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Parafermionic observables and their applications to planar statistical physics models

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Abstract. This volume is based on the PhD thesis of the author. Through the examples of the self-avoiding walk, the random-cluster model, the Ising model and others, the book explores in details two important techniques:

1. *Discrete holomorphicity and parafermionic observables*, which have been used in the past few years to study planar models of statistical physics (in particular their conformal invariance), such as *random-cluster models* and *loop $O(n)$-models.*

2. The *Russo-Seymour-Welsh theory* for percolation-type models with dependence. This technique was initially available for Bernoulli percolation only. Recently, it has been extended to models with dependence, thus opening the way to a deeper study of their critical regime.

The book is organized as follows. The first part provides a general introduction to planar statistical physics, as well as a first example of the parafermionic observable and its application to the computation of the connective constant for the self-avoiding walk on the hexagonal lattice.

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The second part deals with the family of random-cluster models. It studies the Russo-Seymour-Welsh theory of crossing probabilities for these models. As an application, the critical point of the random-cluster model is computed on the square lattice. Then, the parafermionic observable is introduced and two of its applications are described in detail. This part contains a chapter describing basic properties of the random-cluster model.

The third part is devoted to the Ising model and its random-cluster representation, the FK-Ising model. After a first chapter gathering the basic properties of the Ising model, the theory of s-holomorphic functions as well as Smirnov and Chelkak-Smirnov's proofs of conformal invariance (for these two models) are presented. Conformal invariance paves the way to a better understanding of the critical phase and the two next chapters are devoted to the study of the geometry of the critical phase, as well as the relation between the critical and near-critical phases.

The last part presents possible directions of future research by describing other models and several open questions.

Acknowledgments

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Hugo Duminil-Copin

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Part I

Introduction

Chapter 1

What is statistical physics?

1.1 Phase transitions

When we heat a block of ice, it turns to water. This very familiar phenomenon hides a rather intricate one: the properties of H_2O molecules do not depend continuously on the temperature. More precisely, macroscopic properties of a large system of H_2O molecules evolve non-continuously when the temperature rises. For instance, when the temperature passes through 0 degree Celsius, the density increases from 0.91 to 1 (it is even more impressive when passing from water to vapor, where the density drops by a factor 1600). This example of everyday life is an instance of *phase transition.* In a system composed of many particles interacting directly only with their neighbors, a phase transition occurs if a macroscopic property of the system changes abruptly as a relevant parameter (temperature, porosity, density) varies continuously through a critical value.

An example of phase transition is given by superconductors. Superconductivity is the phenomenon of exact zero electrical resistance occurring in special materials at very low temperature. It was discovered by Heike Kamerlingh Onnes and his student Gilles Holst in 1911 when studying solid mercury at very low temperature (liquid helium had been recently discovered, allowing to work with cryogenic temperatures). Below a certain critical temperature $T_c = 4.2\ K$, the mercury loses its resistance abruptly (they also discovered the superfluid transition of helium at $T_c = 2.2\ K$). Since then, superconductivity has been studied extensively, and the number of examples of superconductors has exploded. Practical applications are numerous, and everyone has the image of a superconductor levitating above a magnet in mind.

Another experiment was performed in 1895 by Pierre Curie. He showed that a ferromagnet loses its magnetization when heated above a critical temperature, now called *Curie temperature*. The experiment is fairly simple theoretically: one attaches a rod of iron to an axis, near a large magnet. At room temperature, the rod is attracted by the magnet. When the rod gets hot enough, the axis abruptly comes back to vertical, indicating a loss of magnetization. In practice, the difficulty of the experiment comes from the fact that this temperature equals 770 degrees Celsius for iron. If the composition of the magnet is different, the critical temperature changes (it can be 30 degrees Celsius only), yet the phenomenon remains the same: it is always possible to demagnetize matter by heating it, which naturally leads to the following question: what is the microscopic phenomenon explaining this macroscopic behavior?

Understanding how local interactions govern the behavior of the whole system is extremely hard in general, and involves all fields of physics. In order to simplify the problem, one can introduce a *model*, *i.e.* an idealized system of particles following elementary rules, which should mimic the behavior of the real model. The area of science in charge of modeling large systems mathematically is called *statistical physics*.

1.2 Three models of statistical physics

The previous examples illustrate that different kinds of phase transitions occur in nature. Before starting, a warning: not everything contained in this section is necessarily proved mathematically! We simply plan to motivate through three examples the introduction of diverse notions, such as *critical exponents*, *universality*, *correlation length*, *order* of a phase transition and *thermodynamical quantities* before we study them thoroughly in the rest of this book.

1.2.1 Percolation

Definition and phase transition. Percolation is probably the model of statistical physics which is easiest to define. It was introduced by Broadbent and Hammersley in 1957 as a model for a fluid in a porous medium [BH57]. The medium contains a network of randomly arranged microscopic pores through which fluid can flow. One can interpret the d-dimensional medium as being a lattice (for instance the hypercubic lattice with $\mathbb{Z}^d$ as sites and edges between nearest neighbors), each edge being a possible hole in the medium. In our setting, an edge is called *open* if it is a hole, and *closed* otherwise. One can then think of the sites of $\mathbb{Z}^d$ together with open edges as a subgraph of $\mathbb{Z}^d$.

In order to model the randomness inside the medium, we simply state that edges are open with probability p, and closed with probability $1-p$, and this independently of each other. The random graph obtained is called ω_p, and the probability measure is denoted by $\mathbb{P}_p$.

For a fluid to flow through the medium there must exist a macroscopic set of connected open edges. The phase transition in this model on $\mathbb{Z}^d$ thus corresponds to the emergence of an infinite connected component (sometimes called *cluster*) of open edges.

Intuitively, there are more and more open edges in the graph when p increases. It is thus not surprising that there exists a critical $p_c = p_c(\mathbb{Z}^d) \in [0,1]$ such that

- for $p < p_c(\mathbb{Z}^d)$, there is no infinite cluster almost surely,
- for $p > p_c(\mathbb{Z}^d)$, there is an infinite cluster almost surely.

The behavior changes drastically when the porosity parameter p evolves continuously through $p_c(\mathbb{Z}^d)$. This is the sign of a phase transition if $p_c(\mathbb{Z}^d)$ lies strictly between 0 and 1. Actually, $p_c(\mathbb{Z})$ equals 1 (when the edge-density equals $p < 1$, there are always closed edges to the right and left of every given site), and there is no phase transition in dimension 1. However, as soon as $d > 1$ the phase transition occurs in the sense that $p_c(\mathbb{Z}^d) \in (0,1)$. Let us mention that $p_c(\mathbb{Z}^2) = 1/2$ (we will present a proof of this fact in this book).

Infinite-cluster density $\boldsymbol{\theta(p)}$ and universality. When $p > p_c(\mathbb{Z}^d)$, there is in fact a unique infinite cluster (this result is non-trivial and will be proved in this book). Via invariance by translation, this cluster has a positive density $\theta(p)$, which can be defined as

$$\theta(p) \;=\; \mathbb{P}_p(0 \text{ belongs to the infinite cluster}).$$

We are interested in the behavior of $\theta(p)$ when $p \searrow p_c(\mathbb{Z}^d)$. This behavior is very similar in every dimension, even though subtle differences do occur. More precisely, $\theta(p)$ is always predicted to follow a power law decay in $p - p_c$. The exponent, usually named β, depends on the dimension in the following way:

$$\theta(p) \;\approx\; (p-p_c)^\beta \qquad \text{where } \beta = \begin{cases} 5/36 & \text{if } d = 2, \\ \text{numerical value} & \text{if } d \in \{3,4,5\}, \\ 1 & \text{if } d \geq 6. \end{cases}$$

The value β is called a *critical exponent.*

As mentioned earlier, one can consider percolation on the hypercubic lattice. Nevertheless, percolation can be defined on any graph or lattice. For instance, it could be defined on the hexagonal lattice or the triangular lattice in dimension two. A striking feature of percolation, and more generally of a relevant statistical model, is that the behavior is *universal*: the microscopic properties of the model depend on the local geometry of the graph, while the macroscopic do not. It mimics real phase transitions: the critical temperature for superconductors ranges from a few degrees Kelvin to thirty or even more degrees Kelvin, yet the phase transition is similar. In the case of percolation, connectivity properties between two neighbors in the square or the hexagonal lattices are not the same, yet the thermodynamical properties, such as the infinite-cluster density, behave similarly and the exponent β is expected to be the same for any lattice of a fixed dimension. For instance, β equals 5/36 for the hexagonal, triangular and square lattices.

Correlation length $\xi(p)$ and order of a phase transition. As a matter of fact, phase transitions occur always in infinite volume. To illustrate this, let us make a brief detour and discuss the physical notion of correlation length. It is also a great opportunity to introduce an additional critical exponent.

Assume that p is unknown and consider one realization of the percolation of parameter p on a box of size $N \in (0, \infty]$. Let us take the point of view of a statistician in this paragraph and try to test whether the unknown parameter p is smaller or larger than $p_c(\mathbb{Z}^d)$. When $N = \infty$ (in other words, we look at the percolation on $\mathbb{Z}^d$ itself), testing the existence or not of an infinite cluster provides us with a perfect test. Now, if N is finite, the situation is more intricate. Indeed, when N is not too large, it is even difficult to give good bounds on p while when N is very large, the configuration looks pretty much like the one on $\mathbb{Z}^d$, and the existence or not of very large clusters is a good test of $p > p_c(\mathbb{Z}^d)$ against $p < p_c(\mathbb{Z}^d)$. Roughly speaking, the *correlation length* is the smallest $N = N(p)$ for which we can recognize with good probability if p is supercritical or not. Similarly, the correlation length in the subcritical phase (when $p < p_c(\mathbb{Z}^d)$) is the smallest $N = N(p)$ for which we can decide if p is subcritical or not.

Mathematically, the correlation length is defined in an *a priori* completely different fashion. When $p < p_c(\mathbb{Z}^d)$, the largest connected components in boxes of size N are typically of size $\log N$. Equivalently, the probability for 0 to be connected by a path of adjacent open edges to distance N decays exponentially fast like

$$\mathbb{P}_p\big(\text{the cluster of 0 is of radius larger than } N\big) = \exp\big[-\tfrac{N}{\xi(p)}\big(1 + o_N(1)\big)\big]$$

where $\xi(p) \in (0, \infty)$ is called the *correlation length*. In the supercritical case, a corresponding definition can be introduced.

In the case of percolation, the correlation length is finite when $p \neq p_c$ and goes to infinity when $p \nearrow p_c$. This is not the case for every model (in general, the divergence of the correlation length is an indicator of a *second order phase transition* which is one among several possible types of phase transitions). Once again, the behavior of $\xi(p)$ is expected to follow a power law governed by a critical exponent:

$$\xi(p) \approx |p - p_c|^{-\nu} \quad \text{where } \nu = \begin{cases} 4/3 & \text{if } d = 2, \\ \text{numerical value} & \text{if } d \in \{3,4,5\}, \\ 1/2 & \text{if } d \geq 6. \end{cases}$$

Remark 1.1. The results above are still conjectural for percolation on $\mathbb{Z}^2$ but they have been proved thanks to the works of Smirnov and Lawler-Schramm-Werner for site percolation on the triangular lattice (see the references in [BDC13]) and by Hara and Slade for $\mathbb{Z}^d$ with $d \geq 19$ in [HS90] (this bound was recently improved to $d \geq 15$ by Fitzner in his PhD thesis [Fit13]).

1.2.2 Ising model

The celebrated Lenz-Ising model is one of the simplest models in statistical physics exhibiting an order-disorder transition. It was introduced by Lenz in [Len20] and studied by his student Ising in his thesis [Isi25]. It is a model for ferromagnetism as an attempt to explain Curie's temperature. See [Nis09] for a historical review of the classical theory.

Definition. The definition is slightly more intricate than for percolation. In the Ising model, iron is modeled as a collection of atoms with fixed positions on a crystalline lattice. In order to simplify, each atom has a magnetic "spin", pointing in one of two possible directions. We set the spin to be equal to 1 or −1 depending on their direction. Each configuration of spins has an intrinsic energy, which takes into account the fact that neighboring sites prefer to be aligned (meaning that they have the same spin), exactly like magnets tend to attract or repel each other.

Formally, fix a box Λ in dimension d. Let $\sigma \in \{-1,1\}^\Lambda$ be a configuration of spins 1 or −1. The energy of the configuration σ is given by the Hamiltonian

$$H_\Lambda^{\mathrm{f}}(\sigma) := -\sum_{x\sim y} \sigma_x \sigma_y$$

where $x \sim y$ means that x and y are neighbors in Λ. Note that up to an additive constant (equal to minus the number of couples $x \sim y$ in Λ), H_Λ^{f} is twice the number of disagreeing neighbors.

Following a fundamental principle of physics, we wish to construct a model of random spin configurations that favors configurations with small energy. A natural choice is to sample a random configuration proportionally to its Boltzman weight: at a temperature T, the probability

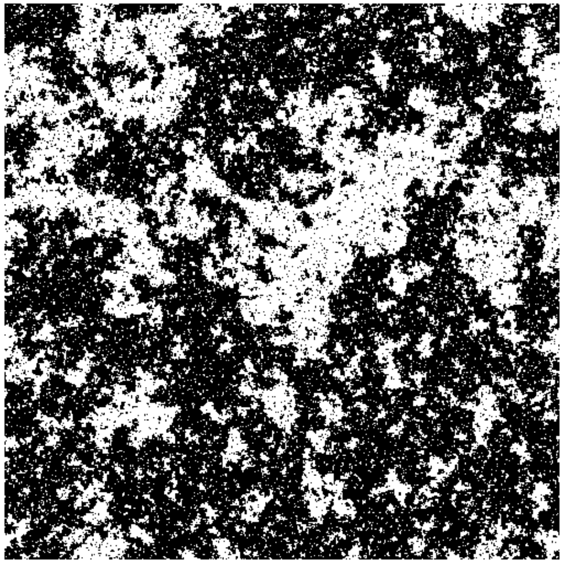


Figure 1.1: A configuration of the Ising model on the square lattice.

$\mu^{\mathrm{f}}_{T,\Lambda}$ of a configuration σ satisfies

$$\mu^{\mathrm{f}}_{T,\Lambda}(\sigma) := \frac{e^{-\frac{1}{T}H^{\mathrm{f}}_{\Lambda}(\sigma)}}{Z^{\mathrm{f}}(T,\Lambda)}$$

where

$$Z^{\mathrm{f}}(T,\Lambda) := \sum_{\tilde\sigma\in\{-1,1\}^{\Lambda}} e^{-\frac{1}{T}H^{\mathrm{f}}_{\Lambda}(\tilde\sigma)}$$

is the so-called *partition function* defined in such a way that the sum of the weights over all possible configurations equals 1.

Note that the configurations minimizing the energy, and therefore the most likely, are the extremal ones: either all +1 or all −1. Nevertheless, there are only two of them, thus the probability to see them in nature is tiny. In other words, there is a competition between *energy* and *entropy*. The number of configurations for some level of energy can balance the decrease of energy. This balance between energy and entropy depends on the temperature. For instance, if T converges to ∞, the configurations become equally likely and the model is almost equivalent to a percolation model (on sites this time) where sites are independent. This phase is called *disordered*. On the contrary, when T goes to 0, the energy outdoes the entropy and configurations with a large majority of +1 (or −1) become typical. This phase is called *ordered*. The existence of two different phases suggests a phase transition.

