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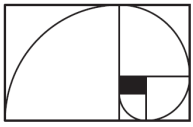

Spread-out oriented percolation and related models above the upper critical dimension: induction and superprocesses

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Abstract. In these notes we give an extensive survey of the recent progress for critical spread-out oriented percolation above the upper critical dimension. We describe the main tools, which are the lace expansion and the inductive method. The lace expansion gives a recursion relation for the two-point functions involved, and the inductive method gives an inductive analysis of the arising recursion relation. These results apply also to self-avoiding walk. We further describe the scaling results for the oriented percolation higher-point functions, and compare these to their branching random walk analogues. Finally, we discuss the relations between scaling limits of critical branching models to super-processes, which are random measures evolving diffusively in time.

Contents

1 Introduction

Random walks and branching random walks are paradigm models appearing in various applications. For example, random walks can be used to model the irregular motion of moving particles, whereas branching random walk are the simplest model for populations where individuals move and produce offspring of their own. In many cases, however, phenomena are more realistically described by adding a kind of *interaction*. For example, for a linear polymer, the different building blocks of the polymer are repelling each other, due to the repellent forces between the building blocks. Such a linear polymer of size n is therefore more realistically modeled by a *self-avoiding walk*, which is an n-step random walk conditioned on not having any self-intersections. Likewise, when one tries to model the spread of a disease in a static population, branching random walk is too crude, since in branching random walk, an individual who is infected from various sources is counted more than once. For *oriented percolation*, this overcounting does not take place, and it therefore makes a more realistic model for the spread of a disease in a static population. Random walk and branching random walk can often be seen as the *mean-field* version of the interacting models, for which the interaction is not present. Needless to say, self-avoiding walk and oriented percolation are still caricatures of the intricate reality of polymers and the spread of diseases, respectively, but already these relatively simple changes to the random walk and branching random walk models make the investigation very difficult indeed.

In many cases, models which can be viewed as interacting versions of random walks and/or branching random walks are expected to have an *upper critical dimension*, above which the scaling ceases to depend on the dimension. For example, for self-avoiding walk, it was long conjectured in the physics community that above 4 dimensions, the scaling limit is Brownian motion, and thus the self-avoiding walk is a small perturbation of the simpler random walk model. In this paper, we describe the *lace expansion*, which has been quite successful in proving that such an upper critical dimension exists, and that the scaling limit is the same as the scaling limit of the mean-field model above the so-called upper critical dimension. This method further allows to prove conjectures long predicted in the physics community, such as the existence of *critical exponents*, which summarize the systems close to criticality, and *universality*, which states that critical exponents and scaling limits should not depend on the precise details of the model. These two predictions from physics have proved to be extremely hard to establish rigorously, so that the cases above the critical dimension are important examples suggesting a more general validity of these physics predictions.

The goal of this survey is to review recent results on the application of the lace expansion. A much more substantial survey has appeared in the Saint-Flour notes of Gordon Slade [62], which covers the entire history of

the method, as well as most of the recent applications, among others to random graphs. Therefore, we have decided to focus on two topics which do not appear in full detail in [62]. The first main topic of this paper is the *inductive method*, appearing for the first time in [33]. The inductive method allows to prove Gaussian scaling for two-point functions, which describe linear structures evolving in time, such as paths in percolation clusters and self-avoiding walks, using induction in time. The inductive method can be used to show that the endpoints of these paths scale to normal distributions, just as for Brownian motion, and suggests scaling to Brownian motion. Naturally, Brownian motion arises as the scaling limit of random walk, the simplest imaginable model of a particle moving randomly in time. The second first main topic of this paper is the application of the results obtained by the inductive method to prove convergence to *super-processes*. Super-processes are measure-valued diffusions, and describe the diffusive behaviour of the measures describing the location of particles as time evolves. The main example of a super-process is *super-Brownian motion*, which arises as the scaling limits of critical branching random walk.

Naturally, in this survey, we cannot cover the wide variety of results obtained using the lace expansion. For a more detailed account of the history of the problem, as well as for more extensive references to the literature, especially the papers prior to 1998, we refer to [62].

This survey consists of three main parts. In Section 2, we state the results for two-point functions and describe the inductive method, in Section 3, we describe the results and proofs for branching structures, and in Section 4, we describe the connection to super-processes.

2 Two-point functions

2.1 Introduction of the models

In this section, we formulate the models, and state the main results on the two-point functions, which describe point-to-point connections in the model. We start with the basic example of spread-out random walks, and then define the spread-out self-avoiding walk and oriented percolation models.

2.1.1 Spread-out random walk

Before moving to self-avoiding walk and oriented percolation, the main statistical mechanical models to be studied in this paper, we describe the simpler situation of random walks. This section allows to introduce notation, and to derive results for random walks that we will later prove for self-avoiding walk and oriented percolation.

We define the *random walk two-point function* by

$$p_n(x) = \sum_{\omega \in \mathcal{W}_n(x)} W(\omega), \tag{2.1.1}$$

where $\mathcal{W}_n(x)$ denotes the set of n-step random walk paths from 0 to x and

$$W(\omega) = \prod_{i=1}^{n} D(\omega(i) - \omega(i-1)) \tag{2.1.2}$$

is the weight of the path $\omega = (\omega(0), \dots, \omega(n))$. It is natural to assume that D is *normalised*, i.e., that

$$\sum_{x \in \mathbb{Z}^d} D(x) = 1. \tag{2.1.3}$$

Then, (2.1.1) is nothing but the probability that a random walker with step distribution $y \mapsto D(y)$ is at $x \in \mathbb{Z}^d$ at time n. We note that p_n satisfies the recursion relation

$$\begin{aligned} p_{n+1}(x) &= \sum_{\omega \in \mathcal{W}_{n+1}(x)} \sum_{y \in \mathbb{Z}^d : \omega(1) = y} W(\omega) \\ &= \sum_{y \in \mathbb{Z}^d} D(y) \sum_{\omega' \in \mathcal{W}_n(x-y)} W(\omega') = (D * p_n)(x), \end{aligned} \tag{2.1.4}$$

where, for two summable functions $f, g \colon \mathbb{Z}^d \to \mathbb{R}$, we define $f * g$ to be the *convolution* of f and g, i.e.,

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x - y). \tag{2.1.5}$$

Clearly, we can solve (2.1.4) to obtain

$$p_n(x) = D^{*n}(x), \tag{2.1.6}$$

where D^{*n} is the n-fold convolution of D. In this paper, we will describe recurrence relations that are similar in spirit to (2.1.4), but which are substantially more involved, so that the above simple solution is no longer feasible. Therefore, we reside to stronger analytical means to study the asymptotics of such recurrence relations.

The function D is assumed to be invariant under the symmetries of $\mathbb{Z}^d$ (permutation of coordinates and replacement of any coordinate by its negative). We further assume that

$$\sigma^2 = \sum_{x \in \mathbb{Z}^d} |x|^2 D(x) < \infty, \tag{2.1.7}$$

where $|\cdot|$ denotes the Euclidian distance in $\mathbb{Z}^d$. It is well know that under the above conditions, the endpoint $\omega(n)$, which has probability mass function p_n, scales as a Gaussian random variable. More precisely, we have that the random variable

$$\frac{\omega(n)}{\sqrt{\sigma^2 n}}, \tag{2.1.8}$$

converges in distribution to a vector of d independent standard normal random variables. One way of showing this is by investigating the *Fourier transform*, which, for a summable function $f\colon \mathbb{Z}^d \to \mathbb{R}$, is defined by

$$\hat{f}(k) = \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} f(x). \tag{2.1.9}$$

We note that the characteristic function of $\omega(n)$, the position of the random walker after n-steps, is equal to

$$\mathbb{E}[e^{ik\cdot\omega(n)}] = \hat{p}_n(k) = \hat{D}(k)^n, \tag{2.1.10}$$

where $\mathbb{E}$ denotes the expectation with respect to the law of the random walk ω. The characteristic function of the rescaled position of the random walker after n steps given in (2.1.8) is equal to

$$\hat{p}_n\big(\frac{k}{\sqrt{\sigma^2 n}}\big). \tag{2.1.11}$$

For k small, when D has $2+2\epsilon$ moments for some $\epsilon > 0$, i.e., when

$$\sum_{x\in\mathbb{Z}^d} |x|^{2+2\epsilon} D(x) < \infty, \tag{2.1.12}$$

then

$$\hat{D}(k) = 1 - \sigma^2 \frac{|k|^2}{2d} + O(|k|^{2+2\epsilon}). \tag{2.1.13}$$

Therefore, we obtain that

$$\begin{aligned} \hat{p}_n\big(\frac{k}{\sqrt{\sigma^2 n}}\big) = \big[\hat{D}\big(\frac{k}{\sqrt{\sigma^2 n}}\big)\big]^n &= \big[1 - \frac{|k|^2}{2dn} + O(|k|^{2+2\epsilon} n^{-(1+\epsilon)})\big]^n \\ &= e^{-\frac{|k|^2}{2d} + O(|k|^{2+2\epsilon} n^{-\epsilon})}. \end{aligned} \tag{2.1.14}$$

Furthermore, we obtain that

$$\sum_{x\in\mathbb{Z}^d} |x|^2 p_n(x) = \sum_{x\in\mathbb{Z}^d} |x|^2 D^{*n}(x) = n \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) = n\sigma^2, \tag{2.1.15}$$

since $\sum_{x\in\mathbb{Z}^d} xD(x) = 0$ by symmetry, and the variance of a sum of n i.i.d. random variables is n times the variance of each of the random variables. Finally, a *local central limit theorem* can be used to show that,

for $x \in \mathbb{R}^d$, the rescaled transition probability $x \mapsto p_n\big(\lceil \frac{x}{\sqrt{n}} \rceil\big)$, where $\lceil x \rceil = (\lceil x_1 \rceil, \dots, \lceil x_d \rceil)$, is close to $n^{-d/2}$ times the density of a normal random variable with variance σ^2. In particular, this shows that there exist constants $0 < C_1 < C_2 < \infty$ such that for all $n \geq 1$

$$C_1 \sigma^{-d} n^{-d/2} \leq \sup_{x \in \mathbb{Z}^d} p_n(x) \leq C_2 \sigma^{-d} n^{-d/2}. \tag{2.1.16}$$

Note that the scaling behaviour described above is insensitive to the precise details of the model, in the sense that scaling and asymptotic behaviour are independent of the precise transition law D, and only depend on the symmetry of the steps with respect to rotations by 90 degrees and the fact that the second moment exists. We conclude that random walk shows *universal behaviour*. This is a weak form of universality. A stronger version of universality would say that also when the steps are weakly dependent, the scaling limit is Brownian motion. There are many examples of such universal results known for weakly-interacting random walks. Examples are m-dependent random walks, where the steps are an m-dependent process, i.e., a process $\omega = (\omega(n))_{n=0}^{\infty}$ where, for every $k \in \mathbb{N}$ and conditionally on $(\omega(n))_{n=k+1}^{k+m}$, the processes $(\omega(n))_{n=0}^{k}$ and $(\omega(n))_{n=k+m+1}^{\infty}$ are independent.

The main goal of this paper is to survey similar results for interacting models, where the simple recurrence relation (2.1.4) is replaced by a more intricate recurrence relation. An example of such a model is the *self-avoiding walk*, which is discussed in more detail in Section 2.1.2. The interaction for self-avoiding walk is *not* weak, so that different scaling can occur. For such models, it will be technically convenient to deal with cases where the interaction is not too strong. This can be achieved by making the range of the random walk step-distribution large, i.e., by making the walk *spread-out*. For example, for self-avoiding walk, if the range of the walk increases, then the self-avoidance constraint deceases in severity. However, spread-out self-avoiding walks are expected to be in the same universality class as nearest-neighbour self-avoiding walks, so that the difficulties in dealing with the interaction persist.

We now formulate the precise assumption on the random walk transition function D. We assume that D obeys the restrictions formulated in (2.1.7) and (2.1.12) above, and to obey Assumption D in [40, Section 1.2]. This assumption entails that there is a constant C such that, for all $L \geq 1$,

$$\|D\|_\infty \leq CL^{-d}, \qquad \sigma^2 = \sum_{x \in \mathbb{Z}^d} |x|^2 D(x) \leq CL^2, \tag{2.1.17}$$

and that there exist constants $\eta, c_1, c_2 > 0$ such that

$$[c_1(Lk)^2] \wedge \eta \leq 1 - \hat{D}(k) \leq [c_2(Lk)^2] \wedge \eta, \qquad \hat{D}(k) > -1 + \eta \quad (k \in [-\pi, \pi]^d). \tag{2.1.18}$$

A simple and basic example is

$$D(x) = \begin{cases} (2L+1)^{-d} & \text{if } \|x\|_\infty \le L, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.19}$$

In this example, the random walker makes a uniform step inside the d-dimensional cube of width $2L+1$.

2.1.2 Self-avoiding walk

An excellent introduction to self-avoiding walks can be found in [49], to which we refer for more details. We begin by introducing the *spread-out self-avoiding walk two-point function*. A path $\omega = (\omega(i))_{i=0}^n$ is called *self-avoiding* when $\omega(i) \neq \omega(j)$ for all $0 \le i < j \le n$. For $x \in \mathbb{Z}^d$, we set $c_0(x) = \delta_{0,x}$ and, for $n \ge 1$, we define

$$c_n(x) \quad = \sum_{\omega \in \mathcal{C}_n(x)} W(\omega), \tag{2.1.20}$$

where $\mathcal{C}_n(x)$ denotes the set of n-step self-avoiding walk paths from 0 to x and we recall the definition of $W(\omega)$ in (2.1.2). The two-point function $c_n(x)$ is the probability that the walker ends at x at time n, without ever revisiting a previously visited site. The self-avoidance constraint is severe, and despite the simple definition of the model, many facets of self-avoiding walks are still not mathematically understood.

The Fourier transform of (2.1.20) is written

$$\hat{c}_n(k) = \sum_{x \in \mathbb{Z}^d} c_n(x) e^{ik\cdot x}, \qquad k \in (-\pi, \pi]^d, \tag{2.1.21}$$

and we use the abbreviation

$$c_n = \hat{c}_n(0) = \sum_{x \in \mathbb{Z}^d} c_n(x). \tag{2.1.22}$$

Self-avoiding walks are a caricature model of polymers in a good solvent, where the self-avoidance constraint models the repulsive forces between the building blocks of the polymer. Self-avoiding walks have attracted a tremendous amount of attention in the literature, both by theoretical physicists with non-rigorous means, as well as by mathematicians with rigorous means.

If we define a path measure $\mathbb{Q}_n$ by

$$\mathbb{Q}_n(\omega) = \frac{1}{c_n} W(\omega), \tag{2.1.23}$$

for all n-step self-avoiding walks, and $\mathbb{Q}_n(\omega) = 0$ otherwise, then the law of $(\omega(1), \ldots, \omega(n))$ is *not* consistent, in the sense that the law of

$(\omega(1),\dots,\omega(n))$ under $\mathbb{Q}_{n+1}$ is not the same as the law of $(\omega(1),\dots,\omega(n))$ under $\mathbb{Q}_n$. Therefore, we cannot think of $(\omega(1),\dots,\omega(n))$ under $\mathbb{Q}_n$ as referring to a stochastic process evolving in time.

In theoretical physics, it is common to describe the scaling behaviour in terms of *critical exponents*, the existence of which is highly non-trivial. A prediction commonly made in theoretical physics is the following:

Conjecture 2.1. *There exist γ,ν such that*

$$c_n \sim An^{\gamma-1}\mu^n, \qquad \frac{1}{c_n}\sum_x |x|^2 c_n(x) \sim Dn^{2\nu} \quad (n\to\infty), \tag{2.1.24}$$

where $\gamma=\gamma(d),\nu=\nu(d)$ are critical exponents that are believed to be independent of L and

$$\begin{array}{llllll} \nu & =1 & \gamma & =1 & d=1, \\ & =\frac{3}{4} & & =\frac{43}{32} & d=2, \\ & =0.588\dots & & =1.162\dots & d=3, \\ & =\frac{1}{2} & & =1 & d\geq 4. \end{array} \tag{2.1.25}$$

For $d=4$, it is believed that there exist logarithmic corrections.

The fact that the critical exponents are independent of the details of the model (such as the range L) is called *universality*. Universality is one of the basic predictions in physics and is expected to hold for most of the paradigm models in statistical mechanics, but mathematical proofs for universality are scarce.

Note that for random walk, the scaling in (2.1.25) holds, with critical exponents $\nu_{\text{RW}}=\frac{1}{2}$ and $\gamma_{\text{RW}}=1$. Therefore, the critical exponents for random walk are universal, and it can be expected that the same holds for related physical models such as the self-avoiding walk. Interestingly, the critical exponents ν and γ for self-avoiding walk are expected to depend on the dimension d. However, when $d\geq 4$, the critical exponents no longer depend on the dimension and take on the values for random walk. In physics, this is termed that the *upper critical dimension* equals 4 and the mean-field model is random walk. Logarithmic corrections are commonly assumed to exist in the critical dimension. For example, in $d=4$, it is expected that $c_n \sim A(\log n)^{1/4}\mu^n$.

Flory [20], see also [49, Chapter 2], has given a very simple argument that suggests that $\nu=\frac{3}{d+2}$ when $d\leq 4$ and $\nu=\frac{1}{2}$ when $d\geq 4$. Despite the simplicity of the argument, the value obtained in the heuristic seems to be the predicted value except for the case where $d=3$. See also [32] for a slightly improved version of the heuristic applying to dimension $d=2$, which is based on scaling for the one-dimensional case and crude heuristics.

Mathematical proofs for Conjecture 2.1 only exist in some cases. Even though nearest-neighbour self-avoiding walk in dimension 1 is rather trivial, and obeys (2.1.24) with $c_n=2$ and $\frac{1}{c_n}\sum_{x\in\mathbb{Z}}|x|^2c_n(x)=n^2$, spread-out

self-avoiding walk is highly non-trivial even in dimension one. In [45], it is shown that $\nu(1) = 1, \gamma(1) = 1$ for a general class of random walk step distributions D, thereby showing universality.

There has been recent progress in the understanding of two-dimensional self-avoiding walk, using the notion of *conformal invariance*. The values of the critical exponents have been predicted by Nienhuis [52], and these values have been confirmed by Monte Carlo simulations. There is also a mathematical explanation for these remarkable values by Werner, Lawler and Schramm. Roughly speaking, it can be expected that important questions for the scaling behavior of two-dimensional self-avoiding walk could be answered if we would know that the scaling limit of self-avoiding walk is conformally invariant. However, even a proper formulation of the latter is not obvious. See [46] for a discussion on the links between Stochastic/Schramm Loewner Evolution and self-avoiding walks, where, among other things, it is shown that *if* the scaling limit exists and is conformally invariant, then it is $\mathrm{SLE}_{\frac{8}{3}}$. See also the review paper [64] and the references therein.

Any understanding of the three-dimensional case is lacking, as even the physics literature does not offer any acceptable heuristics. There are even no conjectures what the values of the critical exponents are, even though there are estimates obtained using substantial Monte Carlo simulations.

An important property used to study self-avoiding walks is its *self-repellence*. We give an example of its use by showing that if γ exists, then $\gamma \geq 1$. This argument is based upon *sub-multiplicativity*, and goes as follows. We note that the set of $n + m$ step self-avoiding walks is a subset of the set of concatenations of n and m step self-avoiding walks. Indeed, the set of $n + m$ step self-avoiding walks is the set of concatenations of n and m step self-avoiding walks, where the two self-avoiding walks are also *mutually avoiding*. This shows that

$$c_{n+m} \leq c_n c_m. \tag{2.1.26}$$

As a result, the sequence $\log c_n$ is *subadditive*, so that the limit

$$\lim_{n\to\infty} \frac{1}{n} \log c_n \equiv \log \mu \tag{2.1.27}$$

exists. Moreover,

$$\log \mu = \inf_{n\geq 1} \frac{1}{n} \log c_n. \tag{2.1.28}$$

As a consequence, we obtain that for all $n \geq 1$,

$$c_n \geq \mu^n, \tag{2.1.29}$$

so that if the critical exponent γ exists, then it must satisfy $\gamma \geq 1$. The above bound is a *mean-field bound*, and similar bounds exist for related percolation models.

The main focus of this paper is the study of the models above the upper-critical dimension. In this case, the existence of the critical exponents ν and γ can be shown. A much stronger result is formulated in the following theorem:

Theorem 2.1. *Let $d > 4$ and $\delta \in (0, \epsilon \wedge 1 \wedge \frac{d-4}{2})$, where ϵ is as in (2.1.12). There is an L_0 such that for $L \geq L_0$ there exist positive constants v, μ, and A (all depending on d and L), and C_1, C_2 (depending on d but not L) such that the following statements hold as $n \to \infty$:*
(a) For all $k \in \mathbb{R}^d$ with $|k|^2$ bounded by a constant,

$$\hat{c}_n(k(\sigma^2 vn)^{-1/2}) = A\mu^n e^{-|k|^2/2d}\big[1+\mathcal{O}(n^{-(d-4)/2})+\mathcal{O}(|k|^2 n^{-\delta})\big]. \quad (2.1.30)$$

(b)

$$\frac{1}{c_n}\sum_{x\in\mathbb{Z}^d} |x|^2\, c_n(x) = \sigma^2 vn\big[1+\mathcal{O}(n^{-\delta})\big]. \quad (2.1.31)$$

(c)

$$C_1\mu^n L^{-d} n^{-d/2} \leq \sup_{x\in\mathbb{Z}^d} c_n(x) \leq C_2\mu^n L^{-d} n^{-d/2}. \quad (2.1.32)$$

Theorem 2.1 proves that Conjecture 2.1 indeed holds, with $\nu = \frac{1}{2}$, $\gamma = 1$, just as for random walk. Equations (2.1.30)–(2.1.31) also provide error estimates. Furthermore, (2.1.30) proves that the endpoint scales to a normal distribution, while (2.1.32) is a statement reminiscent to (2.1.16), when we note that σ is proportional to L. Therefore, we can summarize Theorem 2.1 by saying that self-avoiding walk above 4 dimensions is a *small perturbation* of random walk. Note that the independence of γ and ν on L prove a weak form of universality, even though universality is expected to hold much more generally indeed.

Theorem 2.1 was proved in [42], using the inductive method in [40]. This proof will be described in some detail below. A version of Theorem 2.1 with somewhat weaker error bounds was first proved in [49, Theorem 6.1.1] using generating functions. See also Section 2.6.1 below for an explanation of this method.

2.1.3 The spread-out oriented percolation two-point function

We now introduce the second key model to be studied in this paper, namely, spread-out oriented percolation. Percolation models have attracted a tremendous amount of attention in the statistical mechanics community, since they are very simple to formulate, yet share interesting characteristics such as phase transitions, predictions of existence of critical exponents, and universal behaviour with other common models in statistical mechanics. In this section, we study *oriented percolation*, which is a percolation model that evolves in time.

Let $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$. Consider the graph with vertices $\mathbb{Z}^d \times \mathbb{Z}_+$ and with directed bonds $((x,n),(y,n+1))$, for $n \in \mathbb{Z}_+$ and $x, y \in \mathbb{Z}^d$. Let D be a fixed function $D\colon \mathbb{Z}^d \to [0,1]$, satisfying $\sum_{x\in\mathbb{Z}^d} D(x) = 1$. Let $p \in [0, \|D\|_\infty^{-1}]$, where $\|\cdot\|_\infty$ denotes the supremum norm, so that $pD(w) \leq 1$ for all $w \in \mathbb{Z}^d$. We associate to each directed bond $((x,n),(y,n+1))$ an independent random variable taking the value 1 with probability $pD(y-x)$ and the value 0 with probability $1 - pD(y-x)$. We say that a bond is *occupied* when the corresponding random variable is 1 and *vacant* when it is 0. Note that p is *not* a probability. Rather, p is the average number of occupied bonds from a given vertex. The joint probability distribution of the bond variables will be denoted by $\mathbb{P}_p$ and the corresponding expectation by $\mathbb{E}_p$.

In the example (2.1.19), the bonds are given by $((x,n),(y,n+1))$ with $\|x-y\|_\infty \leq L$, and a bond is occupied with probability $p(2L+1)^{-d}$. Assumption D also allows for certain infinite range models.

One can think of oriented percolation as a caricature model to describe the evolution of a disease in a static population. In this description, we start with a single infected individual at time 0 located at the origin in $\mathbb{Z}^d$. This individual can infect other individuals which are located close to him/her with certain probabilities that are indicated by pD, and every individual recuperates from the disease with probability $1 - pD(0)$. Alternatively, one can think of oriented percolation as a version of percolation with a preferred direction.

We now introduce some notation. We say that (x,n) is *connected* to (y,m), and write $(x,n) \longrightarrow (y,m)$, if there is an oriented path from (x,n) to (y,m) consisting of occupied bonds. Note that this is only possible when $m \geq n$. By convention, (x,n) is connected to itself. We write $C(x,n) = \{(y,m) : (x,n) \longrightarrow (y,m)\}$ to denote the forward cluster of (x,n).

Oriented percolation has a *phase transition* when p increases. Indeed, with

$$\theta(p) = \mathbb{P}_p(|C(0,0)| = \infty), \tag{2.1.33}$$

there exists a p_c such that

$$\theta(p) = 0 \quad \text{for } p < p_c, \qquad \theta(p) > 0 \quad \text{for } p > p_c. \tag{2.1.34}$$

Moreover, it is shown in [2] that critical oriented percolation does not percolate, i.e., $\theta(p_c) = 0$.

We define the *spread-out oriented percolation two-point function* by

$$\tau_n(x) = \mathbb{P}_p((0,0) \longrightarrow (x,n)). \tag{2.1.35}$$

The two-point function describes path in percolation clusters, and has its most intricate behaviour when $p = p_c$, i.e., for critical oriented percolation. Indeed, when $p < p_c$, it can be shown that $\sum_{x\in\mathbb{Z}^d} \tau_n(x)$ decays exponentially in n when $n \to \infty$, while for $p > p_c$, $\sum_{x\in\mathbb{Z}^d} \tau_n(x)$ grows proportionally

to n^d. The above behaviour is similar in all dimensions d. At criticality, scaling is expected to depend sensitively on the dimension, as is the case for self-avoiding walk as formulated in Conjecture 2.1.

The main result in this section is the following theorem, which is proved in [40]:

Theorem 2.2. *Let $d > 4$, $p = p_c$, and $\delta \in (0, 1 \wedge \epsilon \wedge \frac{d-4}{2})$. There is an $L_0 = L_0(d)$ such that for $L \geq L_0$ there exist positive constants v and A (depending on d and L), and C_1, C_2 (depending only on d), such that the following statements hold as $n \to \infty$:*
(a)

$$\hat{\tau}_n(k/\sqrt{v\sigma^2 n}) = Ae^{-\frac{|k|^2}{2d}}[1 + \mathcal{O}(|k|^2 n^{-\delta}) + \mathcal{O}(n^{-(d-4)/2})], \tag{2.1.36}$$

(b)

$$\frac{1}{\hat{\tau}_n(0)} \sum_x |x|^2 \tau_n(x) = v\sigma^2 n[1 + \mathcal{O}(n^{-\delta})], \tag{2.1.37}$$

(c)

$$C_1 L^{-d} n^{-d/2} \leq \sup_{x \in \mathbb{Z}^d} \tau_n(x) \leq C_2 L^{-d} n^{-d/2}, \tag{2.1.38}$$

with the error estimate in (a) uniform in $k \in \mathbb{R}^d$ with $|k|^2 (\log n)^{-1}$ sufficiently small.

Note that Theorem 2.2 proves a similar result for the critical spread-out oriented percolation two-point function as for the spread-out self-avoiding walk two-point function in Theorem 2.1. We can interpret Theorem 2.2 by saying that paths in critical oriented percolation clusters are like random walk paths.

Parts (a) and (b) of Theorem 2.2 were first proved by Nguyen and Yang [51] using generating function methods, with somewhat weaker error estimates.

2.2 The lace expansion for two-point functions

The proof of Theorems 2.1 and 2.2 make use of an expansion technique called the *lace expansion*, which proves a recurrence relation for the two-point functions of self-avoiding walk and oriented percolation. The lace expansion has been used to study several statistical mechanical models above the upper critical dimension, and we will describe some related results below.

Any successful application of the lace expansion consists of three main steps:

1. The derivation of the expansion.
2. The bounds on the expansion coefficients.

3. The analysis of the recurrence relation.

The expansions for self-avoiding walk and oriented percolation will be described in this section, the bounds on the coefficients in Section 2.3, and the analysis of the recurrence relation in Section 2.4. As we will explain in more detail below, the three main steps depend sensitively on each other. There are many models for which we can derive some form of an expansion, but since bounds on the coefficients are missing, no results can be obtained. Examples of such cases are self-avoiding walk and oriented percolation in dimension $d \leq 4$, for which the expansions apply, but the bounds on the coefficients fail. In these examples, Gaussian behaviour as formulated in Theorems 2.1 and 2.2 is even not expected to hold. Another example for which Theorem 2.1 is expected to hold, but the lace expansion fails is for nearest-neighbour self-avoiding walk with weak nearest-neighbour attraction. See also Section 2.5.3 below. We will see that the use of the lace expansion is quite delicate and technically challenging.

2.2.1 The expansion of the self-avoiding walk two-point function

The lace expansion is a combinatorial identity for $\hat{c}_n$ in terms of a function $\hat{\pi}_m(k)$, defined in (2.2.22) below, stating that

$$\hat{c}_{n+1}(k) = \hat{D}(k)\hat{c}_n(k) + \sum_{m=2}^{n+1} \hat{\pi}_m(k)\hat{c}_{n+1-m}(k). \tag{2.2.1}$$

For self-avoiding walk, there are two substantially different derivations of the lace expansion. We will describe both of these derivations below. The first is based upon *inclusion-exclusion*, and obtains the recursion relation by ignoring interaction and making up for the arising error (see also [60]). This method is quite powerful, and will also be used to deal with oriented percolation below. The second derivation uses a rewrite in terms of *graphs*, and in this expansion the laces that give the lace expansion its name, appear (see [6]). This derivation can also be adapted to lattice trees and lattice animals, as well as to oriented percolation.

The inclusion-exclusion derivation of the expansion.

