_\eta^\lambda\left[\sum_{I\subset[n]} (-\theta)^{n-|I|}\, \eta_t^{(|I|)}(\mathrm{d}\boldsymbol{z}_I)\, m^{\otimes^{n-|I|}}(\mathrm{d}\boldsymbol{z}_{[n]\setminus I})\right]$$

$$
\begin{aligned}
&= \sum_{I\subset[n]} (-\theta)^{n-|I|} \mathbb{E}^{\lambda}_{\eta}\left[\eta_t^{(|I|)}(d\boldsymbol{z}_I)\right] m^{\otimes^{n-|I|}}(\mathrm{d}\boldsymbol{z}_{[n]\setminus I}) \\
&= \sum_{I\subset[n]} (-\theta)^{n-|I|} \Bigg(\sum_{J\subset I} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\
&\qquad\qquad \times \int_{E^{|I|-|J|}} \mathfrak{p}_t^{(|I|-|J|)}(\boldsymbol{z}_{I\setminus J},\boldsymbol{y})\, \eta^{(|I|-|J|)}(\mathrm{d}\boldsymbol{y}) \Bigg) m^{\otimes^{n-|J|}}(\mathrm{d}\boldsymbol{z}_{[n]\setminus J})
\end{aligned}
$$

and by exchanging the order of the summation in the last expression above we obtain

$$
\begin{aligned}
&\mathbb{E}^{\lambda}_{\eta}[\eta_t^{(n),\theta}](\mathrm{d}\boldsymbol{z}) \\
&= \sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\
&\times \left(\sum_{I\subset[n]\setminus J} (-\theta)^{n-|I|-|J|} \int_{E^{|I|}} \mathfrak{p}_t^{(|I|)}(\boldsymbol{z}_I,\boldsymbol{y})\, \eta^{(|I|)}(\mathrm{d}\boldsymbol{y}) \right) m^{\otimes^{n-|J|}}(\mathrm{d}\boldsymbol{z}_{[n]\setminus J})\,.
\end{aligned}
$$

We conclude by noticing that

$$
\begin{aligned}
&\sum_{I\subset[n]\setminus J} (-\theta)^{n-|I|-|J|} \int_{E^{|I|}} \mathfrak{p}_t^{(|I|)}(\boldsymbol{z}_I,\boldsymbol{y})\, \eta^{(|I|)}(\mathrm{d}\boldsymbol{y}) \\
&\qquad = \int_{E^{n-|J|}} \mathfrak{p}^{(n-|J|)}(\boldsymbol{z}_{[n]\setminus J},\boldsymbol{y})\, \eta^{[n-|J|],\theta}(\mathrm{d}\boldsymbol{y}),
\end{aligned}
$$

which can be proved using (6.9). □

Notice that the same result holds for any boundary driven system of strongly reversible Markov processes as the ones treated in Section 4.2 and for the discrete system defined in (5.1).

We thus conclude the section by showing that indeed Theorem 6.1 generalizes the duality relation for the discrete system w.r.t. the orthogonal dualities given in (6.4). Notice, indeed, that we have, from (6.8) and (6.4),

$$
\mathbb{E}^{\lambda}_{\eta}\left[\eta_t^{(n),\theta}(\{z_1,\ldots,z_n\})\right] = \mathbb{E}^{\lambda}_{\eta}\left[D^{\mathrm{or}}_{\theta}(\xi,\eta_t)\right].
$$

It thus remains to prove the following.

Proposition 6.2. *Let η_t denote the process defined in* (5.1)*. Then for all $n\in\mathbb{N}$ and $z_1,\ldots,z_n\in V_N$, denoting $\sum_{i=1}^n \delta_{z_i}=\xi$, we have*

$$\sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right]\int \mathfrak{p}_t^{(n-|J|)}(\boldsymbol{z}_{[n]\setminus J},\boldsymbol{y})\,\eta^{(n-|J|),\theta}(d\boldsymbol{y}) = \mathbb{E}^{\mathrm{abs}}_{\xi}\left[D^{\mathrm{or}}_{\mathrm{res},\theta}\left(\xi_t,\eta\right)\right].$$