Phase transition of the Ising model. Assume that spins on the boundary of the box Λ are conditioned to be +1 — we denote the measure thus obtained by $\mu^+_{T,\Lambda}$ — and define the *magnetization* at the origin in the box Λ by

$$M_\Lambda(T) \;:=\; \mu^+_{T,\Lambda}(\sigma_0),$$

where σ_0 is the spin at 0 ($\mu^+_{T,\Lambda}$ also denotes the expectation with respect to the measure $\mu^+_{T,\Lambda}$: the magnetization is therefore the average value of the spin at 0). Since the boundary favors pluses, this magnetization is positive. When letting the size of the box go to infinity, the magnetization decreases and converges to a limit, called the *spontaneous magnetization* $M(T) := \lim_{\Lambda\nearrow\mathbb{Z}^d} M_\Lambda(T)$.

The phase transition in dimension $d \geq 2$ can now be formulated: there exists a *critical temperature* $T_c = T_c(d) \in (0,\infty)$ such that

- when $T > T_c$, $M(T) = 0$,
- when $T < T_c$, $M(T) > 0$.

In other words, when the temperature is large, the correlation between the spin at the origin and the boundary conditions tends to 0: there is no long-range memory. When the temperature is low, the spin keeps track of the boundary conditions at infinity and is still plus with probability larger than 1/2.

We are now in a position to explain Curie's experiment. A magnet imposes an exterior field on an iron rod, forcing exterior sites to be aligned with it. At low temperature, sites deep inside "remember" that boundary sites are aligned, while at high temperature, they do not. Therefore, sites become globally aligned at low temperature, hence explaining the magnetization and the attraction.

In his thesis, Ising proved that there is no phase transition when $d = 1$. In other words, at any positive temperature, the spontaneous magnetization equals 0. He predicted the absence of a phase transition to be the norm in every dimension. This belief was widely shared, and motivated Heisenberg to introduce a famous alternative model where spins take value in the sphere $\mathbb{S}^3$ in 3d (in fact, this is the classical counterpart, first studied in [Hei28] of the quantum Heisenberg model).

However, some years later Peierls [Pei36] used estimates on the length of interfaces between spin clusters to disprove the conjecture, showing a phase transition in the two dimensional case. In fact, a phase transition occurs in every dimension $d \geq 2$, thus proving the prediction of Ising to be wrong. The name "Ising model" was actually coined by Peierls in his publication. Ising retired from academia, discovering 25 years later that his model had become one of the most famous models of statistical physics.

Physical phase transition. Fixing boundary conditions to be +1 or −1 is not completely satisfying physically. In order to mimic the real life experiment, let us add a magnetic field h in the following way: redefine the energy to be

$$H^{\mathrm{f}}_{\Lambda,h}(\sigma) := -\sum_{x\sim y}\sigma_x\sigma_y \; - \; h\sum_{x\in\Lambda}\sigma_x.$$

Obviously, h favors pluses when it is positive (the energy decreases for each spin +1), and minuses when it is negative. Exactly as before, the measure $\mu_{\Lambda,T,h}$ is defined by assigning to each configuration a weight proportional to $\exp[-\frac{1}{T}H^{\mathrm{f}}_{\Lambda,h}(\sigma)]$. As expected, $M(T,h) := \mu_{\Lambda,T,h}(\sigma_0)$ is strictly positive when $h > 0$ and strictly negative when $h < 0$, but what about h going to 0? This operation corresponds to removing the magnetic field in the model. A phase transition occurs in infinite volume, at the same critical temperature T_c as above in the following way:

- When $T > T_c$, $M(T,h)$ goes to 0 as h goes to 0.
- When $T < T_c$, $M(T,h)$ goes to $M(T) > 0$ as h goes to 0 from above, and to $-M(T)$ as h goes to 0 from below.

Therefore, at low temperature, the magnet keeps a spontaneous magnetization of the sign of the magnetic field that was surrounding it.

Can we find the equivalent of the percolation critical exponent $\boldsymbol{\beta}$? Let us study the phase transition, and in particular try to find the equivalent of percolation critical exponents. Exactly as in the percolation case, the behavior of the magnetization $M(T)$ when T approaches T_c from below follows a power law:

$$M(T) \approx (T_c - T)^{\beta} \quad \text{where } \beta = \begin{cases} 1/8 & \text{if } d = 2, \\ 0.3269\ldots & \text{if } d = 3, \\ 1/2 & \text{if } d \geq 4. \end{cases}$$

The critical exponent β can be related to the exponent for the infinite-cluster density of percolation via the class of random-cluster models (see Chapter 7). We may also define the exponent ν as follows. First, when $T < T_c$, it is predicted that

$$\mu^{\mathrm{f}}_T(\sigma_0\sigma_x) = \exp\Big[-\frac{|x|}{\xi(T)}(1 + o_{|x|}(1))\Big],$$

where $\xi(T)$ is called the correlation length. The exponent ν is then defined by the formula $\xi(T) \approx (T_c - T)^{-\nu}$ as $T \nearrow T_c$ (see Chapter 11 for more details).

1.2.3 Self-avoiding walks

Around the middle of the twentieth century, Flory and Orr introduced *self-avoiding walks* (SAW) as a model for ideal polymers [Flo53, Orr47].

Consider a lattice $\mathbb{L}$ (for instance $\mathbb{Z}^d$ or the hexagonal lattice $\mathbb{H}$): a self-avoiding walk is a self-avoiding sequence of neighboring sites. More formally, a *walk* of length $n \in \mathbb{N}$ is a map $\gamma : \{0, \dots, n\} \to \mathbb{L}$ such that γ_i and γ_{i+1} are nearest neighbors for each $i \in \{0, \dots, n-1\}$. An injective walk is called *self-avoiding*.

Enumeration of self-avoiding walks. The first question that pops to mind is the question of the enumeration of self-avoiding walks of length n:

What is the number c_n of SAWs of length n (on the lattice $\mathbb{L}$) that start from the origin?

While computations for small values of n can be made by hand (Orr found $c_6 = 16\,926$ on $\mathbb{L} = \mathbb{Z}^3$), they quickly become impossible to perform, due to the fact that c_n grows exponentially fast. With today's technology and efficient algorithms, one may enumerate walks up to length 71 on $\mathbb{Z}^2$ (see [Cli13]) and 36 on $\mathbb{Z}^3$ (see [SBB11] where a new algorithm is used together with 50 000 hours of computing time to get $c_{36} = 2\,941\,370\,856\,334\,701\,726\,560\,670$).

No exact formula is expected to hold for general values of n but it is still possible to study the asymptotic behavior of c_n as n becomes large. Since a $(n+m)$-step SAW can be uniquely cut into a n-step SAW and a parallel translation of a m-step SAW, we infer that

$$c_{n+m} \le c_n c_m,$$

from which it follows that there exists $\mu_c(\mathbb{L}) \in [1, +\infty)$ such that

$$\mu_c(\mathbb{L}) := \lim_{n\to\infty} c_n^{\frac{1}{n}}.$$

The positive real number $\mu_c(\mathbb{L})$ is called the *connective constant* of the lattice. We thus obtain that $c_n = \mu_c(\mathbb{L})^{n+o(n)}$ and the computation of the connective constant becomes the first step towards the understanding of the asymptotic behavior of c_n.

Unfortunately, explicit formulæ for $\mu_c(\mathbb{L})$ are not expected to be frequent, and mathematicians and physicists only possess numerical predictions for the most common lattices[1] with the notable exception of the hexagonal lattice $\mathbb{H}$, for which $\mu_c(\mathbb{H})$ is exactly equal to $\sqrt{2+\sqrt{2}}$ (see the next chapter).

Overcoming the deception due to the absence (in general) of an explicit formula for $\mu_c(\mathbb{L})$, one can use this quantity to get sharper predictions

[1]for instance $\mu_c(\mathbb{Z}^2) = 2.638\,158\,530\,35(2)$ [CJ12] and $\mu(\mathbb{Z}^3) = 4.684039931(27)$ [Cli13], where the parentheses correspond to the margin errors.

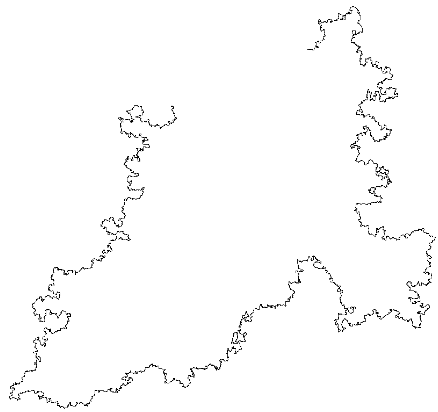


Figure 1.2: A 1000-step self-avoiding walk on the square lattice (© Vincent Beffara).

on the behavior of c_n. Physicists (always one step ahead) conjecture that

$$c_n \approx n^{\gamma-1}\mu_c(\mathbb{L})^n \text{ where } \gamma = \begin{cases} 43/32 & \text{if } d = 2, \\ 1.162\ldots & \text{if } d = 3, \\ 1 & \text{if } d \geq 4 \text{ with log corrections for } d = 4. \end{cases}$$

Above, d refers to the dimension of the lattice. Once again, γ is therefore a universal exponent depending only on the dimension of the lattice. In this context, universality seems even more surprising: it implies that even though the number of SAWs is growing exponentially at different speeds for say the hexagonal and the square lattice, the correction to the exponential growth is the same for both lattices.

Mean-square displacement. Flory was not interested in the combinatorial aspect of SAWs but rather in its geometry. He predicted that the averaged squared Euclidean distance between the ending point and the origin for SAWs of length n

$$\langle|\gamma(n)|^2\rangle := \frac{1}{c_n}\sum_{\gamma \text{ of length } n}|\gamma(n)|^2$$

behaves like $n^{3/2}$ in dimension 2, where $\gamma(n)$ is the last step of an n-steps SAW. Later, physicists provided strong evidence that

$$\langle|\gamma(n)|^2\rangle \approx n^{2\nu} \quad \text{where } \nu = \begin{cases} 3/4 & \text{if } d = 2, \\ 0.59\ldots & \text{if } d = 3, \\ 1/2 & \text{if } d \geq 4. \end{cases}$$

It is now a good place to compare SAWs to the *simple random walks model* on d-dimensional lattices. A *walk* is a trajectory which is possibly self-crossing. The number of walks of length n is obviously D^n, where D is the degree of the lattice, and the uniform measure on the family of walks of length n has a nice interpretation: it corresponds to the random walk constructed as follows: every step, the walker chooses a neighbor uniformly at random. This model is much better understood than the SAW (for instance, $\langle|\gamma(n)|^2\rangle$ behaves asymptotically like n).

SAWs are more spread (they go further) than simple random walks in dimensions 2 and 3. This fact is expected since a self-avoiding trajectory repulses itself. Interestingly, it is no longer true when the dimension becomes larger. It is actually possible to guess that this would occur, since the simple random walk itself becomes macroscopically self-avoiding at large scales when $d \geq 4$.

Phase transition for SAWs. So far, the SAW is not fitting in the framework of statistical physics since it does not depend on any parameter and does not exhibit a phase transition. For this reason, let us restate the model in a slightly different way.

Imagine we are now modeling a polymer in a solvent tied between two points a and b on the boundary of a domain Ω. We can model these polymers by SAWs on a fine lattice $\Omega_\delta := \delta\mathbb{L} \cap \Omega$ of mesh size $\delta \ll 1$. In order to take into consideration the properties of the solvent, let x be a real positive number. Our polymer will be a curve picked at random among every possible SAWs in Ω_δ from a_δ to b_δ (a_δ and b_δ are the closest points to a and b on Ω_δ), with probability proportional to $x^{|\gamma|}$, where $|\gamma|$ is the length of the SAW γ. More precisely, let $\Gamma_\delta(\Omega, a, b)$ be the set of self-avoiding trajectories from a_δ to b_δ in Ω_δ. The random polymer will have the law

$$\mathbb{P}_{\mu,\delta}(\gamma_\delta) := \frac{x^{|\gamma_\delta|}}{\sum_{\gamma \in \Gamma_\delta(\Omega,a,b)} x^{|\gamma|}}.$$

This model of random interface exhibits a phase transition when x varies[2]. On the one hand, when x is very small, the walk is penalized very much by its length, and it tends to be as straight as possible. On the other hand, if x is very large, the walk is favored by its length and tends to be as long as possible. Therefore, there exists x_c such that:

- When $x < x_c$, γ_δ (which is a random curve) becomes ballistic when δ goes to 0: it converges to the (deterministic) geodesic between a and b in Ω [Iof98].

[2] Here, $\delta \to 0$ replaces the passage to the infinite-volume $n \to \infty$ for percolation and Ising.

- When $x > x_c$, γ_δ converges to a random continuous curve filling the whole domain Ω when δ goes to 0 [DCKY11].

It is possible to prove that $x_c = 1/\mu_c(\mathbb{L})$. In other words, in order to obtain a critical model, one should penalize a walk of length n by $\mu_c(\mathbb{L})^{-n}$ (which is somewhat intuitive, since there are roughly $\mu_c(\mathbb{L})^n$ of them). When $x = x_c$, the sequence (γ_δ) should converge in the space of random continuous curves when δ goes to 0. In particular, the possible limit curves should be invariant under scaling. Typical objects having the scale-invariance property are called fractals, and it is conjectured that the scaling limit of SAWs at $x = x_c$ is indeed a random fractal.

Flory's exponents and mean-field approximation. Since it is of historical interest, let us sketch Flory's original determination of ν (a little bit of sweetness in the hostile world of critical exponents). We wish to identify the typical distance N of the last site $\gamma(n)$ of a n-step self-avoiding walk. In order to do so, we compute the probability of $|\gamma(n)| = N$ in two different ways.

First, let us make the assumption that sites are roughly spread on the box of size N (actually one could take $c_{\text{st}} \cdot N$ with a very large constant instead of N, but this would not matter), and that all sites play symmetric roles with respect to each other. We thus know that at each step $k+1 \le n$, a random walker must avoid the k previous sites if it wants to remain self-avoiding, so that it must choose one of the $N^d - k$ available sites. Thus, the probability that γ is still self-avoiding after n steps is of order

$$\prod_{k=0}^{n-1} \left(\frac{N^d - k}{N^d} \right) \approx \exp\left(-\sum_{k=0}^{n-1} k/N^d \right) \approx \exp\left(-\frac{n^2}{2N^d} \right)$$

as long as $n \ll N^d$. The assumption consisting in forgetting geometry (we do not require that the $(k+1)$-th site is a neighbor of the k-th one) is called the *mean-field approximation*.

Second, make the natural assumption that the end-point of the walk is distributed as a Gaussian, the probability for a walk to be at x after n steps would then be of the order of

$$\frac{1}{n^{d/2}} \exp(-\|x\|^2/n).$$

Therefore, the probability that it ends at distance N from the origin is then of the order of

$$N^{d-1} \cdot \frac{1}{n^{d/2}} \exp(-N^2/n).$$

(The term N^{d-1} comes from the fact that there are of order N^{d-1} sites at distance N from the origin.) Equaling the two quantities, we find that $n^3 \approx N^{d+2}$ *i.e.* $N \approx n^{3/(d+2)}$. It gives the following predictions for $d = 1, 2, 3, 4$:

$$\nu_{\text{Flory}} = \begin{cases} 1 \text{ if } d = 1, \\ 3/4 \text{ if } d = 2, \\ 3/5 \text{ if } d = 3, \\ 1/2 \text{ if } d = 4. \end{cases}$$

Flory's argument is slightly more involved and checks in particular that the reasoning cannot be valid when $d > 4$. Surprisingly, the prediction is true for $d = 1, 2$ and 4. It is slightly off for $d = 3$. In fact, the prediction is obvious when $d = 1$. For $d = 4$, the mean-field approximation is valid, even though its rigorous justification is a very hard problem which is currently under investigation [BIS09, BDS11]. Interestingly enough, the prediction in dimension 2 is saved by the surprising cancellation of two large mistakes. The probability to be self-avoiding is much smaller than the one described above. In the same time the Gaussian behavior of the walk is also completely wrong.