Define $R^{(1)}_{n+1}(x)$ by

$$c_{n+1}(x) = \sum_{y\in\mathbb{Z}^d} D(y)c_n(x-y) - R^{(1)}_{n+1}(x). \tag{2.2.2}$$

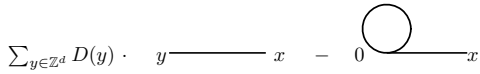
The term $R^{(1)}_{n+1}(x)$ is the contribution of walks that contribute to the first term on the right-hand side of (2.2.2), but not on the left-hand side. Therefore, this contribution is due to paths that have at least one self-intersection. Since the first term on the right-hand side of (2.2.2) can alternatively be

seen as the contribution from concatenations of a step from 0 to some y and a self-avoiding walk from y to x, this self-intersection must be at the origin. The inclusion-exclusion derivation of the lace expansion studies the correction term $R^{(1)}_{n+1}(x)$ in more detail by using inclusion-exclusion on the avoidance properties of the paths involved.

Let $\mathcal{P}^{(1)}_{n+1}(x)$ to be the set of paths $\omega \in \mathcal{W}_{n+1}(x)$ which contribute to $R^{(1)}_{n+1}(x)$, i.e., the walks ω for which there exists an $l \in \{2,\dots,n+1\}$ (depending on ω) with $\omega(l) = 0$ and $\omega(i) \neq \omega(j)$ for all $i \neq j$ with $\{i,j\} \neq \{0,l\}$. For the special case $x = 0$, $\mathcal{P}_{n+1}(0)$ is the set of $(n+1)$-step self-avoiding polygons. For general x, $\mathcal{P}^{(1)}_{n+1}(x)$ is the set of self-avoiding polygons followed by a self-avoiding walk from 0 to x, with the total length being $n+1$ and with the walk and polygon mutually avoiding. Then, by definition,

$$R^{(1)}_{n+1}(x) = \sum_{\omega \in \mathcal{P}^{(1)}_{n+1}(x)} W(\omega). \tag{2.2.3}$$

Diagrammatically the right-hand side of (2.2.2) can be represented by



In the first term on the right side the line indicates an n-step walk from y to x which is unconstrained, apart from the fact that it should be self-avoiding.

We proceed by applying the inclusion-exclusion relation again to $R^{(1)}_{n+1}(x)$. Indeed, we ignore the mutual avoidance constraint of the polygon and self-avoiding walk that together form $\omega \in \mathcal{P}^{(1)}_{n+1}(x)$, and then make up for the overcounted paths by excluding the walks where the polygon and the self-avoiding walk do intersect. For $y \in \mathbb{Z}^d$, let

$$\pi^{(1)}_m(y) = \delta_{0,y} \sum_{\omega \in \mathcal{P}^{(1)}_m(0)} W(\omega), \tag{2.2.4}$$

and define $R^{(2)}_{n+1}(x)$ by

$$R^{(1)}_{n+1}(x) = \sum_{y\in\mathbb{Z}^d} \sum_{m=2}^{n+1} \pi^{(1)}_m(y) c_{n+1-m}(x-y) - R^{(2)}_{n+1}(x). \tag{2.2.5}$$

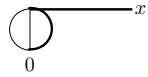
The next step is to investigate $R^{(2)}_{n+1}(x)$, which involves walks consisting of a self-avoiding polygon and a self-avoiding walk from 0 to x, of total length $n+1$, where the self-avoiding polygon and the self-avoiding walk have an intersection point additional to their intersection at the origin. Let $\mathcal{P}^{(2)}_{n+1}(x)$

be the subset of walks of $\mathcal{W}_{n+1}(x)$ satisfying these requirement. Then we clearly have

$$R^{(2)}_{n+1}(x) = \sum_{\omega \in \mathcal{P}^{(2)}_{n+1}(x)} W(\omega). \tag{2.2.6}$$

Diagrammatically, we can represent (2.2.5) as follows:

$$R^{(1)}_{n+1}(x) \quad = \quad \sum_{m=2}^{n+1} (\pi^{(1)}_m * c_{n+1-m})(x) \qquad - \qquad$$



The two think lines are mutually avoiding, so that they together form a self-avoiding walk. The walk and polygon may intersect more than once, and we focus on the *first* intersection point.

We then perform inclusion-exclusion again, neglecting the avoidance between the portions of the self-avoiding walk before and after this first intersection, and again subtracting a correction term. Due to the fact that we look at the *first* intersection point of the self-avoiding walk and the self-avoiding polygon, the three self-avoiding walks in the Θ-shaped diagram are also *mutually avoiding* each other. We define $R^{(3)}_{n+1}(x)$ by

$$R^{(2)}_{n+1}(x) = \sum_{y \in \mathbb{Z}^d} \sum_{m=2}^{n+1} \pi^{(2)}_m(y) c_{n+1-m}(x-y) - R^{(3)}_{n+1}(x), \tag{2.2.7}$$

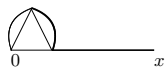
where $\pi^{(2)}_m(y)$ is defined by

$$\pi^{(2)}_m(x) = \sum_{\substack{m_1, m_2, m_3 \geq 1 \\ m_1 + m_2 + m_3 = m}} \prod_{j=1}^{3} \sum_{\omega_j \in \mathcal{C}_{m_j}(x)} W(\omega_i) I(\omega_1, \omega_2, \omega_3), \tag{2.2.8}$$

and $I(\omega_1, \omega_2, \omega_3)$ is equal to 1 if the ω_i are pairwise mutually avoiding apart from their common endpoints, and otherwise equals 0. We do not write down an explicit formula for $R^{(3)}_{n+1}(x)$, as this already gets quite involved.

This inclusion-exclusion step can be diagrammatically represented as

$$R^{(2)}_{n+1}(x) \quad = \quad \sum_{m=2}^{n+1} (\pi^{(2)}_m * c_{n+1-m})(x) \qquad - \qquad$$



The process of using inclusion-exclusion is continued indefinitely, and leads to

$$c_{n+1}(x) = \sum_{y \in \mathbb{Z}^d} D(y) c_n(x-y) + \sum_{y \in \mathbb{Z}^d} \sum_{m=2}^{n+1} \pi_m(y) c_{n+1-m}(x-y), \tag{2.2.9}$$

where

$$\pi_m(y) = \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(y). \tag{2.2.10}$$

We will give explicit expressions for the $\pi_m^{(N)}(y)$ for $N \geq 3$ below. Taking the Fourier transform in (2.2.9), we obtain (2.2.1).

The algebraic derivation of the expansion. We see from the above derivation using inclusion-exclusion that the formulas for $\pi_m^{(N)}$ get more and more involved as N becomes larger. In this section, we derive the lace expansion for self-avoiding walk using an algebraic approach which relies on a rewrite in terms of *graphs*, and is due to Brydges and Spencer [6]. This derivation makes it much more simple to write down formulas for $\pi_m^{(N)}$, and also applies quite generally when the interaction is different than the self-avoidance interaction.

We rewrite

$$c_n(x) = \sum_{\omega \in \mathcal{W}_n(x)} \prod_{0 \leq s < t \leq n} (1 - U_{st}(\omega)) W(\omega), \tag{2.2.11}$$

where

$$U_{st}(\omega) = \begin{cases} 1 & \text{if } \omega(s) = \omega(t), \\ 0 & \text{if } \omega(s) \neq \omega(t). \end{cases} \tag{2.2.12}$$

Indeed, we note that, for $\omega \in \mathcal{W}_n(x)$,

$$\prod_{0 \leq s < t \leq n} (1 - U_{st}(\omega)) = I[\omega \text{ self-avoiding}] = I[\omega \in \mathcal{C}_n(x)]. \tag{2.2.13}$$

In this section, we define $\hat{\pi}_m(k)$ and prove (2.2.1). We will derive the expansion in a slightly more general setting. To introduce this setting, we write, for integers $0 \leq a < b$,

$$K[a,b](\omega) = \prod_{a \leq s < t \leq b} (1 - V_{st}(\omega)), \tag{2.2.14}$$

where $V_{st}(\omega)$ are certain numbers depending on the path ω. We further define

$$c_n(x) = \sum_{\omega \in \mathcal{W}_n(x)} K[0,n](\omega) W(\omega), \tag{2.2.15}$$

where the sum is over all n-step paths from 0 to x. We note that when

$$V_{st}(\omega) = U_{st}(\omega) = I[\omega(s) = \omega(t)], \tag{2.2.16}$$

$K[0,n](\omega)$ is the indicator that the path ω of length n does not have any self-intersections, i.e., the path is self-avoiding. However, when $V_{st}(\omega)$ is

another function of the path, then the expansion remains valid. We will see examples of other $V_{st}(\omega)$ in Section 2.5 below.

Given an interval $I = [a, b]$ of integers with $0 \le a \le b$, we refer to a pair $\{s, t\}$ $(s < t)$ of elements of I as an *edge*. To abbreviate the notation, we write st for $\{s, t\}$. A set of edges is called a *graph*. A graph Γ on $[a, b]$ is said to be *connected* if both a and b are endpoints of edges in Γ and if, in addition, for any $c \in (a, b)$ there is an edge $st \in \Gamma$ such that $s < c < t$. The set of all graphs on $[a, b]$ is denoted $\mathcal{B}[a, b]$, and the subset consisting of all connected graphs is denoted $\mathcal{G}[a, b]$.

The expansion is crucially based on the expansion of large products. In general, for any set of indices $\mathcal{I}$, we have that

$$\prod_{i\in\mathcal{I}}(a_i + b_i) = \sum_{I\subseteq\mathcal{I}} \prod_{i\in I} a_i \prod_{j\in\mathcal{I}\setminus I} b_i. \tag{2.2.17}$$

Applying this to to $\mathcal{I} = \mathcal{B}[a, b]$, $a_{st} = -V_{st}, b_{st} = 1$, for which the product over b_{st} is identically equal to 1, we get

$$K[a, b](\omega) = \sum_{\Gamma\in\mathcal{B}[a,b]} \prod_{st\in\Gamma} (-V_{st}(\omega)). \tag{2.2.18}$$

Note that the size of $\mathcal{B}[0, n]$ is equal to $2^{\binom{n}{2}}$, which is huge. For $0 \le a < b$, we define an analogous quantity, in which the sum over graphs is restricted to connected graphs, namely,

$$J[a, b](\omega) = \sum_{\Gamma\in\mathcal{G}[a,b]} \prod_{st\in\Gamma} (-V_{st}(\omega)). \tag{2.2.19}$$

We will now suppress ω in the notation. We claim that

$$K[0, n + 1] = K[1, n + 1] + \sum_{m=1}^{n+1} J[0, m]K[m, n + 1]. \tag{2.2.20}$$

Indeed, to prove (2.2.20), we note from (2.2.18) that the contribution to $K[0, n + 1]$ from all graphs Γ for which 0 is not in an edge is exactly $K[1, n + 1]$. To resum the contribution from the remaining graphs, we proceed as follows.

When Γ does contain an edge ending at 0, we let $m(\Gamma)$ denote the largest value of m such that the set of edges in Γ with at least one end in the interval $[0, m]$ forms a connected graph on $[0, m]$. Then resummation over graphs on $[m, n + 1]$ gives

$$K[0, n + 1] = K[1, n + 1] + \sum_{m=1}^{n+1} \sum_{\Gamma\in\mathcal{G}[0,m]} \prod_{st\in\Gamma} (-V_{st})\, K[m, n + 1], \tag{2.2.21}$$

which with (2.2.19) proves (2.2.20).

Define

$$\pi_m(x) = \sum_{\omega \in \mathcal{W}_m(x)} J[0,m](\omega) W(\omega), \qquad m \geq 1. \tag{2.2.22}$$

The quantity $\pi_m(x)$ is sometimes called the *irreducible two-point function*. Then (2.2.9) is obtained after insertion of (2.2.20) into (2.2.15) followed by factorisation of the sum over ω. Since the sum over connected graphs is smaller than the sum over all graphs, the coefficient $\pi_m(x)$ may be expected to be smaller then the two-point function $c_m(x)$. However, since the number of connected graphs in $\mathcal{G}[0,n]$ is still at least $2^{\binom{n}{2}-1}$ (since all graphs containing $0n$ are elements of $\mathcal{G}[0,n]$), the sum over connected graph is still huge.

A *lace* is a minimally connected graph, i.e., a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a,b]$ is denoted $\mathcal{L}[a,b]$.

Given a connected graph Γ, the following prescription associates to Γ a unique lace L_Γ: The lace L_Γ consists of edges $s_1t_1, s_2t_2, \ldots$, with $t_1, s_1, t_2, s_2, \ldots$ determined, in that order, by

$$t_1 = \max\{t : at \in \Gamma\}, \quad s_1 = a, \tag{2.2.23}$$

$$t_{i+1} = \max\{t : \exists s < t_i \text{ such that } st \in \Gamma\}, \quad s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}. \tag{2.2.24}$$

Given a lace L, the set of all edges $st \notin L$ such that $\mathsf{L}_{L\cup\{st\}} = L$ is denoted $\mathcal{C}(L)$. Edges in $\mathcal{C}(L)$ are said to be *compatible* with L.

We next rewrite (2.2.22) in a form that can be used to obtain good bounds on $\pi_m(x)$. For this, we note that we can classify

$$\mathsf{L}_\Gamma = L \qquad \Longleftrightarrow \qquad \Gamma = L \cup C, \quad \text{with } C \subseteq \mathcal{C}(L). \tag{2.2.25}$$

This is due to the fact that the lace L_Γ is obtained by checking maxima and minima criteria. In fact, $\mathsf{L}_\Gamma = L$ is equivalent to the statement that an edge that is *not* in L is never chosen in (2.2.23) and (2.2.24), for which it suffices to check each of the edges individually.

Using (2.2.25), we can partially resum the right-hand side of (2.2.19), to obtain

$$\begin{aligned} J[a,b] &= \sum_{L\in\mathcal{L}[a,b]} \sum_{\Gamma:\mathsf{L}_\Gamma = L} \prod_{st\in L} (-V_{st}) \prod_{s't'\in\Gamma\setminus L} (-V_{s't'}) \\ &= \sum_{L\in\mathcal{L}[a,b]} \prod_{st\in L} (-V_{st}) \sum_{C\subseteq\mathcal{C}(L)} \prod_{s't'\in C} (-V_{s't'}). \end{aligned} \tag{2.2.26}$$

Since we can unexpand the sum over compatibles using (2.2.17) as

$$\sum_{C\subseteq\mathcal{C}(L)} \prod_{s't'\in C} (-V_{s't'}) = \prod_{s't'\in\mathcal{C}(L)} (1 - V_{s't'}), \tag{2.2.27}$$

we finally arrive at

$$J[a,b] \quad = \quad \sum_{L\in\mathcal{L}[a,b]} \prod_{st\in L} (-V_{st}) \prod_{s't'\in\mathcal{C}(L)} (1-V_{s't'}). \tag{2.2.28}$$

This rewrite has two main advantages. The first advantage is that the sum over laces is much smaller than the sum over connected graphs. In fact, the total number of laces on $[0,n]$ is bounded from above by e^{2n}. Indeed, we note that the number of laces of size N on $[0,n]$ is bounded above by the number of possible choices of $0 = s_1 < s_2 < s_3 < \ldots < s_N < n$ times the number of possible choices $0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = n$, which is bounded above by

$$\frac{n^{2(N-1)}}{((N-1)!)^2} = \frac{1}{(2(N-1))!} n^{2(N-1)} \binom{2(N-1)}{N-1} \leq \frac{(2n)^{2(N-1)}}{(2(N-1))!}. \tag{2.2.29}$$

Summing over $N \geq 1$ gives a bound e^{2n}, which is *much* smaller than $2^{\binom{n}{2}}$.

The second advantage is that the interaction is restored along the compatible edges. In particular, for $V_{st}(\omega) = U_{st}(\omega)$ in (2.2.12), we have that $\omega(s) \neq \omega(t)$ for all $st \in \mathcal{C}(L)$.

We finally identify $\pi_m^{(N)}(x)$. For $0 \leq a < b$, we define $J^{(N)}[a,b]$ to be the contribution to (2.2.26) coming from laces consisting of exactly N edges, i.e.,

$$J^{(N)}[a,b] = \sum_{L\in\mathcal{L}^{(N)}[a,b]} \prod_{st\in L} V_{st} \prod_{s't'\in\mathcal{C}(L)} (1-V_{s't'}), \qquad N \geq 1, \tag{2.2.30}$$

where $\mathcal{L}^{(N)}[a,b]$ is the set of laces consisting of precisely N edges. Then

$$J[a,b] = \sum_{N=1}^{\infty} (-1)^N J^{(N)}[a,b] \tag{2.2.31}$$

and by (2.2.22),

$$\pi_m(x) = \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(x), \tag{2.2.32}$$

where we define

$$\begin{aligned} \pi_m^{(N)}(x) \quad &= \quad \sum_{\omega\in\mathcal{W}_m(x)} J^{(N)}[0,m](\omega) W(\omega) \qquad (2.2.33) \\ &= \quad \sum_{\omega\in\mathcal{W}_m(x)} W(\omega) \sum_{L\in\mathcal{L}^{(N)}[0,m]} \prod_{st\in L} V_{st}(\omega) \prod_{s't'\in\mathcal{C}(L)} (1-V_{s't'}(\omega)). \end{aligned}$$

This completes the algebraic derivation of the lace expansion.

We finally investigate $\pi_m^{(1)}(x)$ and $\pi_m^{(2)}(x)$. We return to the choice U_{st} in (2.2.12). For this choice, and using that $U_{0,1} = 0$ since a random walk

cannot return to the same site in one step when $D(0) = 0$, we note that we can restrict to $m \geq 2$ for $\pi_m^{(N)}(x)$. When $N = 1$, there is only one lace $L = \{0m\}$, and all other edges are compatible. Therefore, $\pi_m^{(1)}(x)$ coincides with (2.2.4). Moreover, when $N = 2$, we have that

$$\mathcal{L}^{(2)}[0, m] = \big\{\{0t_1, s_2m\} : 0 < s_2 < t_1 < m\big\}. \tag{2.2.34}$$

Therefore, with $L = \{0t_1, s_2m\}$,

$$\prod_{st\in L} U_{st}(\omega) = I[\omega(t_1) = 0, \omega(s_2) = \omega(m)]. \tag{2.2.35}$$

Also, it can be seen that ω_1, ω_2 and ω_3, given by $\omega_1 = (0, \omega(1), \ldots, \omega(s_2))$, $\omega_2 = (\omega(t_1), \omega(t_1-1), \ldots, \omega(s_2))$ and $\omega_3 = (\omega(t_1), \omega(t_1+1), \ldots, \omega(m))$, are three walks starting and ending at the same site. Since

$$\prod_{s't'\in\mathcal{C}(L)} (1 - V_{s't'}(\omega)) = I(\omega_1, \omega_2, \omega_3)K[0, s_2]K[s_2, t_1]K[t_1, m], \tag{2.2.36}$$

where we recall the definition of $I(\omega_1, \omega_2, \omega_3)$ in (2.2.8), the walks ω_1, ω_2 and ω_3 are three mutually avoiding self-avoiding walks. Therefore, also $\pi_m^{(2)}(x)$ coincides with (2.2.8). Similarly, it can be seen that the higher order lace expansion coefficients $\pi_m^{(N)}(x)$ in the algebraic derivation and the inclusion-exclusion derivation agree.

2.2.2 The lace expansion for the oriented percolation two-point function

In this section, we will derive the lace expansion for the oriented percolation two-point function. This expansion is reminiscent of the inclusion-exclusion expansion for self-avoiding walk in Section 2.2.1 above. The lace expansion for the oriented percolation two-point function gives that there exist lace expansion coefficients $\{\pi_m(x)\}_{m\geq 0, x\in\mathbb{Z}^d}$ such that for all $n \geq 0$

$$\tau_{n+1}(x) = \sum_{m=0}^{n} \sum_{u,v\in\mathbb{Z}^d} \pi_m(u)pD(v - u)\tau_{n-m}(x - v) + \pi_{n+1}(x). \tag{2.2.37}$$

The result of the expansion in (2.2.37) is quite close to the result of the expansion for self-avoiding walk in (2.2.9). We stress that even though the notation $\pi_m(x)$ appears in both expansions, the lace expansion coefficients for oriented percolation and self-avoiding walk are *different*. However, even though the lace expansion coefficients are different, they play an identical role in the analysis which we describe later on.

There are several possible expansions that lead to (2.2.37). In [40], an expansion was chosen that relies on the independence of bonds, and applies to oriented percolation as well as to (unoriented) percolation. For percolation, this expansion was first derived in [25]. In [50, 51], an expansion

was used that was based on the algebraic derivation for self-avoiding walks in [6]. In [55], an expansion was performed using inclusion-exclusion. The inclusion-exclusion expansion was rederived in [37], where it was applied to oriented percolation and the contact process simultaneously. The lace expansion coefficients $\pi_m(y)$ in each of these expansion are the *same*, since (2.2.37) uniquely identifies $\pi_m(y)$ for every m and y, which can be seen easily by using induction on m. On the other hand, in each of the expansions, we find that

$$\pi_m(y) = \sum_{N=0}^{\infty} (-1)^N \pi_m^{(N)}(y), \tag{2.2.38}$$

and we expect the formulas for $\pi_m^{(N)}(y)$ to be different in the Hara-Slade expansion used in [40] and in the Nguyen-Yang and the Sakai expansion, while they are identical in the Nguyen-Yang and the Sakai expansion. We now describe the simplest of these expansions, the Sakai expansion, following the presentation in [37].

In the expansion, it is convenient to abbreviate $\Lambda = \mathbb{Z}^d \times \mathbb{Z}_+$, and to use bold letters such as $\boldsymbol{o} = (o, 0)$ and $\boldsymbol{x} = (x, n)$, and to write $\tau(\boldsymbol{x}) = \tau_n(x)$, $\pi^{(N)}(\boldsymbol{x}) = \pi_n^{(N)}(x)$, and so on. Then, the lace expansion in (2.2.37) can be rewritten as

$$\tau(\boldsymbol{x}) = \pi(\boldsymbol{x}) + (\pi \star pD \star \tau)(\boldsymbol{x}), \tag{2.2.39}$$

where $\star$ denotes convolution in Λ, i.e.,

$$(f \star g)(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \Lambda} f(\boldsymbol{y})\, g(\boldsymbol{x} - \boldsymbol{y}), \tag{2.2.40}$$

where we recall that the two-point function is given by $\tau(\boldsymbol{x}) = \mathbb{P}_p(\boldsymbol{o} \longrightarrow \boldsymbol{x})$. In the sequel, we will drop the subscript p from the notation, and write $\mathbb{P}$ instead of $\mathbb{P}_p$.

While the lace expansion for self-avoiding walks quantifies the statement that self-avoiding walk is a small perturbation of random walk, we can intuitively think of (2.2.37) as saying that occupied paths in oriented percolation clusters are like random walk paths. To make this picture more precise, we note that we can summarise the paths from $\boldsymbol{o}$ to $\boldsymbol{x}$ as a *string of sausages*. In this picture, we say that a bond is *pivotal* for the connection from $\boldsymbol{x}$ to $\boldsymbol{y}$ when $\boldsymbol{x} \longrightarrow \boldsymbol{y}$ in the (possibly modified) configuration where the bond is made occupied, while $\boldsymbol{x}$ is not connected to $\boldsymbol{y}$ in the (possibly modified) configuration where the bond is made vacant. The pivotal bonds are ordered in time, and each occupied path from $\boldsymbol{o}$ to $\boldsymbol{x}$ must pass through all pivotal bonds. The pivotals are the *strings* in the sausage and strings picture, while the sausages are the pieces of the cluster of $\boldsymbol{o}$ between the pivotal bonds. The string and sausages picture turns the part of the cluster between $\boldsymbol{o}$ and $\boldsymbol{x}$ in a linear structure.

In turn, the sausage in between pivotal bond b_i and b_{i+1} can be divided into two disjoint pieces. The first is the *backbone*, which are the sites $\boldsymbol{y}$

such that $\overline{b}_i \longrightarrow \boldsymbol{y} \longrightarrow \underline{b}_{i+1}$, where, for a bond $b = (\boldsymbol{u}, \boldsymbol{v})$, we write $\underline{b} = \boldsymbol{u}$ and $\overline{b} = \boldsymbol{v}$. The second part are the *hairs*, and these are all the sites of the sausage which are not in the backbone. Then, the fact that b_i is pivotal is equivalent to saying that all sausages below b_i do not intersect the backbones above b_i. Since, by time-orientation, the different backbones cannot intersect, this is equivalent to the statement that none of the hairs of the previous sausages share sites with the later backbones. This introduces an interaction between the sausages and the backbones, and to derive the expansion, we will use inclusion-exclusion to effectively deal with this interaction.

We split, depending on whether there is a pivotal bond for $\boldsymbol{o} \longrightarrow \boldsymbol{x}$, to obtain

$$\tau(\boldsymbol{x}) = \mathbb{P}(\boldsymbol{o} \Longrightarrow \boldsymbol{x}) + \sum_b \mathbb{P}(\boldsymbol{o} \Longrightarrow \underline{b},\ b \text{ occupied \& pivotal for } \boldsymbol{o} \longrightarrow \boldsymbol{x}), \tag{2.2.41}$$

where we write $\boldsymbol{v} \Longrightarrow \boldsymbol{x}$ when there are at least two bond-disjoint paths from $\boldsymbol{v}$ to $\boldsymbol{x}$ consisting of occupied bonds, and, by convention, we say that $\boldsymbol{x} \Longrightarrow \boldsymbol{x}$ for all $\boldsymbol{x}$.

We let

$$\pi^{(0)}(\boldsymbol{x}) = \mathbb{P}(\boldsymbol{o} \Longrightarrow \boldsymbol{x}) \tag{2.2.42}$$

denote the contribution to $\tau(\boldsymbol{x})$ of configurations where no pivotal bond exists, so that we can rewrite (2.2.41) as

$$\tau(\boldsymbol{x}) = \pi^{(0)}(\boldsymbol{x}) + \sum_b \mathbb{P}(\boldsymbol{o} \Longrightarrow \underline{b},\ b \longrightarrow \boldsymbol{x},\ b \text{ pivotal for } \boldsymbol{o} \longrightarrow \boldsymbol{x}), \tag{2.2.43}$$

where we further abbreviate $b \longrightarrow \boldsymbol{x}$ for the statement that b is occupied and $\overline{b} \longrightarrow \boldsymbol{x}$.

Define

$$R^{(1)}(\boldsymbol{x}) = \sum_b \mathbb{P}(\boldsymbol{o} \Longrightarrow \underline{b},\ b \longrightarrow \boldsymbol{x},\ b \text{ not pivotal for } \boldsymbol{o} \longrightarrow \boldsymbol{x}), \tag{2.2.44}$$

then, by inclusion-exclusion on the event that b is pivotal for $\boldsymbol{o} \longrightarrow \boldsymbol{x}$, we arrive at

$$\tau(\boldsymbol{x}) = \pi^{(0)}(\boldsymbol{x}) + \sum_b \mathbb{P}(\boldsymbol{o} \Longrightarrow \underline{b},\ b \longrightarrow \boldsymbol{x}) - R^{(1)}(\boldsymbol{x}). \tag{2.2.45}$$

The event $\boldsymbol{o} \Longrightarrow \underline{b}$ only depends on bonds with time variables less than or equal to the one of $\underline{b}$, while the event $b \longrightarrow \boldsymbol{x}$ only depends on bonds with time variables larger than or equal to the one of $\underline{b}$. Therefore, by the Markov property, we obtain

$$\begin{aligned} \mathbb{P}(\boldsymbol{o} \Longrightarrow \underline{b},\ b \longrightarrow \boldsymbol{x}) &= \mathbb{P}(\boldsymbol{o} \Longrightarrow \underline{b})\, \mathbb{P}(b \text{ occupied})\, \mathbb{P}(\overline{b} \longrightarrow \boldsymbol{x}) \\ &= \pi^{(0)}(\underline{b})\, pD(b)\, \tau(\boldsymbol{x} - \overline{b}), \end{aligned} \tag{2.2.46}$$

where we abuse notation and write $D((x,n),(y,n+1)) = D(y-x)$. Using (2.2.46), we arrive at

$$\tau(\boldsymbol{x}) = \pi^{(0)}(\boldsymbol{x}) + (\pi^{(0)} \star pD \star \tau)(\boldsymbol{x}) - R^{(1)}(\boldsymbol{x}). \tag{2.2.47}$$

This completes the first step of the expansion, and we are left to investigate $R^{(1)}(\boldsymbol{x})$. For this, given a bond b, we write $\tilde{C}^b(\boldsymbol{x})$ for the set of sites to which $\boldsymbol{x}$ is connected in the (possibly modified) configuration in which b is made vacant.

We then note that

$$\begin{aligned}\{\boldsymbol{v} \longrightarrow \underline{b},\ b \longrightarrow \boldsymbol{x},\ b \text{ not pivotal for } \boldsymbol{v} \longrightarrow \boldsymbol{x}\}\\ = \big\{\boldsymbol{v} \longrightarrow \underline{b},\ b \text{ occupied}, \overline{b} \xrightarrow{\tilde{C}^b(\boldsymbol{v})} \boldsymbol{x}\big\},\end{aligned} \tag{2.2.48}$$

where, given a site set C, we say that $\boldsymbol{v}$ is connected to $\boldsymbol{x}$ *through* C, and write $\boldsymbol{v} \xrightarrow{C} \boldsymbol{x}$, if every occupied path connecting $\boldsymbol{v}$ to $\boldsymbol{x}$ has at least one bond with an endpoint in C. We often abbreviate $\{b \xrightarrow{C} \boldsymbol{x}\} = \{b \text{ occupied}\} \cap \{\overline{b} \xrightarrow{C} \boldsymbol{x}\}$.

Using the above notation, we can rewrite

$$R^{(1)}(\boldsymbol{x}) = \sum_b \mathbb{P}\big(\boldsymbol{o} \Longrightarrow \underline{b},\ b \xrightarrow{\tilde{C}^b(\boldsymbol{o})} \boldsymbol{x}\big). \tag{2.2.49}$$

The event $\{\boldsymbol{v} \xrightarrow{C} \boldsymbol{x}\}$ can be decomposed into two cases depending on whether there is or is not a pivotal bond b for $\boldsymbol{v} \longrightarrow \boldsymbol{x}$ such that $\boldsymbol{v} \xrightarrow{C} \underline{b}$. Let

$$E(b,\boldsymbol{y};C) = \{b \xrightarrow{C} \boldsymbol{y}\} \cap \big\{\nexists\, b' \text{ pivotal for } \overline{b} \longrightarrow \boldsymbol{y} \text{ s.t. } \overline{b} \xrightarrow{C} \underline{b}'\big\}. \tag{2.2.50}$$

See Figure 1 for a schematic representation of the event $E(b,\boldsymbol{x};C)$.