Proof. By the definition of $\eta^{(n-|J|),\theta}$ we obtain

$$\begin{aligned}
&\sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\
&\qquad\cdot\int \mathfrak{p}_t^{(n-|J|)}(\boldsymbol{z}_{[n]\setminus J},\boldsymbol{y})\,\eta^{(n-|J|),\theta}(d\boldsymbol{y}) \qquad (6.10)\\
&= \sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\
&\qquad\cdot\sum_{I\subset[n]\setminus J}(-\theta)^{n-|J|-|I|}\int_{E^{|I|}}\mathfrak{p}_t^{(|I|)}(\boldsymbol{z}_I,\boldsymbol{y})\,\eta^{(|I|)}(d\boldsymbol{y})\\
&= \sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\
&\qquad\cdot\sum_{I\subset[n]\setminus J}(-\theta)^{n-|J|-|I|}\,\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_I}\left[\prod_{x=1}^{N} d(\xi_t(\{x\}),\eta(\{x\}))\right]\\
&= \sum_{J\subset[n]}\sum_{I\subset[n]\setminus J}(-\theta)^{n-|J|-|I|}\,\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right]\\
&\qquad\cdot\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_I}\left[\prod_{x=1}^{N} d(\xi_t(\{x\}),\eta(\{x\}))\right]\\
&= \sum_{U\subset[n]}(-\theta)^{n-|U|}\,\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_U}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\prod_{x=1}^{N} d(\xi_t(\{x\}),\eta(\{x\}))\right], \qquad (6.11)
\end{aligned}$$

where in the last line we used the independence of the particles. Combining (6.10), (3.7) and (1.6) we get

$$\begin{aligned}
&\sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\
&\qquad\cdot\int \mathfrak{p}_t^{(n-|J|)}(\boldsymbol{z}_{[n]\setminus J},\boldsymbol{y})\,\eta^{(n-|J|),\theta}(d\boldsymbol{y})\\
&= \sum_{U\subset[n]}(-\theta)^{n-|U|}\,\mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_U}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}D^{\mathrm{cl}}(\xi_t,\eta)\right]
\end{aligned}$$

$$= \sum_{U \subset [n]} (-\theta)^{n-|U|} \, \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_U} \left[D^{\lambda_L, \lambda_R}(\xi_t, \eta) \right].$$

We then have, using again the independence of particles,

$$\begin{aligned}
&\sum_{U \subset [n]} (-\theta)^{n-|U|} \, \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}_U} \left[\lambda_L^{\xi_t(\{0\})} \lambda_R^{\xi_t(\{N+1\})} D^{\mathrm{cl}}(\xi_t, \eta) \right] \\
&= \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}} \left[\sum_{\xi' \leq \xi_t} \binom{\xi_t}{\xi'} (-\theta)^{n-\xi'(\tilde{V}_N)} \, \lambda_L^{\xi'(\{0\})} \, \lambda_R^{\xi'(\{N+1\})} \, D^{\mathrm{cl}} \left(\xi' \big|_{V_N}, \eta \right) \right] \\
&= \mathbb{E}^{\mathrm{abs}}_{\boldsymbol{z}} \Bigg[\left(\sum_{\ell=0}^{\xi_t(\{0\})} \binom{\xi_t(\{0\})}{\ell} (-\theta)^{\xi_t(\{0\})-\ell} \, \lambda_L^{\ell} \right) \\
&\qquad \times \left(\sum_{\xi' \leq \xi_t|_{V_N}} \binom{\xi_t|_{V_N}}{\xi'} (-\theta)^{\xi_t(V_N)-\xi'(V_N)} \, D^{\mathrm{cl}} \left(\xi', \eta \right) \right) \\
&\qquad \times \left(\sum_{r=0}^{\xi_t(\{N+1\})} \binom{\xi_t(\{N+1\})}{r} (-\theta)^{\xi_t(\{N+1\})-r} \, \lambda_R^{r} \right) \Bigg] \\
&= \mathbb{E}^{\mathrm{abs}}_{\xi} \left[(\lambda_L - \theta)^{\xi_t(\{0\})} (\lambda_R - \theta)^{\xi_t(\{N+1\})} D^{\mathrm{or}}_{\theta} \left(\xi_t \big|_{V_N}, \eta \right) \right]
\end{aligned}$$

where the third identity follows from (6.3). The proof is concluded by noticing that

$$(\lambda_L - \theta)^{\xi_t(\{0\})} (\lambda_R - \theta)^{\xi_t(\{N+1\})} D^{\mathrm{or}}_{\theta} \left(\xi_t \big|_{V_N}, \eta \right) = D^{\mathrm{or}}_{\mathrm{res},\theta} \left(\xi_t, \eta \right).$$

□

A Markov property of the boundary driven independent particles

In this appendix we prove the Markovianity, under the reversibility condition (A.1), of the boundary driven Markov gas defined in Section 4.3. We remind that the *boundary driven Markov gas* in a regular domain $\mathfrak{D}$ of $\mathbb{R}^d$ *with reservoir intensity* λ is the process defined in (4.25) and denoted by $\{\eta_t, t \geq 0\}$. We remind moreover that the reservoir intensity is a function $\lambda : \mathfrak{D}^{\mathrm{ext}} \to \mathbb{R}^+$, where $\mathfrak{D}^{\mathrm{ext}} := \mathbb{R}^d \setminus \mathfrak{D}$. See Section 4.3 for further details.