Flory's argumentation (especially in dimension 4) emphasizes an important fact of statistical physics: the mean-field approximation (*i.e.* assuming that the system lives on the complete graph) provides tractable ways to predict values for critical exponents and in large enough dimensions, these predictions are right. The reason for this connection is actually much deeper than Flory's argument. Roughly speaking, high-dimensional lattices behave with respect to statistical models like trees or complete graphs (in such case we speak of mean-field behavior). The dimension at which lattice exponents start to equal mean-field exponents is called the *upper critical dimension* d_c. It is equal to 4 for the self-avoiding walk and the Ising model, while it is 6 for percolation.

On the contrary in low dimensions, the behavior does not correspond to the mean-field one. Interestingly, the critical exponents in this case are all rational and fairly simple, which suggests a specific feature of two-dimensions that we shall discuss now.

1.3 Why two dimensions?

In the previous section, we studied three very different models of statistical physics which shared properties concerning their phase transitions. On the one hand, critical exponents become independent of the dimension when exceeding the upper critical dimension of the model. On the other hand, exponents have rational values in two dimensions, which suggests the existence of a deep underlying mechanism coming from physical laws. Our goal is to understand the phase transition in the latter case and **we now fix $d = 2$ for the rest of the book**.

In the next paragraphs, we will restrict our attention to critical models for the following reason. The critical exponents related to the thermodynamical quantities describing the phase transition are not independent: they are connected via so-called *scaling relations*, which do not depend on the model. For instance, one example of scaling relation is given by $\beta = \nu\eta$, where β and ν were defined in the context of percolation and the Ising model (they also exist for other statistical models), and η is the one-arm critical exponent, which is defined as follows:

- for percolation at criticality, there is no infinite cluster and the probability for 0 and x to be connected converges to 0 when x tends to ∞. In fact, the behavior should be

$$\mathbb{P}_{p_c}(0 \leftrightarrow x) \approx \frac{1}{|x|^{d-2+\eta}},$$

- for the Ising model, the magnetization equals 0 and we have

$$\mu_{T_c}(\sigma_0\sigma_x) \approx \frac{1}{|x|^{d-2+\eta}}.$$

The relation $\beta = \nu\eta$ provides one relation between exponents but there are other such relations (see e.g. [Kes87, BCKS99] for the fundamental example of percolation, and Sections 11.4 and 13.2.3 for more details). The important feature of these relations is that they relate exponents defined away from criticality (for instance ν and β) to fractal properties of the critical regime. *In other words, the behavior of a model through its phase transition is intimately related to its behavior at criticality.* It is therefore natural to focus on the critical phase, which has a rich geometry that we now discuss.

1.3.1 Exactly solvable models and Conformal Field Theory:

The planar Ising model has been the subject of experimentations for both mathematical and physical theories for almost a century. Through a short history of this model, we shall explain two physical perspectives on statistical physics.

Exactly solvable models. After Peierls' proof of the existence of a phase transition, the next step in the understanding of the Ising model was achieved by Onsager in 1944. In a series of seminal papers [Ons44, KO50], Onsager and Kaufman computed the *free energy* of the model. The formula led to an explosion in the number of results on the planar Ising model (papers published on the Ising model can now be counted by thousands). Among the most noteworthy results:

- the two-point function was proved to decay as the distance to the power $\frac{1}{4}$ by Onsager and Kaufman[3] (i.e. $\eta = \frac{1}{4}$ with the definition of the previous page);
- Yang clarified the connection between the spontaneous magnetization and the two-point function [Yan52] (the result was derived non rigorously by Onsager himself);
- McCoy and Wu [MW73] computed many important quantities of the Ising model including several critical exponents. The study culminated with the exact derivation of two-point correlations $\mu_T(\sigma_0\sigma_x)$ between sites 0 and $x = (n, n)$ in the whole plane.

See the more recent book of Palmer [Pal07] for an exposition of these and other results and for precise references.

The computation of the partition function was accomplished later by several other methods and the model became the most prominent example of an *exactly solvable* model. The most classical techniques include the transfer-matrices technique introduced by Kramers and Wannier (they were also used by Onsager and then developed by Lieb and Baxter [Lie67, Bax71] for more general models), the Pfaffian method, initiated by Fisher and Kasteleyn, using a connection with dimer models [Fis66, Kas61], and the combinatorial approach to the Ising model, initiated by Kac and Ward [KW52] and then developed by Sherman [She60] and Vdovichenko [Vdo65], see also the more recent [DZM+99, Cim12, KLM13].

Despite the number of results that can be obtained using the free energy, the impossibility to compute it explicitly enough in finite volume makes the geometric study of the model very hard to perform using the classical methods. The lack of understanding of the geometric nature of the model remained unsatisfying for years.

Renormalization Group and Conformal Field Theory. The arrival of the Renormalization Group (see [Fis98] for a historical exposition) led to a better physical and geometrical understanding, albeit mostly non-rigorous. It suggests that block-spin renormalization transformation (coarse-graining, *e.g.* replacing a block of neighboring sites by one site having a spin equal to the dominant spin in the block) corresponds to appropriately changing the scale and the temperature of the model. The critical point arises then as the fixed point of the renormalization transformations. In particular, under simple rescaling the Ising model at the critical temperature should converge to a scaling limit, a "continuous" version of the originally discrete Ising model, corresponding to a quantum field theory. This continuous model leads naturally to the concept of universality: the Ising models on different regular lattices or even more

[3] This result represented a shock for the community: it was the first mathematical evidence that the mean-field behavior was inaccurate in low dimensions!

general planar graphs belong to the same renormalization space, with a unique critical point, and so at criticality the scaling limit of the Ising model should always be the same: it should be independent of the lattice while the critical temperature depends on it[4].

Being unique, the scaling limit at the critical point must be invariant under translations, rotations and scaling. This prediction enabled [PP66, Kad66] to deduce some information about correlations.

In [BPZ84b, BPZ84a] Belavin, Polyakov and Zamolodchikov suggested a much stronger invariance of the model. Since the scaling-limit quantum field theory is a local field, it should be invariant by any map which is *locally* a composition of translations, rotations and homotheties. Thus it becomes natural to postulate full conformal invariance (under all conformal transformations[5] of subregions). This prediction generated an explosion of activity in conformal field theory[6], allowing for non rigorous explanations of many phenomena, see [ISZ88] for a collection of the original papers of the subject.

Note that planarity enters into consideration through the fact that conformal maps form a rich family of operators: conformal maps in dimension $d \geq 3$ are simply compositions of translations, rotations and inversions, while many other conformal maps can be found in two dimensions.

Where are we now? The above exposition shows two different approaches to the same problem relying heavily on two-dimensionality:

- The exact solvability of the (discrete) planar Ising model which allows rigorous derivations of important quantities yet at the same time provides a poor geometric understanding.
- The non-rigorous conformal field theory approach, with the postulate of a "continuum limit" invariant under many geometric transformations, which allows a deep geometric understanding of the model.

1.3.2 A mathematical setting for conformal invariance of lattice models

To summarize, Conformal Field Theory asserts that a planar statistical model, such as percolation, Ising or self-avoiding walk, admits a "scaling

[4]The same phenomenon occurs for self-avoiding walks: the connective constant depends on the lattice, while the polynomial correction to the exponential term does not.

[5]Conformal maps are maps on open sets of $\mathbb{C}$ conserving the angles. Equivalently, they are the one-to-one holomorphic maps.

[6]Conformal field theory is the domain of physics studying quantum field theories which are invariant under conformal transformations.

limit" at criticality, and that this scaling limit is a conformally invariant object. From a mathematical perspective, the notion of conformal invariance of an entire model is ill-posed, since the meaning of scaling limit depends on the object we wish to study (interfaces, size of clusters, crossings, etc). Nevertheless, a mathematical setting for studying scaling limits of interfaces has been developed in recent years, and for this reason we choose to focus on this aspect in this document.

Let us start with the study of one interface, meaning one curve separating two phases of the model. For pedagogical reasons, we simplify the presentation as much as possible by providing three examples in elementary cases. Fix a simply connected domain (Ω, a, b) with two points on the boundary and consider discretizations $(\Omega_\delta, a_\delta, b_\delta)$ of (Ω, a, b) by an hexagonal lattice of mesh size δ. The clockwise boundary arc of Ω_δ from a_δ to b_δ is denoted by $a_\delta b_\delta$, and the one from b_δ to a_δ by $b_\delta a_\delta$.

- The simplest model to start with is the critical SAW. The model of random polymer between a_δ and b_δ contains by definition only one interface (the walk itself), denoted by $\gamma_\delta^{\mathrm{SAW}}$.
- Let us now turn our interest to the critical Ising model on the triangular lattice (the definition is similar to the definition on the square lattice). Sites of the triangular lattice can be seen as faces of the hexagonal one, and we may therefore see this model as a random assignment of spins -1 and $+1$ on faces of the hexagonal lattice. Assume now that we fix the spins to be $+1$ on the faces outside Ω_δ and adjacent to $a_\delta b_\delta$ and -1 on the faces outside Ω_δ and adjacent to $b_\delta a_\delta$. With this convention, there exists a unique interface on the hexagonal lattice between $+1$ and -1 going from a_δ to b_δ. We denote this interface by $\gamma_\delta^{\mathrm{Ising}}$.
- We may also consider a percolation model defined as follows. Every face of the hexagonal lattice is open with probability $1/2$, and closed with probability $1/2$. If we fix faces outside Ω_δ and adjacent to $a_\delta b_\delta$ to be open, and faces outside Ω_δ and adjacent to $b_\delta a_\delta$ to be closed, we obtain a unique interface between closed and open faces going from a_δ to b_δ. This interface is called $\gamma_\delta^{\mathrm{perco}}$.

Conformal field theory leads to the prediction that $\gamma_\delta^{\mathrm{SAW}}$, $\gamma_\delta^{\mathrm{Ising}}$ and $\gamma_\delta^{\mathrm{perco}}$ converge as $\delta \to 0$ to a random, continuous, non-self-crossing curve from a to b staying in Ω, and which is expected to be conformally invariant in the following sense.

Definition 1.2. A family of random non-self-crossing continuous curves $\gamma_{(\Omega,a,b)}$, going from a to b and contained in Ω, indexed by simply connected domains with two marked points on the boundary (Ω, a, b) is *conformally invariant* if for any (Ω, a, b) and any conformal map $\psi : \Omega \to \mathbb{C}$,

$$\psi(\gamma_{(\Omega,a,b)}) \text{ has the same law as } \gamma_{(\psi(\Omega),\psi(a),\psi(b))}.$$

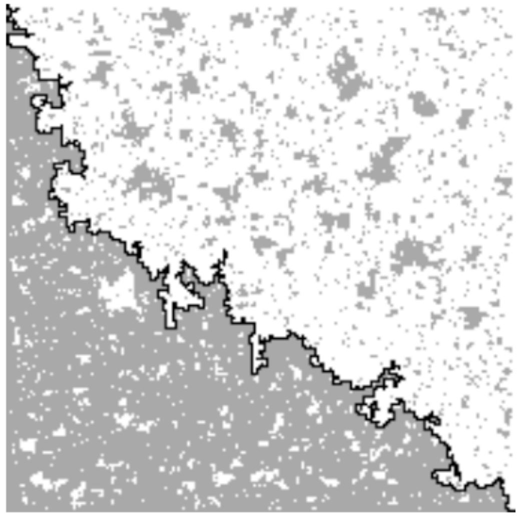


Figure 1.3: The interface of an Ising model at critical temperature (© Stanislav Smirnov).

In words, the random curve obtained by taking the scaling limit of SAWs on $(\psi(\Omega), \psi(a), \psi(b))$ has the same law as the image by ψ of the scaling limit of SAWs on (Ω, a, b) (and similarly for percolation and the Ising model). Let us emphasize how powerful this prediction is: it is clear, when working on the hexagonal lattice, that rotations by an angle $\pi/3$ are preserving the model. Conformal Field Theory predicts that the model possesses much more symmetries, such as rotations by any angle, as soon as we consider the scaling limit.

In 1999, Schramm proposed a natural candidate for the possible conformally invariant families of continuous non-self-crossing curves. He noticed that interfaces of models further satisfy the *domain Markov property*[7] which, together with the assumption of conformal invariance, determine a one-parameter families of possible curves. In [Sch00], he introduced the Stochastic Loewner evolution (SLE for short) which is now known as the Schramm–Loewner evolution. For $\kappa > 0$, a domain Ω and two points a and b on its boundary, SLE(κ) is the random Loewner evolution in Ω from a to b with driving process $\sqrt{\kappa}B_t$, where (B_t) is a standard Brownian motion[8]. By construction, the process is conformally invariant, random and fractal. In addition, it is possible to study quite precisely the behavior of SLEs using stochastic calculus and to derive path

[7]See Section 9.2 for a formal definition.

[8]The precise definition of SLE is presented in Section 9.2.

properties such as the Hausdorff dimension, intersection exponents, etc... Depending on κ, the behavior of the process is very different, as one can see on Fig. 1.4. The prediction of Conformal Field Theory then translates into the following predictions for models: $\gamma_\delta^{\mathrm{SAW}}$, $\gamma_\delta^{\mathrm{Ising}}$ and $\gamma_\delta^{\mathrm{perco}}$ converge as $\delta \to 0$ to Schramm-Loewner Evolutions[9].

The parameter κ depends on the model. It is usually possible to guess which one it should be and for instance, self-avoiding walks should converge to SLE(8/3), while Ising interfaces should converge to SLE(3) and percolation interfaces to SLE(6).

For completeness, let us mention that when considering not only a single curve but multiple interfaces, families of interfaces in a model are also expected to converge in the scaling limit to a conformally invariant family of non-intersecting loops. In the case of self-avoiding walks, the problem does not make sense, yet for the Ising or percolation models, there are many interfaces. For instance, consider the Ising model with +1 boundary conditions in an approximation of Ω. Interfaces between +1s and −1s now form a family of loops. By consistency, each loop should look like a SLE(3). Sheffield and Werner (see e.g. [SW10, SW12]) introduced a one-parameter family of processes of non-intersecting loops which are conformally invariant. These processes are called the Conformal Loop Ensembles CLE(κ) for $\kappa > 8/3$. The CLE(κ) process is related to the SLE(κ) in the following manner: the loops of CLE(κ) are locally similar to SLE(κ).

1.3.3 Conformal invariance of an observable in percolation and Ising models

Even though we now have a mathematical framework for conformal invariance, proving convergence of the interfaces in $(\Omega_\delta, a_\delta, b_\delta)$ to SLE remains an extremely hard task. Nevertheless, working with interfaces offers an important simplification that we illustrate in the cases of percolation and the Ising model.