If there are pivotal bonds for $\boldsymbol{v} \longrightarrow \boldsymbol{x}$, then we take the *first* such pivotal bond b for which $\boldsymbol{v} \xrightarrow{C} \underline{b}$. Then we have the partition

$$\begin{aligned}\{\boldsymbol{v} \xrightarrow{C} \boldsymbol{x}\} = E'(\boldsymbol{v},\boldsymbol{x};C)\\ \dot{\cup} \dot{\bigcup_b} \big\{E'(\boldsymbol{v},\underline{b};C) \cap \{b \text{ occupied \& pivotal for } \boldsymbol{v} \longrightarrow \boldsymbol{x}\}\big\}.\end{aligned} \tag{2.2.51}$$

Defining

$$\pi^{(1)}(\boldsymbol{y}) = \sum_b \mathbb{P}\big(\{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap E(b,\boldsymbol{y};\tilde{C}^b(\boldsymbol{o}))\big), \tag{2.2.52}$$

we obtain

$$\begin{aligned}R^{(1)}(\boldsymbol{x}) = \pi^{(1)}(\boldsymbol{x}) + \sum_{b_1,b_2} \mathbb{P}\big(\{\boldsymbol{o} \Longrightarrow \underline{b}_1\} \cap E(b_1,\underline{b}_2;\tilde{C}^{b_1}(\boldsymbol{o}))\\ \cap \{b_2 \text{ occupied \& pivotal for } \overline{b}_1 \longrightarrow \boldsymbol{x}\}\big).\end{aligned} \tag{2.2.53}$$

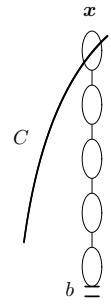


Figure 1: Schematic representation of the event $E(b, \boldsymbol{x}; C)$.

To the second term, we apply the inclusion-exclusion relation

$$\begin{aligned}&\{b \text{ occupied \& pivotal for } \boldsymbol{v} \longrightarrow \boldsymbol{x}\}\\ &\quad = \{\boldsymbol{v} \longrightarrow \underline{b},\ b \longrightarrow \boldsymbol{x}\} \setminus \{\boldsymbol{v} \longrightarrow \underline{b},\ b \xrightarrow{\tilde{C}^b(\boldsymbol{v})} \boldsymbol{x}\}.\end{aligned} \tag{2.2.54}$$

We will be able to use the Markov property for the contribution of the first event, and, to denote the contribution due to the overcounted event $\overline{b}_1 \longrightarrow \underline{b}_2,\ b_2 \xrightarrow{\tilde{C}^{b_2}(\overline{b}_1)} \boldsymbol{x}$, we define

$$R^{(2)}(\boldsymbol{x}) = \sum_{b_1,b_2} \mathbb{P}\big(\{\boldsymbol{o} \Longrightarrow \underline{b}_1\} \cap E(b_1, \underline{b}_2; \tilde{C}^{b_1}(\boldsymbol{o})) \cap \{b_2 \xrightarrow{\tilde{C}^{b_2}(\overline{b}_1)} \boldsymbol{x}\}\big). \tag{2.2.55}$$

Since $E(b_1, \underline{b}_2; C) \cap \{\overline{b}_1 \longrightarrow \underline{b}_2\} = E(b_1, \underline{b}_2; C)$, we obtain

$$\begin{aligned}R^{(1)}(\boldsymbol{x}) = \pi^{(1)}(\boldsymbol{x}) + \sum_{b_1,b_2} &\mathbb{P}\big(\{\boldsymbol{o} \Longrightarrow \underline{b}_1\} \cap E(b_1, \underline{b}_2; \tilde{C}^{b_1}(\boldsymbol{o})) \cap \{b_2 \longrightarrow \boldsymbol{x}\}\big)\\ &- R^{(2)}(\boldsymbol{x}).\end{aligned} \tag{2.2.56}$$

To use the Markov property, we note that the event $\{\boldsymbol{o} \Longrightarrow \underline{b}_1\} \cap E(b_1, \underline{b}_2; \tilde{C}^{b_1}(\boldsymbol{o}))$ depends only on bonds before $\underline{b}_2$, while $\{b_2 \longrightarrow \boldsymbol{x}\}$ depends only on bonds after $\underline{b}_2$. By the Markov property, we end up with

$$\begin{aligned}R^{(1)}(\boldsymbol{x}) &= \pi^{(1)}(\boldsymbol{x}) + \sum_{b_2} \pi^{(1)}(\underline{b}_2)\, pD(b_2)\, \tau(\boldsymbol{x} - \overline{b}_2) - R^{(2)}(\boldsymbol{x})\\ &= \pi^{(1)}(\boldsymbol{x}) + (\pi^{(1)} \star pD \star \tau)(\boldsymbol{x}) - R^{(2)}(\boldsymbol{x}),\end{aligned} \tag{2.2.57}$$

so that

$$\tau(\boldsymbol{x}) = \pi^{(0)}(\boldsymbol{x}) - \pi^{(1)}(\boldsymbol{x}) + \big((\pi^{(0)} - \pi^{(1)}) \star pD \star \tau\big)(\boldsymbol{x}) + R^{(2)}(\boldsymbol{x}). \quad (2.2.58)$$

This completes the second step of the expansion.

To complete the expansion for $\tau(\boldsymbol{x})$, we must investigate $R^{(2)}(\boldsymbol{x})$ in more detail by repeated use of inclusion-exclusion. For this, we note that $R^{(2)}(\boldsymbol{x})$ involves the probability of a subset of $\big\{b_2 \xrightarrow{\tilde{C}^{b_2}(\bar{b}_1)} \boldsymbol{x}\big\}$, to which we will use (2.2.51) and (2.2.54) repeatedly, and use the Markov property. We will not describe the repeated use of inclusion-exclusion, and merely state the result. For the details of the expansion, please see [37, Section 3].

To state the result of the expansion, we make a few more definitions. We let

$$\tilde{E}^{(0)}_{\vec{b}_0}(\boldsymbol{x}) = \{\boldsymbol{o} \Longrightarrow \boldsymbol{x}\}, \quad (2.2.59)$$

and, for $\vec{b}_N = (b_1, \dots, b_N)$ with $N \geq 1$, we define

$$\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x}) = \tilde{E}^{(N-1)}_{\vec{b}_{N-1}}(\underline{b}_N) \cap E\big(b_N, \boldsymbol{x}; \tilde{C}^{b_N}(\bar{b}_{N-1})\big). \quad (2.2.60)$$

Using this notation, we define

$$\pi^{(N)}(\boldsymbol{x}) = \sum_{\vec{b}_N} \mathbb{P}\big(\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x})\big), \qquad \text{and} \qquad \pi(\boldsymbol{x}) = \sum_{N=0}^{\infty} (-1)^N \pi^{(N)}(\boldsymbol{x}). \quad (2.2.61)$$

Then, for $N = 0, 1$, (2.2.61) coincides with (2.2.42) and (2.2.52), respectively. Note that the sum over N in (2.2.61) is a finite sum, as long as $n_{\boldsymbol{x}}$ is finite, where $n_{\boldsymbol{x}}$ denotes the time coordinate of $\boldsymbol{x}$, since each of the bonds $b_1, \dots, b_N$ eats up at least one time-unit, so that $\pi^{(N)}(\boldsymbol{x}) = 0$ for $N > n_{\boldsymbol{x}}$. Then, (2.2.39) follows. This completes the derivation of the lace expansion for the oriented percolation two-point function.

2.3 Bounds on the lace expansion coefficients

As explained in Section 2.2, any lace expansion analysis comes in three steps. The first is the derivation of the expansion, which was explained in detail in Section 2.2, both for self-avoiding walk and for oriented percolation. The second step is the bound on the lace expansion coefficients π, which we explain in this section. In Section 2.3.1, we bound the coefficients for self-avoiding walk, in Section 2.3.2 for oriented percolation.

2.3.1 Bounds on the self-avoiding walk coefficients

In this section, we indicate how we can bound $\sum_{x \in \mathbb{Z}^d} |\pi_m(x)|$ in terms of $\|c_j\|_\infty$ and $\|c_j\|_1$ with $j < m$. Here we use the notation $\|h\|_1 = \sum_{x \in \mathbb{Z}^d} |h(x)|$

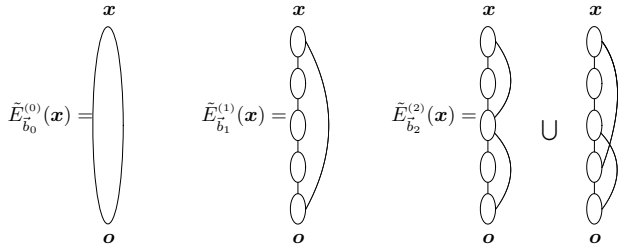


Figure 2: Schematic representations of the events in the definition of $\pi^{(0)}(\boldsymbol{x})$, $\pi^{(1)}(\boldsymbol{x})$ and $\pi^{(2)}(\boldsymbol{x})$.

and $\|h\|_\infty = \sup_x |h(x)|$ for functions $h\colon \mathbb{Z}^d \to \mathbb{R}$. These bounds are formulated in [42, Proposition 4.1], with slightly different notation ($\pi_m^{(N)}(x)$ is denoted there by $\pi_{m,N}^{(1)}(x)$).

To initiate the bounds, we note that

$$\sum_{x\in\mathbb{Z}^d} |\pi_m(x)| \le \sum_{N=1}^{\infty} \sum_{x\in\mathbb{Z}^d} \pi_m^{(N)}(x). \tag{2.3.1}$$

To convey the strategy behind the bounds on π_m, we will prove the bounds on $\sum_{x\in\mathbb{Z}^d} \pi^{(1)}(x)$ and $\sum_{x\in\mathbb{Z}^d} \pi^{(2)}(x)$. Recall from (2.2.4) that

$$\sum_{x\in\mathbb{Z}^d} \pi_m^{(1)}(x) = \sum_{\omega\in\mathcal{P}_m^{(1)}(0)} W(\omega) = \sum_{y\in\mathbb{Z}^d} D(y) c_{m-1}(y) \le \|c_{m-1}\|_\infty. \tag{2.3.2}$$

This bounds $\sum_{x\in\mathbb{Z}^d} \pi_m^{(1)}(x)$ in terms of $\|c_{m-1}\|_\infty$.

We proceed with $N = 2$. Recall (2.2.8), and bound $I(\omega_1, \omega_2, \omega_3) \le 1$, which neglects the mutual avoidance between ω_1, ω_2 and ω_3, to obtain

$$\begin{aligned}\sum_x \pi_m^{(2)}(x) &\le \sum_x \sum_{\substack{m_1,m_2,m_3\ge 1\\ m_1+m_2+m_3=m}} \prod_{j=1}^{3} c_{m_j}(x)\\ &\le 3! \sum_{\substack{m_1\ge m_2\ge m_3\ge 1\\ m_1+m_2+m_3=m}} \|c_{m_1}\|_\infty \|c_{m_2}\|_\infty \|c_{m_3}\|_1. \end{aligned}\tag{2.3.3}$$

Thus $\pi_m^{(2)}$ can be bounded in terms of norms of c_l with $l < m$. Similar estimates are possible for $\pi_m^{(N)}$ with $N \ge 3$. The fact that l here is *strictly* less than m is what enables the inductive approach to succeed.

In the setting of self-avoiding walks in Section 2.1.2, the two-point function $c_n(x)$ is exponentially small when n grows large. Indeed,

$$c_n^{\frac{1}{n}} \approx \mu < 1. \tag{2.3.4}$$

Therefore, to prove the asymptotics for c_n in Theorem 2.1, we need to cancel out the exponential decay of the two-point function, i.e., to multiply c_n by μ^{-n}. One of the goals of the analysis of the lace expansion is to give a representation for μ.

Fix some z, which we think to be close to μ^{-1}, and suppose that the bounds

$$\|c_l\|_1 z^l \le K, \qquad \text{and} \qquad \|c_l\|_\infty z^l \le K\beta(l+1)^{-d/2} \tag{2.3.5}$$

were true, for all $l < m$, where we define

$$\beta = L^{-d}, \tag{2.3.6}$$

to indicate the inverse of the range of the model. We will take the model to be *spread-out*, which means that β is a small parameter. If c_l were replaced with p_l, the random walk two-point function, these bounds, with $z = 1$, follow from the fact that $\|p_l\|_1 = 1$ and from the bounds following from the local central limit theorem (see (2.1.16)). Therefore, these bounds are consistent with the Gaussian behaviour of self-avoiding walk. Applying these bounds to (2.3.3) then gives

$$\sum_{x\in\mathbb{Z}^d} \pi_m^{(2)}(x) z^m \le 3! \sum_{\substack{m_1 \ge m_2 \ge m_3 \ge 1 \\ m_1+m_2+m_3 = m}} \|c_{m_1}\|_\infty z^{m_1} \|c_{m_2}\|_\infty z^{m_2} \|c_{m_3}\|_1 z^{m_3} \tag{2.3.7}$$

$$\le 3!K^3\beta^2 \sum_{\substack{m_1 \ge m_2 \ge m_3 \ge 1 \\ m_1+m_2+m_3 = m}} (m_1+1)^{-d/2}(m_2+1)^{-d/2}$$

$$\le 3!K^3\beta^2 \Big(\frac{m}{3}+1\Big)^{-d/2} \sum_{m_2 \ge m_3 \ge 1} (m_2+1)^{-d/2} \tag{2.3.8}$$

$$\le CK^3\beta^2(m+1)^{-d/2},$$

where the last inequality uses $d > 4$. The bounds on $\sum_{x\in\mathbb{Z}^d} \pi_m^{(N)}(x)$ for $N \ge 3$ are similar, as shown in [42, Section 5], and use induction on N. The main result is the following:

Proposition 2.3 (Bounds on the lace expansion). *Assume* (2.3.5) *for some z and all $m \le n$. Then there exist $\beta_0 > 0$ and $C_K < \infty$ (both depending on d, but not on n and L) such that for $\beta < \beta_0$, $m \in \mathbb{Z}_+$ with $2 \le m \le n+1$, $q = 0, 2, 4$ and $\delta' \in [0, 1 \wedge \delta]$,*

$$\sum_x |x|^q \, |\pi_m(x)| z^m \le \frac{C_K \sigma^q \beta}{(m+1)^{(d-q)/2}}. \tag{2.3.9}$$

In the analysis, related results (such as Taylor expansions for $\hat{\pi}_m(k)$ for k small) are also necessary, but we refrain from stating these. The important message in Proposition 2.3 is that bounds on the two-point function c_j for all $j \le n$, as formulated in (2.3.5), imply bounds on the lace expansion coefficients $\pi_m(x)$ for $m \le n+1$. In turn, in the lace expansion equation (2.2.9) for c_{n+1}, only the lace expansion coefficients $\pi_m(x)$ for $m \le n+1$ appear, as well as c_j for all $j \le n$. Therefore, the induction hypotheses should give us the bounds that allow us in turn to advance the induction hypotheses. For this, we note that we have assumed bounds on $\|c_l\|_1$ and $\|c_l\|_\infty$, so the induction hypotheses must imply the bounds on these norms formulated in (2.3.5).

The fact that bounds on c_j for $j \le n$ imply bounds on π_m for $m \le n+1$, which in turn imply bounds on c_j for $j \le n+1$ is a kind of *consistency* which is present in the lace expansion. This consistency is crucial in the analysis of the lace expansion recurrence relation, as will be explained in more detail in Section 2.4 below.

2.3.2 Bounds on the oriented percolation coefficients

For the bounds on π for oriented percolation, the strategy is similar to the one sketched in Section 2.3.1. We indicate this by bounding $\pi^{(0)}$.

Recall from (2.2.42) that

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d} \pi_m^{(0)}(x) &= \sum_{x\in\mathbb{Z}^d} \mathbb{P}\big(((0,0) \longrightarrow (x,m)) \circ ((0,0) \longrightarrow (x,m))\big) \\ &\le \sum_{x\in\mathbb{Z}^d} \mathbb{P}\big((0,0) \longrightarrow (x,m)\big)^2 = \tau_m(x)^2,\end{aligned} \tag{2.3.10}$$

where $((0,0) \longrightarrow (x,m)) \circ ((0,0) \longrightarrow (x,m))$ is the event that there are two bond-disjoint paths from $(0,0)$ to (x,m). The bound follows from the *Van den Berg-Kesten inequality* or BK inequality [21], which states that the probability of paths existing disjointly is bounded from above by the product of the probabilities of the occurrence of these paths. We obtain that for any $p \in [0,1]$,

$$\sum_{x\in\mathbb{Z}^d} \pi_m^{(0)}(x) \le \sum_{x\in\mathbb{Z}^d} \tau_m(x)^2. \tag{2.3.11}$$

Then we obtain that

$$\sum_{x\in\mathbb{Z}^d} \pi_m^{(0)}(x) \le \|\tau_m\|_1 \|\tau_m\|_\infty. \tag{2.3.12}$$

Fix some p, which we think to be close to p_c, and suppose that the bounds

$$\|\tau_l\|_1 \le K, \qquad \text{and} \qquad \|\tau_l\|_\infty \le K\beta(l+1)^{-d/2} \tag{2.3.13}$$

were true, for all $l < m$. Then, we cannot directly bound $\sum_{x\in\mathbb{Z}^d} \pi_m^{(0)}(x)$, since τ_m appears, rather than τ_j for some $j < m$. For this, we use Boole's inequality and the Markov property to obtain

$$\tau_m(x) = \mathbb{P}_p\big(\cup_y \{(0,0) \longrightarrow (y,1) \longrightarrow (x,m)\}\big) \le \sum_y pD(y)\tau_{m-1}(x-y). \tag{2.3.14}$$

These bounds immediately imply that

$$\|\tau_m\|_1 \le p\|\tau_{m-1}\|_1 \qquad \text{and} \qquad \|\tau_m\|_\infty \le p\|\tau_{m-1}\|_\infty. \tag{2.3.15}$$

Therefore, we obtain that

$$\sum_{x\in\mathbb{Z}^d} \pi_m^{(0)}(x) \le p^2\|\tau_{m-1}\|_1\|\tau_{m-1}\|_\infty \le p^2K^2\beta m^{-d/2} \le Cp^2K^2\beta(m+1)^{-d/2}. \tag{2.3.16}$$

It can be seen that similar bounds are valid for the higher-order contributions to the lace expansion coefficients $\pi_m^{(N)}$. Therefore, the bounds in (2.3.13) for all $j < m$ imply bounds on π_m, as proved in [41, Section 4]. These bounds are formulated in the following proposition, which is similar to Proposition 2.3:

Proposition 2.4 (Bounds on the lace expansion). *Assume* (2.3.13) *for some p and all $m \le n$. Then there exist $\beta_0 > 0$ and $C_K < \infty$ (both depending on d, but not on n and L) such that for $\beta < \beta_0$, $m \in \mathbb{Z}_+$ with $2 \le m \le n+1$, $q = 0, 2, 4$ and $\delta' \in [0, 1 \wedge \delta]$,*

$$\sum_x |x|^q\,|\pi_m(x)| \le \frac{C_K\sigma^q\beta}{(m+1)^{(d-q)/2}}. \tag{2.3.17}$$

Again several related bounds on the lace expansion are required in the analysis. The most difficult of these is the bound on the derivative of $\sum_x \pi_m(x)$ with respect to p. See [41, Section 4] or [37, Section 4] for details.

2.4 Induction for two-point functions

As explained in Section 2.2, any lace expansion analysis requires three steps. The first step is the derivation of the expansion, which is described in detail in Section 2.2. The second step is the derivation of the bounds on the lace expansion coefficients, and these bounds were sketched and formulated in Section 2.3. The bounds on the lace expansion coefficients are intimately related to the form of the expansion. The final step in a lace expansion analysis is the analysis of the recursion relation. This analysis boils down to the proof of the asymptotics of the recursion relation, using the bounds on the coefficients. In this section, we describe the *inductive method*, which

allows to derive the asymptotics for two-point functions which have a time variable. The inductive method was first used in [33] to prove diffusive behaviour for a certain class of weakly self-avoiding walks. This model is described in more detail in Section 2.5.1 below. In [40], the method was generalised to general lace expansion equations, where the consistency of the recurrence relation was formulated as an assumption on the model and the expansion, and the diffusive results in Theorems 2.1 and 2.2 were shown under this assumption. Therefore, to prove results of the form in Theorems 2.1 and 2.2, it suffices to verify the consistency assumption, which is a consequence of the bounds on the lace expansion coefficients. Apart from its applications to the oriented percolation and self-avoiding walk two-point function, the generalised induction was also applied to a particular version of *self-avoiding walks with nearest-neighbour attraction* in [63]. See Section 2.5.3 for more details on the model and results. In this section, we describe the generalised inductive approach in [40].

When applied to self-avoiding walks or oriented percolation, the lace expansion gives rise to a recursion relation of the form

$$f_{n+1}(k;z) = \sum_{m=1}^{n+1} g_m(k;z) f_{n+1-m}(k;z) + e_{n+1}(k;z) \qquad (n \geq 0), \quad (2.4.1)$$

with $f_0(k;z) = 1$, and where z plays the role of the parameter in the model which needs to be tuned to the critical value. For example, for oriented percolation, z is equal to the percolation parameter p. Here, $k \in [-\pi, \pi]^d$ is the Fourier variable, which is dual to the spatial lattice variable $x \in \mathbb{Z}^d$. We think of the functions g_m and e_m as given, and the goal is to understand the behaviour of the solution $f_n(k;z)$ of (2.4.1). We first rewrite the lace expansion equation for self-avoiding walk and oriented percolation in the form in (2.4.1).

For self-avoiding walk, the lace expansion in (2.2.1) yields that, for $n \geq 0$, after multiplication by z^{n+1},

$$\hat{c}_{n+1}(k) z^{n+1} = z\hat{D}(k)\hat{c}_n(k) z^n + \sum_{m=2}^{n+1} \hat{\pi}_m(k) z^m \hat{c}_{n+1-m}(k) z^{n+1-m} \quad (n \geq 0). \quad (2.4.2)$$

This is in the form of (2.4.1), with $f_n(k;z) = \hat{c}_n(k)z^n$, $g_1(k;z) = z\hat{D}(k)$, $g_m(k;z) = \hat{\pi}_m(k)z^m$ for $m \geq 2$, and $e_n(k;z) = 0$.

For oriented percolation, on the other hand, the lace expansion yields that, now writing $\hat{\tau}_{n+1}(k;z)$ for the Fourier transform of the two-point function when the percolation parameter is equal to z,

$$\begin{aligned} \hat{\tau}_{n+1}(k;z) = z\hat{D}(k)\hat{\tau}_n(k;z) + z\hat{D}(k) \sum_{m=2}^{n} \hat{\pi}_m(k;z)\hat{\tau}_{n-m}(k;z) \\ + \hat{\pi}_{n+1}(k;z) \quad (n \geq 0). \end{aligned} \quad (2.4.3)$$

Equation (2.4.3) is a special case of (2.4.1) with the choices $f_n(k;z) = \hat{\tau}_n(k;z)$, $e_n(k;z) = \hat{\pi}_n(k;z)$, $g_1(k;z) = z\hat{D}(k)$, $g_2(k;z) = 0$, $g_m(k;z) = z\hat{D}(k)\hat{\pi}_{m-1}(k;z)$ for $m \geq 3$.

The consistency in the recurrence relation is in general formulated by two assumptions on the coefficients g_m and e_m in (2.4.1). To be able to describe these assumptions, we write, for a function $f\colon [-\pi,\pi]^d \to \mathbb{R}$,

$$\|f\|_1 = \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} |f(k)|. \tag{2.4.4}$$

Then, the assumptions on e_m and g_m state that bounds on $f_j(0)$ and $\|f_j\|_1$ for $j \leq n$ and fixed z imply bounds on e_m and g_m for $m \leq n+1$. In particular, these bounds imply that

$$|g_m(k;z)| \leq \frac{C_K\beta}{(m+1)^{d/2}} \qquad \text{and} \qquad |\partial_z g_m(k;z)| \leq \frac{C_K\beta}{(m+1)^{(d-2)/2}}. \tag{2.4.5}$$

The second bound shows that the coefficients are sufficiently smooth in the parameter z. For self-avoiding walk, $\partial_z g_m(k;z) = mz^{-1}g_m(k;z)$, so the second bound in (2.4.5) follows immediately from the first in (2.4.5).

The proof of these assumptions for self-avoiding walk and oriented percolation follows from Propositions 2.3–2.4. However, Propositions 2.3–2.4 involve supremum-norms of the two-point function rather than the integral norm of its Fourier transform. These can be related as we explain now. We prove the relation for c_n, but the bounds are valid rather generally. We first use that $\|c_n\|_1 = \sum_x c_n(x) = \hat{c}_n(0)$, so that bounds on $\|c_n\|_1$ follow from the bounds on $\hat{c}_n(k)$ for $k = 0$. We further note that, by the Fourier inversion formula,

$$c_n(x) = \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} e^{-ik\cdot x}\hat{c}_n(k). \tag{2.4.6}$$

Therefore, we arrive at

$$\|c_n\|_\infty = \sup_{x\in\mathbb{Z}^d} \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} e^{-ik\cdot x}\hat{c}_n(k) \leq \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} |\hat{c}_n(k)| = \|\hat{c}_n\|_1. \tag{2.4.7}$$

In particular, Propositions 2.3 can be reformulated as saying that bounds on $\hat{c}_j(0)$ and $\|\hat{c}_j\|_1$ for $j \leq n$ imply bounds on $\hat{\pi}_m$ for $m \leq n+1$.

One of the main difficulties is that the parameter z should be carefully tuned to a critical value. For self-avoiding walks, this critical value is equal to $z_c = \mu^{-1}$, where μ describes the exponential decay of c_n. For oriented percolation, this critical value is equal to $z_c = p_c$, the critical percolation threshold. The main difficulty, as we now explain, is that this critical value can be uniquely defined in terms of the lace expansion coefficients π_m, but

for this, all coefficients are necessary. This makes the use of induction more intricate.

We now show how we can identify the critical value z_c and the diffusion constant v in terms of the lace expansion coefficients. For simplicity, we only treat self-avoiding walk. We let $G_z(x)$ be the generating function of $c_n(x)$, i.e.,

$$G_z(x) = \sum_{n=0}^{\infty} z^n c_n(x). \tag{2.4.8}$$

We let $\hat{G}_z(k)$ denote the Fourier transform of $G_z(x)$. Then, $\hat{G}_z(0)$ is the generating function of $c_n = \sum_{x\in\mathbb{Z}^d} c_n(x)$. By (2.1.27), it is clear that $\hat{G}_z(0) < \infty$ when $z < \mu^{-1}$, while $\hat{G}_z(0) = \infty$ when $z > \mu^{-1}$. Moreover, by (2.1.29), we have that $\hat{G}_{\mu^{-1}}(0) = \infty$. Therefore, $z_c = \mu^{-1}$ can be uniquely identified as the radius of convergence of the generating function $\hat{G}_z(0)$.

By (2.4.2), we see that we have the linear equation

$$\hat{G}_z(k) = 1 + [z\hat{D}(k) + \hat{\Pi}_z(k)]\hat{G}_z(k), \tag{2.4.9}$$

where the 1 on the right-hand side is the contribution due to the zero-step walk, for which $c_n(x) = \delta_{0,x}$, and where

$$\hat{\Pi}_z(k) = \sum_{m=0}^{\infty} \hat{\pi}_m(k) z^m. \tag{2.4.10}$$

We can solve this equation explicitly to yield

$$\hat{G}_z(k) = \frac{1}{1 - [z\hat{D}(k) + \hat{\Pi}_z(k)]}. \tag{2.4.11}$$

Taking $k = 0$ and using (2.1.3), we obtain that

$$\hat{G}_z(0) = \frac{1}{1 - [z + \hat{\Pi}_z(0)]}. \tag{2.4.12}$$

Since $z_c = \mu^{-1}$ is such that $\hat{G}_z(0) = \infty$, the critical value $z_c = \mu^{-1}$ must obey the implicit equation

$$z_c + \hat{\Pi}_{z_c}(0) = 1, \qquad \text{or} \qquad z_c = 1 - \sum_{m=2}^{\infty} \hat{\pi}_m(0) z_c^m. \tag{2.4.13}$$

The main difficulty is that (2.4.13) involves $\hat{\pi}_m(0)$ for *all* $m \geq 2$. In a similar way, we see that, by differentiating with respect to z and using (2.4.13),

$$\begin{aligned} \hat{G}_z(0) = \frac{1}{1 - [z + \hat{\Pi}_z(0)]} &= \frac{1}{z_c - z + [\hat{\Pi}_{z_c}(0) - \hat{\Pi}_z(0)]} \\ &\approx \frac{1}{(z_c - z)[1 + \partial_z \hat{\Pi}_z(0)]_{z=z_c}}, \end{aligned} \tag{2.4.14}$$

if we assume that sufficiently many derivatives exist. When we compare with Theorem 2.1(a) with $k = 0$, we see that

$$\hat{G}_z(0) = \sum_{n=0}^{\infty} z^n c_n \approx \sum_{n=0}^{\infty} z^n z_c^{-n} A = \frac{A z_c}{z_c - z}, \tag{2.4.15}$$

so that we conclude that

$$A = \frac{1}{z_c[1 + \partial_z \hat{\Pi}_z(0)]_{z=z_c}} = \frac{1}{z_c + \sum_{m=2}^{\infty} m \hat{\pi}_m(0) z_c^m}. \tag{2.4.16}$$

In the more general context of (2.4.1), it is not hard to verify that

$$A = \frac{1 + \sum_{m=1}^{\infty} e_m(0; z_c)}{1 + \sum_{m=2}^{\infty} \partial_z g(0; z_c)}. \tag{2.4.17}$$

Finally, assuming that $\hat{c}_n(k) \approx A e^{-\sigma^2 v |k|^2/2d}$, a similar computation yields that

$$v = \frac{-\frac{1}{\sigma^2} \sum_{m=1}^{n} \nabla^2 g_m(0; z)}{1 + \sum_{m=1}^{n} (m-1) g_m(0; z)}. \tag{2.4.18}$$

We conclude that the constants z_c, A and v involve $\hat{\pi}_m(0)$ for *all* $m \geq 2$. In the next section, we show how this problem can be resolved.