Theorem A.1. *Assume that*

$$p_t(x, \mathrm{d}y) m(\mathrm{d}x) = p_t(y, \mathrm{d}x) m(\mathrm{d}y) \qquad \textit{on} \qquad \mathfrak{D} \times \mathfrak{D} \tag{A.1}$$

for some finite measure $m(\mathrm{d}x)$ on $\mathfrak{D}$. Denote by $\mathscr{P}_{\lambda_t}$ the law of a Poisson point process with intensity measure given in (4.24) with the measure m in place of μ and let $P_t^{\mathrm{res}} : \Omega \times \mathscr{B}(\Omega) \to [0,1]$, $t \geq 0$ defined by

$$P_t^{\mathrm{res}}(\eta, B) := \int_{\Theta+\xi \in B} \mathscr{P}_{\lambda_t}(\mathrm{d}\Theta) P_t(\eta, \mathrm{d}\xi), \tag{A.2}$$

where P_t denotes the semigroup of the process $\{\xi_t,\, t \geq 0\}$. Then, the family P_t^{res}, $t \geq 0$ is a time homogeneous transition function on $(\Omega, \mathscr{B}(\Omega))$ and there exists a Markov family with transition function P_t^{res}.

Proof. We need to show that P_t^{res} satisfies the Chapman-Kolmogorv equation, which, due to [5, Lemma A.3], boils down to check that for any continuous function ψ with compact support strictly contained in $\mathfrak{D}$ and for any $s, t > 0$

$$\int P_{s+t}^{\mathrm{res}}(\eta, \mathrm{d}v) e^{i \int \psi \mathrm{d}v} = \int \int P_s^{\mathrm{res}}(\eta, \mathrm{d}\zeta) P_t^{\mathrm{res}}(\zeta, \mathrm{d}v) e^{i \int \psi \mathrm{d}v}.$$

By the definition of P_{t+s}^{res} and using (3.12), we have that the left hand side is equal to

$$\exp\left\{ \int (e^{i\psi} - 1) \mathrm{d}\lambda_{t+s} \right\} \int P_{s+t}(\eta, \mathrm{d}\xi) e^{i \int \psi \mathrm{d}\xi}.$$

On the other hand, for the right hand side we have,

$$\begin{aligned}
&\int \int P_s^{\mathrm{res}}(\eta, \mathrm{d}\zeta) P_t^{\mathrm{res}}(\zeta, \mathrm{d}v) e^{i \int \psi \mathrm{d}v} \\
&= \int \mathscr{P}_{\lambda_s}(\mathrm{d}\Theta_1) \int P_s(\eta, \mathrm{d}\xi_1) \int \mathscr{P}_{\lambda_t}(\mathrm{d}\Theta_2) \int P_t(\Theta_1 + \xi_1, \mathrm{d}\xi_2) e^{i \int \psi \mathrm{d}(\xi_2 + \Theta_2)} \\
&= \exp\left\{ \int (e^{i\psi} - 1) \mathrm{d}\lambda_t \right\} \\
&\qquad \times \int \mathscr{P}_{\lambda_s}(\mathrm{d}\Theta_1) \int P_s(\eta, \mathrm{d}\xi_1) \int P_t(\Theta_1 + \xi_1, \mathrm{d}\xi_2) e^{i \int \psi \mathrm{d}\xi_2} \\
&= \exp\left\{ \int (e^{i\psi} - 1) \mathrm{d}\lambda_t \right\} \\
&\qquad \times \int \mathscr{P}_{\lambda_s}(\mathrm{d}\Theta_1) \int P_s(\eta, \mathrm{d}\xi_1) \int P_t(\Theta_1, \mathrm{d}\xi_{2,1}) e^{i \int \psi \mathrm{d}\xi_{2,1}} \\
&\qquad\qquad \times \int P_t(\xi_1, \mathrm{d}\xi_{2,2}) e^{i \int \psi \mathrm{d}\xi_{2,2}} \\
&= \exp\left\{ \int (e^{i\psi} - 1) \mathrm{d}\lambda_t \right\}
\end{aligned}$$