In 1992, the observation that properties of interfaces should also be conformally invariant led Langlands, Pouliot and Saint-Aubin [LPSA94] to publish numerical values in agreement with the conformal invariance in the scaling limit of crossing probabilities in percolation[10]. More precisely, consider a Jordan domain Ω with four points A, B, C and D on the boundary. The 5-tuple (Ω, A, B, C, D) is called a *topological rectangle*. The authors checked numerically that the probability $\mathcal{C}_\delta(\Omega, A, B, C, D)$ of having a path of adjacent open sites between the boundary arcs AB and CD converges as δ goes to 0 towards a limit which is the same for

[9] See Section 9.2 for more details on the notion of convergence considered here.

[10] The authors attribute the conjecture on conformal invariance of the limit of crossing probabilities to Aizenman.

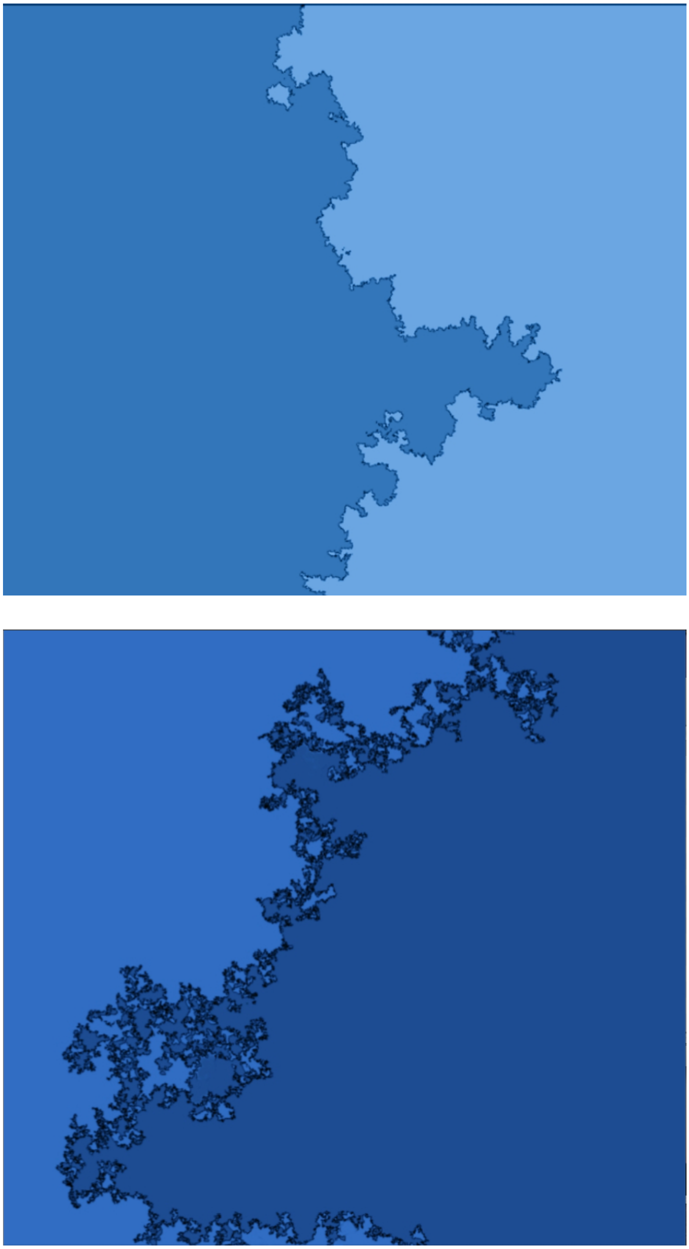


Figure 1.4: Two examples of Schramm-Loewner Evolutions (SLE(8/3) and SLE(6)). The behavior is very different: the first one is almost surely a simple curve (i.e. non intersecting) while the second one has self-touching points. The Haussdorff dimensions are also different. (© V. Beffara).

(Ω, A, B, C, D) and $(\Omega', A', B', C', D')$ if they are images of each other by a conformal map. Notice that the existence of such a crossing property can be expressed in terms of properties of a well-chosen interface, thus keeping this discussion in the frame proposed earlier.

The paper [LPSA94], while only numerical, attracted many mathematicians to the domain. The same year (1992), Cardy [Car92] proposed an explicit formula for the limit. In 2001, Smirnov [Smi01] proved Cardy's formula rigorously for critical site percolation on the triangular lattice, hence rigorously providing a concrete example of a conformally invariant property of the model. A remarkable consequence of this theorem is that, even though Cardy's formula provides information on crossing probabilities only, it can in fact be used to prove much more. In particular, it implies the convergence of interfaces to the *trace* of SLE(6). In other words, conformal invariance of one well-chosen quantity can be sufficient to prove conformal invariance of interfaces.

This phenomenon is not expected to be restricted to the percolation case. In 2010, Smirnov struck a second time by exhibiting conformally covariant (see Chapter 9 for a definition of this concept) observables for the so-called FK-Ising [Smi10] and Ising [CS12] models. Nonetheless, in this case the study of the critical regime is harder than in the percolation case: long-range dependence at criticality makes the mathematical understanding more involved and even proving convergence of interfaces to SLEs is difficult. However, the philosophy remains the same and full conformal invariance follows from conformal covariance of these observables.

We conclude this paragraph with a warning (or a touch of hope, depending on personal opinion): there are very few models which have been proved to be conformally invariant. For instance, the self-avoiding walk does not belong to this restricted club and it remains a very important open problem to prove convergence of self-avoiding walks to SLE(8/3).

1.3.4 Discrete holomorphicity and statistical models

The previous section explained that it is sufficient to prove convergence of discrete observables to conformally covariant objects in order to understand the critical phase, but how do we do it? Archetypical examples of conformally covariant objects are holomorphic solutions to boundary value problems such as Dirichlet or Riemann problems. It becomes natural that discrete observables which are conformally covariant in the scaling limit are naturally preharmonic or preholomorphic functions, *i.e.* relevant discretizations of harmonic and holomorphic functions, which are solutions of discretization of classical Boundary Value Problems. It therefore comes as no surprise that proofs of conformal invariance are based on *discrete complex analysis* in a substantial way.

The use of discrete holomorphicity appeared first in the case of dimers [Ken00] and has been extended to several statistical physics models since then. Other than being interesting in themselves, preholomorphic functions have found several applications in geometry, analysis, combinatorics, probability, and we refer the interested reader to the expositions by Lovász [Lov04], Stephenson [Ste05], Mercat [Mer01], Bobenko and Suris [BS08].

To conclude this section, we are now in possession of a natural mathematical framework to prove conformal invariance of a model: one needs to prove conformal covariance of an observable. Proving this requires a deep understanding of discrete complex analysis, and of its connections to the model. Very often, the integrability properties of the underlying model are at the heart of this connection, thus exhibiting a new link between exactly solvable models and Conformal Field Theory.

1.4 A model to rule them all: the random-cluster model

Percolation, Ising and self-avoiding walks provide us with three examples of models which are conformally invariant in the scaling limit (only conjecturally for the self-avoiding walk). They correspond to three values of the Schramm-Loewner Evolution (κ equals 6, 3 and 8/3 respectively). But what about other values of κ? Is it always possible to find a conformally invariant model whose interfaces converge to SLE(κ)? More importantly, can these seemingly very different models be related to each other? At last, can this relation explain the similarities between the different models? The answer to these questions come from the existence of two families of models, the random-cluster model and the $O(n)$-models. These models will be at the heart of this book and we would like to briefly present the random-cluster model now to motivate the next chapters.

Fortuin and Kasteleyn introduced the random-cluster model in 1969. Roughly speaking, the *random-cluster model* (it is also named *Fortuin-Kasteleyn percolation*) on a graph G is also a percolation model, in the sense that the output is a random subgraph of G with the same set of sites and a subset of its edges, but no longer independent.

More precisely, let $p \in [0,1]$ and $q \in (0,\infty)$. An edge of a finite graph G is either *open* or *closed*. The random-cluster configuration ω is the graph obtained by keeping only the open edges. The probability of ω for the random-cluster model on G with parameters p, q is given by

$$\phi_{p,q}(\omega) := \frac{1}{Z_{G,p,q}}\, p^{\#\text{ open edges}}(1-p)^{\#\text{ closed edges}}q^{\#\text{ connected components}}$$

where $Z_{G,p,q}$ is once again a normalizing factor called the partition function of the model. When $q = 1$, the model is simply percolation. When $q \neq 1$, the model is different and exhibits long range dependence.

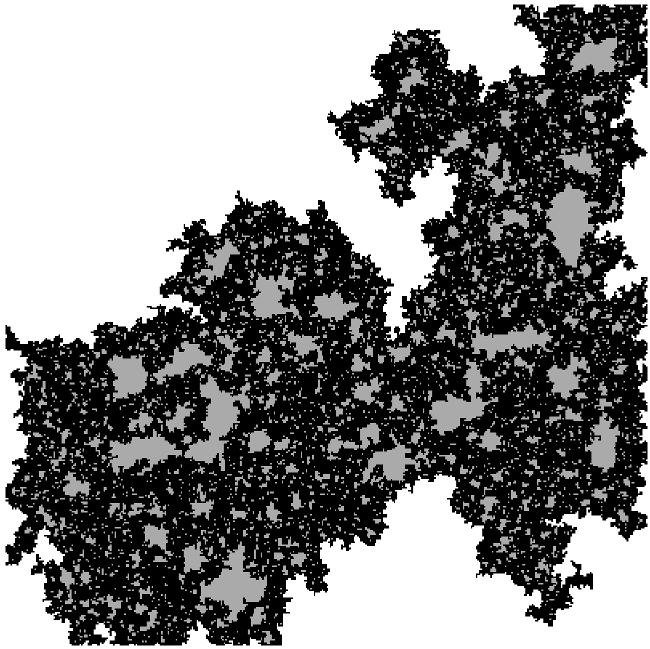


Figure 1.5: A macroscopic cluster in a critical percolation configuration with $p = 1/2$.

The previous measures are a priori defined on finite subgraphs of $\mathbb{Z}^2$, however it is possible to extend the model to $\mathbb{Z}^2$. As for percolation, the random-cluster model with fixed $q > 0$ should encounter a phase transition in p. Below some critical parameter $p_c(q)$, there is no infinite cluster, while above it, there exists a unique infinite cluster.

The phase transition is different when q varies, and the richness of this behavior is one of the successes of random-cluster models. More precisely,

- when $q \in (0, 4]$, the transition is expected to be continuous, in the sense that the density $\theta(p, q)$ of the infinite cluster converges to 0 when $p \searrow p_c(q)$. The critical phase should also be conformally invariant, and the collection of interfaces at criticality[11] should

[11] We did not describe interfaces in bond percolation on $\mathbb{Z}^2$ or the random-cluster model, yet one can consider the boundary of connected components for instance. We will provide more details in the next chapters.

converge to CLE(κ), where

$$\kappa = 4\pi/\arccos(-\sqrt{q}/2).$$

- when $q > 4$, the phase transition becomes *first order* and $p \mapsto \theta(p,q)$ does not converge to 0 when p goes down to $p_c(q)$.

Another important advantage of the random-cluster model is its connection to other models. When $p \to 0$ with $q/p \to 0$, we obtain a model of a random connected graph, called the uniform spanning tree, see [LSW11]. When q is an integer, one can play the following game. Color independently each connected component of a (p,q)-random-cluster configuration ω with one of q fixed colors chosen uniformly[12]. We obtain a random coloring $\sigma \in \{1,\ldots,q\}^G$ of G. The probability measure P is a Boltzman measure with energy given by

$$H_{q,G}(\sigma) \ := \ 2\sum_{x\sim y} 1_{\sigma_x \neq \sigma_y}.$$

The random coloring of the lattice with law P is called the *Potts* model with q colors at temperature T. When $q = 2$, it corresponds to the Ising model (simply call one color +1 and the other −1). Therefore, there exists a *coupling* of the Ising model with the $q = 2$ random-cluster model. This property links the Ising model to random-cluster models and thus to percolation.

Conclusion

We presented several aspects of planar statistical physics and we sketched important links between physics and mathematics. Nevertheless, most of what we presented is still conjectural. In this book, we make some of the connections between physics and mathematics rigorous by studying random-cluster and $O(n)$-models.

In particular, we will focus on two important theories: the so-called Russo-Seymour-Welsh theory of crossing events for random-cluster models, and the discrete holomorphicity of so-called parafermionic observables. In the specific case of the Ising model and its random-cluster representation (i.e. with cluster-weight $q = 2$), these two tools will lead to the rigorous proof of conformal invariance. For more general cluster-weights, conformal invariance remains out of reach, but the observable can still be used to discriminate between second-order and first-order phase transitions (we will define these concepts later) and to formulate precise conjectures.

[12]By this we mean that we choose a color for each cluster, and we color every site of the cluster in this color.

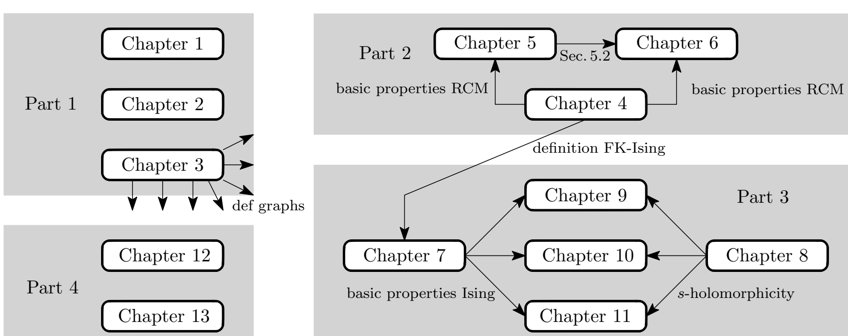
1.5 Organization of the book

The book is organized as follows. Chapter 2 should be understood as a warm-up: it provides a typical example of the application of parafermionic observables in the simplest context of self-avoiding walks. Chapter 3 describes the definitions for graphs that will be used in the reminder of the book.

We then devote an important part of this book to random-cluster models with cluster-weights $q \geq 1$ which are treated in Chapters 4, 5 and 6. Chapter 4 begins with a description of basic properties of the random-cluster model. Chapter 5 is devoted to the Russo-Seymour-Welsh theory and its applications (computation of the critical point, mixing properties, etc). Chapter 6 deals with the other important tool described in this book, namely the parafermionic observable. We define the observable and we describe two of its applications.

The third part of the book focuses on the Ising model and its random-cluster representation, the FK-Ising model. In this case, the observable can be proved to be discrete holomorphic. Chapter 7 gathers classical features of the Ising model. Chapter 8 develops the theory of discrete and s-holomorphic functions, two crucial concepts for the study of the critical Ising model. Chapter 9 presents the proofs of conformal invariance of Ising and FK-Ising models. Chapter 10 dives further into the study of crossings for the FK-Ising model and their applications to arm-events. Chapter 11 concludes this part of the book by discussing the non-critical Ising model (in particular we will compute the correlation length).

The last part of the book is composed of two chapters opening new perspectives. The first one describes parafermionic observables and their applications to other models. The last one lists important open problems.



Chapter 2

A warm-up: the connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$

The present chapter is intended to offer an elementary application of the parafermionic observable. It will provide us with the first example of a parafermionic observable, which can in fact be very easily defined in this context.

Let $\mathbb{H}$ be the hexagonal lattice of mesh size 1, translated and rotated in such a way that 0 is the center of an horizontal edge. Consider self-avoiding walks between *mid-edges* of $\mathbb{H}$, *i.e.* centers of edges of $\mathbb{H}$ (the set of mid-edges will be denoted by H). The *length* $\ell(\gamma)$ of the walk is the number of vertices belonging to γ. Note that it is equal to the number of mid-edges visited by the walk minus 1. In particular, a singleton is a walk of length zero. We wish to estimate the number of self-avoiding walks of length n starting from the origin[1]. Let c_n be the number of self-avoiding walks of length n starting from 0.