2.4.1 Statement of the induction hypotheses

In the frame work of (2.4.1), it can be seen that the critical value z_c is given by

$$z_c = 1 - \sum_{m=2}^{\infty} g_m(0; z_c). \tag{2.4.19}$$

To circumvent the difficulty that the critical value z_c is given in terms of the lace expansion coefficients $\hat{\pi}_m$ for all $m \geq 1$, we use a recursion, which tunes into the solution to (2.4.19). We will do this in the general frame work of (2.4.1). For this, we let $z_0 = z_1 = 1$, and define z_n recursively by

$$z_{n+1} = 1 - \sum_{m=2}^{n+1} g_m(0; z_n), \qquad n \geq 1. \tag{2.4.20}$$

When the sequence z_n converges, using the bound in (2.4.5), as well as the bound on the derivative in (2.4.5), it follows that the limit z_∞ of z_n satisfies

$$z_\infty = 1 - \sum_{m=2}^{\infty} g_m(0; z_\infty). \tag{2.4.21}$$

Moreover, from the assumptions on g_m, it follows that the solution to (2.4.19) is *unique*, so that $z_\infty = z_c$. Note, however, that this equality can only be established *after* the convergence of z_n has been verified.

We now state the induction hypotheses. For this, we need some further notation. We abbreviate

$$a(k) = 1 - \hat{D}(k). \tag{2.4.22}$$

The induction hypotheses involve a sequence v_n, which tunes into the variance factor v computed in (2.4.18), and which is defined as follows. We set $v_0 = b_0 = 1$, and, for $n \geq 1$, we define

$$b_n = -\frac{1}{\sigma^2} \sum_{m=1}^{n} \nabla^2 g_m(0;z), \quad c_n = \sum_{m=1}^{n} (m-1) g_m(0;z), \quad v_n = \frac{b_n}{1+c_n}. \tag{2.4.23}$$

We see that v_n is an approximation to v in (2.4.18), by neglecting the contributions of g_m for $m \geq n+1$. The diffusion constant $\sigma^2 v$ of Theorem 2.1–2.2 will turn out to be given by $\sigma^2 v_\infty(z_c)$. However, we have not yet proved that both series in the definition of v_∞ converge. This fact, as well as the convergence of z_n to z_c, will be proved in the course of the induction.

The z–dependence of b_n, c_n, v_n will usually be left implicit in the notation. We will often simplify the notation by dropping z also from e_n, f_n and g_n, and write, e.g., $f_n(k) = f_n(k;z)$.

The induction hypotheses also involve several constants. Let $d > 4$, and recall that ϵ was specified in (2.1.12). We fix $\gamma, \delta, \rho > 0$ according to

$$0 < \tfrac{d-4}{2} - \rho < \gamma < \gamma + \delta < 1 \wedge \tfrac{d-4}{2} \wedge \epsilon. \tag{2.4.24}$$

This can be done by first fixing $\gamma \in (0, 1 \wedge \frac{d-4}{2} \wedge \epsilon)$ and then choosing δ and ρ accordingly. We also introduce constants $K_1, \ldots, K_5$, which are independent of β. To advance the induction, we will need to assume that

$$K_3 \gg K_1 \gg K_4 \gg 1, \quad K_2 \gg K_1, K_4, \quad K_5 \gg K_4. \tag{2.4.25}$$

Here $a \gg b$ denotes the statement that a/b is sufficiently large. The amount by which, for instance, K_3 must exceed K_1 is independent of β and is determined in the course of the advancement of the induction.

For $n \geq 1$, we define intervals

$$I_n = [z_n - K_1 \beta n^{-(d-2)/2}, z_n + K_1 \beta n^{-(d-2)/2}]. \tag{2.4.26}$$

The induction hypotheses are that the following four statements hold for all $z \in I_n$ and all $1 \leq j \leq n$:

(H1) $|z_j - z_{j-1}| \leq K_1 \beta j^{-d/2}$.

(H2) $|v_j - v_{j-1}| \leq K_2 \beta j^{-(d-2)/2}$.

(H3) For k such that $a(k) \le \gamma j^{-1} \log j$, $f_j(k;z)$ can be written in the form

$$f_j(k;z) = \prod_{i=1}^{j} [1 - v_i a(k) + r_i(k)] ,$$

with $r_i(k) = r_i(k;z)$ obeying

$$|r_i(0)| \le K_3 \beta i^{-(d-2)/2}, \quad |r_i(k) - r_i(0)| \le K_3 \beta a(k) i^{-\delta}.$$

(H4) For k such that $a(k) > \gamma j^{-1} \log j$, $f_j(k;z)$ obeys the bounds

$$|f_j(k;z)| \le K_4 a(k)^{-2-\rho} j^{-d/2},$$

$$|f_j(k;z) - f_{j-1}(k;z)| \le K_5 a(k)^{-1-\rho} j^{-d/2}.$$

2.4.2 Discussion of the induction hypotheses

In this section, we discuss the induction hypotheses in some detail. We will indicate how the induction hypotheses imply the results in Theorems 2.1–2.2, and how the induction hypotheses can be used to obtain the bounds on the lace expansion coefficients in Propositions 2.3–2.4. For simplicity, we will focus on the self-avoiding walk case in Theorem 2.1, the oriented percolation case is similar.

Induction hypothesis (H1) and convergence of z_n. Note from (2.4.20) and (2.4.26) that for $z \in I_n$,

$$\begin{aligned} |z_{n-1} - z| \le |z_n - z| + |z_n - z_{n-1}| &\le K_1 \beta n^{-(d-2)/2} + K_1 \beta n^{-d/2} \\ &\le K_1 \beta (n-1)^{-(d-2)/2}, \end{aligned} \tag{2.4.27}$$

so that $I_n \subset I_{n-1}$. More generally, $I_1 \supset I_2 \supset \cdots \supset I_n$. This is crucial, as it allows us to use the induction hypotheses while advancing them. Also, I_n is a sequence of nested closed intervals of which the width converges to zero. It follows that there is a unique limit, which we denote by z_c. In particular, $z_c \in I_n$ for every $n \ge 1$, so that all induction hypotheses apply to z_c. From (2.4.1), it then follows that z_c satisfies (2.4.21). As a consequence, in the remainder of this section, we will use the results of (H1-H4) for $z = z_c$.

Hypothesis (H3) and Theorem 2.1. Note that, for $k = 0$, (H3) reduces to $f_j(0) = \prod_{i=1}^{j} [1 + r_i(0)]$. Therefore, the bound on $r_j(0)$ in (H3) implies that

$$f_n(0) \approx \prod_{i=1}^{\infty} [1 + r_i(0)], \tag{2.4.28}$$

and this product is well-defined. We conclude that

$$A = \prod_{i=1}^{\infty}[1 + r_i(0)]. \tag{2.4.29}$$

Using the fact that, after the advancement of the induction hypotheses are completed, the sums in (2.4.17) are all well-defined, it is not hard to conclude (2.4.17) from (2.4.1). This shows that Theorem 2.1(a) for $k = 0$ follows from (H3) with $k = 0$. We will now indicate that the statement in (H3) for k with $a(k) \le \gamma n^{-1} \log n$ implies Theorem 2.1(a). Since $a(k) \approx \sigma^2 \frac{|k|^2}{2d}$, the bound $a(k) \le \gamma n^{-1} \log n$ is implied by $|k|^2 \le \gamma' n^{-1} \log n$ for some γ' sufficiently small. In particular, we can take k to be proportional to $\frac{1}{\sqrt{n}}$, as in Theorem 2.1(a). In the latter case, one can verify that

$$\begin{aligned} f_n(k; z_c) \approx \prod_{i=1}^{n} [1 - v_i a(k) + r_i(0)] &\approx \prod_{i=1}^{n} [1 + r_i(0)] \prod_{i=1}^{n} [1 - v_i a(k)] \\ &\approx A \prod_{i=1}^{n} [1 - v_i a(k)] \,, \end{aligned} \tag{2.4.30}$$

where we use (2.4.29). Continuing by using $1 - v_i a(k) \approx e^{-v_i \sigma^2 \frac{|k|^2}{2d}}$, as well as $v_i(z_c) \approx v$, we arrive at

$$f_n(k; z_c) \approx A e^{-nv\sigma^2 \frac{|k|^2}{2d}}. \tag{2.4.31}$$

Theorem 2.1(a) follows by investigating the error terms in more detail, while Theorem 2.1(b) follows by a Taylor expansion of $f_n(k; z_c)$ for $|k|$ small, together with the bounds in (H3).

The induction hypotheses and the bounds on the lace expansion coefficients. In order to advance the induction hypotheses, it is crucial to have bounds on e_{n+1} and g_{n+1}. For this, we use Propositions 2.3 and 2.4. In order to apply these results, we need to have bounds on $\|c_j\|_1 z^j$ and $\|c_j\|_\infty z^j$ for all $j \le n$ and for all $z \in I_n$. These bounds follow from the induction hypotheses, as we will now show. The bound on $\|c_n\|_\infty z_c^j$ will also prove the upper bound in Theorem 2.1(c). The lower bound in Theorem 2.1(c) is a consequence of Theorem 2.1(a).

We now check whether the bounds (2.3.5) and (2.3.13), which are necessary to apply Propositions 2.3–2.4, follow from the induction hypothesis. For $\|c_j\|_1 z^j$, we note that by (H3) for $k = 0$,

$$\|c_j\|_1 z^j = f_j(0; z) = \prod_{i=1}^{j} [1 + r_i(0)] \le \prod_{i=1}^{j} \left[1 + K_3 \beta i^{-(d-2)/2}\right] \le K', \tag{2.4.32}$$

when $d > 4$, where $K' = 1 + CK_3\beta \leq 2$ when β is sufficiently small. Therefore, the first bounds in (2.3.5) and (2.3.13) follow from (H3) with $k = 0$ for sufficiently small β.

Furthermore, for $z \in I_n$ and assuming (H2), (H3) and (H4), one can see that (see [41, Lemma 2.3]) for $1 \leq j \leq n$,

$$\|\hat{D}^2 f_j(\cdot; z)\|_1 \leq C(1 + K_4)\beta j^{-d/2}. \tag{2.4.33}$$

Indeed, the bound can be obtained by splitting the integration domain into several regions, depending on whether (H3) or (H4) applies, and depending on the magnitude of $|k|^2$ and the bounds on $\hat{D}(k)$ in (2.1.17). This gives four regions of integration, some of which may be empty. On each of these regions, we can use the bounds on f_j provided by either (H3) or (H4), and the bounds on $a(k)$ provided by (2.1.17). Here we see that the bounds in (H4) are essential in the lace expansion analysis.

When applied to self-avoiding walk, we have that

$$c_{n+1}(x) \leq (D * c_n)(x). \tag{2.4.34}$$

Using this bound twice, we obtain that

$$\|c_n\|_\infty \leq \|D^{*2} * c_{n-2}\|_\infty \leq \|\hat{D}^2 \hat{c}_{n-2}\|_1, \tag{2.4.35}$$

so that

$$\|c_n\|_\infty z^n \leq z^2 \|\hat{D}^2 \hat{f}_{n-2}(\cdot; z)\|_1, \tag{2.4.36}$$

and (2.4.33) allows us to apply Proposition 2.3 with $K = C(1 + K_4)$. This bound is crucial to obtain bounds on $\hat{\pi}_m(k)$, which in turn will allow us to advance the induction hypotheses. We conclude that the induction hypotheses for $j \leq n$ imply bounds on the lace expansion coefficients $\hat{\pi}_m(k)$ for $m \leq n + 1$, which are necessary to advance the induction hypotheses. Similar arguments apply to oriented percolation.

We now sketch the advancement of (H1) and of (H3). We only give the complete advancement of (H3) for $k = 0$, which gives the idea of the proof. The advancement of the induction hypotheses (H1)–(H4) is technically quite involved, so we will not give all details.

2.4.3 Induction hypothesis (H1) advanced

By (2.4.20) and the mean-value theorem,

$$\begin{aligned} z_{n+1} - z_n &= -\sum_{m=2}^{n} [g_m(0; z_n) - g_m(0; z_{n-1})] - g_{n+1}(0; z_n) \\ &= -(z_n - z_{n-1}) \sum_{m=2}^{n} \partial_z g_m(0; y_n) - g_{n+1}(0; z_n), \end{aligned} \tag{2.4.37}$$

for some y_n between z_n and z_{n-1}. By (H1) and (2.4.26), $y_n \in I_n$. Using Propositions 2.3–2.4, in particular (2.4.5), as well as (H1), it follows that

$$\begin{aligned} |z_{n+1}-z_n| &\leq K_1\beta n^{-d/2}\sum_{m=2}^{n} C_K\beta m^{-(d-2)/2} + C_K\beta(n+1)^{-d/2} \\ &\leq C_K\beta(1+CK_1\beta)(n+1)^{-d/2}. \end{aligned} \tag{2.4.38}$$

Thus (H1) holds for $n+1$, for β small and $K_1 > C_K$, and we have advanced induction hypothesis (H1). □

Having advanced (H1) to $n+1$, it then follows from (2.4.27) that $I_1 \supset I_2 \supset \cdots \supset I_{n+1}$.

For $n \geq 0$, define

$$\zeta_{n+1} = \zeta_{n+1}(z) = \sum_{m=1}^{n+1} g_m(0;z) - 1 = \sum_{m=2}^{n+1} g_m(0;z) + z - 1. \tag{2.4.39}$$

Note that for $z = z_c$, by (2.4.19), we should have that

$$\zeta_{n+1} = -\sum_{m=n+2}^{\infty} g_m(0;z_c), \tag{2.4.40}$$

so that, when (2.4.5) holds,

$$|\zeta_{n+1}| \leq \sum_{m=n+2}^{\infty} |g_m(0;z_c)| \leq K(n+1)^{-(d-2)/2}. \tag{2.4.41}$$

Unfortunately, the above argument cannot be made rigorous at this stage, since we cannot derive bounds on g_m for general $m \geq n$ from the induction hypotheses (H1)–(H4). The following lemma, whose proof makes use of (H1) for $n+1$, establishes (2.4.41) by only using the recursion for z_n:

Lemma 2.5. *For all $z \in I_{n+1}$,*

$$|\zeta_{n+1}| \leq CK_1\beta(n+1)^{-(d-2)/2}. \tag{2.4.42}$$

Proof. By (2.4.20) and the mean-value theorem,

$$\begin{aligned} |\zeta_{n+1}| &= \Big|(z-z_{n+1}) + \sum_{m=2}^{n+1}[g_m(0;z) - g_m(0;z_n)]\Big| \\ &= \Big|(z-z_{n+1}) + (z-z_n)\sum_{m=2}^{n+1}\partial_z g_m(0;y_n)\Big|, \end{aligned} \tag{2.4.43}$$

for some y_n between z and z_n. Since $z \in I_{n+1} \subset I_n$ and $z_n \in I_n$, we have $y_n \in I_n$. Then,

$$\begin{aligned}|\zeta_{n+1}| &\le K_1\beta(n+1)^{-(d-2)/2} + K_1\beta n^{-(d-2)/2}\sum_{m=2}^{n+1} C_\kappa \beta m^{-(d-2)/2} \\ &\le K_1\beta(1+CC_\kappa\beta)(n+1)^{-(d-2)/2}.\end{aligned} \tag{2.4.44}$$

The lemma then follows, for β sufficiently small. □

2.4.4 Induction hypothesis (H3) advanced

The advancements of the induction hypotheses (H3–H4) are the most technical part of the proof. In this section, we sketch the advancement of (H3), and complete the advancement for $k = 0$. The details of the advancement can be found in [40, Section 3].

For (H3), we fix k with $a(k) \le \gamma(n+1)^{-1}\log(n+1)$, and $z \in I_{n+1}$. We write

$$\frac{f_{n+1}(k)}{f_n(k)} = 1 - v_{n+1}a(k) + r_{n+1}(k). \tag{2.4.45}$$

Then the advancement follows if $r_{n+1}(0)$ and $r_{n+1}(k) - r_{n+1}(0)$ satisfy the bounds required by (H3).

To begin, we divide the recursion relation (2.4.1) by $f_n(k)$, and use (2.4.39), to obtain

$$\begin{aligned}\frac{f_{n+1}(k)}{f_n(k)} = 1 + \sum_{m=1}^{n+1}\Big[g_m(k)\frac{f_{n+1-m}(k)}{f_n(k)} - g_m(0)\Big] + \zeta_{n+1} \\ + \frac{e_{n+1}(k)}{f_n(k)}.\end{aligned} \tag{2.4.46}$$

By (2.4.23),

$$v_{n+1} = b_{n+1} - v_{n+1}c_{n+1} = -\sigma^{-2}\sum_{m=1}^{n+1}\nabla^2 g_m(0) - v_{n+1}\sum_{m=1}^{n+1}(m-1)g_m(0). \tag{2.4.47}$$

Thus we can rewrite (2.4.46) as

$$\frac{f_{n+1}(k)}{f_n(k)} = 1 - v_{n+1}a(k) + r_{n+1}(k), \tag{2.4.48}$$

where

$$r_{n+1}(k) = X(k) + Y(k) + Z(k) + \zeta_{n+1} \tag{2.4.49}$$

with

$$X(k) = \sum_{m=2}^{n+1} \Big[\big(g_m(k) - g_m(0)\big) \frac{f_{n+1-m}(k)}{f_n(k)} - a(k)\sigma^{-2}\nabla^2 g_m(0) \Big], \quad (2.4.50)$$

$$Y(k) = \sum_{m=2}^{n+1} g_m(0) \left[\frac{f_{n+1-m}(k)}{f_n(k)} - 1 - (m-1)v_{n+1}a(k) \right], \quad (2.4.51)$$

$$Z(k) = \frac{e_{n+1}(k)}{f_n(k)}. \quad (2.4.52)$$

The $m = 1$ terms in X and Y vanish and have not been included. For simplicity, we will only sketch the advancement of the induction hypothesis (H3) for $k = 0$, for which $X(0) = 0$.

For $k = 0$, we will prove that

$$|r_{n+1}(0)| \le \frac{C(K_1 + C_\kappa)\beta}{(n+1)^{(d-2)/2}}. \quad (2.4.53)$$

This gives (H3) for $n+1$, provided we assume that $K_3 \gg K_1$ and $K_3 \gg C_\kappa$. To prove the bounds on r_{n+1} of (2.4.53), it will be convenient to make use of an elementary convolution bounds which states that for $n \ge 2$,

$$\sum_{m=2}^{n} \frac{1}{m^a} \sum_{j=n-m+1}^{n} \frac{1}{j^b} \le \quad Cn^{-(a-1)\wedge b} \quad \text{for } a > 2, b > 1. \quad (2.4.54)$$

The induction step. By definition,

$$r_{n+1}(0) = Y(0) + Z(0) + \zeta_{n+1}. \quad (2.4.55)$$

Since $|\zeta_{n+1}| \le CK_1\beta(n+1)^{-(d-2)/2}$ by Lemma 2.5, to prove (2.4.53) it suffices to show that

$$|Y(0)| \le CC_\kappa\beta(n+1)^{-(d-2)/2}, \qquad |Z(0)| \le CC_\kappa\beta(n+1)^{-(d-2)/2}. \quad (2.4.56)$$

For the bound on Y, we note that

$$Y(0) = \sum_{m=2}^{n+1} g_m(0) \left[\frac{f_{n+1-m}(0)}{f_n(0)} - 1 \right], \quad (2.4.57)$$

which we can bound, using (2.4.54), to obtain

$$|Y(0)| \le \sum_{m=2}^{n+1} \frac{C_\kappa\beta}{m^{d/2}} \sum_{j=n+2-m}^{n} \frac{CK_3\beta}{j^{(d-2)/2}} \le \frac{CC_\kappa K_3\beta^2}{(n+1)^{(d-2)/2}}, \quad (2.4.58)$$

where we use that (H3) implies that

$$\left|\frac{f_{n+1-m}(0)}{f_n(0)} - 1\right| = \left|\prod_{j=n+2-m}^{n} [1 + r_j(0)]^{-1} - 1\right| \leq \sum_{j=n+2-m}^{n} \frac{CK_3\beta}{j^{(d-2)/2}}. \tag{2.4.59}$$

Taking β small then gives the desired bound on $Y(0)$ of (2.4.56).

For the bound on Z, we recall that $Z(0) = \frac{e_{n+1}(0)}{f_n(0)}$. Using (2.4.5) and the fact that $f_n(0) \geq 1 - CK_3\beta$, we obtain

$$|Z(0)| \leq CC_K\beta(n+1)^{-d/2}. \tag{2.4.60}$$

This completes the proof of (2.4.53), and hence completes the advancement of (H3) to $n+1$ when $k = 0$. The proof for $k \neq 0$ is similar, but substantially more involved.

2.5 The inductive method applied to related models

In this section, we survey some results that show Gaussian behaviour for related models, using adaptations of the inductive method. These results show that the inductive method is a flexible tool, that can be adapted to various different settings.

2.5.1 Diffusive behaviour for the forgetful weakly self-avoiding walk

In this section, we describe the results in [33], in which the inductive method was applied for the first time to a model related to self-avoiding walk.

For $x \in \mathbb{Z}^d$, we set $c_0(x) = \delta_{0,x}$ and, for $n \geq 1$, $p \in \mathbb{R}$, $\beta \geq 0$, we define

$$c_n(x) \quad = \quad \sum_{\omega \in \mathcal{W}_n(x)} \prod_{0 \leq s < t \leq n} (1 - \lambda_{st} U_{st}(\omega)) W(\omega), \tag{2.5.1}$$

where U_{st} is defined in (2.2.12) and

$$\lambda_{st} = \lambda_{st}(\beta, p) = 1 - e^{-\frac{\beta}{|s-t|^p}}. \tag{2.5.2}$$

In this model, the walk receives a penalty $e^{-\frac{\beta}{m^p}}$ for every self-intersection for which the time elapsed between the two visits of the same site is equal to m. When $p > 0$, the interaction decreases with time, so that the walker 'forgets' its past. Therefore this model is sometimes called the *forgetful weakly self-avoiding walk*. When $p = 0$, we retrieve the weakly self-avoiding walk for which every self-intersection receives a constant penalty $e^{-\beta}$. When $\beta \to \infty$, the forgetful weakly self-avoiding walk converges to the self-avoiding walk.

In this section, we take D to be the nearest-neighbour step distribution, i.e.,

$$D(x) = \frac{1}{2d} I[|x| = 1]. \tag{2.5.3}$$

We again use the abbreviation in (2.1.22). Define

$$\epsilon = p + \frac{d-4}{2} > 0, \tag{2.5.4}$$

which turns out to be the key parameter in the model. The following theorem from [33] shows that the model is diffusive when $p \geq 0$ and $\epsilon > 0$:

Theorem 2.6. *Suppose that either $d > 4$, $p \geq 0$ or $d \leq 4$, $p > \frac{4-d}{2}$. Then there is a $\beta_0 = \beta_0(d,p) > 0$ such that for $\beta < \beta_0$,*
(a)

$$c_n = A\mu^n[1 + \mathcal{O}(n^{-\epsilon})], \tag{2.5.5}$$

(b)

$$\frac{1}{c_n}\sum_x |x|^2 c_n(x) = \begin{cases} vn[1+\mathcal{O}(n^{-1\wedge\epsilon})] & \epsilon \neq 1 \\ vn[1+\mathcal{O}(n^{-1}\log n)] & \epsilon = 1 \end{cases}, \tag{2.5.6}$$

(c)

$$\frac{1}{c_n}\hat{c}_n\Big(\frac{k}{\sqrt{vn}}\Big) = e^{-\frac{k^2}{2d}[1+\mathcal{O}(n^{-\delta'})]}, \tag{2.5.7}$$

where $\mu, A, v > 0$ are constants (depending on d, p, β), ϵ is given by (2.5.4), $\delta' \in (0, 1\wedge\epsilon)$ is arbitrary, and the error estimate in (c) is uniform in $k \in \mathbb{R}^d$ provided $\|k\|_1 (\log n)^{-1/2}$ is sufficiently small.

2.5.2 Ballistic behaviour for the forgetful weakly self-avoiding walk

We continue with the forgetful weakly self-avoiding walk defined in the previous section, but now study it in dimension 1 when $p \leq 1$, where the behaviour is rather different.

To be able to state that result, we define the Fourier transform of (2.1.20) restricted to the non-negative integers to be

$$\hat{c}_n^+(k) = \sum_{x\geq 0} c_n(x) e^{ikx}, \qquad k \in [-\pi, \pi]. \tag{2.5.8}$$

We will use the abbreviation

$$c_n^+ = \hat{c}_n^+(0) = \sum_{x\geq 0} c_n(x). \tag{2.5.9}$$

In dimension 1, when $p > \frac{3}{2}$ the self-repellence is weak and the behaviour is diffusive. On the other hand, when $p \leq 1$, the self-repellence is strong, and the behaviour is *ballistic*, as shown in [30]:

Theorem 2.7. *Fix $d = 1$ and $p \in [0, 1]$. Then there exist $\beta_0 = \beta_0(p) > 0$ and $\epsilon > 0$ such that for $\beta > \beta_0$,*

(a)

$$c_n = \begin{cases} A\mu^n[1 + \mathcal{O}(e^{-\epsilon\beta n^\gamma})] & \text{for } p \in [0, 1), \\ A\mu^n[1 + \mathcal{O}(n^{-\epsilon\beta})] & \text{for } p = 1. \end{cases} \tag{2.5.10}$$

(b)

$$\frac{1}{c_n^+} \sum_{x \geq 0} x c_n(x) = \theta n[1 + \mathcal{O}(\frac{1}{n})]. \tag{2.5.11}$$

(c)

$$\frac{1}{c_n^+} \sum_{x \geq 0} x^2 c_n(x) - \Big(\frac{1}{c_n^+} \sum_{x \geq 0} x c_n(x)\Big)^2 = \sigma^2 n[1 + \mathcal{O}(\frac{1}{n})]. \tag{2.5.12}$$

(d)

$$\frac{1}{c_n^+} e^{-ik\frac{g}{\sigma}\sqrt{n}} \hat{c}_n^+\Big(\frac{k}{\sigma\sqrt{n}}\Big) = e^{-\frac{k^2}{2}}[1 + \mathcal{O}(n^{-\alpha})], \tag{2.5.13}$$

where $\gamma = \gamma(p) = \frac{(1-p)(1+p-p^2)}{1+p}$ and $\mu, A, \theta, \sigma > 0$ are constants (depending on p, β), $\alpha \in (0, \frac{1}{2})$ is arbitrary and the error estimate in (d) is uniform in $k \in \mathbb{R}$ provided $k^2 \leq \delta \log n$, where $\delta > 0$ is sufficiently small.

Part (d) of Theorem 2.7 shows the central limit theorem. Indeed, we have that conditionally on the endpoint being non-negative, it is approximately normally distributed with mean θn and variance $\sigma^2 n$. We can conclude from Theorems 2.6 and 2.7 that the dependence on p is rather sensitive. Interestingly, it has been predicted that the scaling of the endpoint in dimension 1 is given by $\nu(p) = 2 - p$ for $p \in [1, \frac{3}{2}]$ [7], which is linear interpolation between the value $\nu = \frac{1}{2}$ for $p = \frac{3}{2}$ and $\nu = 1$ for $p = 1$. The fact that ballistic behavour pertains when $p = 1$ and β is sufficiently large is interesting, and it raises the question whether ballistic behaviour persists when $p = 1$ for all $\beta > 0$.

The proof of Theorem 2.7 makes use of the inductive method, adapted so as to deal with the ballistic behaviour.

2.5.3 Self-avoiding walk with nearest-neighbour attraction

An interesting application of the inductive method for the lace expansion can be found in [63]. Ueltschi studies self-avoiding walk with nearest-neighbour attraction, which is a model for a polymer in a repulsive solution. The two-point function of this model is given by (2.2.11), where now we define

$$U_{st}(\omega) = I[\omega(s) = \omega(t)] - \kappa I[|\omega(s) - \omega(t)| = 1], \tag{2.5.14}$$

In this model, each nearest-neighbour contact of the path is rewarded with a factor $1 + \kappa$. When κ is small, it can be expected that the model is

close to self-avoiding walk, but for κ large, it may show different scaling behaviour.

Ueltschi [63] studies a model in which the step distribution D satisfies

$$\inf_{x,y:|x-y|=1} \frac{D(x)}{D(y)} = \Delta > 0, \tag{2.5.15}$$

under the assumption that

$$(1+\kappa)^{2d} \leq 1 + \frac{\Delta^2}{2d(1+\kappa)^{2d-1}}. \tag{2.5.16}$$

Note that the model with D in (2.1.19) does not satisfy this assumption. In fact, no finite range model satisfies the assumption in (2.5.15). Models satisfying (2.5.15) must be such that the tails of D decay not faster than exponentially.

Among other things, Ueltschi proves that the model is self-repellent in the sense that (2.1.26) also holds for the sum over the two-point function for self-avoiding walk with nearest-neighbour attraction and κ sufficiently small. Moreover, he shows that under (2.5.16), and for every D satisfying the assumptions in Section 2.1.1, that the assumptions for the general inductive method in [40] are satisfied, so that the same results hold for $\hat{c}_n(k)$ as for the self-avoiding walk two-point function as formulated in Theorem 2.1. This result is remarkable, since it applies to a model which is not strictly self-repellent. General results for the model of (weakly) self-avoiding walk with nearest-neighbour attraction are still missing, for example for the model where D is given by (2.1.19), even when the attraction parameter κ is extremely small.

