SOCIEDADE BRASILEIRA DE MATEMÁTICA ENSAIOS MATEMÁTICOS

Corrigendum to

“The Dynamics of Inner Functions”

Ensaios Matemáticos, Volume 3, 1991

Claus I. Doering

Ricardo Mañé

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $f : \overline{\mathbb{C}} \hookleftarrow$ a rational map. Let $U \subset \overline{\mathbb{C}}$ be a fixed parabolic basin of f and, for $z \in U$, let λ_z be the harmonic probability on the Borel σ-algebra of ∂U with respect to z.

In our memoir “The Dynamics of Inner Functions”, Ensaios de Matemática (SBM), Volume 3 (1991), pp 1-79, we stated (see Theorem J, or Theorem 6.2) that if $q \in U$ and $\varphi \in L^\infty(\lambda_q)$ is ≥ 0 and not a.e. zero, then

$$\lim_{n \to +\infty} \frac{\displaystyle\sum_{i=0}^{n} \int (\varphi \circ f^i) \psi \, d\lambda_q}{\displaystyle\sum_{i=0}^{n} \int (\varphi \circ f^i) \, d\lambda_q} = \int \psi \, d\lambda_q \qquad (*)$$

holds for every $\psi \in L^1(\lambda_q)$.

Jon Aaronson observed to us that this contradicts a result of his. In fact, our proof has a mistake (to be explained below). The mistake, as we shall see, disappears *replacing* $\varphi \geq 0$ *by* $\inf \varphi > 0$. However, *a much more interesting* substitute for the above property is the following theorem, that is the original one with the hypotheses $\varphi \in L^\infty, \psi \in L^1$ interchanged.

Claus I. Doering
Instituto de Matemática e Estatística - UFRGS
cdoering@mat.ufrgs.br

Theorem 6.3. (*) *holds for every* $q \in U$ *and* $\varphi \in L^1(\lambda_q)$ *that is* ≥ 0 *and not a.e. zero, and every* $\psi \in L^\infty(\lambda_q)$.

First we shall prove the theorem. Afterwards we shall show where the published mistake is and how to trivially circumvent it when $\inf \varphi > 0$.

Lift $f|U \colon U \hookleftarrow$ to an inner function $\hat{f} : D \hookleftarrow$, where $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ is the open unit disk. We shall prove that if $q \in D$, then

$$\lim_{n \to +\infty} \frac{\displaystyle\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i) \psi d\lambda_q}{\displaystyle\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i) d\lambda_q} = \int \psi d\lambda_q \qquad (**)$$

for functions $\psi, \varphi : \partial D \to \mathbb{R}$ satisfying $\varphi \in L^1(\lambda_q)$, $\varphi \geq 0$ and not a.e. zero, and $\psi \in L^\infty(\lambda_q)$. Clearly (**) implies (*). To prove (**) we begin assuming that ψ is a Radon-Nykodim derivative $\psi = d\lambda_x / d\lambda_q$. Then we have to show

$$\lim_{n \to +\infty} \frac{\displaystyle\sum_{i=0}^{n} \int \varphi d\lambda_{\hat{f}^i(x)}}{\displaystyle\sum_{i=0}^{n} \int \varphi d\lambda_{\hat{f}^i(q)}} = 1. \qquad (***)$$

But if $\tilde{\varphi}$ is the harmonic extension of φ, this is equivalent to

$$\lim_{n \to +\infty} \frac{\displaystyle\sum_{i=0}^{n} \tilde{\varphi}\big(\hat{f}^i(x)\big)}{\displaystyle\sum_{i=0}^{n} \tilde{\varphi}\big(\hat{f}^i(q)\big)} = 1.$$

Since $\hat{f}$ is the lifting of a rational map restricted to a fixed parabolic basin, we know from our Theorem 6.1 that given $\alpha > 1/2$, the inequality

$$1 - \left|\hat{f}^i(q)\right| \geq \frac{1}{i^\alpha} \qquad (1)$$

holds for every sufficiently large i. This implies that $\hat{f}$ is recurrent, hence ergodic, and then

$$\lim_{i \to +\infty} d_P\big(\hat{f}^i(x), \hat{f}^i(q)\big) = 0, \qquad (2)$$

where $d_P(\cdot,\cdot)$ is the Poincaré metric. Since $\varphi \geq 0$, it follows from (1) that

$$\tilde{\varphi}\big(\hat{f}^i(q)\big) \geq \frac{C}{i^\alpha}$$

for some $C > 0$ and large i. Hence

$$\sum_i \tilde{\varphi}\big(\hat{f}^i(q)\big) = +\infty. \tag{3}$$

From (2) we get

$$\lim_{i\to+\infty} \frac{\tilde{\varphi}\big(\hat{f}^i(x)\big)}{\tilde{\varphi}\big(\hat{f}^i(q)\big)} = 1. \tag{4}$$