Lemma 2.1 (Hammersley)**.** *There exists* $\mu_c \in [\sqrt{2}, 2]$ *such that*

$$\lim_{n\to\infty} c_n^{1/n} = \mu_c(\mathbb{H}).$$

[1]The number of SAWs of length n between two vertices is related to the number of self-avoiding walks of length n starting from mid-edges. The formula is not very explicit but the ratio of these two quantities is clearly between 1 and 4. Therefore, studying SAWs starting from vertices or mid-edges will be equivalent as long as we are interested in a rough estimation of the number of such walks.

Proof. As explained in the introduction, a $(n+m)$-step SAW can be uniquely cut into a n-step SAW and a parallel translation of a m-step SAW. Hence,

$$c_{n+m} \le c_n c_m,$$

from which it follows (by a classical lemma on sub-multiplicative sequences of real numbers) that there exists $\mu_c(\mathbb{H}) \in [1, +\infty)$ such that

$$\mu_c(\mathbb{H}) := \lim_{n\to\infty} c_n^{\frac{1}{n}}.$$

Now, a SAW is in particular non-backtracking, and therefore, $c_n \le 3\times 2^{n-1}$. On the other hand, if we force the walk to take a step to the right every two steps, we necessarily obtain a self-avoiding trajectory, and therefore $c_n \ge 2^{\lfloor n/2 \rfloor}$. □

The previous lemma illustrates the fact that one may estimate $\mu_c(\mathbb{H})$ by adding more and more conditions on the local geometry of the walk. The values $\sqrt{2}$ and 2 can obviously be improved, since for instance a self-avoiding walk is not only non-backtracking, but it also does not contain any cycle of length 6, a fact which prevents many more walks, and shows that $\mu_c(\mathbb{H}) < 2$. There is *a priori* no good reason for being able to compute $\mu_c(\mathbb{H})$ explicitly. Nevertheless, the hexagonal lattice possesses special properties which make such a derivation possible. More precisely, B. Nienhuis [Nie82, Nie84] used the Coulomb gas formalism to predict that $\mu_c(\mathbb{H})$ is equal to $\sqrt{2+\sqrt{2}}$. Unfortunately, Nienhuis's derivation is based on assumptions that seem difficult to justify. In this chapter, we propose an alternative way of approaching the problem and we rigorously prove the following statement.

Theorem 2.2 (Duminil-Copin, Smirnov [DCS12b]). *For the hexagonal lattice, $\mu_c(\mathbb{H}) = \sqrt{2+\sqrt{2}}$.*

We will write $\gamma : a \to E$ if a walk γ starts at a and ends at some mid-edge of $E \subset H$. In the case $E = \{b\}$, we simply write $\gamma : a \to b$. Let $x > 0$. We will work with the (increasing in x) sum

$$Z(x) = \sum_{\gamma : a \to H} x^{\ell(\gamma)} \quad \in (0, +\infty].$$

This sum does not depend on the choice of a. Establishing $\mu = \sqrt{2+\sqrt{2}}$ is equivalent to showing that $Z(x) = +\infty$ for $x > 1/\sqrt{2+\sqrt{2}}$ and $Z(x) < +\infty$ for $x < 1/\sqrt{2+\sqrt{2}}$. To this effect, we first restrict walks to bounded domains and weigh them counting their windings. The vertex operator obtained like that leads to a *parafermionic*[2] *observable* which is a generalization of the spin fermionic observable. To simplify formulæ below, we set $x_c := 1/\sqrt{2+\sqrt{2}}$.

The chapter is organized as follows. In Section 2.1, the parafermionic observable is introduced and its principal property is derived. Section 2.2 contains the proof of Theorem 2.2.

2.1 Parafermionic observable

A (hexagonal lattice) *domain* $\Omega \subset H$ is a union of all mid-edges emanating from a given collection of vertices $V(\Omega)$ (see Fig. 2.1): a mid-edge z belongs to Ω if at least one end-point of its associated edge is in $V(\Omega)$, it belongs to $\partial\Omega$ if only one of them is in $V(\Omega)$. We further assume Ω to be simply connected, *i.e.* being connected and having a connected complement.

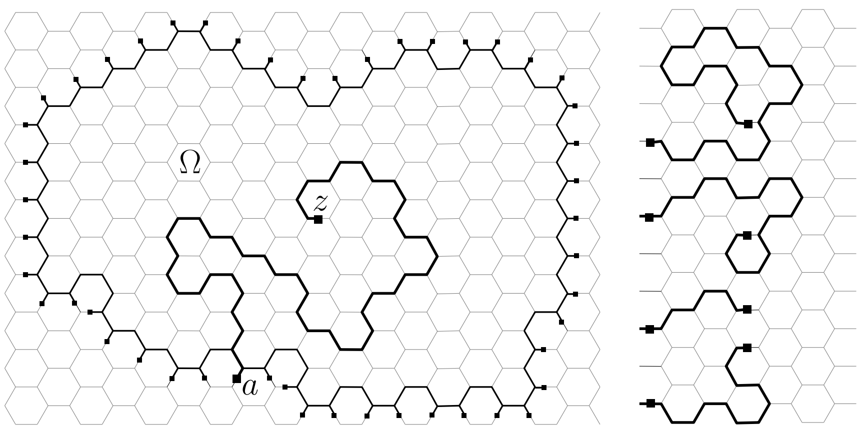


Figure 2.1: **Left.** A domain Ω with two marked mid-edges a and z. **Right.** The winding of a curve γ can also be seen as the number of left turns minus the number of right turns times $\frac{\pi}{3}$. We deduce that on the top, the winding equals 2π (17 left turns and 11 right turns), in the middle -2π (6 left turns and 12 right turns) and for the two bottom examples, 0 (respectively 3 and 6 left and right turns).

Definition 2.3. The winding $\mathrm{W}_\gamma(a,b)$ of a self-avoiding walk γ between mid-edges a and b (not necessarily the start and the end) is the total rotation of the direction in radians when γ is traversed from a to b, see Fig. 2.1.

[2] The name parafermionic will be justified later in the book.

The *parafermionic observable* is defined as follows: for $a \in \partial\Omega$ and $z \in \Omega$, set

$$F(z) = F(a,z,x,\sigma) = \sum_{\gamma\subset\Omega:\ a\to z} \mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)} x^{\ell(\gamma)}.$$

Lemma 2.4. *If $x = x_c (= 1/\sqrt{2+\sqrt{2}})$ and $\sigma = \frac{5}{8}$, then F satisfies the following relation for every vertex $v \in V(\Omega)$:*

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0, \tag{2.1}$$

where p, q, r are the mid-edges of the three edges adjacent to v.

Note that with $\sigma = 5/8$, the term $\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)}$ gives a weight λ or $\bar\lambda$ per left or right turn of γ, where

$$\lambda = \exp\Big(-\mathrm{i}\frac{5}{8}\cdot\frac{\pi}{3}\Big) = \exp\Big(-\mathrm{i}\frac{5\pi}{24}\Big).$$

Proof. In this proof, we further assume that the mid-edges p, q and r are oriented counterclockwise around v. Note that $(p-v)F(p)+(q-v)F(q)+(r-v)F(r)$ is a sum of "contributions"

$$c(\gamma) = (z-v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)} x_c^{\ell(\gamma)}$$

over all possible walks γ finishing at $z \in \{p,q,r\}$. The set of walks γ finishing at p, q or r can be partitioned into pairs and triplets of walks in the following way, see Fig 2.2:

- If a walk γ_1 visits all three mid-edges p, q and r, it means that the edges belonging to γ_1 form a self-avoiding path plus (up to a half-edge) a self-avoiding loop from v to v. One can associate to γ_1 the walk passing through the same edges, but exploring the loop from v to v in the other direction. Hence, walks visiting the three mid-edges can be grouped in pairs.

- If a walk γ_1 visits only one mid-edge, it can be grouped with two walks γ_2 and γ_3 that visit exactly two mid-edges by prolonging the walk one step further (there are two possible choices). The reverse is true: a walk visiting exactly two mid-edges belongs to the group of a walk visiting only one mid-edge (this walk is obtained by erasing the last step). Hence, walks visiting one or two mid-edges can be grouped in triplets.

If the sum of contributions for each pair and each triplet vanishes, then the total sum is zero. We now intend to show that this is the case.

Let γ_1 and γ_2 be two walks that are grouped as in the first case. Without loss of generality, we assume that γ_1 ends at q and γ_2 ends at r. Note that

γ_1 and γ_2 coincide up to the mid-edge p since (γ_1, γ_2) are matched together. We deduce that $\ell(\gamma_1) = \ell(\gamma_2)$ and

$$\begin{cases} \mathrm{W}_{\gamma_1}(a,q) = \mathrm{W}_{\gamma_1}(a,p) + \mathrm{W}_{\gamma_1}(p,q) = \mathrm{W}_{\gamma_1}(a,p) - \frac{4\pi}{3}, \\ \mathrm{W}_{\gamma_2}(a,r) = \mathrm{W}_{\gamma_2}(a,p) + \mathrm{W}_{\gamma_2}(p,r) = \mathrm{W}_{\gamma_1}(a,p) + \frac{4\pi}{3}. \end{cases}$$

In order to evaluate the winding of γ_1 between p and q, we used the fact that a is on the boundary and Ω is simply connected. Altogether,

$$\begin{aligned} c(\gamma_1) + c(\gamma_2) &= (q - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,q)} x_c^{\ell(\gamma_1)} + (r - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_2}(a,r)} x_c^{\ell(\gamma_2)} \\ &= (p - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,p)} x_c^{\ell(\gamma_1)} \left(j\bar{\lambda}^4 + \bar{j}\lambda^4 \right) = 0 \end{aligned}$$

where $j = \mathrm{e}^{\mathrm{i}2\pi/3}$. The last equality is due to the chosen value $\lambda = \exp(-\mathrm{i}5\pi/24)$.

Let $\gamma_1, \gamma_2, \gamma_3$ be three walks matched as in the second case. Without loss of generality, we assume that γ_1 ends at p and that γ_2 and γ_3 extend γ_1 to q and r respectively. As before, we easily find that $\ell(\gamma_2) = \ell(\gamma_3) = \ell(\gamma_1) + 1$ and

$$\begin{cases} \mathrm{W}_{\gamma_2}(a,q) = \mathrm{W}_{\gamma_2}(a,p) + \mathrm{W}_{\gamma_2}(p,q) = \mathrm{W}_{\gamma_1}(a,p) - \frac{\pi}{3}, \\ \mathrm{W}_{\gamma_3}(a,r) = \mathrm{W}_{\gamma_3}(a,p) + \mathrm{W}_{\gamma_3}(p,r) = \mathrm{W}_{\gamma_1}(a,p) + \frac{\pi}{3}. \end{cases}$$

Following the same steps as above, we obtain

$$c(\gamma_1) + c(\gamma_2) + c(\gamma_3) = (p - v)e^{-\mathrm{i}\sigma \mathrm{W}_{\gamma_1}(a,p)} x_c^{\ell(\gamma_1)} \left(1 + x_c j\bar{\lambda} + x_c \bar{j}\lambda \right) = 0.$$

Here is the *only* place where we use the crucial fact that $x_c^{-1} = \sqrt{2 + \sqrt{2}} = 2\cos\frac{\pi}{8}$. The claim follows readily by summing over all pairs and triplets. □

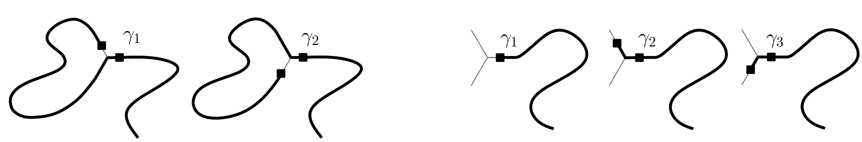


Figure 2.2: **Left:** a pair of walks visiting the three mid-edges and matched together. **Right:** a triplet of walks, one visiting one mid-edge, the other twos visiting two mid-edges, which are matched together.

Remark 2.5. Coefficients above are three cube roots of unity multiplied by $p - v$, so that the left-hand side can be seen as a discrete integral along an elementary contour on the dual lattice in the following sense. Let $\mathbb{H}^*$ be the triangular lattice constructed as follows: put a dual vertex v in the center of each face of $\mathbb{H}$, and edges between nearest neighbors. For a path $\gamma : \{0, \ldots, n\} \longrightarrow \mathbb{H}^\star$, where γ_i and γ_{i+1} are neighbors for every $0 \le i < n$, the discrete integral of a function $F : H \to \mathbb{C}$ is defined by

$$\oint_\gamma F(z)dz := \sum_{i=0}^{n-1} F\left(\frac{\gamma_i + \gamma_{i+1}}{2}\right)(\gamma_{i+1} - \gamma_i). \tag{2.2}$$

Equation (2.1) implies that for any $v \in V(\Omega)$, the integral of F along the path $\gamma_0 = a$, $\gamma_1 = b$, $\gamma_2 = c$ and $\gamma_3 = a$, where a, b and c are the three dual vertices corresponding to faces around v, is zero. More generally, let $\gamma : \{0, \ldots, n\} \longrightarrow \mathbb{H}^\star$ be a self-avoiding path with $\gamma_n = \gamma_0$. By summing the previous relation over vertices surrounded by γ, the discrete integral along γ also vanishes.

The fact that the integral of the parafermionic observable along discrete contours vanishes is a glimpse of conformal invariance of the model in the sense that the observable satisfies a weak notion of discrete holomorphicity, see Chapter 13 for more details. Nevertheless, these relations do not uniquely determine F: given a function f on the boundary, the solution of the Boundary Value Problem: discrete contours of F vanish and $F(z) = f(z)$ for any z on the boundary of Ω is not unique. Indeed, the number of mid-edges (and therefore of unknown variables) exceeds the number of linear relations (2.1) (which corresponds to the number of vertices). (We will see later that stronger notions of discrete holomorphicity will satisfy this crucial property that boundary value problems possess unique discrete holomorphic solutions.)

Let us conclude this remark by mentioning that these relations may also be understood as discrete Cauchy-Riemann equations around vertices of $\mathbb{H}$. We will provide more details in the case of the square lattice later in the book.

2.2 Proof of Theorem 2.2

Counting argument in a strip domain. We consider a vertical strip domain S_T composed of T strips of hexagons, and its finite version $S_{T,L}$ cut at height L at an angle of $\pi/3$, see Fig. 2.3. Then

$$V(S_T) = \left\{ z \in V(\mathbb{H}) : 0 \le \mathrm{Re}(z) \le \frac{3T+1}{2} \right\},$$
$$V(S_{T,L}) = \left\{ z \in V(S_T) : |\sqrt{3}\mathrm{Im}(z) - \mathrm{Re}(z)| \le 3L \right\}.$$

Denote by α the left boundary of S_T and by β the right one. Symbols ε and $\bar{\varepsilon}$ denote the top and bottom boundaries of $S_{T,L}$. Introduce the following positive quantities:

$$A^x_{T,L} := \sum_{\gamma\subset S_{T,L}:\, a\to\alpha\smallsetminus\{a\}} x^{\ell(\gamma)},$$
$$B^x_{T,L} := \sum_{\gamma\subset S_{T,L}:\, a\to\beta} x^{\ell(\gamma)},$$
$$E^x_{T,L} := \sum_{\gamma\subset S_{T,L}:\, a\to\varepsilon\cup\bar\varepsilon} x^{\ell(\gamma)}.$$

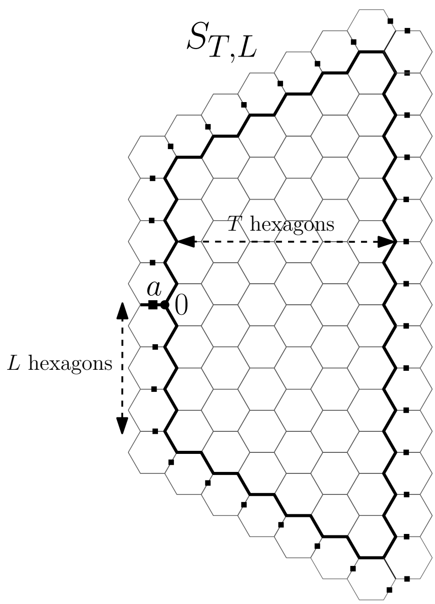


Figure 2.3: The domain $S_{T,L}$ and its boundary parts α, β, ε and $\bar\varepsilon$.