2.5.4 The spread-out contact process above 4 dimensions

The contact process is a simple model for the spread of an infection in continuous time in a static population. The spread-out contact process is defined as follows. Let $C_t \subset \mathbb{Z}^d$ be the set of infected individuals at time $t \in \mathbb{R}_+$, and let $C_0 = \{o\}$. An infected site x *recovers* at rate 1 independently of t, while a healthy site x is *infected*, depending on the status of its neighbours, at rate $\lambda \sum_{y \in C_t} D(x-y)$, where $\lambda \geq 0$ is the infection rate and $D(x-y)$ represents the strength of the interaction between x and y. Let $\mathbb{P}^\lambda$ be the law of the contact process with infection rate λ. We denote the *two-point function* by

$$\tau_t^\lambda(x) = \mathbb{P}^\lambda(x \in C_t). \tag{2.5.17}$$

By an extension of the results in [3, 22] to the spread-out contact process,

there exists a unique critical value $\lambda_c \in (0,\infty)$ such that

$$\begin{aligned}\chi(\lambda) &= \int_0^\infty dt\ \hat\tau_t^\lambda(0) \begin{cases} <\infty, & \text{if } \lambda<\lambda_c,\\ =\infty, & \text{if } \lambda\ge\lambda_c,\end{cases}\\ \theta(\lambda) &\equiv \lim_{t\uparrow\infty} \mathbb{P}^\lambda(C_t\neq\varnothing) \begin{cases} =0, & \text{if } \lambda\le\lambda_c,\\ >0, & \text{if } \lambda>\lambda_c,\end{cases}\end{aligned} \tag{2.5.18}$$

where we denote the Fourier transform of a summable function $f\colon \mathbb{Z}^d\to\mathbb{R}$ by

$$\hat f(k) = \sum_{x\in\mathbb{Z}^d} f(x)\, e^{ik\cdot x} \qquad (k\in[-\pi,\pi]^d). \tag{2.5.19}$$

We assume that D satisfies the restrictions in Section 2.1.1. The main result for the sufficiently spread-out contact process at $\lambda=\lambda_c$ for $d>4$ proved in [38] is the following:

Theorem 2.8. *Let $d>4$ and $\delta\in(0,1\wedge\epsilon\wedge\frac{d-4}{2})$. There is an $L_0=L_0(d)$ such that, for $L\ge L_0$, there are positive and finite constants $v=v(d,L)$, $A=A(d,L)$, $C_1=C_1(d)$ and $C_2=C_2(d)$ such that*

$$\hat\tau_t^{\lambda_c}(\tfrac{k}{\sqrt{v\sigma^2 t}}) = A\, e^{-\frac{|k|^2}{2d}}\left[1+O(|k|^2(1+t)^{-\delta})+O((1+t)^{-(d-4)/2})\right], \tag{2.5.20}$$

$$\frac{1}{\hat\tau_t^{\lambda_c}(0)}\sum_{x\in\mathbb{Z}^d}|x|^2\tau_t^{\lambda_c}(x) = v\sigma^2 t\left[1+O((1+t)^{-\delta})\right], \tag{2.5.21}$$

$$C_1L^{-d}(1+t)^{-d/2} \le \|\tau_t^{\lambda_c}\|_\infty \le e^{-t}+C_2L^{-d}(1+t)^{-d/2}, \tag{2.5.22}$$

with the error estimate in (2.5.20) *uniform in $k\in\mathbb{R}^d$ with $|k|^2/\log(2+t)$ sufficiently small.*

In [38], also an extension to a local mean-field limit for spread-out oriented percolation and the contact process below and at the critical dimension $d_c=4$ is proved. In this case, these models are *not* believed to exhibit the mean-field behaviour as long as L remains finite, and Gaussian asymptotics are not expected to hold in this case. We specialize to oriented percolation. Then, we can study oriented percolation with range $L=L_n$ and at times proportional to n, where $L_n\to\infty$ as $L_n=Ln^b$, where b is such that $bd>(4-d)/2$. A similar approach was used by Durrett and Perkins [16], with somewhat different restrictions on b. See [38] and the references therein for details.

The strategy of proof of Theorem 2.8 is to use *time-discretization*. Instead of investigating the contact process in continuous time, we investigate an oriented percolation approximation, where time is in $\varepsilon\mathbb{N}$, and where the

occupation probability of bonds $b = ((u,t),(v,t+\varepsilon))$, for $u, v \in \mathbb{Z}^d$ and $t \in \varepsilon\mathbb{N}$, is equal to

$$p_\varepsilon(v-u) = \begin{cases} 1-\varepsilon & \text{when } u = v, \\ \varepsilon\lambda D(v-u) & \text{when } u \neq v. \end{cases} \tag{2.5.23}$$

In [3], it has been shown that the discretized contact process converges to the contact process, and in [56], it has been shown that also the critical point converges as $\varepsilon \downarrow 0$. This result was extended in [38] to the statement that the two-point function and its Fourier transform converge as $\varepsilon \downarrow 0$ to their equivalents for the contact process. Then, Theorem 2.8 follows from a version of Theorem 2.2 where the error terms are uniform in ε. The time-discretization has two main advantages. The first is that it allows to derive the lace expansion. Indeed, any derivation of the lace expansion in continuous time is missing, and this necessitates to discretize time. The second advantage is that the time-discretization allows to perform induction in time along the multiples of ε.

In [38], also the continuum limit was taken, and the lace expansion recursion relation in discrete time was shown to converge to a continuum equation involving $\partial_t \tau_t^\lambda(x)$. It is unclear, though, how this relation can be used effectively.

2.5.5 Long-lived lattice trees above 8 dimensions

Let us introduce some notation. A *lattice tree* is a tree embedded in $\mathbb{Z}^d$ containing no cycles. We give uniform weight to lattice trees with a fixed number of bonds, and assume that the bonds are either nearest-neighbour, or spread-out (as in (2.1.19)). In general, the number of lattice trees of fixed size grows exponentially with the size. Denote by t_N the total number of lattice trees of size N containing 0. Then, it is known that

$$\lim_{N\to\infty} t_N^{1/N} = \lambda \in (0,\infty). \tag{2.5.24}$$

In fact, (2.5.24) follows from the *super-multiplicativity relation*

$$(N+M)t_{N+M} \geq N t_N M t_M, \tag{2.5.25}$$

which follows from the fact that we can glue two lattice trees together at the rightmost corner of the one, and the leftmost corner of the other. This shows that for the number of lattice trees τ_N where the left corner is located at the origin, we have $\tau_{N+M} \geq \tau_N \tau_M$. Equation (2.5.25) follows by noting that $t_N = N\tau_N$.

We define

$$t_n(x) = \sum_{N=1}^{\infty} t_{n;N}(x)\lambda^{-N} \tag{2.5.26}$$

to be the two-point function for lattice trees, where $t_{n;N}(x)$ is the number of lattice trees of size N such that the length of the unique path along the lattice tree connecting 0 to x equals n.

Lattice trees in high dimensions have been studied in [12, 13, 27], where several versions of mean-field behaviour were proved to hold using generating function techniques. In this section, we describe the results in [43], where a version of the inductive method is used.

Of course, the existence of the sum in (2.5.26) is a non-trivial result, and follows from [24] for the spread-out model, and from [23] for the nearest-neighbour model with d sufficiently large. In [43], Holmes uses the inductive method for the lace expansion to prove Theorem 2.1 for $t_n(x)$ when the dimension d is larger than the upper critical dimension $d_c = 8$. In this case, the exponent of the error term is $(d-8)/2$, which is $(d-d_c)/2$, similarly to (2.1.30).

This application of the inductive method is somewhat unusual, since the lace expansion coefficient $\hat{\pi}_m(k)$ for lattice trees *cannot* be bounded by $t_j(x)$ with $j \leq m$. However, the main result in [24] is a bound in x-space of the form (see Theorem 2.11 below)

$$G(x) = \sum_{n=0}^{\infty} t_n(x) \approx C(|x|+1)^{-(d-2)}, \tag{2.5.27}$$

and Holmes makes essential use of this result to be able to bound the lace expansion diagrams, and thus to apply the inductive method.

2.6 Comparison to other methods

In this section, we describe two different methods to analyse the lace expansion equation. We have described the inductive method in detail, and in this section, we will describe the original method used to analyse the lace expansion, generating functions, and a method using Banach fixed point theorems.

2.6.1 Generating functions

In the papers up to 1998, and also many papers after 1998, the lace expansion was always analysed using *generating function methodology*. The main ideas are already present in [6], and we sketch the method now. The method analyses the Green's function $G_z(x)$ in (2.4.8) or its Fourier transform. A key ingredient in the generating function approach is the following lemma, which replaces the consistency property that is crucial in the inductive method (see also the end of Section 2.3.1):

Lemma 2.9. *Let $f : [z_1, z_c) \to \mathbb{R}$ and $a \in (0,1)$ be given. Suppose that*

(i) *f is continuous on the interval $[z_1, z_c)$.*

(ii) $f(z_1) \leq a$.

(iii) *For each* $z \in (z_1, z_c)$, *if* $f(z) \leq 1$ *then in fact* $f(z) \leq a$. *(In other words, one inequality implies a stronger inequality.)*

Then $f(z) \leq a$ *for all* $z \in [z_1, z_c)$.

Lemma 2.9 states that if a function is continuous on an open interval, and an inequality implies a stronger inequality, then the stronger inequality is true for all elements in the open interval. When applied to a generating function, the Monotone Convergence Theorem implies that the bound is then also true at the right-end of the interval, which is typically the radius of convergence of the generating function. Here we are relying on the fact that the functions f in the analysis are typically monotone increasing in the parameter z.

The important ingredient in Lemma 2.9 is the implication that a weak bound implies a stronger bound. The proof of this fact is then based on the lace expansion. For example, for self-avoiding walk, we define $H_z(x) = G_z(x) - \delta_{0,x}$. Then

$$|\hat{\Pi}_z^{(2)}(k)| \leq \|H_z\|_\infty \|H_z\|_2^2. \tag{2.6.1}$$

Similar bounds turn out to be true for all $N \geq 1$, as well as for the derivatives of $\hat{\Pi}_z^{(N)}(k)$ with respect to z. The strategy is then to choose $f(z)$ appropriately in such a way that the bound $f(z) \leq a$ implies bounds on $\|H_z\|_\infty$ and $\|H_z\|_2^2$ where an extra factor $\beta = L^{-d}$ appears. By choosing L large, we can then make $\|H_z\|_\infty$ and $\|H_z\|_2^2$ small, which in turn implies that $\hat{G}_z(k)$ is a small perturbation of the Fourier transform of the random walk Green's function. This allows for an improvement from a bound that $f(z) \leq 1$ to $f(z) \leq a$ when β is sufficiently small. Lemma 2.9 then implies the improved bound for $\hat{G}_z(k)$ for *all* $z \leq z_c$.

The above strategy is used in all early works on the lace expansion, and its application to self-avoiding walk in the most modern version of the method is explained in detail in [62, Sections 3-6]. Before comparing the generating function approach and the inductive method, we state two results that are shown using the generating function approach. We recall that for the nearest-neighbour model, D is given by (2.5.3).

Theorem 2.10. *For nearest-neighbour self-avoiding walk and* $d \geq 5$, *the rescaled path* $\left\{\frac{1}{\sqrt{n}}\omega(\lfloor nt \rfloor)\right\}_{t\in[0,1]}$ *converges weakly to a Brownian motion.*

Theorem 2.10 is a seminal result. Indeed, it was shown using the lace expansion, for which typically a small parameter is needed. In the case of nearest-neighbour self-avoiding walk, this role of this small parameter is played by $1/(2d)$, which is not very small when $d = 5$. In [58], it was shown that the endpoint converges to a normal distribution when the dimension is sufficiently high. In [59], this was improved to the scaling of the entire path to Brownian motion, still for sufficiently high dimensions. In [26], this was

improved to Theorem 2.10, using a computer-assisted proof. These proofs all rely on the generating function approach, and it is not clear whether similar results would be possible using the inductive method.

A second result proved using the generating function method proves scaling in x-space for the self-avoiding walk Green's function:

Theorem 2.11. *For nearest-neighbour self-avoiding walk for $d \geq 5$, or for sufficiently spread-out self-avoiding walks for $d > 4$, there exists a constant B and a $\delta > 0$ such that*

$$G_{z_c}(x) = \frac{B}{\sigma^2(|x|+1)^{d-2}}\big(1 + O(|x|^{-\delta})\big). \tag{2.6.2}$$

The nearest-neighbour result is proved in [23], the spread-out result in [24]. These results also apply to the percolation and lattice tree and animal two-point functions.

We now discuss the generating function method. To obtain improved asymptotic results, rather than bounds, the second step is to improve the bounds to asymptotics when z is close to the critical value z_c, in combination with a Tauberian Theorem to turn asymptotics of the generating functions for z close to z_c to asymptotics for the coefficients c_n for n large. We will now compare the generating function method to the induction method.

The key advantage of the generating function approach is that it provides the simplest way to prove convergence of the lace expansion. Convergence means that the lace expansion coefficients $\pi_m^{(N)}$ are absolutely summable in N. In particular, one can solve for $\hat{G}_z(k)$, as was done in (2.4.11), and this solution writes $\hat{G}_z(k)$, which is divergent at the critical point for $k = 0$, in terms of $\hat{\Pi}_z(k)$, which is convergent at $z = z_c$ for all k. A second crucial advantage is that the generating function method can also be used when an explicit time variable is absent, such as for lattice trees and percolation clusters. On the other hand, due to the fact that two transforms are used (the generating function and the Fourier transform), work is needed to go from statements on the generating function to statements on its coefficients. Tauberian Theorems are hard to apply, and sometimes fractional derivatives need to be used. This implies in particular that it is difficult to obtain good error estimates, as we will indicate now.

The generating function method uses Tauberian Theorems to transform statements of the form

$$\hat{G}_z(0) = \frac{A\mu}{z_c - z} + O\big((z_c - z)^{\epsilon-1}\big), \tag{2.6.3}$$

into statements of the form

$$c_n = A\mu^n[1 + O(n^{-\epsilon})]. \tag{2.6.4}$$

In the reverse direction, (2.6.3) immediately follows from (2.6.4), and we will now investigate the implications of the results proved using the inductive method. Indeed, Theorem 2.1 implies that

$$c_n = A\mu^n[1 + O(n^{-(d-4)/2})]. \tag{2.6.5}$$

Therefore, we have that, for $z < z_c = \mu^{-1}$,

$$\begin{aligned} \hat{G}_z(0) = \sum_{n=0}^{\infty} z^n c_n &= \sum_{n=0}^{\infty} A(z\mu)^n[1 + O(n^{-(d-4)/2})] \\ &= \frac{A\mu}{z_c - z} + O\big((z_c - z)^{(d-6)/2}\big). \end{aligned} \tag{2.6.6}$$

By (2.4.12) and (2.4.13), this is equivalent to the statement that

$$\begin{aligned} 1 - z - \hat{\Pi}_z(0) &= (z_c - z) + \hat{\Pi}_{z_c}(0) - \hat{\Pi}_z(0) \\ &= (z_c - z)[1 + \partial_z \hat{\Pi}_{z_c}(0)] + O\big((z_c - z)^{(d-2)/2}\big). \end{aligned} \tag{2.6.7}$$

However, for the reverse statement, when $d = 10$, we need to show that $\partial_z^2 \hat{\Pi}_{z_c}(0) = \partial_z^3 \hat{\Pi}_{z_c}(0) = 0$, and $|\partial_z^4 \hat{\Pi}_{z_c}(0)| < \infty$. In the inductive method, the improved error estimates are more easily verified.

A second advantage of the inductive method is that it also allows to prove error estimates on $\hat{c}_n(k)\mu^{-n} - \hat{c}_{n-m}(k)\mu^{-(n-m)}$ that are valid for any n and m. These bounds for self-avoiding walk are convenient to prove convergence of network of mutually self-avoiding walks to networks of Brownian motions. For oriented percolation, the bounds are essential to prove convergence of the higher-point functions, as we will see in Section 3 below.

2.6.2 Banach fixed-point theorems

An interesting approach to convolution equations arising through the lace expansion is given by Bolthausen and Ritzmann [5]. They consider the lace expansion equation (2.2.9) as a *fixed-point equation* in an appropriately chosen Banach space. For example, the lace expansion for c_n gives

$$c_n z_c^n = c_{n-1} z_c^{n-1} + \sum_{m=2}^{n} c_{n-m} z_c^{n-m} \hat{\pi}_m(0) z_c^m, \tag{2.6.8}$$

which, using (2.4.13) and abbreviating $a_n = c_n z_c^n$ with $z_c = \mu^{-1}$, can be rewritten as

$$a_n = a_{n-1} + \sum_{m=2}^{n} (a_{n-m} - a_{n-1}) \hat{\pi}_m(0) z_c^m - a_{n-1} \sum_{m=n+1}^{\infty} \hat{\pi}_m(0) z_c^m. \tag{2.6.9}$$

Define $b_m = \hat{\pi}_m(0) z_c^m a_m^{-1}$, then (2.6.9) is equivalent to

$$a_n = a_{n-1} + \sum_{m=2}^{n} (a_{n-m} - a_{n-1}) a_m b_m - a_{n-1} \sum_{m=n+1}^{\infty} a_m b_m. \tag{2.6.10}$$

Define the operator F on sequences $g = (g_n)_{n=0}^{\infty}$ by

$$(Fg)_n = g_{n-1} + \sum_{m=2}^{n} (g_{n-m} - g_{n-1}) g_m b_m - g_{n-1} \sum_{m=n+1}^{\infty} g_m b_m. \quad (2.6.11)$$

Then, the sequence $a = (a_n)_{n=0}^{\infty}$ is the fixed point of the operator F. Bolthausen and Ritzmann prove that for nearest-neighbour weakly self-avoiding walk with small enough self-repellence parameter β, the operator F is a *contraction* on the sequence space where the norm is defined by

$$\|g\|_{\triangleright} = g_0 + \sum_{n=1}^{\infty} |g_n - g_{n-1}|. \quad (2.6.12)$$

Moreover, they prove that the fixed point of the operator F is *unique*. In particular, this proves that $a = (a_n)_{n=0}^{\infty}$ satisfies $\|a\|_{\triangleright} < \infty$, so that $a = (a_n)_{n=0}^{\infty}$ is a Cauchy sequence and consequently converges. One particular difficulty that Bolthausen and Ritzmann need to overcome is that the operator F involves b_m for *all* $m \geq 2$, that is, $\hat{\pi}_m(0)$ for all $m \geq 2$. This is overcome with an ingenious induction scheme reminiscent of the inductive method. Bolthausen and Ritzmann use an extension of the above idea to show that $c_n(x)$ satisfies certain pointwise bounds. Before stating these bounds, let

$$\varphi_t(x) = \frac{d^{d/2}}{(2\pi t)^{d/2}} e^{-\frac{d|x|^2}{2t}} \quad (2.6.13)$$

denote the density of the normal distribution with mean 0 and variance t. Then

Theorem 2.12. *Let $d > 4$. There exists a $\beta_0 > 0$ such that for all $0 \leq \beta \leq \beta_0$, there exist positive constants v, μ, A, δ and K (all depending on d and β) such that the following statements hold as $n \to \infty$:*
(a)

$$c_n = A\mu^n [1 + O(n^{-1/2})]. \quad (2.6.14)$$

(b) For all $x \in \mathbb{Z}^d$ such that $n - \|x\|_1$ is even

$$\Big| \frac{c_n(x)}{c_n} - 2\varphi_{vn}(x) \Big| \leq K \Big[\varphi_{\delta n}(x) n^{-1/2} + n^{-d/2} \sum_{j=1}^{n/2} \varphi_{\delta j}(x) \Big]. \quad (2.6.15)$$

As indicated in [62, Section 6.1], the bound in (2.6.15) implies pointwise bounds on $c_n(x)$. For example, (2.6.15) implies that $G(x) = \sum_{n=0}^{\infty} c_n(x)\mu^{-n}$ decays proportionally to $|x|^{-(d-2)}$ when $|x| \to \infty$. This allows for a proof of Theorem 2.11 in the nearest-neighbour case for weakly self-avoiding walk.

We note further that it can be seen that (2.6.15) implies a *local central limit theorem* for x depending on n such that $x = O(n^{1/2})$ with $|x|n^{-\eta} \to \infty$ for some $\eta > 0$. The estimate in (2.6.15) is proved by showing that the lace

expansion recurrence relation can be rewritten as the fixed-point equation of some operator acting on sequences of functions, and this operator is again shown to be a contraction. The method of Bolthausen and Ritzmann is quite innovative, for example since all estimates are performed directly in x-space, rather than by using the Fourier transform. The Banach fixed point method has the potential to become equally useful as the inductive method, but, so far, the weakly self-avoiding walk is its only application.

2.7 Conclusion and open problems

In Section 2, we have sketched the inductive method applied to the lace expansion equations for oriented percolation and self-avoiding walk. We have indicated the different applications in the various models to which the lace expansion has been successfully applied. There are many more applications of the lace expansion, for example to percolation [25]. For a more detailed account of the different results obtained using the various approaches that can be used to analyse the lace expansion, we refer to the extensive account [62] and the references therein.

Needles to say, there are many more models of linear structures which are believed to have an upper critical dimension with Gaussian behaviour expected above the upper critical dimension. Many important examples are *self-interacting stochastic processes*. Examples are reinforced random walks, random walks in random environment, loop-erased random walk, true self-avoiding walk, etc. These models are all entirely different from the examples sketched in Section 2, in the sense that they are not strictly self-repellent. Indeed, denote the two-point function in such models by $q_m(x)$, so that $q_m(x)$ is the probability that the walk is at x after m steps. Then, it is impossible that, for all x and $m, n \geq 0$,

$$q_{m+n}(x) \leq (q_n * q_m)(x), \tag{2.7.1}$$

since both sides sum up to 1. Therefore, if the bound does hold, both sides are equal for all n, m and x, and this implies that $q_n(x) = q_1^{*n}(x)$, so that the model is random walk. It would be of interest to investigate whether inclusion/exclusion type arguments can be used for such models. For example, the transient behaviour of edge-reinforced random walks on the nearest-neighbour lattice in sufficiently high dimensions has not been understood using the classical means, and it could be that lace expansion techniques could improve our understanding of the methods. For example, when we can show that $\sup_{x\in\mathbb{Z}^d} q_n(x) \leq Cn^{-d/2}$, as in Theorem 2.1(a), then we obtain that the process visits its starting point only finitely often. To apply the lace expansion to such self-interacting stochastic processes, we need to overcome the difficulty that the walker decides where to step based on its entire *history*, and this intricate dependence on its own past makes an application of the lace expansion quite challenging.

3 Higher point functions

In Section 2, we have described the results of the inductive method, focussing on self-avoiding walk and oriented percolation. Even though the two-point function results for these two models are identical, oriented percolation is clearly a *richer* model, in the sense that clusters contain much more information than the paths that the two-point function describes. To describe the clusters in more detail, we will investigate the *branching structure* of oriented percolation clusters. One way to do this is by investigating the *higher-point functions* $\tau^{(r)}_{n_1,\dots,n_{r-1}}(x_1,\dots,x_{r-1})$, which are given by

$$\tau^{(r)}_{n_1,\dots,n_{r-1}}(x_1,\dots,x_{r-1}) = \mathbb{P}_{p_c}((0,0) \longrightarrow (x_1,n_1),\dots,(x_{r-1},n_{r-1}). \tag{3.0.1}$$

For $r = 2$, we retrieve the two-point function, while for $r \geq 3$, the higher-point functions give us information on how occupied paths from a single starting point go to several destinations. We will often abbreviate

$$\tau_{\vec{n}}(\vec{x}) = \tau^{(r)}_{n_1,\dots,n_{r-1}}(x_1,\dots,x_{r-1}), \tag{3.0.2}$$

where $\vec{n} = (n_1,\dots,n_{r-1})$ and $\vec{x} = (x_1,\dots,x_{r-1})$.

In this section, we will describe the lace expansion results for the higher-point functions. We will prove that the Fourier transforms of these higher-point functions, when properly rescaled, converge to some limiting objects. In order to define these limiting objects, and to sharpen our intuition for the behaviour of the r-point functions, we will start by investigating the r-point functions of *branching random walk*.

3.1 Critical branching random walk

We start by introducing branching random walk. We follow the construction in [1] and describe some of the results derived in [31]. Branching random walk is defined in terms of embeddings of abstract trees into $\mathbb{Z}^d$. The abstract trees are the family trees of the critical branching process with a critical offspring distribution $(q_m)_{m=0}^{\infty}$ with finite variance. For simplicity, we will assume that $(q_m)_{m=0}^{\infty}$ has all moments.

In more detail, we begin with a single individual having ξ offspring, where ξ is a random variable with distribution $(q_m)_{m=0}^{\infty}$, *i.e.*, $\mathbb{P}(\xi = m) = q_m$ with

$$\sum_m m q_m = 1, \qquad \sigma_q^2 = \sum_m m(m-1)q_m < \infty. \tag{3.1.1}$$

Each of the offspring then independently has offspring of its own, with the same critical distribution $(q_m)_{m=0}^{\infty}$. For a tree T, with the i^{th} individual having ξ_i offspring, the probability of the tree T is equal to

$$\mathbb{P}(T) = \prod_{i \in T} q_{\xi_i}. \tag{3.1.2}$$

The product is over the vertices of T, and we emphasize that $(\xi_i)_{i\in T}$ is determined by the tree T.

It is important to be clear about when two trees T are the same and when they are not. For this, we introduce a description of T in terms of *words*. These words arise inductively as follows. The root is the word 0. The children of the root are the words $01, 02, \ldots, 0\xi_0$, where we recall that ξ_0 is the number of children of the root 0. The children of 01 are the words $011, \ldots, 01\xi_{01}$, and so on. The family tree is then uniquely represented by a set of words. Two trees are the same if and only if they are represented by the same set of words.

We define an embedding ϕ of T into $\mathbb{Z}^d$ to be a mapping from the vertices of T into $\mathbb{Z}^d$ such that the root is mapped to the origin and, given that i is mapped to $x \in \mathbb{Z}^d$, the child j of i is mapped to $y \in \mathbb{Z}^d$ with probability $D(y-x)$. We always assume that D satisfies the restrictions in Section 2.1.1.

Branching random walk is then defined to be the set of configurations (T, ϕ), with probabilities

$$\mathbb{P}(T,\phi) = \mathbb{P}(T) \prod_{ij\in T} D(\phi(j) - \phi(i)). \tag{3.1.3}$$

Here $ij \in T$ means that j is the child of i in the tree T. In particular, the path in $\mathbb{Z}^d$ from the origin to $\phi(i)$, for $i \in T$, is a random walk path of length $|i|$ with transition probabilities given by D. Here $|i|$ denotes the generation of i in T, which is the same as the graph distance between the root of T and i.

Branching random walk (BRW) is a caricature model for a population where individuals give rise to offspring, and the offspring moves to settle and produce offspring of its own. Needles to say, in true populations, the behaviour of the population is much more intricate than in this model.

A convenient way to describe the distribution of a discrete random measure is by using the *r-point functions*. In this section, we will assume that $(q_m)_{m=0}^{\infty}$ has all moments. The r-point function describe the numbers and locations of particles present at various times. Denote the branching random walk r-point functions by

$$p_{\vec{n}}(\vec{x}) = \sum_{i_1,\ldots,i_{r-1}} \mathbb{P}(i_j \in T, \phi(i_j) = x_j, |i_j| = n_j \text{ for each } j = 1, \ldots, r-1) \tag{3.1.4}$$

where, an in (3.0.2), we abbreviate

$$p_{\vec{n}}(\vec{x}) = p^{(r)}_{n_1,\ldots,n_{r-1}}(x_1, \ldots, x_{r-1}). \tag{3.1.5}$$

Note that for $r = 2$, we are abusing notation and using $p_n(x)$ both for the random walk two-point function and its branching random walk equivalent. In Section 3.2, we will see that, in fact, this is not an abuse of notation.

The r-point functions give rise to measures, and these measure are called *mean-moment measures*, or just *moment measures*. To understand this terminology, let $n \geq 0$, and define the random measures $\{\mu_n\}_{n=0}^{\infty}$ by

$$\mu_n(x) = \sum_{i \in T : |i| = n} I[\phi(i) = x]. \tag{3.1.6}$$

The stochastic process $\{\mu_n\}_{n=0}^{\infty}$ is a Markov process of random measures evolving in time. The measure μ_n describes the amount of mass and the spatial location of the mass of the BRW at time n. We start with a single particle at time 0 located at the origin, so that

$$\mu_0(x) = \delta_{x,0}. \tag{3.1.7}$$

Then, the law of the discrete measured-valued process $\{\mu_n\}_{n=0}^{\infty}$ can be computed in terms of the joint moments $\mathbb{E}[\prod_{i=1}^{l} \mu_{m_i}(y_i)^{a_i}]$, where $l \in \mathbb{N}, m_i \in \mathbb{N}, y_i \in \mathbb{Z}^d, a_i \in \mathbb{N}$. The r-point functions appear explicitly in this description, since

$$\mathbb{E}[\prod_{i=1}^{l} \mu_{m_i}(y_i)^{a_i}] = p_{\vec{n}}(\vec{x}), \tag{3.1.8}$$

where (x_j, n_j) equals (y_i, m_i) precisely a_i times, and $r - 1 = \sum_{i=1}^{l} a_i$. Thus, the joint moments of the random measures $\{\mu_n\}_{n=0}^{\infty}$ are equal to the r-point functions. In the remainder of this section, we will derive recursive formulas for $p_{\vec{n}}(\vec{x})$.

We first introduce some notation. Let $(f_j)_{j=0}^{\infty}$ denote the factorial moments of the distribution $(q_m)_{m=0}^{\infty}$, i.e.,

$$f_j = \sum_{m=j}^{\infty} \frac{m!}{(m-j)!} q_m. \tag{3.1.9}$$

When $\mathbb{P}(\xi \leq 2) = 1$, then $f_j = 0$ for all $j \geq 3$. Also, when the branching process is critical, then $f_1 = 1$ and $f_2 = \sigma_q^2$. Recall that we have assumed that q has all moments, so that f_j is finite for all j.

One way to visualize a branching random walk configuration contributing to the BRW r-point function is as a *tree of random walk paths*. For this, we will introduce the (x, n)-sites in the BRW configuration, which are the vertices in the BRW configuration for which the spatial position is equal to $x \in \mathbb{Z}^d$, while the tree distance from the root is equal to n. Note that there may be more than one (x, n)-site. When considering the r-point function, we must consider the (x_j, n_j)-sites in the BRW configuration, for all $j \in J$, where we write

$$J = \{1, \ldots, r-1\}. \tag{3.1.10}$$

The path connecting the root to each of the (x_j, n_j)-sites are random walk paths, possibly sharing some steps. Therefore, a configuration contributing

to the r-point function can be thought of as a tree of random walk paths. The r-point function is obtained by summing each configuration over the different possible choices of the (x_j, n_j)-sites in the branching random walk configuration (see also (3.1.4)), multiplied by the probability of the configuration. The tree of random walks picture is useful in the understanding of the higher-point functions, as we shall see now.