$$\times \int \mathscr{P}_{\lambda_s}(\mathrm{d}\Theta_1) \int P_t(\Theta_1, \mathrm{d}\xi_{2,1}) e^{i\int \psi \mathrm{d}\xi_{2,1}} \int P_{t+s}(\eta, \mathrm{d}\xi) e^{i\int \psi \mathrm{d}\xi}$$

where we used the definition of P_t^{res} first, the independence of the particles after and finally the Champan-Kolmogorov equation for P_t. Thus, it remains to show that

$$\begin{aligned} &\int \mathscr{P}_{\lambda_s}(\mathrm{d}\Theta_1) \int P_t(\Theta_1, \mathrm{d}\xi_{2,1}) e^{i\int \psi \mathrm{d}\xi_{2,1}} \\ &= \exp\left\{ \int (e^{i\psi} - 1)\mathrm{d}\lambda_{t+s} - \int (e^{i\psi} - 1)\mathrm{d}\lambda_t \right\}. \end{aligned} \tag{A.3}$$

By the independence of the particles and (3.12) follows that

$$\int \mathscr{P}_{\lambda_s}(\mathrm{d}\Theta_1) \int P_t(\Theta_1, \mathrm{d}\xi_{2,1}) e^{i\int \psi \mathrm{d}\xi_{2,1}} = \exp\left\{ \int S_t(e^{i\psi} - 1)(x)\lambda_s(\mathrm{d}x) \right\},$$

where S_t denotes the semigroup of the absorbed Markov process upon hitting $\mathfrak{D}^{\mathrm{ext}}$ and which is given by

$$S_t f(x) = \int_{\mathfrak{D}} p_t(x, \mathrm{d}y) f(y) + \int_{\mathfrak{D}^{\mathrm{ext}}} f(z) \mathbb{P}_x(\tau_{\mathfrak{D}^{\mathrm{ext}}} \le t,\, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z)$$

for any $f : \mathbb{R}^d \to \mathbb{R}$ bounded function. Since ψ is equal to zero at $\mathfrak{D}^{\mathrm{ext}}$, we have that

$$\int S_t(e^{i\psi} - 1)(x)\lambda_s(\mathrm{d}x) = \int \left(\int p_t(x, \mathrm{d}y)(e^{i\psi(y)} - 1) \right) \lambda_s(\mathrm{d}x)$$

and thus, (A.3) is given if one proves that

$$\int p_t(x, \mathrm{d}y)\lambda_s(\mathrm{d}x) = \lambda_{t+s}(\mathrm{d}y) - \lambda_t(\mathrm{d}y). \tag{A.4}$$

Using the definition of λ_s given in (4.24), we have that the left hand side of (A.4) is equal to

$$\left[\int p_t(x, \mathrm{d}y) \left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(z)\, \mathbb{P}_x(\tau_{\mathfrak{D}^{\mathrm{ext}}} \le s,\, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z) \right) m(\mathrm{d}x) \right].$$

Using the strong Markov property of the absorbed Markov process, we have

$$\begin{aligned} &\lambda_{t+s}(\mathrm{d}y) \\ &= \left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(z) \mathbb{P}_y(\tau_{\mathfrak{D}^{\mathrm{ext}}} \le t+s,\, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z) \right) m(\mathrm{d}y) \end{aligned}$$

$$
\begin{aligned}
=&\left(\int_{\mathfrak{D}^{\rm ext}} \lambda(z)\mathbb{P}_y(\tau_{\mathfrak{D}^{\rm ext}} \le t,\, X_{\tau_{\mathfrak{D}^{\rm ext}}} \in \mathrm{d}z)\right) m(\mathrm{d}y)\\
&+\left[\int p_t(y,\mathrm{d}x)\left(\int_{\mathfrak{D}^{\rm ext}} \lambda(z)\mathbb{P}_x(\tau_{\mathfrak{D}^{\rm ext}} \le s,\, X_{\tau_{\mathfrak{D}^{\rm ext}}} \in \mathrm{d}z)\right) m(\mathrm{d}y)\right]\\
=&\lambda_t(\mathrm{d}y)+\left[\int p_t(x,\mathrm{d}y)\left(\int_{\mathfrak{D}^{\rm ext}} \lambda(z)\mathbb{P}_x(\tau_{\mathfrak{D}^{\rm ext}} \le s,\, X_{\tau_{\mathfrak{D}^{\rm ext}}} \in \mathrm{d}z)\right) m(\mathrm{d}x)\right],
\end{aligned}
$$

where in the last identity we used the condition (A.1). Thus (A.4) follows, concluding the proof. □

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