Lemma 2.6. *When $x = x_c$, we have*

$$1 = c_\alpha A^{x_c}_{T,L} + B^{x_c}_{T,L} + c_\varepsilon E^{x_c}_{T,L}, \tag{2.3}$$

where $c_\alpha = \cos\left(\frac{3\pi}{8}\right)$ and $c_\varepsilon = \cos\left(\frac{\pi}{4}\right)$.

Remark 2.7. The proof (see below) of this lemma can be understood in the following way: we used the fact that the discrete integral along the exterior boundary of $S_{T,L}$ vanishes. Then, we add the information that the winding of self-avoiding walks ending at boundary mid-edges is deterministic and explicit. Miraculously, even if the fact that discrete contour integrals vanish does not determine a function from boundary values, it is still sufficient to study their "average boundary value" and to obtain highly non-trivial relations like (2.3).

Proof. Sum the relation (2.1) over all vertices in $V(S_{T,L})$. Values at interior mid-edges disappear and we arrive at

$$0 = -\sum_{z\in\alpha} F(z) + \sum_{z\in\beta} F(z) + j\sum_{z\in\varepsilon} F(z) + \bar{j}\sum_{z\in\bar{\varepsilon}} F(z), \tag{2.4}$$

where $j = e^{2i\pi/3}$ again. Using the symmetry mirror image of the domain, we deduce that $F(\bar{z}) = \bar{F}(z)$, where $\bar{z}$ is the symmetric of z with respect to the real axis. Observe that the winding of any self-avoiding walk from a to the bottom part of α is $-\pi$ while the winding to the top part is π. We conclude

$$\begin{aligned}\sum_{z\in\alpha} F(z) &= F(a) + \sum_{z\in\alpha\setminus\{a\}} F(z) = 1 + \frac{e^{-i\sigma\pi} + e^{i\sigma\pi}}{2} A^x_{T,L} \\ &= 1 - \cos\left(\frac{3\pi}{8}\right) A^x_{T,L} = 1 - c_\alpha A^x_{T,L}.\end{aligned}$$

Above, we have used the fact that the only walk from a to a is of length 0. Similarly, the winding from a to any half-edge in β (resp. ε and $\bar{\varepsilon}$) is 0 (resp. $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$), therefore

$$\sum_{z\in\beta} F(z) = B^x_{T,L} \quad \text{and} \quad j\sum_{z\in\varepsilon} F(z) + \bar{j}\sum_{z\in\bar{\varepsilon}} F(z) = \cos\left(\frac{\pi}{4}\right) E^x_{T,L} = c_\varepsilon E^x_{T,L}.$$

The lemma follows readily by plugging these three formulæ in (2.4). □

Observe that sequences $(A^x_{T,L})_{L>0}$ and $(B^x_{T,L})_{L>0}$ are increasing in L and are bounded for $x \le x_c$ thanks to (2.3) and the monotonicity in x. Thus, they have limits

$$\begin{aligned} A^x_T &:= \lim_{L\to\infty} A^x_{T,L} = \sum_{\gamma\subset S_T:\, a\to\alpha\setminus\{a\}} x^{\ell(\gamma)}, \\ B^x_T &:= \lim_{L\to\infty} B^x_{T,L} = \sum_{\gamma\subset S_T:\, a\to\beta} x^{\ell(\gamma)}. \end{aligned}$$

When $x = x_c$, via (2.3) again, we conclude that $(E^{x_c}_{T,L})_{L>0}$ decreases and converges to a limit $E^{x_c}_T := \lim_{L\to\infty} E^{x_c}_{T,L}$. Then, (2.3) implies

$$1 = c_\alpha A^{x_c}_T + B^{x_c}_T + c_\varepsilon E^{x_c}_T. \tag{2.5}$$

Proof of Theorem 2.2. Let us first prove that $Z(x_c) = +\infty$, which implies $\mu \ge \sqrt{2+\sqrt{2}}$. Suppose that for some T, $E^{x_c}_T > 0$. As noted before, $E^{x_c}_{T,L}$ decreases in L and therefore

$$Z(x_c) \ge \sum_{L>0} E^{x_c}_{T,L} \ge \sum_{L>0} E^{x_c}_T = +\infty,$$

which completes the proof. Assume on the contrary that $E_T^{x_c} = 0$, then (2.5) simplifies to

$$1 = c_\alpha A_T^{x_c} + B_T^{x_c}. \tag{2.6}$$

Observe that walks entering into account for $A_{T+1}^{x_c}$ and not for $A_T^{x_c}$ have to visit some vertex adjacent to the right edge of S_{T+1}. Cutting such a walk at the first such point (and adding half-edges to the two halves), we obtain two walks "crossing" S_{T+1} (these walks are sometimes called bridges). We conclude that

$$A_{T+1}^{x_c} - A_T^{x_c} \le x_c \left(B_{T+1}^{x_c}\right)^2. \tag{2.7}$$

Combining (2.6) for T and $T+1$ with (2.7), we can write

$$\begin{aligned} 0 = 1 - 1 &= \left(c_\alpha A_{T+1}^{x_c} + B_{T+1}^{x_c}\right) - \left(c_\alpha A_T^{x_c} + B_T^{x_c}\right) \\ &= c_\alpha\left(A_{T+1}^{x_c} - A_T^{x_c}\right) + B_{T+1}^{x_c} - B_T^{x_c} \\ &\le c_\alpha x_c \left(B_{T+1}^{x_c}\right)^2 + B_{T+1}^{x_c} - B_T^{x_c}, \end{aligned}$$

so

$$c_\alpha x_c \left(B_{T+1}^{x_c}\right)^2 + B_{T+1}^{x_c} \ge B_T^{x_c}.$$

By induction, it is easy to check that

$$B_T^{x_c} \ge \frac{\min[B_1^{x_c}, 1/(c_\alpha x_c)]}{T}$$

for every $T \ge 1$. This implies

$$Z(x_c) \ge \sum_{T>0} B_T^{x_c} = +\infty.$$

This completes the proof of the inequality $\mu \ge x_c^{-1} = \sqrt{2+\sqrt{2}}$.

Let us turn to the other needed inequality $\mu \le x_c^{-1}$. First of all, let us restrict our attention to self-avoiding walks starting and ending at the mid-edge of an horizontal edge. Let $\tilde{Z}(x)$ be the partition function of such walks. The reader may easily check that the number of self-avoiding walks of length n starting and ending at a mid-edge of an horizontal edge is within a bounded (in n) multiplicative factor of c_n, and that therefore $Z(x) < \infty$ if and only if $\tilde{Z}(x) < \infty$. A *bridge* of width T is a self-avoiding walk in S_T from one side to the opposite side, defined up to vertical translation. The partition function of bridges of width T is exactly B_T^x. Using (2.5), we can bound $B_T^{x_c}$ by 1. Noting that a bridge of width T has length at least T, we obtain for $x < x_c$

$$B_T^x \le \left(\frac{x}{x_c}\right)^T B_T^{x_c} \le \left(\frac{x}{x_c}\right)^T.$$

Let us now state the following lemma.

Lemma 2.8. *For any $x > 0$,* $\tilde{Z}(x) \le \frac{1}{x^3} \prod_{T \ge 0} (1 + xB_T^x)^2.$

Before proving this lemma, let us conclude the proof of the theorem. The series $\sum_{T\ge0} B_T^x$ converges whenever $x < x_c$ and so does the product $\prod_{T\ge0}(1 + xB_T^x)$. This implies that $\tilde{Z}(x) < \infty$ and thus $\mu \le x_c^{-1} = \sqrt{2+\sqrt{2}}$. In conclusion, we only need to prove the previous lemma in order to finish the proof of the theorem. □

Proof of the lemma. Let us show that a self-avoiding walk starting and ending at the mid-edge of an horizontal edge can be canonically divided into two so-called "half-space walks", each of which can be decomposed into bridges in a canonical way. This decomposition was first introduced in the case of the square lattice by Hammersley and Welsh in [HW62] (for a modern treatment, see [MS93, Section 3.1] or [BDCGS12]).

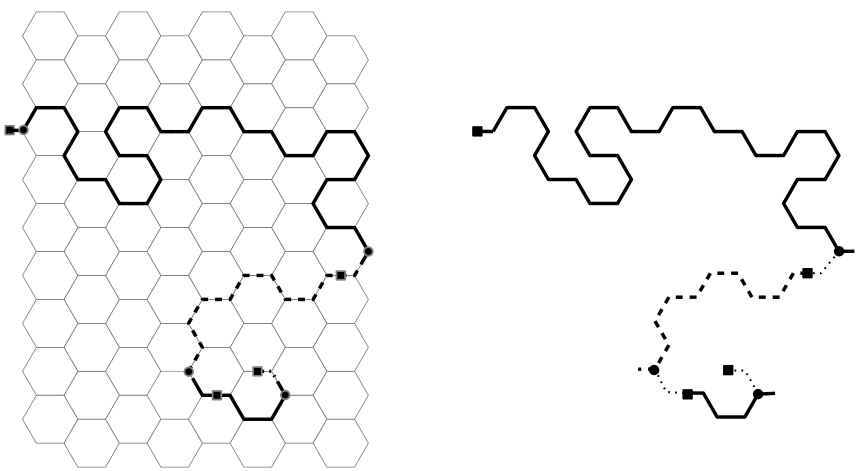


Figure 2.4: **Left:** Decomposition of a half-plane walk into four bridges with widths $8 > 3 > 1 > 0$. The first bridge corresponds to the maximal bridge containing the origin. Note that the decomposition contains one bridge of width 0 (the walk corresponding to the decomposition without this last bridge would not contain the last dotted steps). **Right:** The reverse procedure. If the starting mid-edge and the first vertex are fixed, the decomposition is unambiguous.

First assume that the first coordinate of the starting mid-edge of γ is extremal. We prove by induction on the width that the walk admits a canonical decomposition into bridges of widths $T_0 > \cdots > T_j$. Without loss of generality, assume that the starting mid-edge is minimal. If γ is a bridge, the decomposition is the walk itself. If this is not the case, proceed as follows. Out of the vertices visited by the walk and having maximal first coordinate, choose the one visited last, say after n steps. The walk up to this vertex together with the additional mid-edge on its right form a bridge $\tilde{\gamma}_1$ (of width T_0). This bridge is the first bridge of our decomposition. Erase $\tilde{\gamma}_1$ and the part of the walk between the n-th vertex and the mid-edge between the $(n+1)$-th and $(n+2)$-th vertices (the removed piece is composed of an edge plus a half-edge). The starting mid-edge of the walk composed of the consequent steps corresponds to an horizontal edge and has now maximum first coordinate among remaining mid-edges. Note that the width T_1 of the remaining walk is strictly smaller than T_0. Using the induction hypothesis, we obtain a decomposition of this new walk into bridges of widths $T_1 > \cdots > T_j$. The decomposition of γ is created by adding $\tilde{\gamma}_1$ to this decomposition.

Let us make two important observations:

- The sum of the lengths of bridges in the decomposition of γ is equal to the length of γ minus j (recall that some steps of the walks are deleted in-between bridges).
- Assume that the starting mid-edge of γ is minimal. The walk is determined by its decomposition. Indeed, there is a natural reconstruction procedure defined inductively as follows, see Fig. 2.4. Consider the first bridge in the decomposition. If there is only this bridge in the decomposition, the walk is simply this bridge. If there is more than one bridge in the decomposition, modify the end of the first bridge as follows. Let v be the last vertex visited by the bridge. By definition, the bridge passes through two mid-edges p and q before and after v, and q is the middle of an horizontal edge. Remove this mid-edge and replace it by the third mid-edge r around v (it corresponds to turning by $\frac{2\pi}{3}$ or $-\frac{2\pi}{3}$ the last half-edge around the last vertex). Then, add to this walk an additional step from r to the middle of the next horizontal edge (there is only one way of doing so if we want the walk to remain self-avoiding). We then concatenate the second bridge to the end of this new walk. If there is no bridge left, we have found γ. Otherwise, we modify the end of the walk like we did for the first bridge, and we concatenate the next bridge, and so on. We proceed like that until we reach the last bridge. The final concatenation gives us γ.

Let $H(x)$ be the generating function of *half-space walks*, i.e. of self-avoiding walks whose starting mid-edge has minimal first coordinate. The two observations above have the following consequence:

$$H(x) \leq \sum_{j=0}^{\infty} \Big(x^j \sum_{T_0 > T_1 > \cdots > T_j} B_{T_0} B_{T_1} \dots B_{T_j} \Big) = \frac{1}{x} \prod_{T \geq 0} (1 + x B_T^x).$$

Now, a self-avoiding walk in the plane can be divided into two pieces γ_1 and γ_2 by cutting at the first vertex v of minimal first coordinate. Add to γ_1 and γ_2 the additional mid-edge on the left of v. The two walks γ_2 and γ_1 are respectively a half-space walk and the time-reversal of one. We deduce that

$$\tilde{Z}(x) \le \tfrac{1}{x} H(x)^2 \le \tfrac{1}{x^3} \prod_{T\ge 0} (1 + x B_t^x)^2$$

and we are done. □

Remark 2.9. There are variations on the observable used above. For instance, let $x, y > 0$ and introduce the observable

$$G(z) = G(a, z, x, y, \sigma) = \sum_{\gamma \subset S_{T,L}:\ a \to z} \mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)} x^{\ell(\gamma)} y^{n(\gamma)}$$

where $n(\gamma)$ is the number of visits of γ to the right boundary of $S_{T,L}$. This observable can be used to show that the so-called critical fugacity y_c for surface adsorption of self-avoiding walks on the hexagonal lattice equals $1 + \sqrt{2}$. Let us add a few words on this. Assume that the weight of a half-space walk of length n is not uniform but proportional to $y^{n(\gamma)}$ where $n(\gamma)$ is the number of intersections with the y-axis. Then,

- For $y > y_c$, there exists $c = c(y) > 0$ such that the probability that a random half-space walk of length n visits the y-axis more than cn times tends to 1 as n tends to infinity;
- For $y < y_c$ and for any $\varepsilon > 0$, the probability that the random half-space walk of length n visits the y-axis more than εn times tends to zero exponentially fast in n.

We refer to [BBMDC$^+$12] for more details.

Conclusion. The proof presented above involves several interesting ingredients to which we will come back in other parts of this book. Let us highlight them one more time.

1. A function expressed in terms of the self-avoiding walk model is defined in finite subgraphs of the lattice. This function satisfies exact local relations at some specific "integrable point" (here $x = x_c$).