We write $\mathcal{P}_j$ for the set of partitions of J into j non-empty subsets, where we order the elements of $\vec{I} \in \mathcal{P}_j$ by ordering the smallest components. Thus, I_1 contains the element 1, while I_2 contains the smallest element that is not in I_1. Finally, for $I = \{i_1, \ldots, i_j\} \subseteq J$, we write $\vec{n}_I = (n_{i_1}, \ldots, n_{i_j})$. The following proposition is proved in [31]:

Proposition 3.1. *For every $\vec{x} \in \mathbb{Z}^{d(r-1)}$ and every $\vec{n} = (n_1, \ldots, n_{r-1})$ with $n_i \geq 1$ for all $i = 1, \ldots, r-1$,*

$$p_{\vec{n}}(\vec{x}) = \sum_{j=1}^{r-1} f_j \sum_{\vec{I} \in \mathcal{P}_j} \prod_{s=1}^{j} (D * p_{\vec{n}_{I_s} - 1})(\vec{x}_{I_s}). \tag{3.1.11}$$

We can interpret the different contributions in (3.1.11) in Proposition 3.1 by investigating the children of the root. The set I_i are those indices j such that the individual located at time n_j at spatial position x_j is the child of the i^{th} child of the root. The proof of Proposition 3.1 makes this intuition precise. Indeed, Proposition 3.1 follows by conditioning on the number of offspring of the individual located at the origin at time zero, and by investigating which of his/her children are ancestors of the individuals located at x_j at time n_j.

We will see that two contributions to (3.1.11) are special. The first is when $I_i = J$ for some i, which is the case where all r points in the r-point function are descendents of the same child of the root. In this case, all random walks in the tree of random walks picture share the same initial step, and the r-point function appears again in $(D * p_{\vec{n}-1})(\vec{x})$. The second is the case where $I_2 = J \backslash I_1$. In this case, there are precisely two children that are the ancestors of the r-points of the r-point function. Then, the random walks to the (x_i, n_i)-site for $i \in I$ share the same initial step, and the random walks to the (x_j, n_j)-site for $j \in J \backslash I$ share the same initial step, but these two initial steps may be *different*. In this case, two s-point functions with $s = |I_1| + 1$, respectively, $s = |I_2| + 1$ appear. It turns out that the other contributions constitute error terms.

The significance of (3.1.11) lies in the fact that we can use it recursively to identify the r-point functions. For example, for $r = 2$, we obtain

$$p_n(x) = f_1(D * p_{n-1})(x), \quad \text{so that} \quad p_n(x) = f_1^n D^{*n}(x) = D^{*n}(x), \tag{3.1.12}$$

where the last equality holds since the branching process is critical, so that $f_1 = 1$. In particular, the branching random walk and ordinary random walk two-point functions agree when the mean offspring equals 1.

A special example arises when we consider binary branching, i.e., $q_m = \frac{1}{2}(\delta_{m,0} + \delta_{m,2})$. For binary branching, $f_1 = f_2 = 1$, and $f_m = 0$ for all $m \geq 3$, so that only the special contributions in (3.1.11) appear. Indeed, we obtain that, writing $I = I_2$, so that $1 \notin I$,

$$p_{\vec{n}}(\vec{x}) = (D * p_{\vec{n}-1})(\vec{x}) + \sum_{I \subseteq J_1 : I \neq \varnothing} (D * p_{\vec{n}_I - 1})(\vec{x}_I)(D * p_{\vec{n}_{J\setminus I}-1})(\vec{x}_{J\setminus I}), \tag{3.1.13}$$

where $J_1 = J\setminus\{1\}$. Iterating the recursion yields

$$\begin{aligned} p_{\vec{n}}(\vec{x}) = \sum_{I \subseteq J_1 : I \neq \varnothing} \sum_{m=0}^{\underline{n}-1} \sum_{y} D^{*m}(y)(D * p_{\vec{n}_I - m - 1})(\vec{x}_I - y) \\ \times (D * p_{\vec{n}_{J\setminus I}-m-1})(\vec{x}_{J\setminus I} - y), \end{aligned} \tag{3.1.14}$$

where $\underline{n}$ denotes the minimal element of $\vec{n} = (n_1, \ldots, n_{r-1})$. Using (3.1.12), we can write (3.1.11) as

$$\begin{aligned} p_{\vec{n}}(\vec{x}) = \sum_{I \subseteq J_1 : I \neq \varnothing} \sum_{m=0}^{\underline{n}-1} \sum_{y} p_m(y)(D * p_{\vec{n}_I - m - 1})(\vec{x}_I - y) \\ \times (D * p_{\vec{n}_{J\setminus I}-m-1})(\vec{x}_{J\setminus I} - y). \end{aligned} \tag{3.1.15}$$

Equation (3.1.15) yields an explicit recursion for the r-point function in terms of r, since on the right-hand side only s-point functions with $s < r$ appear. For different offspring distributions, (3.1.11) is not so easily solved, but one can identify the scaling limit of $p_{\vec{n}}(\vec{x})$ for general offspring distributions, by proving that the contribution due to $j \geq 3$ in (3.1.11) is an error term. The result of this analysis is described in the next section.

3.2 Convergence of BRW r-point functions

We now identify the scaling limit of (3.1.15) when $n \to \infty$. We note that when the BRW is still alive at time n, then the individuals at time n have arrived there by a random walk path of n steps. Therefore, it follows that at time n, the spatial scaling should be proportional to $\sqrt{n}$. We now describe the scaling in more detail.

Since $p_n(x) = D^{*n}(x)$, we have that

$$\hat{p}_n\big(\frac{k}{\sqrt{\sigma^2 n}}\big) = \hat{D}\big(\frac{k}{\sqrt{\sigma^2 n}}\big) = e^{-\frac{|k|^2}{2d}}[1 + o(1)]. \tag{3.2.1}$$

It turns out that also the higher-point functions converge when we substitute a Fourier variable which decays as $n^{-1/2}$. To identify these limits, we define, for $l = 1$,

$$\hat{M}_t^{(1)}(k) = e^{-\frac{|k|^2 t}{2d}}, \tag{3.2.2}$$

so that

$$\hat{p}_n\big(\frac{k}{\sqrt{\sigma^2 n}}\big) = \hat{M}_1^{(1)}(k)[1+o(1)]. \tag{3.2.3}$$

The recursion in (3.1.15) can be used to prove by induction on the number of points $r-1$ that the rescaled r-point function $n^{-(r-2)}\hat{p}_{\lceil n\vec{t}\rceil}\big(\frac{\vec{k}}{\sqrt{\sigma^2 n}}\big)$ converges for any $r \geq 2$. To describe its limit, we introduce some more notation. We write

$$k_I = \sum_{i\in I} k_i, \qquad \vec{k}_I = (k_i; i \in I). \tag{3.2.4}$$

Then, for $l > 1$, the $\hat{M}^{(l)}_{\vec{t}}(\vec{k})$ are given recursively by

$$\hat{M}^{(l)}_{\vec{t}}(\vec{k}) = \int_0^{\underline{t}} dt\; \hat{M}^{(1)}_t(k_J) \sum_{I\subseteq J_1 : |I|\geq 1} \hat{M}^{(i)}_{\vec{t}_I - t}(\vec{k}_I)\hat{M}^{(l-i)}_{\vec{t}_{J\setminus I}-t}(\vec{k}_{J\setminus I}), \tag{3.2.5}$$

where $i = |I|$, $J = \{1,\dots,l\}, J_1 = J\backslash\{1\}$, $\underline{t} = \min_i t_i$, $\vec{t}_I = (t_i; i \in I)$, and $\vec{t}_I - t$ denotes subtraction of t from each component of $\vec{t}_I$. The explicit solution to the recursive formula (3.2.5) can be found in [40, (1.25)]. For example,

$$\hat{M}^{(2)}_{t_1,t_2}(k_1,k_2) = \int_0^{t_1\wedge t_2} e^{-\frac{|k_1+k_2|^2 t}{2d}} e^{-\frac{|k_1|^2(t_1-t)}{2d}} e^{-\frac{|k_2|^2(t_2-t)}{2d}} dt. \tag{3.2.6}$$

Then, the scaling limits of the BRW r-point functions are given as follows (see [31, Theorem 3.2]):

Theorem 3.2. *Fix an offspring distribution $(q_m)_{m=0}^\infty$ such that all moments are finite, and assume that* (2.1.12) *holds. Then,*

$$\hat{p}_{\vec{n}}(\vec{k}/\sqrt{\sigma^2 n}) = (\sigma_q^2 n)^{r-2}\Big[\hat{M}^{(r-1)}_{\vec{n}/n}(\vec{k}) + \mathcal{O}((n_{(2)}+1)^{-\delta})\Big] \quad (r\geq 2) \tag{3.2.7}$$

holds uniformly in $n \geq n_{(2)}$, where $n_{(2)}$ denotes the second largest component of $\vec{n}$.

We now intuitively explain this result. We can see that the recursion formula in (3.2.5) is closely related to the one in (3.1.15). The factor n^{r-2} in (3.2.7) is equal to the number of summands, and can be used to replace the sum in (3.1.15) by a Riemann sum approximation of the integral in (3.2.5). We can also give an alternative interpretation of the power of n by using scaling of branching random walk quantities, as we will now explain. It is well-known that for the branching process survival probability

$$\theta_n = \mathbb{P}(\exists i \in T : |i| = n), \tag{3.2.8}$$

we have the asymptotics

$$\lim_{n\to\infty} n\theta_n = \frac{2}{\sigma_q^2}. \tag{3.2.9}$$

Then, consider $\vec{n} = (n, \dots, n)$ and $\vec{k} = \vec{0}$, for which we have that

$$\hat{p}_{\vec{n}}(\vec{0}) = \mathbb{E}[N_n^{r-1}], \tag{3.2.10}$$

where N_n is the number of individuals alive at time n. First take $r = 2$, and note that

$$1 = \hat{p}_n(0) = \mathbb{E}[N_n] = \theta_n \mathbb{E}[N_n | N_n > 0]. \tag{3.2.11}$$

Since θ_n decays to zero like $1/n$, we must have that N_n is of the order n when $N_n > 0$. This suggests that

$$\hat{p}_{\vec{n}}(\vec{0}) = \mathbb{E}[N_n^{r-1}] = \theta_n \mathbb{E}[N_n^{r-1} | N_n > 0] \tag{3.2.12}$$

grows like n^{r-2}, as in (3.2.7).

We finally indicate how we can obtain (3.2.9). We condition on the offspring of the root to obtain, using that the third moment of $(q_m)_{m=0}^{\infty}$ is finite, to obtain

$$\begin{aligned}
\theta_n &= \sum_m q_m \big[1 - \mathbb{P}(\text{all } m \text{ children of the root die before time } n)\big] \\
&= \sum_m q_m \big[1 - (1 - \theta_{n-1})^m\big] \\
&= \theta_{n-1}(\sum_m m q_m) - \theta_{n-1}^2 \frac{1}{2}(\sum_m m(m-1) q_m) + \mathcal{O}(\theta_{n-1}^3) \\
&= \theta_{n-1} - \frac{1}{2}\sigma_q^2 \theta_{n-1}^2 + \mathcal{O}(\theta_{n-1}^3).
\end{aligned} \tag{3.2.13}$$

It is not so hard to see that the above recursion relation has an asymptotic solution given by (3.2.9). See e.g., [4, Section 8.5]. For oriented percolation, the above argument does not really work, and in the next section, we discuss the convergence of the critical oriented percolation r-point functions when $d > 4$, in the spirit of Theorem 3.2.

3.3 The oriented percolation higher-point functions

Oriented percolation is closely related to branching random walk. Indeed, if, for branching random walk, we let particles that arrive at the same spatial location at the same time *merge* or *coalesce*, then oriented percolation arises rather than branching random walk. Indeed, in oriented percolation, there is dependence between the offspring of sites in the oriented percolation cluster, due to the fact that different sites can infect the same site in the next generation. We also see that the dependence or interaction is due to *loops*, more precisely, due to sites that are infected at least twice from different sources. Since loops in high dimensions are rare, it may be expected that the scaling of the r-point functions is close to the scaling of the r-point functions for BRW. Theorem 2.2 proves this for the two-point function, and we now extend this analogy to all higher-point functions:

Theorem 3.3. *Let $d > 4$, $p = p_c$, $\delta \in (0, 1 \wedge \epsilon \wedge \frac{d-4}{2})$, $r \geq 3$, $\vec{t} = (t_1, \ldots, t_{r-1}) \in (0,\infty)^{r-1}$, and $\vec{k} = (k_1, \ldots, k_{r-1}) \in \mathbb{R}^{(r-1)d}$. There is a constant V and an $L_0 = L_0(d)$ (independent of r) such that for $L \geq L_0$,*

$$\hat{\tau}^{(r)}_{\lfloor n\vec{t}\rfloor}(\vec{k}/\sqrt{v\sigma^2 n}) = n^{r-2}V^{r-2}A^{2r-3}\left[\hat{M}^{(r-1)}_{\vec{t}}(\vec{k}) + \mathcal{O}(n^{-\delta})\right], \qquad (3.3.1)$$

with the error estimate uniform in $\vec{k}$ in a bounded subset of $\mathbb{R}^{(r-1)d}$, and where A is given in Theorem 2.2.

Theorem 3.3 can be summarised by saying that, at least on the level of r-point function, oriented percolation above 4 spatial dimensions is a small perturbation of branching random walk.

Theorem 3.3 is proved in [41]. The remainder of Section 3 is devoted to an outline of the proof Theorem 3.3 for oriented percolation above 4 spatial dimensions. We first sketch the strategy of its proof. We try to follow the strategy for branching random walk as closely as possible. For BRW, the recurrence relation in (3.1.15) plays a central role, and the proof for oriented percolation will involve proving a similar, though more complicated, relation for critical oriented percolation. We start by introducing some notation. Recall that $J = \{1, \ldots, r-1\}$. For $I = \{i_1, \ldots, i_s\} \subset J$, we write $\vec{\boldsymbol{x}}_I = \{\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_s}\}$ and $\vec{\boldsymbol{x}}_I - \boldsymbol{y} = \{\boldsymbol{x}_{i_1} - \boldsymbol{y}, \ldots, \boldsymbol{x}_{i_s} - \boldsymbol{y}\}$. Given a subset $I \subseteq J_1$, we let $r_1 = |J \setminus I| + 1$ and $r_2 = |I| + 1$.

To visualize the higher-point functions, we again use the strings of sausages picture for the connections between $\boldsymbol{o}$ and $\boldsymbol{x}_j$ for the different $j \in J$. The strings of sausages replace the random walk paths underlying the trees of random walk paths picture for the BRW higher-point function in Section 3.1. The union of the strings of sausages from $\boldsymbol{o}$ to the various $\boldsymbol{x}_j$ give rise to a tree of sausages. Depending on the number of sausages that contain the origin in the different strings of sausages from $\boldsymbol{o}$ to $\boldsymbol{x}_j$ for $j \in J$, there are different contributions to the r-point function. The first contribution is when the sausage containing $\boldsymbol{o}$ is the same for all $\boldsymbol{x}_j$, which happens precisely when the first pivotal for $\boldsymbol{o} \longrightarrow \boldsymbol{x}_j$ is the same for all $j \in J$, and, intuitively, corresponds to the $(D * p_{\vec{n}-1})(\vec{x})$ contribution in Proposition 3.1. A configuration of this type contributing to the three-point function is drawn in Figure 3. The second main contribution is when there are precisely two different sausages that contain $\boldsymbol{o}$ for the connections from $\boldsymbol{o}$ to $\boldsymbol{x}_j$ for the different $j \in J$, which happens precisely when there are two bonds, one is pivotal for $\boldsymbol{o} \longrightarrow \boldsymbol{x}_i$ for all $i \in I$ for some $I \subsetneq J$ with $I \neq \varnothing$, while the other is pivotal for $\boldsymbol{o} \longrightarrow \boldsymbol{x}_j$ for all $j \in J \setminus I$. The contribution from the cases where there are more different pivotal bonds for the connections $\boldsymbol{o} \longrightarrow \boldsymbol{x}_j$ for $j \in J$ will turn out to be an error term.

We will prove that the spread-out oriented percolation r-point function satisfies a recurrence relation similar to the one in (3.1.15) for BRW. This equation is shown using the lace expansion for the r-point function, as derived in [40, Sections 5 and 6]. We will derive a related expansion,

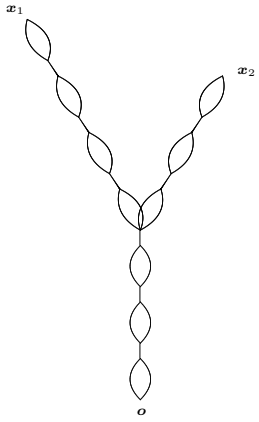


Figure 3: Schematic representation of a configuration contributing to the three-point function.

which is closer in spirit to the lace expansion for the two-point function as described in Section 2.2.2. This expansion contains two main steps: the first and the second expansion, which we now briefly summarise. The expansion is explained in more detail in Sections 3.4–3.7 below.

The first expansion deals with the case where there is a single sausage containing $\boldsymbol{o}$, and decouples the interaction between that single sausage and the other sausages of the tree of sausages leading to $\vec{\boldsymbol{x}}_J$. We write the subscript J explicitly in $\vec{\boldsymbol{x}}_J$ to indicate what the components of $\vec{\boldsymbol{x}}_J$ are.

The expansion writes $\tau(\vec{\boldsymbol{x}}_J)$ in the form

$$\tau(\vec{\boldsymbol{x}}_J) = A(\vec{\boldsymbol{x}}_J) + (B \star \tau)(\vec{\boldsymbol{x}}_J) = A(\vec{\boldsymbol{x}}_J) + \sum_{\boldsymbol{v}} B(\boldsymbol{v})\ \tau(\vec{\boldsymbol{x}}_J - \boldsymbol{v}), \quad (3.3.2)$$

where we recall that $\star$ represents the space-time convolution

$$(f \star g)(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \Lambda} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}). \quad (3.3.3)$$

We have that

$$B(\boldsymbol{x}) = (\pi \star pD)(\boldsymbol{x}), \qquad A(\boldsymbol{x}) = \pi(\boldsymbol{x}) \quad (3.3.4)$$

where $\pi(\boldsymbol{x})$ is the expansion coefficient for the two-point function as derived in Section 2.2.2. In particular, when $r = 2$, (3.3.2) becomes

$$\tau(\boldsymbol{x}) = \pi(\boldsymbol{x}) + (\pi \star pD \star \tau)(\boldsymbol{x}). \quad (3.3.5)$$

We first use the first expansion in a similar way as for BRW in (3.1.13)–(3.1.15). For this, we define

$$\nu(\boldsymbol{x}) = \sum_{n=0}^{\infty} B^{\star n}(\boldsymbol{x}), \tag{3.3.6}$$

where $B^{\star n}$ denotes the n-fold space-time convolution of B with itself, with $B^{\star 0}(\boldsymbol{x}) = \delta_{\boldsymbol{o},\boldsymbol{x}}$. Then (3.3.2) can be solved to give

$$\tau(\vec{\boldsymbol{x}}_J) = (\nu \star A)(\vec{\boldsymbol{x}}_J). \tag{3.3.7}$$

The function ν can be identified as follows. We note that iterating (3.3.7) applied to $r = 2$ yields that

$$\tau(\boldsymbol{x}) = (\nu * A)(\boldsymbol{x}). \tag{3.3.8}$$

Thus, extracting the $n = 0$ term from (3.3.6) and using (3.3.4) to write one factor of B as $A \star pD$ for the terms with $n \geq 1$, it follows from (3.3.8) that

$$\nu(\boldsymbol{x}) = \delta_{\boldsymbol{o},\boldsymbol{x}} + (pD \star \nu \star A)(\boldsymbol{x}) = \delta_{\boldsymbol{o},\boldsymbol{x}} + (pD \star \tau)(\boldsymbol{x}). \tag{3.3.9}$$

Substituting (3.3.9) into (3.3.7), we obtain

$$\tau(\vec{\boldsymbol{x}}_J) = A(\vec{\boldsymbol{x}}_J) + (\tau \star pD \star A)(\vec{\boldsymbol{x}}_J), \tag{3.3.10}$$

which recovers (3.3.5) when $r = 2$, using (3.3.4). We note that (3.3.10) is reminiscent of (3.1.15), apart from the fact that $A(\vec{\boldsymbol{x}}_J - \boldsymbol{y})$, which appears in (3.3.10), is replaced with

$$\sum_{I \subseteq J_1 : I \neq \varnothing} (D \star \tau)(\vec{\boldsymbol{x}}_I - \boldsymbol{y})(D \star \tau)(\vec{\boldsymbol{x}}_{J \setminus I} - \boldsymbol{y}) \tag{3.3.11}$$

in (3.1.15), and the extra term $A(\vec{\boldsymbol{x}}_J)$ appearing in (3.3.10), which will turn out to be an error term. The remainder of the expansion writes $A(\vec{\boldsymbol{x}}_J)$ in a form that is closer to (3.3.11). In this way, we can see the lace expansion for the higher-point function as a perturbation of the corresponding equation for BRW.

To achieve this, the next step in the expansion is to write $A(\vec{\boldsymbol{x}}_J)$ as

$$A(\vec{\boldsymbol{x}}_J) = \sum_{I \subseteq J_1 : I \neq \varnothing} A(\vec{\boldsymbol{x}}_{J \setminus I}, \vec{\boldsymbol{x}}_I) + a(\vec{\boldsymbol{x}}_J; 1), \tag{3.3.12}$$

where $J \setminus I$ consists of those j for which the first pivotal of $\boldsymbol{x}_j$ is the same as the one for $\boldsymbol{x}_1$, while for $i \in I$, this first pivotal is different. The contribution $a(\vec{\boldsymbol{x}}_J; 1)$ will turn out to be an error term, and corresponds for example to contributions where there are at least first three pivotal bonds for the connections $\boldsymbol{o} \longrightarrow \boldsymbol{x}_j$ for the various $j \in J$.

To leading order, $A(\vec{\boldsymbol{x}}_{J\setminus I}, \vec{\boldsymbol{x}}_I)$ represents the probability of observing a configuration for which we have: (a) a first sausage which ends with a bond b, which is pivotal for $\boldsymbol{o} \longrightarrow \boldsymbol{x}_{J\setminus I}$; (b) occupied paths from the sausage leading to $\vec{\boldsymbol{x}}_I$; and (c) connections from $\bar{b}$ to $\vec{\boldsymbol{x}}_{J\setminus I}$. In the second expansion, we wish to treat these connections and the sausage as being approximately independent. This procedure is the second expansion, and it leads to a result of the form

$$A(\vec{\boldsymbol{x}}_{J\setminus I}, \vec{\boldsymbol{x}}_I) = \sum_{\boldsymbol{y}_1,\boldsymbol{y}_2} C(\boldsymbol{y}_1,\boldsymbol{y}_2)\, \tau(\vec{\boldsymbol{x}}_{J\setminus I} - \boldsymbol{y}_1)\, \tau(\vec{\boldsymbol{x}}_I - \boldsymbol{y}_2) + a(\vec{\boldsymbol{x}}_{J\setminus I}, \vec{\boldsymbol{x}}_I), \tag{3.3.13}$$

where $a(\vec{\boldsymbol{x}}_{J\setminus I}, \vec{\boldsymbol{x}}_I)$ is an error term, and, to first approximation, $C(\boldsymbol{y},\boldsymbol{z})$ represents the sausage at $\boldsymbol{o}$ together with the pivotal bonds ending at $\boldsymbol{y}$ and $\boldsymbol{z}$, with the two branches removed. In particular, $C(\boldsymbol{y},\boldsymbol{z})$ is independent of I. The leading contribution to $C(\boldsymbol{y},\boldsymbol{z})$ is $pD(\boldsymbol{y})pD(\boldsymbol{z})$ with $\boldsymbol{y} \neq \boldsymbol{z}$, corresponding to the case where the sausage at $\boldsymbol{o}$ is the single point $\boldsymbol{o}$.

For $r \geq 3$, we further substitute (3.3.12)–(3.3.13) into (3.3.10). Let

$$\psi(\boldsymbol{y}_1,\boldsymbol{y}_2) = \sum_{\boldsymbol{v}} pD(\boldsymbol{v})\, C(\boldsymbol{y}_1 - \boldsymbol{v}, \boldsymbol{y}_2 - \boldsymbol{v}), \tag{3.3.14}$$

$$\varphi^{(r)}(\vec{\boldsymbol{x}}_J) = A(\vec{\boldsymbol{x}}_J) + (\tau \star pD \star a)(\vec{\boldsymbol{x}}_J), \tag{3.3.15}$$

where

$$a(\vec{\boldsymbol{x}}_J) = a(\vec{\boldsymbol{x}}_J; 1) + \sum_{I\subseteq J_1 : I\neq\varnothing} a(\vec{\boldsymbol{x}}_{J\setminus I}, \vec{\boldsymbol{x}}_I). \tag{3.3.16}$$

Then, (3.3.10) becomes

$$\begin{aligned}\tau^{(r)}(\vec{\boldsymbol{x}}_J) = &\sum_{\boldsymbol{v},\boldsymbol{y}_1,\boldsymbol{y}_2} \tau^{(2)}(\boldsymbol{v})\, \psi(\boldsymbol{y}_1 - \boldsymbol{v}, \boldsymbol{y}_2 - \boldsymbol{v}) \\ &\times \sum_{I\subseteq J_1 : I\neq\varnothing} \tau^{(r_1)}(\vec{\boldsymbol{x}}_{J\setminus I} - \boldsymbol{y}_1)\, \tau^{(r_2)}(\vec{\boldsymbol{x}}_I - \boldsymbol{y}_2) + \varphi^{(r)}(\vec{\boldsymbol{x}}_J),\end{aligned} \tag{3.3.17}$$

where we recall that $r_1 = |J \setminus I| + 1$ and $r_2 = |I| + 1$, and we write the superscripts of the r-point functions explicitly. Since $1 \leq |I| \leq r-2$, we have that $r_1, r_2 \leq r-1$, which opens up the possibility for induction in r. Equation (3.3.17) is closely related to the recurrence relation for the BRW r-point functions in (3.1.11).

We recall (3.2.4) and write $k = \sum_{i=1}^{r-1} k_i$. For $I \subseteq J$, we also write $\underline{n}_I = \min_{i\in I} n_i$ and $\underline{n} = \underline{n}_J$. Then, the Fourier transform of (3.3.17) becomes

$$\begin{aligned}\hat{\tau}^{(r)}_{\vec{n}}(\vec{k}) = &\sum_{m_0=0}^{\underline{t}} \hat{\tau}^{(2)}_{m_0}(k) \sum_{I\subseteq J_1 : |I|\geq 1} \sum_{m_1=2}^{\underline{n}_{J\setminus I}-m_0} \sum_{m_2=m_1}^{\underline{n}_I - m_0} \hat{\psi}_{m_1,m_2}(k_{J\setminus I}, k_I) \\ &\times\ \hat{\tau}^{(r_1)}_{\vec{n}_{J\setminus I}-m_1-m_0}(\vec{k}_{J\setminus I}) \hat{\tau}^{(r_2)}_{\vec{n}_I - m_2 - m_0}(\vec{k}_I) \ \ + \ \ \hat{\varphi}^{(r)}_{\vec{n}}(\vec{k}).\end{aligned} \tag{3.3.18}$$

The identity (3.3.18) is the point of departure for analysing the r-point functions for $r \geq 3$. Apart from ψ and $\varphi^{(r)}$, the right-hand side of (3.3.17) involves the s-point functions with $s = 2, r_1, r_2$. Since $r_1 + r_2 = r + 1$ and $r_1, r_2 \geq 2$, it follows that r_1 and r_2 are both strictly less than r. This allows for an analysis by induction in r, with the $r = 2$ case given by the result of Theorem 2.2 for $r = 2$ proved in [41]. The term involving ψ is the main term, whereas $\varphi^{(r)}$ will turn out to be an error term.

3.4 The first expansion for the r-point function

In this section, we derive the expansion (3.3.2) which extracts an explicit r-point function $\tau(\vec{\boldsymbol{x}}_J - \boldsymbol{v})$, and an unexpanded contribution $A(\vec{\boldsymbol{x}}_J)$. The expansion is very close in nature to the expansion for the two-point function. There are many different possible expansions. The expansion in [40] expands the r-point functions using the Hara-Slade expansion. In [38], we choose a related expansion for which the bounds are more easily derived, and for which the expansion coefficients are closely related to the lace expansion coefficients for the survival probability. See Section 3.8.2 below for more details concerning the expansion of the survival probability. In this section, we describe an expansion which is more based upon the inclusion-exclusion approach for the two-point function, and we will refer to this expansion as the Sakai expansion. We emphasize that the expansion we sketch here is *different* from the ones in either [41] or [38].