From (3) and (4) follows (***). Now let us prove (**) assuming that ψ is continuous. Let C^0 be the space of continuous functions $\psi : \partial D \to \mathbb{R}$ with the maximum norm $\|\psi\|_0 = \max_z |\psi(z)|$. Observe that $\varphi \geq 0$ implies

$$\left| \frac{\displaystyle\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i)\psi d\lambda_q}{\displaystyle\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i) d\lambda_q} \right| \leq \|\psi\|_0.$$

Hence, if we prove (*) for a dense set of $\psi \in C^0$, it will follow for every $\psi \in C^0$. But finite linear combinations of functions of the form $d\lambda_x/d\lambda_q$, $x \in D$, are a dense subset of C^0, and for them we have already checked (**). This completes the proof of (**) assuming ψ continuous. Now we want to prove it for $\psi \in L^\infty(\lambda_q)$. This will follow from an approximation procedure that requires the following remark.

Lemma 1. *For every Borel set $A \subset \partial D$ we have*

$$\limsup_{n\to+\infty} \frac{\displaystyle\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i)\psi_A d\lambda_q}{\displaystyle\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i) d\lambda_q} \leq \lambda_q(A), \tag{5}$$

where ψ_A denotes the characteristic function of A.

To prove the lemma, simply observe that for every Borel set $A \subset \partial D$ and every $\varepsilon > 0$ there exists a continuous function $\psi \geq \psi_A$ with

$$\int \psi d\lambda_q < \lambda_q(A) + \varepsilon.$$

Then the lim sup in (5) is bounded by (**) for the selected ψ. Hence the lim sup in (5) is $\leq \lambda_q(A) + \varepsilon$. Since ε is arbitrary, the lemma is proved.

Now let us prove (**) when ψ is the characteristic function ψ_A of a Borel set $A \subset \partial D$. Take a sequence of compact sets $K_m \subset A$ and open sets $V_m \supset A$ such that $\lim_{m\to+\infty} \lambda_q(V_m - K_m) = 0$. Take continuous functions $\psi_m : \partial D \to \mathbb{R}$ with $\|\psi_m\|_0 = 1, \psi_m/K_m \equiv 1, \psi_m/V_m^c \equiv 0$. Then

$$\limsup_{n\to+\infty} \frac{\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i)(\psi_A - \psi_m) d\lambda_q}{\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i) d\lambda_q}$$

$$\leq \limsup_{n\to+\infty} \frac{\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i)\psi_{V_m - K_m} d\lambda_q}{\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i) d\lambda_q} \leq \lambda_q(V_m - K_m).$$

Hence, since (**) holds for ψ_m, we have

$$\limsup_{n\to+\infty} \left| \frac{\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i)\psi_A d\lambda_q}{\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i) d\lambda_q} - \int \psi_m d\lambda_q \right| \leq \lambda_q(V_m - K_m).$$

Taking limit when $m \to +\infty$,

$$\lim_{n\to+\infty} \frac{\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i)\psi_A d\lambda_q}{\sum_{i=0}^{n} \int (\varphi \circ \hat{f}^i) d\lambda_q} = \int \psi_A d\lambda_q \,.$$

This proves (**) when $\psi = \psi_A$, and then (**) follows if ψ is a finite linear combination of characteristic functions. From this the final case $\psi \in L^\infty(\lambda_q)$ follows observing that for every $\varepsilon > 0$, there exist finite linear combinations of characteristic functions $\psi_1 \leq \psi \leq \psi_2$ with $\|\psi_2 - \psi_1\|_\infty < \varepsilon$, and applying the previous case to ψ_1 and ψ_2.

Finally, the error in the proof of Theorem 6.2 (or Theorem J) lies in the first inequality of its proof, where we essentially asserted the existence of $C > 0$ such that

$$\left| \frac{\sum_{i=0}^{n} \int (\varphi \circ f^i)\psi d\lambda_q}{\sum_{i=0}^{n} \int (\varphi \circ f^i) d\lambda_q} \right| \leq C\|\psi\|_1$$

for all q and $\psi \in L^1(\lambda_q)$. But when $\inf \varphi > 0$, this is indeed true, taking $C = \dfrac{\|\varphi\|_\infty}{\inf \varphi}$, and then the rest of the proof of that theorem remains correct.