2. While the local relations do not determine the function explicitly (in the sense that the boundary conditions are not sufficient to reconstruct the function inside the domain), they are still sufficient to understand the average boundary behavior.

3. This behavior enables us to exhibit a specific set of properties satisfied at $x = x_c$.

4. A completely different argument shows that these properties can only be satisfied at the critical point of the model, thus identifying x_c as this critical point.

We will see variations around this strategy. The goal of the program will often be different, and therefore Step 4 will be very different. In some cases, Step 3 will be much more evolved, while in other cases, Step 1 will be vastly improved, thus leading to a strong notion of discrete holomorphicity.

Chapter 3

Notation and definitions for the graphs

We work with subsets of the plane. Points will therefore be considered as elements of $\mathbb{R}^2$ as well as elements of $\mathbb{C}$ depending on the context. Seeing points as complex numbers will have its advantages. For instance, an oriented edge of the medial lattice naturally gives rise to a complex number. The distance between two points x and y will be measured by the complex modulus $|x-y|$. Equivalently, it corresponds to the Euclidean norm on $\mathbb{R}^2$. The distance between a point x and a closed set F is defined by $d(x,F) := \inf\{|x-y| : y \in F\}$.

3.1 Primal, dual and medial lattices

We will work with subgraphs of the following lattices, see Fig. 3.1.

- The *square lattice* $(\mathbb{Z}^2, \mathbb{E})$ is the graph with vertex set $\mathbb{Z}^2 = \{(n,m) : n,m \in \mathbb{Z}\}$ and edge set $\mathbb{E}$ given by edges between nearest neighbors. The square lattice will be identified with the set of vertices, i.e. $\mathbb{Z}^2$.
- The *dual square lattice* $(\mathbb{Z}^2)^\star$ is the dual graph of $\mathbb{Z}^2$. The vertex set is $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ and the edges are given by nearest neighbors. The vertices and edges of $(\mathbb{Z}^2)^\star$ are called *dual-vertices* and *dual-edges*. In particular, every edge e of $\mathbb{Z}^2$ is naturally associated to a dual-edge, denoted by $e^\star$, that it crosses in its center.
- The *medial lattice* $(\mathbb{Z}^2)^\diamond$ is the graph with the centers of edges of $\mathbb{Z}^2$ as vertex set, and edges connecting nearest vertices. This lattice is a rotated and rescaled (by a factor $1/\sqrt{2}$) version of $\mathbb{Z}^2$. The vertices and edges of $(\mathbb{Z}^2)^\diamond$ are called *medial-vertices* and *medial-edges*. We will often identify the faces of $(\mathbb{Z}^2)^\diamond$ with the vertices

of $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^\star$. For instance, we may speak of the medial-edge bordering a vertex or a dual-vertex: by this, we mean bordering the face of $(\mathbb{Z}^2)^\diamond$ associated to this vertex or dual-vertex. A face of the medial lattice is said to be *black* if it corresponds to a vertex of $\mathbb{Z}^2$, and *white* otherwise. Edges of $(\mathbb{Z}^2)^\diamond$ are oriented counterclockwise around black faces, so that the medial lattice can sometimes be seen as an oriented graph.

We will only consider subgraphs of $\mathbb{Z}^2$, $(\mathbb{Z}^2)^\star$ or $(\mathbb{Z}^2)^\diamond$, and use the following notations. For a graph G, we denote by V_G its vertex set and by E_G its edge set. If we do not need to put a special emphasis on V_G, we will simply write G instead of V_G. An edge e of G with endpoints x and y is denoted by $[xy]$; we then say that x and y are *neighbors (in G)* and write $x \sim y$. Furthermore, if x is an end-point of e, we say that e is *incident* to x. Finally, the *boundary* of G, denoted by ∂G, is the set of vertices of G with strictly fewer than four incident edges in E_G.

Let Λ_n be the subgraph of $\mathbb{Z}^2$ induced by $[-n,n]^2$.

3.2 Discrete domains

In this section and the next one, we encourage the reader to look at pictures as much as possible in order to get a better intuition of the definitions.

Discrete domains provide a discrete analog of simply connected open sets. They are defined as follows.

Consider a sequence $\partial = \{v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_{n-1} \sim v_n \sim v_0\}$ of neighboring medial-edges satisfying the following conditions:

- The path ∂ is *edge-avoiding*, i.e. it does not use the same medial-edge twice.
- The path follows the orientation of the medial lattice, i.e. $[v_i v_{i+1}]$ is oriented from v_i to v_{i+1}.
- The corresponding oriented path is going counterclockwise.

We do not assume that all the end-points of edges of ∂ are distinct: the path may visit the same medial-vertex twice. Nonetheless, the path is necessarily non-self-crossing since it follows the orientation of the medial lattice. Also observe that the orientation of the lattice determines an interior and an exterior.

Definition 3.1 (medial discrete domain)**.** Let ∂ be a path as above, and $\Omega^\diamond$ the subgraph of $(\mathbb{Z}^2)^\diamond$ induced by the medial-vertices that are either enclosed by ∂, or endpoints of an edge of ∂. Such a graph $\Omega^\diamond$ is said to be a *medial discrete domain*, see Fig. 3.2.

The graph $\Omega^\diamond$ is necessarily connected as a subgraph of $(\mathbb{Z}^2)^\diamond$.

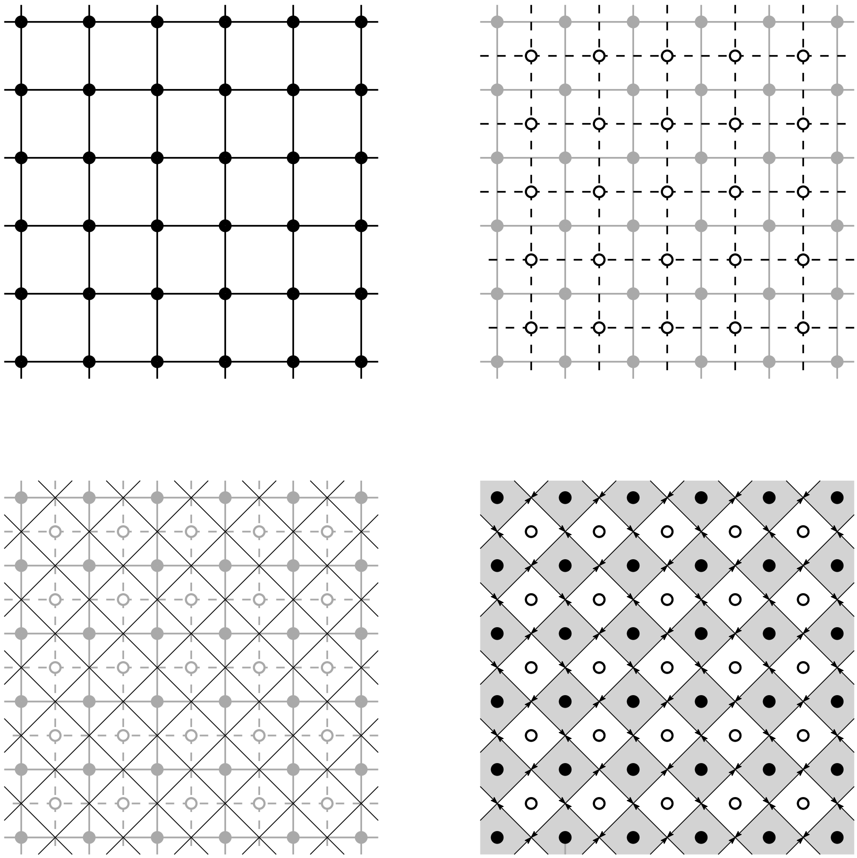


Figure 3.1: **Top left.** Square lattice. By convention, edges will always be drawn with plain lines. **Top right.** Dual of the square lattice. By convention, dual-edges will always be drawn in dashed lines. **Bottom left.** Medial lattice. **Bottom right.** Medial lattice with the orientation on edges. The faces corresponding to vertices of $\mathbb{Z}^2$ are gray, the others are white.

Remark 3.2. The graph $\Omega^\diamond$ may be seen as a closed subset F of the plane by taking the union of the medial faces enclosed by ∂. Some points are *pinched points* (i.e. that removing them disconnects the set). We may see $\Omega^\diamond$ as a simply connected domain of the plane by taking the interior of the union of F and small balls of radius $\varepsilon \ll 1$ around the pinched points, see Fig. 3.3.

Pinched points are medial-vertices of ∂ visited twice by the path. In the reverse direction, medial-vertices visited twice are not necessarily pinched

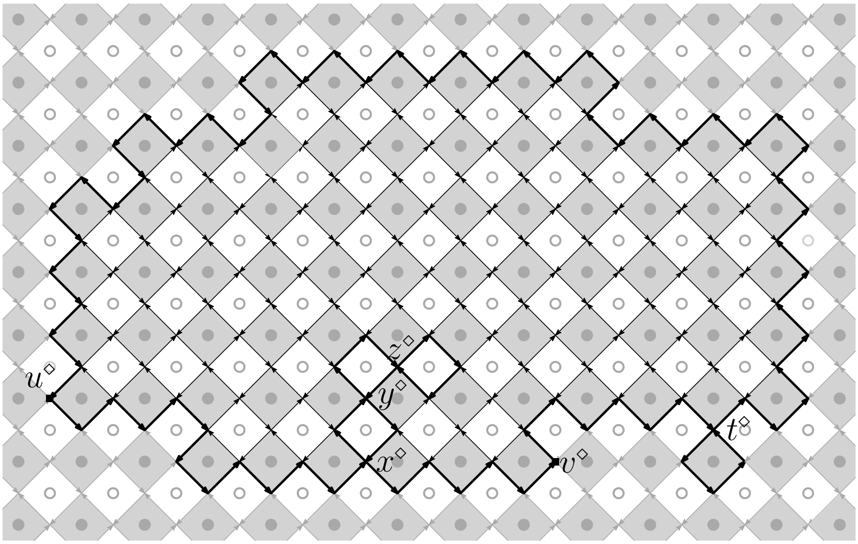


Figure 3.2: A medial discrete domain with two marked points on its boundary. The medial-vertices $x^\diamond$, $y^\diamond$ and $z^\diamond$ provide examples of medial-vertices visited twice that correspond to two prime-ends. The medial-vertex $t^\diamond$ is another example of medial-vertex visited twice. This one corresponds to a pinched point of the domain.

points. They can on the contrary correspond to "two points" of the boundary. It will be important to distinguish between these two points and we do so as follows. Consider the conformal map from the unit disk onto the open domain enclosed by ∂. Then, two different points of the unit circle may be mapped to a single medial-vertex of ∂ visited twice. In such case, the medial-vertex corresponds to two distinct *prime ends* in the standard complex analysis sense. Therefore, instead of considering such a medial-vertex as one medial-vertex of degree 4, it is natural to consider it as two distinct prime ends of degree 2. *In particular, these vertices belong to* $\partial\Omega^\diamond$.

In conclusion, medial-vertices of ∂ visited twice are either pinched points or medial-vertices corresponding to two prime-ends.

Remark 3.3. Since the boundary $\partial\Omega^\diamond$ of $\Omega^\diamond$ is the set of medial-vertices having less than four incident medial-edges in $E_{\Omega^\diamond}$, it corresponds to only roughly half of the end-points of ∂.

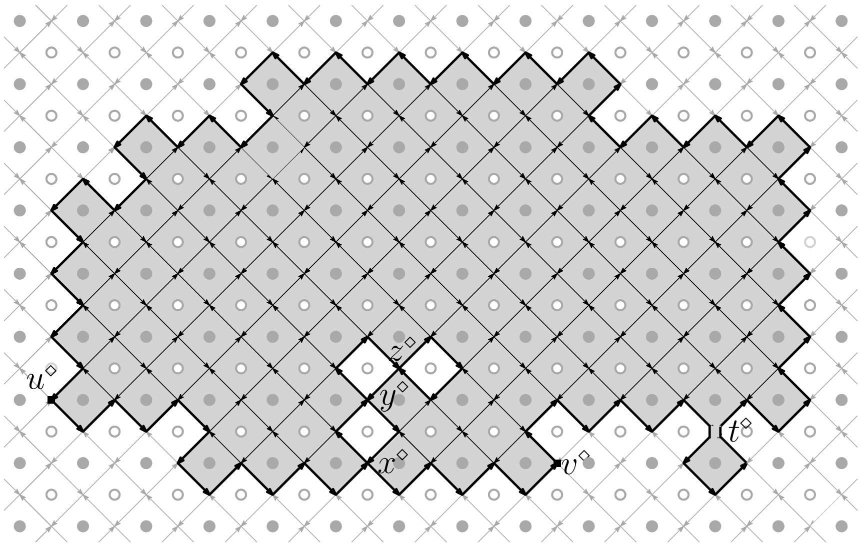


Figure 3.3: One example of the open domain associated to a medial discrete domain. Observe the additional small patch near $t^\diamond$, which guarantees that the domain is simply connected.

Definition 3.4 (primal and dual discrete domains)**.** Let $\Omega^\diamond$ be a medial discrete domain. Let Ω be the subgraph of $\mathbb{Z}^2$ with edge-set given by edges corresponding to medial-vertices of $\Omega^\diamond \setminus \partial\Omega^\diamond$[1], and vertex-set given by the end-points of these edges. The graph Ω is said to be a *primal discrete domain*, see Fig. 3.4. Let $\Omega^\star$ be the subgraph of $(\mathbb{Z}^2)^\star$ with edge-set given by dual-edges corresponding to medial-vertices of $\Omega^\diamond$ and vertex-set given by the end-points of these dual-edges. The graph $\Omega^\star$ is said to be a *dual discrete domain*, see Fig. 3.4.

The notations Ω, $\Omega^\star$ and $\Omega^\diamond$ will always refer to graphs that are primal, dual and medial discrete domains, respectively. We drop the reference to primal, dual and medial, and simply speak of *discrete domain.*

3.3 Dobrushin domains

Dobrushin domains are introduced to provide a discrete analogue of simply connected domains with two marked points on their boundary. They are

[1]Recall that doubly-visited vertices are either pinched points, in which case they do not belong to $\partial\Omega^\diamond$, or two prime-ends, in such case they do belong to the boundary, and therefore primal edges going through such medial-vertices are not included.

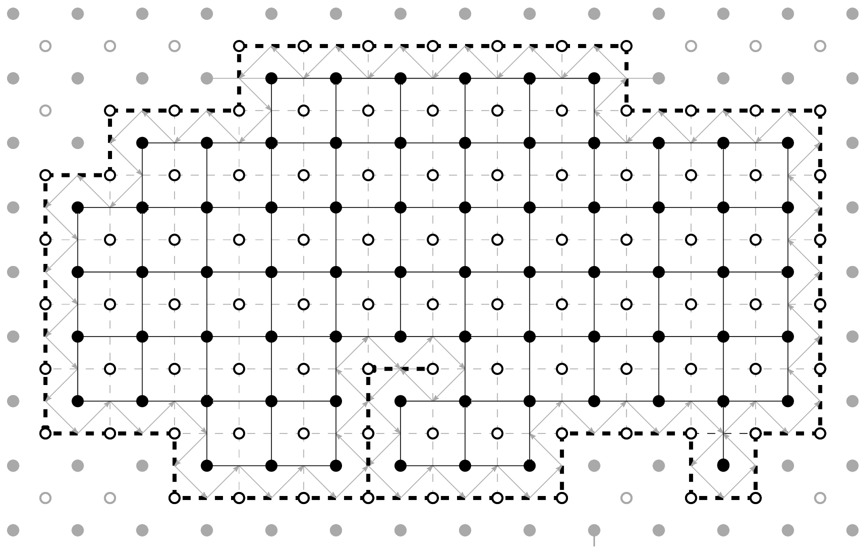


Figure 3.4: The primal and dual discrete domains associated to the medial discrete domain drawn in Fig. 3.2.

defined as follows.