The r-point function is $\tau(\vec{\boldsymbol{x}}_J) = \mathbb{P}(\boldsymbol{o} \longrightarrow \vec{\boldsymbol{x}}_J)$, where we abbreviate

$$\{\boldsymbol{v} \longrightarrow \vec{\boldsymbol{x}}_J\} = \{\boldsymbol{v} \longrightarrow \boldsymbol{x}_j \ \forall j \in J\}. \tag{3.4.1}$$

We take the first occupied pivotal bonds b for $\boldsymbol{o} \longrightarrow \boldsymbol{x}_j$ for all $j \in J$, if this exists. Then, we have the partition

$$\begin{aligned} &\big\{\boldsymbol{o} \longrightarrow \vec{\boldsymbol{x}}_J\big\} = E'(\vec{\boldsymbol{x}}_J) \\ &\dot\cup \dot{\bigcup_b} \Big\{\{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap \{b \text{ is occ. \& piv. for } \boldsymbol{o} \longrightarrow \boldsymbol{x}_j \ \forall j \in J\}\Big\}, \end{aligned} \tag{3.4.2}$$

where

$$E'(\vec{\boldsymbol{x}}_J) = \big\{\boldsymbol{o} \longrightarrow \vec{\boldsymbol{x}}_J\big\} \cap \big\{\nexists b : b \text{ is occ. \& piv. for } \boldsymbol{o} \longrightarrow \boldsymbol{x}_j \ \forall j \in J\big\}. \tag{3.4.3}$$

In particular,

$$E'(\{\boldsymbol{x}_1, \boldsymbol{x}_2\}) = \{\boldsymbol{o} \longrightarrow \boldsymbol{x}_1\} \circ \{\boldsymbol{o} \longrightarrow \boldsymbol{x}_2\}. \tag{3.4.4}$$

Defining

$$A^{(0)}(\vec{\boldsymbol{x}}_J) = \mathbb{P}(E'(\vec{\boldsymbol{x}}_J)), \tag{3.4.5}$$

we obtain

$$\tau(\vec{\boldsymbol{x}}_J) = A^{(0)}(\vec{\boldsymbol{x}}_J) + \sum_b \mathbb{P}\Big(\big\{\boldsymbol{o} \longrightarrow \vec{\boldsymbol{x}}_J\big\} \cap \{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap \{b \text{ is occ. piv. for } \boldsymbol{o} \longrightarrow \boldsymbol{x}_j \ \forall j \in J\}\Big). \tag{3.4.6}$$

To the sum in (3.4.6), we apply an inclusion-exclusion relation which is reminiscent of (2.2.54), and reads

$$\begin{aligned} &\{b \text{ is occupied \& pivotal for } \boldsymbol{v} \longrightarrow \boldsymbol{x}_j \ \forall j \in J\} \\ &= \{\boldsymbol{v} \longrightarrow \underline{b},\ b \longrightarrow \vec{\boldsymbol{x}}_J\} \setminus \big\{\boldsymbol{v} \longrightarrow \underline{b},\ b \xrightarrow{\tilde{C}^b(\boldsymbol{v})} \vec{\boldsymbol{x}}_J\big\}, \end{aligned} \tag{3.4.7}$$

where we abbreviate

$$\big\{\boldsymbol{y} \xrightarrow{C} \vec{\boldsymbol{x}}_J\big\} = \{\boldsymbol{y} \longrightarrow \vec{\boldsymbol{x}}_J\} \cap \big\{\exists\, j \in J,\ \boldsymbol{y} \xrightarrow{C} \boldsymbol{x}_j\big\}. \tag{3.4.8}$$

Let

$$B^{(0)}(\boldsymbol{y}) = \sum_{\boldsymbol{u}} \mathbb{P}(\boldsymbol{o} \Longrightarrow \boldsymbol{u}) pD(\boldsymbol{y} - \boldsymbol{u}), \tag{3.4.9}$$

$$R^{(1)}(\vec{\boldsymbol{x}}_J) = \sum_b \mathbb{P}\Big(\{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap \big\{b \xrightarrow{\tilde{C}^b(\boldsymbol{o})} \vec{\boldsymbol{x}}_J\big\}\Big). \tag{3.4.10}$$

By (3.4.7) and the Markov property, we end up with

$$\tau(\vec{\boldsymbol{x}}_J) = A^{(0)}(\vec{\boldsymbol{x}}_J) + \sum_{\boldsymbol{y}} B^{(0)}(\boldsymbol{y})\, \tau(\vec{\boldsymbol{x}}_J - \boldsymbol{y}) - R^{(1)}(\vec{\boldsymbol{x}}_J). \tag{3.4.11}$$

This completes the first step of the expansion.

To complete the expansion for $\tau(\boldsymbol{x}_J)$, we need to investigate $R^{(1)}(\vec{\boldsymbol{x}}_J)$ in more detail. Note that $R^{(1)}(\vec{\boldsymbol{x}}_J)$ involves the probability of a subset of $\big\{b \xrightarrow{\tilde{C}^b(\boldsymbol{v})} \vec{\boldsymbol{x}}_J\big\}$. For $C \subseteq \Lambda$, we let

$$\begin{aligned} &E'(\boldsymbol{v}, \vec{\boldsymbol{x}}_J; C) = \big\{\boldsymbol{v} \xrightarrow{C} \vec{\boldsymbol{x}}_J\big\} \\ &\cap \big\{\nexists \text{ pivotal bond } b \text{ for } \boldsymbol{v} \longrightarrow \boldsymbol{x}_j \ \forall j \in J \text{ such that } \boldsymbol{v} \xrightarrow{C} \underline{b}\big\}, \end{aligned} \tag{3.4.12}$$

$$E(b, \vec{\boldsymbol{x}}_J; C) = \{b \text{ is occupied}\} \cap E'(\overline{b}, \vec{\boldsymbol{x}}_J; C). \tag{3.4.13}$$

See Figure 4 for a schematic representation of $E'(\boldsymbol{v}, \vec{\boldsymbol{x}}_J; C)$. Then, we have the partition, for general sets of vertices $C \subseteq \Lambda$,

$$\begin{aligned} &\big\{\boldsymbol{v} \xrightarrow{C} \vec{\boldsymbol{x}}_J\big\} = E'(\boldsymbol{v}, \vec{\boldsymbol{x}}_J; C) \\ &\dot{\cup} \dot{\bigcup_b} \Big\{E'(\boldsymbol{v}, \underline{b}; C) \cap \{b \text{ is occ. \& piv. for } \boldsymbol{v} \longrightarrow \boldsymbol{x}_j \ \forall j \in J\}\Big\}. \end{aligned} \tag{3.4.14}$$

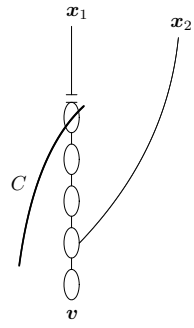


Figure 4: Schematic representation of $E'(\boldsymbol{v}, \{\boldsymbol{x}_1, \boldsymbol{x}_2\}; C)$.

Define

$$A^{(1)}(\vec{\boldsymbol{x}}_J) = \sum_b \mathbb{P}\Big(\{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap E'(b, \vec{\boldsymbol{x}}_J; \tilde{C}^b(\boldsymbol{o}))\Big). \tag{3.4.15}$$

Then, we obtain

$$\begin{aligned}\tau(\vec{\boldsymbol{x}}_J) = A^{(0)}(\vec{\boldsymbol{x}}_J) - A^{(1)}(\vec{\boldsymbol{x}}_J) + \sum_{\boldsymbol{y}} B^{(0)}(\boldsymbol{y})\, \tau(\vec{\boldsymbol{x}}_J - \boldsymbol{y}) \\ - \sum_{b_1, b_2} \mathbb{P}\Big(\{\boldsymbol{o} \Longrightarrow \underline{b}_1\} \cap E'(b_1, \underline{b}_2; \tilde{C}^{b_1}(\boldsymbol{o})) \\ \cap \{b_2 \text{ is occ. \& piv. for } \overline{b}_1 \longrightarrow \boldsymbol{x}_j \ \forall j \in J\}\Big).\end{aligned} \tag{3.4.16}$$

Now, we use (2.2.54) again to obtain, with

$$B^{(1)}(\boldsymbol{y}) = \sum_{b_1, b_2} pD(b_2) \mathbb{P}\Big(\{\boldsymbol{o} \Longrightarrow \underline{b}_1\} \cap E'(b_1, \underline{b}_2; \tilde{C}^{b_1}(\boldsymbol{o}))\Big), \tag{3.4.17}$$

to arrive at

$$\tau(\vec{\boldsymbol{x}}_J) = A^{(0)}(\vec{\boldsymbol{x}}_J) - A^{(1)}(\vec{\boldsymbol{x}}_J) + \sum_{\boldsymbol{y}} [B^{(0)}(\boldsymbol{y}) - B^{(1)}(\boldsymbol{y})] \tau(\vec{\boldsymbol{x}}_J - \boldsymbol{y}) + R^{(2)}(\vec{\boldsymbol{x}}_J), \tag{3.4.18}$$

where

$$R^{(2)}(\vec{\boldsymbol{x}}_J) = \sum_{b_1, b_2} \mathbb{P}\Big(\{\boldsymbol{o} \Longrightarrow \underline{b}_1\} \cap E'(b_1, \underline{b}_2; \tilde{C}^{b_1}(\boldsymbol{o})) \cap \big\{b_2 \xrightarrow{\tilde{C}^{b_2}(\overline{b}_1)} \vec{\boldsymbol{x}}_J\big\}\Big).$$

This completes the second step of the expansion.

For $R^{(2)}(\vec{\boldsymbol{x}}_J)$, we must repeat the above steps indefinitely. We omit the details here, and only give the result of the expansion. To write down the result of the expansion, we make a few more definitions. For $\vec{b}_N = (b_1, \dots, b_N)$ for $N \geq 1$, we define recursively

$$\tilde{E}^{(N)}_{\vec{b}_N}(\vec{\boldsymbol{x}}_J) = \tilde{E}^{(N-1)}_{\vec{b}_{N-1}}(\underline{b}_N) \cap E(b_N, \vec{\boldsymbol{x}}_J; \tilde{C}^{b_N}(\bar{b}_{N-1})), \tag{3.4.19}$$

where we let

$$\tilde{E}^{(0)}_{\vec{b}_0}(\vec{\boldsymbol{x}}_J) = E'(\vec{\boldsymbol{x}}_J). \tag{3.4.20}$$

Using this notation, we define

$$A^{(N)}(\vec{\boldsymbol{x}}_J) = \sum_{\vec{b}_N} \mathbb{P}\big(\tilde{E}^{(N)}_{\vec{b}_N}(\vec{\boldsymbol{x}}_J)\big), \quad B^{(N)}(\boldsymbol{y}) = \sum_{\boldsymbol{u}, \vec{b}_N} \mathbb{P}\big(\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{u})\big)\, pD(\boldsymbol{y} - \boldsymbol{u}), \tag{3.4.21}$$

which agree with (3.4.5) and (3.4.9) when $N = 0$, and with (3.4.15) and (3.4.17) when $N = 1$, and denote their respective alternating sums by

$$A(\vec{\boldsymbol{x}}_J) = \sum_{N=0}^{\infty} (-1)^N A^{(N)}(\vec{\boldsymbol{x}}_J), \quad B(\boldsymbol{y}) = \sum_{N=0}^{\infty} (-1)^N B^{(N)}(\boldsymbol{y}). \tag{3.4.22}$$

Then, (3.3.2) follows. This completes the derivation of the first expansion, which is an adaptation of the expansion for the oriented percolation two-point function in Section 2.2.2.

3.5 The Factorisation Lemma

Recall (3.4.4) and (3.4.5) in the first expansion for the three-point function. The problem in obtaining two independent two-point function from the two disjoint connections is that these connections use bonds with the *same* time coordinates. Therefore, we will never be able to use the Markov property effectively. In order to obtain a form of independence, we will use the fact that disjoint sets of bonds are independent. This will be formulated in the Factorisation Lemma, Lemma 3.5, below. Lemma 3.5 and the definitions below apply to general percolation models. Thus, for instance, they also apply to unoriented percolation, where $\Lambda = \mathbb{Z}^d$. The Factorisation Lemma is the key ingredient in the lace expansion for the (unoriented) percolation two-point function, and was also used for the expansion of the oriented percolation two-point function in [41].

To be able to state the Factorisation Lemma, we need some definitions.

Definition 3.4. (i) Given a (deterministic or random) set of vertices A and a bond configuration ω, we define ω_A, the restriction of ω to A, to be

$$\omega_A(\{\boldsymbol{x},\boldsymbol{y}\}) = \begin{cases} \omega(\{\boldsymbol{x},\boldsymbol{y}\}) & \text{if } \boldsymbol{x},\boldsymbol{y} \in A, \\ 0 & \text{otherwise,} \end{cases} \tag{3.5.1}$$

for every $\boldsymbol{x},\boldsymbol{y}$ such that $\{\boldsymbol{x},\boldsymbol{y}\}$ is a bond. In words, ω_A is obtained from ω by making every bond that does not have both endpoints in A vacant.

(ii) Given a (deterministic or random) set of vertices A and an event E, we say that E *occurs via the bonds in* A, and write E in A, if $\omega_A \in E$. In other words, E occurs on the (possibly modified) configuration in which every bond that does not have both endpoints in A is made vacant.

The notion of "occurs via the bonds in" is convenient to describe pivotal bonds. Indeed, it is easy to see that

$$\begin{aligned} &\{b \text{ pivotal for } \boldsymbol{v} \longrightarrow \boldsymbol{x}\} \\ &= \big\{\{\boldsymbol{v} \longrightarrow \underline{b}, \overline{b} \not\in \tilde{C}^b(\boldsymbol{v})\} \text{ in } \tilde{C}^b(\boldsymbol{v})\big\} \cap \{\overline{b} \longrightarrow \boldsymbol{x} \text{ in } \Lambda \backslash \tilde{C}^b(\boldsymbol{v})\}. \end{aligned} \tag{3.5.2}$$

In terms of the above definition, we have the following *Factorisation Lemma*, which will allow us to prove that the two events on the right-hand side of (3.5.2) are independent conditionally on $\tilde{C}^b(\boldsymbol{v})$. In its statement, we write $\theta(p) = \mathbb{P}_p(|C(\boldsymbol{x})| = \infty)$ for the probability that the cluster of $\boldsymbol{x}$ is infinite.

Lemma 3.5 (Factorisation Lemma). *For any p such that $\theta(p) = 0$ and any bond b, vertex $\boldsymbol{v}$ and events E, F*

$$\begin{aligned} &\mathbb{E}\Big(I[E \text{ in } \tilde{C}^b(\boldsymbol{v}), F \text{ in } \Lambda \backslash \tilde{C}^b(\boldsymbol{v})]\Big) \\ &= \mathbb{E}_0\Big(I[E \text{ in } \tilde{C}_0^b(\boldsymbol{v})] \mathbb{E}_1\big(I[F \text{ in } \Lambda \backslash \tilde{C}_0^b(\boldsymbol{v})]\big)\Big). \end{aligned} \tag{3.5.3}$$

The proof follows by conditioning on $\tilde{C}^b(\boldsymbol{y})$, which is finite almost surely, and the fact that disjoint sets of bonds are independent. Lemma 3.5 also applies to unoriented percolation. The first version of Lemma 3.5 appeared in [25].

In the nested expectation on the right-hand side of (3.5.3), the set $\tilde{C}_0^b(\boldsymbol{v})$ is *random* with respect to the outer expectation, but *deterministic* with respect to the inner expectation. We have added a subscript 0 to $\tilde{C}_0^b(\boldsymbol{v})$ and subscripts 0 and 1 to the expectations on the right-hand side of (3.5.3) to emphasize this distinction. The inner expectation on the right-hand side effectively introduces a second percolation model on a second lattice, which is coupled to the first percolation model via the set $\tilde{C}_0^b(\boldsymbol{v})$.

We will refer to the bond b to which we can effectively apply Lemma 3.5 as a *cutting bond*.

3.6 The first expansion for $A(\vec{\boldsymbol{x}}_J)$

In this section, we use the Factorisation Lemma to derive the expansion (3.3.12). To obtain (3.3.12), we define the notion of a *cutting bond* for $\boldsymbol{o} \longrightarrow \vec{\boldsymbol{x}}_J$. To simplify the explanation, we will go to $N = 0$ and $r = 3$, and only study

$$A^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathbb{P}(E'(\{\boldsymbol{x}_1, \boldsymbol{x}_2\})) = \mathbb{P}(\{\boldsymbol{o} \longrightarrow \boldsymbol{x}_1\} \circ \{\boldsymbol{o} \longrightarrow \boldsymbol{x}_2\}). \quad (3.6.1)$$

We say that a bond b is the $\boldsymbol{x}_1$-*cutting bond* if b is the *first* occupied pivotal bond for $\boldsymbol{o} \longrightarrow \boldsymbol{x}_1$. We abbreviate the statement that b is the $\boldsymbol{x}_1$-*cutting bond* for $\boldsymbol{o} \longrightarrow \{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ to the statement that b is $\boldsymbol{x}_1$-cutting.

Let

$$\tilde{A}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_b \mathbb{P}\Big(\{b \text{ is } \boldsymbol{x}_1\text{-cutting}\} \cap \big\{(\boldsymbol{o} \longrightarrow \boldsymbol{x}_1) \circ (\boldsymbol{o} \longrightarrow \boldsymbol{x}_2)\big\}\Big), \quad (3.6.2)$$

$$a^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2; 1) = \mathbb{P}\Big(\{\nexists\, \boldsymbol{x}_1\text{-cutting bond}\} \cap \big\{(\boldsymbol{o} \longrightarrow \boldsymbol{x}_1) \circ (\boldsymbol{o} \longrightarrow \boldsymbol{x}_2)\big\}\Big). \quad (3.6.3)$$

Then, we obtain

$$A^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \tilde{A}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2) + a_1^{(0)}, \quad (3.6.4)$$

where we abbreviate $a_1^{(0)} = a^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2; 1)$. To apply the Factorisation Lemma to $\tilde{A}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in (3.6.2), we will use a rewrite similar to (3.5.2) as follows:

$$\begin{aligned} &\{b \text{ is } \boldsymbol{x}_1\text{-cutting}\} \cap \big\{(\boldsymbol{o} \longrightarrow \boldsymbol{x}_1) \circ (\boldsymbol{o} \longrightarrow \boldsymbol{x}_2)\big\} \\ &\quad = \big\{\{\boldsymbol{o} \Longrightarrow \underline{b}, \boldsymbol{o} \longrightarrow \boldsymbol{x}_2, \overline{b} \not\longrightarrow \tilde{C}^b(\boldsymbol{o})\} \text{ in } \tilde{C}^b(\boldsymbol{o})\big\} \cap \{b \text{ occ.}\} \\ &\qquad \cap \big\{\overline{b} \longrightarrow \boldsymbol{x}_1 \text{ in } \Lambda \setminus \tilde{C}^b(\boldsymbol{o})\big\}, \end{aligned} \quad (3.6.5)$$

where, by definition, $\{b \text{ is occ.}\}$ is independent of the other two events on the right-hand side.

We now apply Lemma 3.5 and (3.6.5) and the independence of the occupation status of b to extract $\tau(\boldsymbol{x}_1 - \boldsymbol{y}_1)$ from $\tilde{A}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in (3.6.2), to obtain

$$\tilde{A}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_b pD(b)\, \mathbb{E}\Big[I[\{\boldsymbol{o} \Longrightarrow \underline{b}, \boldsymbol{o} \longrightarrow \boldsymbol{x}_2\} \text{ in } \tilde{C}^b(\boldsymbol{o})]\, \tau^{\tilde{C}^b(\boldsymbol{o})}(\overline{b}, \boldsymbol{x}_1)\Big], \quad (3.6.6)$$

where we define the *restricted r-point function* $\tau^C(\boldsymbol{v}, \vec{\boldsymbol{x}}_I)$ as

$$\tau^C(\boldsymbol{v}, \vec{\boldsymbol{x}}_I) = \begin{cases} \mathbb{P}(\boldsymbol{v} \longrightarrow \vec{\boldsymbol{x}}_I \text{ in } \Lambda \setminus C), & \text{if } \boldsymbol{v} \notin C, \\ 0, & \text{if } \boldsymbol{v} \in C. \end{cases} \quad (3.6.7)$$

We note that we can omit "in $\tilde{C}^b(\boldsymbol{o})$" in (3.6.6) by the fact that when $\boldsymbol{o} \longrightarrow \boldsymbol{x}_2$ occurs, but $\{\boldsymbol{o} \longrightarrow \boldsymbol{x}_2 \text{ in } \tilde{C}^b(\boldsymbol{o})\}^c$, then $\overline{b} \in \tilde{C}^b(\boldsymbol{o})$, so that $\tau^{\tilde{C}^{b}(\boldsymbol{o})}(\overline{b}, \vec{\boldsymbol{x}}_2) = 0$ (see (3.6.7)). We next abbreviate $\tilde{C}^b = \tilde{C}^b(\boldsymbol{o})$.

We use the *inclusion-exclusion relation* following from (3.6.7)

$$\tau^{\tilde{C}^b}(\overline{b}, \boldsymbol{x}_1) = \tau(\boldsymbol{x}_1 - \overline{b}) - \mathbb{P}\big(\overline{b} \xrightarrow{\tilde{C}^b} \boldsymbol{x}_1\big). \tag{3.6.8}$$

Similarly to the expansion for the two-point function $\tau(\boldsymbol{x})$ described in Section 2.2.2, we can derive an expansion for the restricted two-point function $\tau^C(\boldsymbol{v}, \boldsymbol{x})$. Indeed, we can show that for any $C \subseteq \Lambda$, $J \neq \varnothing$, $p \in [0,1]$ and $\vec{\boldsymbol{x}}_J \in \Lambda^{|J|}$, there exist $A(\boldsymbol{v}, \vec{\boldsymbol{x}}_J; C)$ and $B(\boldsymbol{v}, \boldsymbol{y}; C)$ such that

$$\mathbb{P}\big(\boldsymbol{v} \xrightarrow{C} \vec{\boldsymbol{x}}_J\big) = A(\boldsymbol{v}, \vec{\boldsymbol{x}}_J; C) + \sum_{\boldsymbol{y}} B(\boldsymbol{v}, \boldsymbol{y}; C)\, \tau(\vec{\boldsymbol{x}}_J - \boldsymbol{y}). \tag{3.6.9}$$

Then, we arrive at

$$\tilde{A}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_{\boldsymbol{y}_1} \tilde{B}^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2)\ \tau(\boldsymbol{x}_1 - \boldsymbol{y}_1) + a_2^{(0)}, \tag{3.6.10}$$

where, writing, $B_\delta(\overline{b}, \boldsymbol{y}; C) = \delta_{\overline{b}, \boldsymbol{y}} - B(\overline{b}, \boldsymbol{y}; C)$, we define

$$\tilde{B}^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2) = \sum_b pD(b)\, \mathbb{E}\Big[I[\boldsymbol{o} \Longrightarrow \underline{b}, \boldsymbol{o} \longrightarrow \boldsymbol{x}_2]\ B_\delta(\overline{b}, \boldsymbol{y}_1; \tilde{C}^b)\Big], \tag{3.6.11}$$

$$a_2^{(0)} = -\sum_b pD(b)\, \mathbb{E}\Big[I[\boldsymbol{o} \Longrightarrow \underline{b}, \boldsymbol{o} \longrightarrow \boldsymbol{x}_2]\ A(\overline{b}, \boldsymbol{x}_1; \tilde{C}^b)\Big], \tag{3.6.12}$$

where we abbreviate $a_2^{(0)} = a^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2; 2)$. This completes the first expansion for $A(\vec{\boldsymbol{x}}_J)$. The result of this expansion is that a single factor $\tau(\boldsymbol{x}_1 - \boldsymbol{y}_1)$ is extracted from $\tilde{A}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$. In the second expansion, we will extract a second factor $\tau(\boldsymbol{x}_2 - \boldsymbol{y}_2)$.

3.7 The second expansion for $A(\vec{\boldsymbol{x}}_J)$

First, we introduce the second cutting bond for $\boldsymbol{o} \longrightarrow \boldsymbol{x}_2$. Using this notion, we then expand $\tilde{B}^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2)$ and extract the factor $\tau(\boldsymbol{x}_2 - \boldsymbol{y}_2)$ for some $\boldsymbol{y}_2 \in \Lambda$.

Recall (3.6.11). Given $m \geq 0$, we say that a bond e is the $(m, \boldsymbol{x}_2)$-*cutting bond* if e is the *first* occupied pivotal bond for $\boldsymbol{o} \longrightarrow \boldsymbol{x}_2$ such that $m_{\bar{e}} \geq m$. Let $a_3^{(0)} = a^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2; 3)$ be the contribution to $\tilde{B}^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2)$ where there does not exist a $(m_{\boldsymbol{y}_1}, \boldsymbol{x}_2)$-cutting bond. Then, we obtain

$$\begin{aligned}\tilde{B}^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2) = \sum_{b,e} pD(b)\, \mathbb{E}\Big[I[\{\boldsymbol{o} \Longrightarrow \underline{b}, \boldsymbol{o} \longrightarrow \boldsymbol{x}_2\} \\ \cap \{e \text{ is } (m_{\boldsymbol{y}_1}, \boldsymbol{x}_2)\text{-cutting}\}]\ B_\delta(\overline{b}, \boldsymbol{y}_1; \tilde{C}^b)\Big] + a_3^{(0)}.\end{aligned} \tag{3.7.1}$$

To investigate the first term in the right-hand side of (3.7.1), we will use a rewrite that is similar to (3.6.5). Then we have

$$\begin{aligned}&\{\boldsymbol{o} \Longrightarrow \underline{b}, \boldsymbol{o} \longrightarrow \boldsymbol{x}_2\} \cap \{e \text{ is } (m_{\boldsymbol{y}_1}, \boldsymbol{x}_2)\text{-cutting}\} \\ &= \big\{\{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap \{\nexists (m_{\boldsymbol{y}_1}, \underline{e})\text{-cutting bond}\} \text{ in } \tilde{C}^e(\boldsymbol{o})\big\} \\ &\qquad \cap \{e \text{ is occ.}\} \cap \big\{\bar{e} \longrightarrow \boldsymbol{x}_2 \text{ in } \Lambda \setminus \tilde{C}^e(\boldsymbol{o})\big\},\end{aligned} \tag{3.7.2}$$

where, again, $\{e \text{ is occupied}\}$ is independent of the other two events in the right-hand side.

We now complete the expansion for $\tilde{B}^{(0)}(\boldsymbol{y}, \boldsymbol{x}_2)$. We apply Lemma 3.5 and (3.7.2) to complete the expansion for $\tilde{B}^{(0)}(\boldsymbol{y}, \boldsymbol{x}_2)$. By (3.7.2) and the independence of the occupation status of e, we obtain

$$\begin{aligned}&\tilde{B}^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2) - a_3^{(0)} \\ &= \sum_{b,e} pD(b)pD(e)\, \mathbb{E}\Big[I[\{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap \{\nexists (m_{\boldsymbol{y}_1}, \underline{e})\text{-cutting bond}\} \text{ in } \tilde{C}^e(\boldsymbol{o})] \\ &\quad \times B_\delta(\bar{b}, \boldsymbol{y}_1; \tilde{C}^b)\; I[\bar{e} \longrightarrow \boldsymbol{x}_2 \text{ in } \Lambda \setminus \tilde{C}^e(\boldsymbol{o})]\Big].\end{aligned} \tag{3.7.3}$$

Note that $B_\delta(\bar{b}, \boldsymbol{y}_1; \tilde{C}^b)$ is random when $m_{\bar{b}} < m_{\boldsymbol{y}_1}$ and depends only on bonds with time variables in between $m_{\bar{b}}$ and $m_{\boldsymbol{y}_1} - 1$.

Then, by conditioning on the bonds that are part of $\tilde{C}^b$ with time variables in between $m_{\bar{b}}$ and $m_{\boldsymbol{y}_1} - 1$, we can show that Lemma 3.5 can also be used to factor expected values of random variables, rather than of indicators. This leads to

$$\begin{aligned}&\tilde{B}^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2) - a_3^{(0)} \\ &\quad = \sum_{b,e} pD(b)pD(e)\, \mathbb{E}\Big[I[\{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap \{\nexists (m_{\boldsymbol{y}_1}, \underline{e})\text{-cutting bond}\} \\ &\qquad \text{in } \tilde{C}^e] B_\delta(\bar{b}, \boldsymbol{y}_1; \tilde{C}^b)\; \tau^{\tilde{C}^e}(\bar{e}, \boldsymbol{x}_2)\Big],\end{aligned} \tag{3.7.4}$$

where we abbreviate $\tilde{C}^e = \tilde{C}^e(\boldsymbol{o})$. We note that, because $\{\boldsymbol{o} \Longrightarrow \underline{b}\}$ and $\{\nexists (m_{\boldsymbol{y}_1}, \underline{e})$-cutting bond$\}$ only depend on bonds with time indices less than or equal to $m_{\underline{e}}$, we can again omit the 'in $\tilde{C}^e$'.

Finally, we use (3.6.8) and (3.6.9) to arrive at

$$\tilde{B}^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2) - a_3^{(0)} = \sum_{\boldsymbol{y}_2} C^{(0)}(\boldsymbol{y}_1, \boldsymbol{y}_2)\; \tau(\boldsymbol{x}_2 - \boldsymbol{y}_2) + a_4^{(0)}, \tag{3.7.5}$$

where $a_4^{(0)} = a^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2; 4)$ is the contribution due to $A(\bar{e}, \boldsymbol{x}_2; \tilde{C}^e)$ and

$$\begin{aligned}C^{(0)}(\boldsymbol{y}_1, \boldsymbol{y}_2) = \sum_{b,e} pD(b)pD(e)\, \mathbb{E}\Big[&I[\{\boldsymbol{o} \Longrightarrow \underline{b}\} \cap \{\nexists (m_{\boldsymbol{y}_1}, \underline{e})\text{-cutting bond}\}] \\ &B_\delta(\bar{b}, \boldsymbol{y}_1; \tilde{C}^b)\; B_\delta(\bar{e}, \boldsymbol{y}_2; \tilde{C}^e)\Big].\end{aligned} \tag{3.7.6}$$

This completes the expansion for $\tilde{B}^{(0)}(\boldsymbol{y}, \boldsymbol{x}_2)$. Summarizing, we obtain that

$$A^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_{\boldsymbol{y}_1, \boldsymbol{y}_2} C^{(0)}(\boldsymbol{y}_1, \boldsymbol{y}_2)\, \tau(\boldsymbol{x}_1 - \boldsymbol{y}_1)\tau(\boldsymbol{x}_2 - \boldsymbol{y}_2) + a^{(0)}, \qquad (3.7.7)$$

where $a^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the contribution due to the error terms $a^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2; 1)$ and $a^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2; 2)$, as well as $a^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2; 3)$ and $a^{(0)}(\boldsymbol{y}_1, \boldsymbol{x}_2; 4)$. A similar strategy works for $A^{(N)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ for $N \geq 1$.

When $r > 3$, then we obtain a related result, where a sum over all non-empty subsets I of J_1 appears, and $\tau(\boldsymbol{x}_1 - \boldsymbol{y}_1)$ in (3.7.7) is replaced by $\tau(\vec{\boldsymbol{x}}_{J\setminus I} - \boldsymbol{y}_1)$ and $\tau(\boldsymbol{x}_2 - \boldsymbol{y}_2)$ by $\tau(\vec{\boldsymbol{x}}_I - \boldsymbol{y}_2)$. This completes the description of the expansion for the r-point function $\tau(\vec{\boldsymbol{x}}_J)$ as formulated in (3.3.17).

3.8 Related work

3.8.1 The contact process

In Section 2.5.4, we have described the results from [37], where Gaussian scaling of the two-point function in Theorem 2.8 was proved for the critical spread-out contact process above 4 spatial dimensions.

In [39], which is in preparation, the aim is to extend this result to convergence of the r-point functions. The methodology should be closely related to the one in [40], but the details change significantly due to the discretization procedure.

3.8.2 The survival probability and the incipient infinite cluster

Denote by

$$\theta_n = \mathbb{P}_{p_c}(\exists x \in \mathbb{Z}^d : (x, n) \in C(0, 0)) \qquad (3.8.1)$$

the *oriented percolation survival probability*. In [35, 36], which are in preparation, the survival probability is investigated for critical spread-out oriented percolation above 4 spatial dimensions. The papers [35, 36] aim to answer the question posed in [34], where it was shown that the *incipient infinite cluster* measure for spread-out oriented percolation above 4 spatial dimensions exists.

The incipient infinite cluster (IIC) describes the spatial structure of large critical clusters. In [34], two constructions where given for the IIC, and these two constructions were shown to agree under the assumption that

$$\theta_n \sim \frac{1}{Bn}. \qquad (3.8.2)$$

In [35, 36], we intend to prove (3.8.2), by an analysis based on induction [35] on a recursion relation for θ_n derived using the lace expansion [36]. This lace expansion is a *point-to-plane* expansion, rather than a point-to-(many) point(s) expansion which is derived for the higher-point functions in [40]. This expansion is more difficult than the one for the higher-point

functions in [40]. Peculiarly, it turns out that the expansions for the r-point function and the expansion for the survival probability can be chosen such that the lace expansion coefficients are (up to a constant factor) *identical*, which will be shown in [38]. In particular, this implies that the constant B in (3.8.2) is given by

$$B = \frac{AV}{2}. \tag{3.8.3}$$

This equality is also of independent interest, since it implies convergence of the rescaled particle numbers $\frac{1}{n}N_n$, where

$$N_n = \#\{x : (0,0) \longrightarrow (x,n)\}, \tag{3.8.4}$$

conditionally on $N_n > 0$, weakly converges to an exponential random variable with parameter $\lambda = \frac{2}{A^2V} = \frac{2}{AB}$. See [34, Theorem 1.5]. In [34, Section 5.3], an independent proof of (3.8.3) is given using the scaling of the r-point functions.

3.8.3 Lattice trees

For lattice trees, similar results for the r-point function can be obtained using the inductive method, as described in Section 2.5.5, together with an analysis of the r-point function using induction on r. This is the main content of [43]. In a way, we can think of the proof of convergence described in this section as a general strategy to prove convergence of the r-point functions, which can be adapted to deal with other settings, such as lattice trees and the contact process. We will describe related results on scaling of lattice trees in Section 4.4.2 below.

4 Super-Brownian motion

4.1 Introduction

In the previous section, we have described the convergence of the critical BRW and oriented percolation r-point functions to some limiting objects. Naturally, one expects these limiting objects to be related to a *limiting stochastic process*. This is the main message in this section. In the following discussion, for simplicity, we will only consider BRW.

In the scaling limit, time is sped up with a factor n. Therefore, between time t and $t + \varepsilon$, there are of the order εn time units or generations of the BRW. Since for BRW, the particles branch and die at every integer time, this means that the scaling limit should branch at *infinitesimal* times. Furthermore, for BRW, the children move away from their ancestors, in such a way that the spatial locations of the ancestors of a single individual are random walks. When scaled appropriately, these paths should converge to Brownian motion paths. The scaling limit of BRW thus describes a

cloud of individuals that branch on an infinitesimal scale, and move diffusively. Such processes are called *super-processes*, and super-Brownian motion (SBM), for which the movement is Brownian, is the prime example of such a process. In the following section, we will describe how super-Brownian motion can be obtained as the *scaling limit* of BRW, when it is appropriately scaled. Super-processes describe the movement of a population of individuals, and take values in the space of measures. Since the individuals these measures describe move diffusively, super-processes are sometimes called *measure-valued diffusions*.

Super-processes have received considerable attention in the probability community in the past decades. See e.g. [14, 18, 19, 48, 53] and the references therein. Super-Brownian motion is the principle example of a measure-valued Markov process in a similar way as Brownian motion is the principle example of a diffusion. As a consequence, one can expect super-Brownian motion to arise as a universal scaling limit for weakly interacting branching and moving population models, in a similar way as Brownian motion arises as the universal scaling limit of weakly interacting random paths. See [61] for an excellent nontechnical introduction describing scaling limits and their relations to super-processes.

There are two natural versions of a super-process. The first is when the initial population is described by a measure. In this case, a super-process is a continuous-time Markov process on the space of measures. The second is when we investigate the behaviour of the offspring of a *single* individual. Roughly speaking, the case where the initial population is a measure can be seen as the sum of many independent super-processes describing the single individuals, so that the case where we start with a single individual is called the *canonical measure*. In fact, any super-process is an infinite divisible process, and the canonical measure of the super-process is the usual canonical measure of the infinite divisible process as for example defined by Kallenberg [44]. The canonical measure of super-Brownian motion is described in more detail in [48, 53], and describes the structure of a single continuum tree embedded into $\mathbb{R}^d$ where particles undergo critical branching at any time scale, and move according to Brownian motions.

The canonical measure is an illusive object, since its law is *not* a probability measure. The reason is that if we start with a *single* individual and critical branching, then the branching process will die out with high probability before the convergence to the super-process sets in. Therefore, the scaling limit has an infinite atom at the empty measure. However, when we look at events which imply that the super-process is alive at some macroscopic time $t > 0$, then the measure of the event is uniformly bounded. To study the scaling limit, we can consider the conditional law of the process up to some time t when we restrict to configurations of the process that are alive at time t.

4.2 The canonical measure of Super-Brownian motion

We first discuss a construction of the canonical measure as a scaling limit of critical branching random walk to explain the canonical measure in more detail. For simplicity, we take an offspring distribution for which $\sigma_q = 1$. Let $n \geq 0$, and recall the definition of the random measures $\{\mu_n\}_{n=0}^{\infty}$ in (3.1.6)–(3.1.7).

We expect that, as $n \to \infty$, the process $\{\mu_n\}_{n=0}^{\infty}$ has a scaling limit. The difficulty in describing this scaling limit, however, is that

$$\mathbb{P}(\exists x \in \mathbb{Z}^d \text{ such that } \mu_n(x) \neq 0) = \mathbb{P}(\exists i \in T : |i| = n) = \theta_n, \qquad (4.2.1)$$

so that by (3.2.9), with probability close to 1 for n large, the random measure μ_n has mass zero. We are interested in the scaling limit of a single critical BRW, and, in particular, in *large* realizations of T for which μ_n is not identically equal to 0. We now describe the construction of the scaling limit in detail.

We define, for $t \geq 0$ and $x \in \mathbb{R}^d$, the random measure-valued Markov process $\{X_{n,t}\}_{t\geq 0}$ by defining its expected value of a bounded function $f\colon \mathbb{R}^d \to \mathbb{R}$ as

$$X_{n,t}(f) = \frac{1}{n} \sum_{x\in\mathbb{Z}^d} f(\sqrt{n\sigma}x)\mu_{\lfloor nt \rfloor}(x). \qquad (4.2.2)$$

We first motivate the scaling in (4.2.2), which we can alternatively write as

$$X_{n,t}(f) = \frac{1}{n} \sum_{y\in\mathbb{Z}^d/\sqrt{n\sigma^2}} f(y)\mu_{\lfloor nt \rfloor}(\frac{y}{\sqrt{n\sigma^2}}). \qquad (4.2.3)$$

It turns out that when there is a particle alive at time $\lfloor nt \rfloor$, then there are in fact *many* particles alive at the same time. Indeed, it can be shown that conditionally on $N_m \geq 1$, the random variable $\frac{N_m}{m}$ weakly converges to an exponential random variable. See [65] or [53, Theorem II.1.1(b)], as well as Section 3.8.2 above. We are particularly interested in branching processes that are alive at time proportional to n, so that we should normalise the number of particles with a factor of $\frac{1}{n}$. This explains the factor $\frac{1}{n}$ in (4.2.3). To explain the scaling in the spatial coordinate, we note that when there is a particle present at some site z at time proportional to n, then this particle has arrived to z by a random walk path of length proportional to n. Therefore, we can expect that z is proportional to $\sqrt{\sigma^2 n}$. This explains the scaling in (4.2.3). We now describe the scaling limit.

For an event E that is a measurable subset of the space of measure-valued paths on $\mathbb{R}^d$, we take the limit

$$\lim_{n\to\infty} n\mathbb{P}(\{X_{n,t}\}_{t\geq 0} \in E). \qquad (4.2.4)$$

It turns out that the above limit exists as an element of $[0, \infty]$ (see e.g., [53, Theorem II.7.3(a)]), and is by definition equal to the measure of the

indicator of the event E under the canonical measure of super-Brownian motion. We denote by $\mathbb{N}_0$ the *canonical measure of super-Brownian motion*, so that the above limit is equal to $\mathbb{N}_0(I[E])$. The canonical measure is a measure on continuous paths from $[0,\infty)$ into non-negative finite measures on $\mathbb{R}^d$. The canonical measure is an elusive object, as it is not a *probability measure*, but rather a σ-finite, non-negative measure. For simplicity, we take $\mathbb{N}_0$ to be normalised to have unit branching and diffusion rates, which corresponds to $\sigma_p^2 = 1$ for BRW. The factor n in (4.2.4) explains that the measure $\mathbb{N}_0$ is not a probability measure, but rather a σ-finite measure.

For a measure μ on $\mathbb{R}^d$, we write $\mu(1) = \int_{\mathbb{R}^d} 1 d\mu$ for its total mass. We write $\{X_t\}_{t\geq 0}$ for the process of non-negative measures under the canonical measure $\mathbb{N}_0$. Note that when $E = \{X_t(1) > 0\}$, then by (3.2.9),

$$\mathbb{N}_0(I[X_t(1)>0]) = \lim_{n\to\infty} n\mathbb{P}(X_{n,t}(1)>0) = \lim_{n\to\infty} n\theta_{\lfloor nt\rfloor} = \frac{2}{\sigma_q^2 t} = \frac{2}{t}, \tag{4.2.5}$$

since we have assumed that $\sigma_q = 1$. Therefore, $\mathbb{N}_0$ is a *finite* measure on events E that imply that $X_t(1) > 0$ for some $t > 0$.

We now discuss this construction of the canonical measure and its relation to super-Brownian motion started from a proper initial measure. Often, super-Brownian motion is considered as starting from a *proper initial measure*. This corresponds to a different scaling limit. Indeed, let the measure $\mu_{0,n}$ be such that $\mu_{0,n}(x)$ takes integer value for every $x \in \mathbb{Z}^d$, and let

$$\nu_n(x) = \frac{1}{n}\mu_{0,n}(\lfloor x\sqrt{\sigma^2 n}\rfloor). \tag{4.2.6}$$

We assume that ν_n is a measure that weakly converges to some limiting measure ν. Then, we let $(T_x^{(j)}, \phi_x^{(j)})$ for $j = 1,\ldots,\mu_{0,n}(x)$ be $\mu_{0,n}(x)$ independent branching random walks started at x, so that $\phi_x^{(j)}(0) = x$, where 0 is the root of the tree $T_x^{(j)}$. Then we consider the evolution of these independent BRW's from the various starting points, and take the sum of the particle numbers of the different trees. More precisely, the spatial locations of the branching random walk particles at time m with initial measure μ_n are then given by

$$\mu_{m,n}(x) = \sum_{y\in\mathbb{Z}^d} \sum_{j=1}^{\mu_{0,n}(y)} \sum_{i\in T_y^{(j)}:|i|=m} I[\phi_y^{(j)}(i) = x]. \tag{4.2.7}$$

In words, the random variable $\mu_{m,n}(x)$ equals the number of particles that are present at time m at the location x when we start with initial measure $\mu_{0,n}$. Then, we define

$$X_{n,t}(f) = \frac{1}{n}\sum_{x\in\mathbb{Z}^d} f(\sqrt{n\sigma}x)\mu_{\lfloor nt\rfloor,n}(x). \tag{4.2.8}$$

Thus, in particular, $X_{n,0}(f) = \nu_n(f)$. Denote the law of $\{X_{n,t}\}_{t=0}^{\infty}$ by $\mathbb{P}_{\nu_n}^{(n)}$. Then, the limit

$$\lim_{n\to\infty} \mathbb{P}_{\nu_n}^{(n)}(\{X_{n,t}\}_{t\geq 0} \in E) \tag{4.2.9}$$

exists. This limit is

$$\mathbb{P}_{\nu}(\{X_t\}_{t\geq 0} \in E), \tag{4.2.10}$$

where $\mathbb{P}_{\nu}$ is the law of *super-Brownian motion with initial measure* ν.

We can think of the law $\mathbb{P}_{\nu_n}^{(n)}$ as being described by the evolution of independent branching random walk copies, where the copies are located at positions described by the initial measure ν_n. In a similar way, we can think of $\mathbb{P}_{\nu}$ as being described by (infinitely) many independent copies of canonical measures according to the initial measure ν (see e.g., [53, Theorem II.7.2]). This intuitive picture can be made precise by noting that $\mathbb{P}_{\nu}$ is *infinitely divisible*, and using the general notion of infinitely divisible measures in e.g., [44]. In fact, in the terminology of infinite divisible measures, the canonical measure of super-Brownian motion is the canonical measure for the infinitely divisible measure $\mathbb{P}_{\nu}$. See e.g. [15, Section 1.3, and, in particular, Corollary 1.3] for more details.

On the other hand, we can also describe the canonical measure in terms of SBM by using the Markov property. Indeed, the law of $\{X_{s+t}\}_{t=0}^{\infty}$ given X_s is the same as the law of $\{X_t\}_{t=0}^{\infty}$ under $\mathbb{P}_{X_s}$. This shows that the laws of SBM with a proper initial measure and the canonical measure are intimately connected.

In this section, we describe the moment measures of the canonical measure of super-Brownian motion. We first discuss what these moment measures are and how they can be characterized.

As we will explain in more detail below, moment measures describe the finite-dimensional distributions of a super-process. Indeed, a measure can be determined by its expectation of a sufficiently rich class of bounded continuous functions. For a random measure X_t, we can thus determine the law of X_t by describing the laws of $X_t(f)$ for a sufficiently rich class of continuous functions, where

$$X_t(f) = \int_{\mathbb{R}^d} f(x) X_t(dx). \tag{4.2.11}$$

We will be using Fourier transforms, so that we take as a class of continuous functions $\{f_k\}_{k\in\mathbb{R}^d}$, where $f_k(x) = e^{ik\cdot x}$, and $k \cdot x$ is the inner product between x and k. Thus, in order to determine the law of super-Brownian motion, it suffices to know the law of $\{X_s(f_k)\}_{s\geq 0, k\in\mathbb{R}^d}$. This law will be uniquely determined by the finite-dimensional distributions $\{X_{s_i}(f_{k_i})\}_{i=1}^{r}$ for any $s = (s_1, \ldots, s_r)$ and $k = (k_1, \ldots, k_r)$. These laws, in turn, will be uniquely determined in terms of the joint moments, that is, for every

vector $(a_1, \ldots, a_r) \in \mathbb{N}^r$,

$$\mathbb{N}_0\big(\prod_{i=1}^{r} X_{s_i}(f_{k_i})^{a_i}\big) = \mathbb{N}_0\big(\prod_{j=1}^{l} X_{t_j}(f_{k_j})\big), \tag{4.2.12}$$

where $l = a_1 + \ldots + a_r$, and the components of $(t_1, \ldots, t_l)$ are equal to s_j precisely a_j times. Thus, we are led to investigate

$$\hat{M}_{\vec{t}}^{(l)}(\vec{k}) = \mathbb{N}_0\big(\prod_{j=1}^{l} X_{t_j}(f_{k_j})\big) = \mathbb{N}_0\big(\int_{\mathbb{R}^{dl}} X_{t_1}(dx_1)\cdots X_{t_l}(dx_l) \prod_{j=1}^{l} e^{ik_j\cdot x_j}\big). \tag{4.2.13}$$

These are the Fourier transforms of the *moment measures of the canonical measure of super-Brownian motion*. It turns out that these moment measures are given by (3.2.5)

Since the moment measures determine the law of the super-process, Theorem 3.2 tells us that the Fourier transforms of the BRW moment measures converge to those of super-Brownian motion. This leads to the following corollary:

Corollary 4.1. *The rescaled probability measure $n\mathbb{P}^{(n)}(\{X_{n,t}\}_{t>0} \in \cdot)$ converges to $\mathbb{N}_0$ in the sense of convergence of finite-dimensional distributions.*

We first give some remarks on extensions and implications of the above result:

Remark 4.2. (i) For BRW, it can be shown that the measures $\mathbb{P}^{(n)}(\{X_{n,t}\}_{t>0} \in \cdot)$ are *tight*, so that the convergence in finite-dimensional distribution actually implies weak convergence on the appropriate function space.

(ii) Since the measure $\mathbb{P}^{(n)}_{\nu_n}$ is obtained as the *independent* sum of the BRWs started from the measure ν_n, the above results can be extended to prove weak convergence of the law $\mathbb{P}^{(n)}_{\nu_n}$ to the law $\mathbb{P}_\nu$.

(iii) The canonical measure of super-Brownian motion is not a probability measure. Indeed, since critical BRW dies out with high probability, the canonical measure of super-Brownian motion has an infinite atom at the empty measure, which corresponds to all BRW configurations that have not survived to a macroscopic time of order n. This means that we cannot quite understand the convergence of $n\mathbb{P}^{(n)}(\{X_{n,t}\}_{t>0} \in \cdot)$ to $\mathbb{N}_0$ as the weak convergence of a usual stochastic process. One possible way of interpreting the convergence result is through *conditioning on survival*. Indeed, we can investigate the laws

$$\mathbb{P}^{(n)}\Big(\{X_{n,s}\}_{s\geq 0} \in \cdot \Big| X_{n,t}(1) > 0\Big), \tag{4.2.14}$$

which is a measure valued process which is forced to survive up to macroscopic time, and investigate whether

$$\mathbb{P}^{(n)}\Big(\{X_{n,s}\}_{s\geq 0} \in E \big| X_{n,t}(1) > 0\Big) \to \frac{\mathbb{N}_0(I[E]I[X_t(1) > 0])}{\mathbb{N}_0(I[X_t(1) > 0])}. \tag{4.2.15}$$

The measure $E \mapsto \frac{\mathbb{N}_0(I[E]I[X_t(1)>0])}{\mathbb{N}_0(I[X_t(1)>0])}$ *is* a probability measure, and the corresponding intuition is that BRW conditionally on survival up to time nt, when appropriately scaled, converges to the canonical measure of super-Brownian motion, conditioned to survive up to time t.

We next investigate the convergence of the appropriately scaled oriented percolation clusters to the canonical measure of super-Brownian motion. For spread-out oriented percolation above 4+1 dimensions, the scaling of the moment measures follows from Theorem 3.3. To formulate it, we define the measure

$$\mu_n(x) = I[(x, n) \in C(0, 0)], \tag{4.2.16}$$

and define its rescaled version $X_{n,t}$ to be the random measure with

$$X_{n,t}(f) = \frac{1}{n} \sum_{x\in\mathbb{Z}^d} f(\sqrt{n\sigma}x)\mu_{\lfloor nt\rfloor}(x). \tag{4.2.17}$$

Similarly to the BRW case, the measure $X_{n,t}$ is a discrete measure on $\mathbb{R}^d$. We will write $\mathbb{N}_{0,\rho}$ for the canonical measure of SBM when the branching rate is equal to ρ, which is the limit of critical BRW when the variance of the branching law is equal to ρ, rather than 1. Theorem 3.3 leads to the following convergence towards super-Brownian motion result:

Corollary 4.3. *Let $d > 4$, $p = p_c$. The rescaled probability measure $nA^{-1}\mathbb{P}_{p_c}(\{X_{n,t}\}_{t>0} \in \cdot)$ converges to $\mathbb{N}_{0,A^2V}$ in the sense of convergence of finite-dimensional distributions.*

4.3 Future challenges and open problems for spread-out oriented percolation

4.3.1 Weak convergence towards super-Brownian motion

As explained in Remark 4.2(i), one can show for BRW that the measures $n\mathbb{P}^{(n)}(\{X_{n,t}\}_{t>0} \in \cdot)$ are *tight*. Weak convergence is much stronger than convergence of the finite dimensional distributions, as weak convergence of the process implies convergence of all continuous functionals of the process

Tightness arguments for BRW typically use *martingale methods*, see e.g., [53, Section II.4]. In particular, tightness arguments are typically not based upon properties of the higher-point functions. For oriented percolation,

it is unclear how martingale methods should be used, and the necessary criteria for the higher-point functions require detailed bounds on differences of the five-point function at different time points. The problem of proving tightness for the rescaled oriented percolation cluster is still open.

4.3.2 Convergence of multiple clusters

As explained in Remark 4.2(ii), for BRW, the convergence of BRW's starting from a single individual to the canonical measure of super-Brownian motion imply the convergence of BRW's started from many individuals to super-Brownian motion started in the weak limit of the starting measure. The proof of this fact relies solely on the fact that the BRW's started in different spatial locations are *independent*.

For oriented percolation, on the other hand, when we would be interested in the the union of clusters of several points, this independence does not follow. More precisely, take a certain set $A \subseteq \mathbb{Z}^d$, and consider

$$C(A) = \cup_{a\in A} C(a, 0). \tag{4.3.1}$$

We will be interested in the scaling limit of the rescaled cluster. In this case, the starting measure is equal to

$$\mu_{n,0}(x) = \frac{1}{n} I[x \in A], \tag{4.3.2}$$

and the measure at time m is

$$\mu_{n,m}(x) = \frac{1}{n} I[(x, m) \in C(A)]. \tag{4.3.3}$$

Then, the rescaled measure is again described by

$$X_{n,t}(f) = \frac{1}{n} \sum_{x\in\mathbb{Z}^d} f(\sqrt{n\sigma}x)\mu_{\lfloor nt\rfloor}(x). \tag{4.3.4}$$

In order to have convergence, we should have convergence at time 0. Therefore, the set $A = A_n$ should be picked such that $|A_n| = n$ and, for every subset $B \subseteq \mathbb{R}^d$,

$$X_{n,0}(B) = \frac{1}{n} \#\{y \in \mathbb{Z}^d : \frac{y}{\sqrt{\sigma^2 vn}} \in B\} \tag{4.3.5}$$

converges to $\nu(B)$ for some initial measure ν. It is natural to conjecture that the process $\{X_{n,t}\}$ converges to super-Brownian motion started with initial measure $A\nu$ and branching rate $\rho = A^2V$.

For oriented percolation, the particles are *not* independent, but rather *positively dependent*, in the sense that, by the FKG-inequality, the event that $(x, m) \in C(A)$ makes it more likely that $(y, l) \in C(A)$. It would be of interest to prove the convergence towards super-Brownian motion starting from a general initial measure.

4.3.3 Convergence of the rescaled clusters conditioned on survival

In Remark 4.2(iii), the convergence of a single critical BRW to the canonical measure of super-Brownian motion was interpreted by investigating the random measures describing the location of the particles alive *conditionally* on the BRW to be alive at time tn. It was noted that this conditional process is a measure-valued stochastic process, that, when (4.2.15) holds, converges as a stochastic process to the canonical measure of super-Brownian motion conditioned to survive up to time t. It would be interesting to prove this statement for the rescaled oriented percolation clusters above 4 spatial dimensions. We note that this approach can only be successful if we know the scaling of the survival probability as discussed in Section 3.8.2.

4.3.4 Integrated super-Brownian excursion

Integrated super-Brownian excursion can be obtained as follows. Let

$$E = \Big\{ \int_0^\infty X_t(1)dt = 1 \Big\} \tag{4.3.6}$$

be the event that the total mass of super-Brownian motion is equal to 1. Then, integrated super-Brownian excursion (ISE) is the process $\{X_t\}_{t\geq 0}$ under the canonical measure of super-Brownian motion, conditioned on the event E. Therefore, it is the canonical measure of super-Brownian motion conditioned on the integral of its total mass being equal to one.

More formally, the measure of ISE corresponds to the measure $\frac{\mathbb{N}_0(\cdot I[E])}{\mathbb{N}_0(I[E])}$. Naturally, the event E has measure zero under $\mathbb{N}_0$. However, we can make sense of this by conditioning on

$$E_\varepsilon = \Big\{ 1 \leq \int_0^\infty X_t(1)dt \leq 1+\varepsilon \Big\}, \tag{4.3.7}$$

which turns out to have finite and positive measure under $\mathbb{N}_0$, and then taking the limit of $\varepsilon \downarrow 0$.

ISE can also be obtained as the scaling limit of BRW, see [1]. Indeed, we can condition on the size of the family tree to be equal to N, and take an appropriate scaling limit. It turns out that, when BRW lives until time n, the number of individuals in each generation is typically proportional to n. As a consequence, the size of the family tree is typically proportional to n^2, and conditioning on the family tree to be equal to N is comparable to conditioning the branching process not to have died out at time $\sqrt{N}$. Since distances in the cluster when it has survived up to time n are proportional to $\sqrt{n}$, this means that the spatial scaling when conditioning on the family tree to have size N are $\sqrt[4]{N}$.

There are many established connections between ISE and statistical mechanical models above the upper critical dimension. See Section 4.4.2 and

4.4.3 below. It is natural to conjecture that for oriented percolation above 4 spatial dimension and $p = p_c$, when we condition the cluster of the origin to have size N, and rescale time by $\sqrt{N}$ and space by $\sqrt[4]{N}$, the rescaled cluster converges to ISE. See also [40, Section 1.3.2], where this conjecture is formulated. However, this is a difficult problem. For one, since we condition on the size of the cluster to be equal to N, we should know the asymptotics of the law of the cluster size distribution. Similarly to BRW, it should be the case that

$$\mathbb{P}_{p_c}(|C(0,0)| = N) \approx CN^{-3/2}, \tag{4.3.8}$$

for some constant C, but, so far, (4.3.8) remains unproven for oriented percolation.

4.3.5 Other models

There are many more models where branching occurs and connections to super-processes can be expected. Examples are uniform spanning forests, invasion percolation, forest fire models, etc. For some of these models, one can expect that above a certain critical dimension, super-Brownian motion arises as the scaling limit. Possibly the lace expansion can be used to make this connection precise. However, the difficulty lies in establishing the lace expansion with appropriate bounds on the coefficients. Naturally, convergence towards super-processes is much more involved than convergence of the two-point function, so that it is natural to start by investigating the two-point functions. See also Section 2.7 for a discussion of the related problem for the two-point function.

4.4 Results for related models

There are several related results that prove relations between the scaling limits of interacting particle systems or statistical mechanical models. In this section, we discuss the voter model, the contact process, lattice trees and percolation.

4.4.1 The voter model and the contact process

In [9], it is shown that the rescaled voter model converges to super-Brownian motion. In the voter model, the variables $\xi_t(x)$ denotes the opinion of x at time t, where the 'opinions' are denoted by 0 and 1. The dynamics is such that neighbours adopt each other's opinion at rate 1 according to some transition probability, which we denote by D.

We start with an initial measure of all individuals having opinion 0, while a finite number of individuals have opinion 1, and are interested in whether the opinion 1 survives, and if so, where the individuals with opinion 1 are located. It is shown in [9] that if we rescale the particle numbers in the

same way as in (4.2.8), then for $d \geq 2$, the rescaled process converges to super-Brownian motion for the finite range model (for example, the nearest-neighbour model). Since weak convergence to SBM is proved, also the issue of tightness is resolved for the voter model.

In [9], also results concerning the convergence of BRW to super-Brownian motion can be found, as well as results for the contact process where the range of D grows simultaneously with time. The method to prove these results rely on *martingale techniques*, rather than on the use of the lace expansion. For the voter model, the proof crucially relies on the fact that the dual process is *coalescing random walks*. In [10], the weak convergence of the voter model is also used to compute the scaling of the survival probability, which is the probability that opinion 1 survives up to time t, when at time 0 there is a single individual with opinion 1, and a version of (3.8.2) for the voter model is shown there.

Similar results are shown in [16] and [11]. In [16], the contact process was studied above 2 dimensions when the range and time grow simultaneously. Again the proof relies on martingale methods. In [38], these results are expected to be extended to the finite-range contact process above 4 dimensions, using the lace expansion and following the strategy described in Section 3. We emphasize, however, that in [9] and [16], weak convergence results are proved, while in the strategy explained in these notes, only convergence of finite dimensional distributions is obtained. In [11], the Lotka-Volterra model was studied, which is a certain predator-prey model. Finally, see also [54] for an expository paper describing super-processes and convergence results to super-Brownian motion, focussing on martingale methods.

4.4.2 Lattice trees

There are many connections between lattice trees and super-Brownian motion. In [12, 13], it was shown that the r-point functions of lattice trees of fixed size, converge to those of ISE. The statements are complete when dealing with the r-point functions where the number of steps between 0 and x along the tree is not fixed, and there are partial results when this number is fixed and scales with the size of the lattice tree.

There is current progress in understanding the connection to SBM [43], see also Section 2.5.5. For lattice trees, similarly to Corollaries 4.1 and 4.3, it can be shown that the finite-dimensional distributions for lattice trees, when appropriately scaled, converge to those of the canonical measure of SBM.

4.4.3 Percolation

Unlike the models described in Sections 4.4.1, percolation does not have an obvious time variable, so that it is not immediately obvious in what way

the rescaled clusters should converge to SBM above 6 dimensions. There are several papers describing the critical exponents for percolation above 6 dimensions, as well as aspects of its scaling limit. See [25, 28, 29] for details. In the latter paper, the connection to integrated super-Brownian excursion (ISE) is explored, and, in particular, (4.3.8) is shown to hold for nearest-neighbour percolation in sufficiently high dimensions. Also, convergence of the percolation two- and three-point functions to the ones of ISE is proved in [29]. The proofs are based on generating function methodology and rely on the lace expansion. It would be of interest to extend these results to convergence to the canonical measure of super-Brownian motion, when the time variable is the shortest-paths distance along the cluster of the origin.

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The expansion for the r-point functions described in Section 3 is joint work with Akira Sakai, but, due to the connection to the survival probability, in [38] we choose a different expansion.

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