Let $a^\diamond$ and $b^\diamond$ be two distinct medial-vertices, and $\partial^\diamond_{ab} = \{v_0 \sim v_1 \sim \dots \sim v_n\}$, $\partial^\diamond_{ba} = \{w_0 \sim w_1 \sim \dots \sim w_m\}$ two paths of neighboring medial-vertices satisfying the following properties:

- The paths start from $a^\diamond$ and end at $b^\diamond$, i.e. $v_0 = w_0 = a^\diamond$ and $v_n = w_m = b^\diamond$.
- The paths follow the orientation of the medial lattice.
- The path $\partial^\diamond_{ab}$ goes counterclockwise, while $\partial^\diamond_{ba}$ goes clockwise.
- The paths are edge-avoiding.
- The paths intersect only at $a^\diamond$ and $b^\diamond$.

Note that $\partial^\diamond_{ab} \cup \partial^\diamond_{ba}$ is a non-self crossing edge-avoiding polygon. However, some vertices might be visited twice.

Definition 3.5 (medial Dobrushin domains)**.** Let $\partial^\diamond_{ab}$ and $\partial^\diamond_{ba}$ be two paths as above, and let $\Omega^\diamond$ be the subgraph of $(\mathbb{Z}^2)^\diamond$ induced by the medial-vertices that are enclosed by or in the path $\partial^\diamond_{ab} \cup \partial^\diamond_{ba}$. Then, $(\Omega^\diamond, a^\diamond, b^\diamond)$ is called a *medial Dobrushin domain.* An example is given in Fig. 3.5.

Remark 3.6. As before, the medial Dobrushin domains give rise to a simply connected domain of the plane by taking the union of faces plus small balls around medial vertices of $\partial^\diamond_{ab} \cup \partial^\diamond_{ba}$ corresponding to pinched

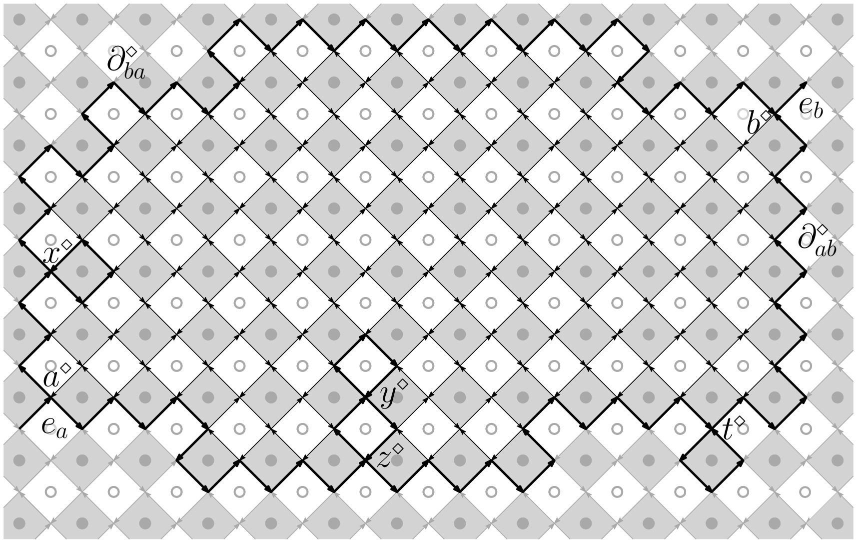


Figure 3.5: A medial Dobrushin domain. Note the position of e_a and e_b.

points of the graph. As before, medial-vertices may be considered as prime-ends.

As it stands, $a^\diamond$ and $b^\diamond$ have three incident medial-edges in $E_{\Omega^\diamond}$. Call e_a and e_b the fourth medial-edges incident to $a^\diamond$ and $b^\diamond$ respectively. In what follows, we will consider that e_a and e_b are also in $E_{\Omega^\diamond}$, but we will not add in $V_{\Omega^\diamond}$ the end-points of e_a and e_b which are not $a^\diamond$ and $b^\diamond$. Therefore, e_a and e_b have only one end-point in $\Omega^\diamond$.

Remark 3.7. Once again, the boundary $\partial\Omega^\diamond$ of $\Omega^\diamond$ does not coincide with all the elements of $\partial^\diamond_{ab}$ and $\partial^\diamond_{ba}$ but only with roughly half of them. Moreover, $a^\diamond$ and $b^\diamond$ do not belong to $\partial\Omega^\diamond$ since with e_a and e_b respectively, they possess four incident medial-edges.

We are now in a position to define the (primal) and (dual) Dobrushin domains, see Fig. 3.6.

Definition 3.8 (primal and dual Dobrushin domains with two marked points)**.** Let $(\Omega^\diamond, a^\diamond, b^\diamond)$ be a medial Dobrushin domain.

Let $\Omega \subset \mathbb{Z}^2$ be the graph with edge set composed of edges passing through end-points of medial-edges in $E_{\Omega^\diamond} \setminus \partial^\diamond_{ab}$ (if a medial-vertex is the end-point of a medial-edge in $E_{\Omega^\diamond} \setminus \partial^\diamond_{ab}$ and one in $\partial^\diamond_{ab}$, it is included) and vertex set given by the end-points of these edges. Let a and b be the two vertices of Ω bordered by e_a and e_b. The triplet (Ω, a, b) is called a *primal Dobrushin*

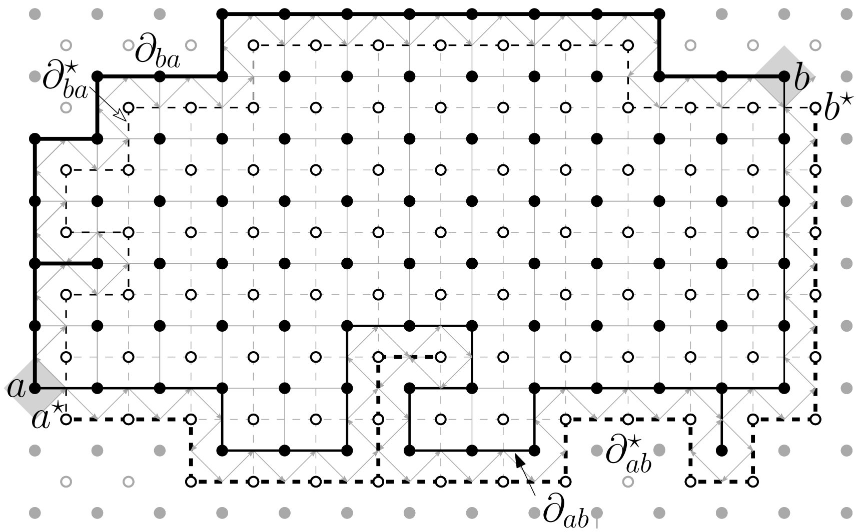


Figure 3.6: The primal and dual Dobrushin domains associated to a medial Dobrushin domain. Note the position of a, $a^\star$, b and $b^\star$.

domain. We denote by ∂_{ba} the set of edges corresponding to medial-vertices in $\partial\Omega^\diamond$, including doubly-visited ones, which are also end-points of medial-edges in $\partial^\diamond_{ba}$, and set $\partial_{ab} = \partial\Omega \smallsetminus \partial_{ba}$.

Let $\Omega^\star \subset (\mathbb{Z}^2)^\star$ be the graph with edge set composed of dual-edges passing through medial-edges in $E_{\Omega^\diamond} \smallsetminus \partial^\diamond_{ba}$ and vertex set given by the end-points of these dual-edges. Let $a^\star$ and $b^\star$ be the two dual-vertices of $\Omega^\star$ bordered by e_a and e_b. The triplet $(\Omega^\star, a^\star, b^\star)$ is called a *dual Dobrushin domain.* We denote by $\partial^\star_{ab}$ the set of dual-edges corresponding to medial-vertices in $\partial\Omega^\diamond$, including doubly-visited ones, which are also end-points of medial-edges in $\partial^\diamond_{ab}$, and set $\partial^\star_{ba} = \partial\Omega^\star \smallsetminus \partial^\star_{ab}$.

The notations (Ω, a, b), $(\Omega^\star, a^\star, b^\star)$ and $(\Omega^\diamond, a^\diamond, b^\diamond)$ will always refer to a primal, dual and medial Dobrushin domain, respectively. The three triplets are in direct correspondence. We will forget about the reference to primal, dual and medial in the future and speak of *Dobrushin domain.*

Remark 3.9. We will often consider discrete domains $\Omega^\diamond$ with two marked points $u^\diamond$ and $v^\diamond$ on $\partial\Omega^\diamond$ (see for instance Fig. 3.2). Note that this graph is not a Dobrushin domain with two marked points on the boundary. Indeed, $u^\diamond$ and $v^\diamond$ have degree 2 (meaning that two medial-edges are incident to them), while $a^\diamond$ and $b^\diamond$ have degree 4 in the case of a Dobrushin domain. Furthermore, ∂ is oriented counter-clockwise, while $\partial^\diamond_{ab}$ and $\partial^\diamond_{ba}$

are oriented from $a^\diamond$ to $b^\diamond$. Conversely, when forgetting about $a^\diamond$ and $b^\diamond$ in a Dobrushin domain $(\Omega^\diamond, a^\diamond, b^\diamond)$, the graph $\Omega^\diamond$ is not a discrete domain for the same reason.

3.4 Discretizations of domains

We will be interested in finer and finer graphs approximating continuous domains. For $\delta > 0$, we consider the rescaled square lattice $\delta\mathbb{Z}^2$. The definitions of dual and medial Dobrushin domains extend to this context. Note that the medial lattice $(\delta\mathbb{Z}^2)^\diamond$ has mesh-size $\delta/\sqrt{2}$.

Generically, discrete domains on $\delta\mathbb{Z}^2$, $(\delta\mathbb{Z}^2)^\star$ and $(\delta\mathbb{Z}^2)^\diamond$ will be denoted by Ω_δ, $\Omega_\delta^\star$ and $\Omega_\delta^\diamond$. Similarly, Dobrushin domains on $\delta\mathbb{Z}^2$, $(\delta\mathbb{Z}^2)^\star$ and $(\delta\mathbb{Z}^2)^\diamond$ will be denoted by $(\Omega_\delta, a_\delta, b_\delta)$, $(\Omega_\delta^\star, a_\delta^\star, b_\delta^\star)$ and $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$.

We wish to speak of discrete domains and Dobrushin domains approximating a continuous domain with marked points on its boundary (by a marked point, we mean a marked prime end). In order to quantify how close a discrete graph is to its continuum counterpart, we introduce the notion of *Carathéodory convergence*. Consider a discrete domain $\Omega_\delta^\diamond$ or a Dobrushin domain $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$ as a simply connected domain as explained previously (in this case, the small additional balls are of size $\varepsilon \ll \delta$). Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half-plane.

Definition 3.10. Let (Ω, a, b) be a simply connected domain with two marked points on its boundary. Consider a sequence of Dobrushin domains (or discrete domains with two marked points on the boundary) $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$. We say that $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$ converges to (Ω, a, b) in the Carathéodory sense if

$$f_\delta \longrightarrow f \qquad \text{on any compact subset } K \subset \mathbb{H},$$

where f_δ (resp. f) is the unique conformal map from $\mathbb{H}$ to $\Omega_\delta^\diamond$ (resp. Ω) satisfying $f_\delta(0) = a_\delta^\diamond$, $f_\delta(\infty) = b_\delta^\diamond$ and $f_\delta'(\infty) = 1$ (resp. $f(0) = a$, $f(\infty) = b$ and $f'(\infty) = 1$).

Let us mention that this notion of convergence coincides with the Haussdorff convergence in the case of smooth domains. Therefore, one may simply think of the Carathéodory convergence as being a very natural notion of convergence and not bother with details. For sufficiently smooth domains, a possible example of a converging sequence is provided by $\Omega_\delta = \Omega \cap (\delta\mathbb{Z}^2)$ with a_δ and b_δ being the closest vertices of Ω_δ to a and b, but we may consider more general approximations.

We will be considering sequences of functions on Ω_δ (more precisely V_{Ω_δ}) for δ going to 0 and we wish to speak of uniform convergence on every compact subset of Ω. In order to do this, we implicitly perform the following operation: for a function f on Ω_δ, choose a diagonal for every (square) face and extend the function to the faces of Ω_δ in a piecewise linear way on the two triangles made of the diagonal and two edges. Since no confusion will be possible, the extension will be denoted by f as well. Constructed like that, the function is not necessarily defined on all of Ω (since the union of faces of Ω_δ may be different from Ω), nevertheless, it is defined on any compact subset of Ω provided that δ is small enough (how small it must be depends on the compact subset). The same procedure will also be applied to functions defined on $\Omega_\delta^\star$ and $\Omega_\delta^\diamond$.

Remark 3.11. The Carathéodory convergence implies that for every compact subset K of Ω, there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$, $K \subset \Omega_\delta$. Therefore, it is possible to speak of uniform convergence on compact subsets of Ω for any sequence of functions (f_δ) defined on a sequence of domains Ω_δ converging to Ω.

Part II

Random-cluster models and parafermionic observables

Chapter 4

Basic properties of the two-dimensional random-cluster model

Before diving into the study of the random-cluster model, and in particular the Russo-Seymour-Welsh theory and the applications of parafermionic observables, we need to gather several classical properties on this model. For additional information, we refer to the extensive literature on the subject, e.g. to [Gri06].

4.1 Formal definition of the random-cluster model

The random-cluster model can be defined on any graph. However, we restrict ourselves to the square lattice. Let G be a finite subgraph of $\mathbb{Z}^2$. A *configuration* ω is an element of $\{0,1\}^{E_G}$. An edge e with $\omega(e) = 1$ is said to be *open*, otherwise it is said to be *closed*. The configuration ω can be seen as a subgraph of G, with vertex set V_G and edge set $\{e \in E_G : \omega(e) = 1\}$.

Two sites a and b are said to be *connected* if there is an *open path*, *i.e.* a path $a = v_0, \dots, v_n = b$ with $e_i = (v_i, v_{i+1}) \in E_G$ and $\omega(e_i) = 1$ for any $0 \le i < n$ (this event will be denoted by $a \leftrightarrow b$). Two sets A and B are *connected* if there exists an open path connecting $a \in A$ and $b \in B$ (denoted $A \leftrightarrow B$). The maximal connected components of ω are called *clusters*.

Boundary conditions ξ are given by a partition $P_1 \sqcup \dots \sqcup P_k$ of ∂G. Two vertices are *wired in* ξ if they belong to the same P_i. The graph obtained from the configuration ω by identifying the wired vertices together is denoted by ω^ξ [1]. Boundary conditions should be understood informally as encoding how sites are connected outside of G. Let $o(\omega)$ and $c(\omega)$ denote the number of open and closed edges of ω and $k(\omega^\xi)$ the number of connected components of the graph ω^ξ.

Definition 4.1. The probability measure $\phi^\xi_{p,q,G}$ of the random-cluster model on G with *edge-weight* $p \in [0,1]$ and *cluster-weight* $q > 0$ and boundary conditions ξ is defined by

$$\phi^\xi_{p,q,G}(\{\omega\}) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega^\xi)}}{Z^\xi_{p,q,G}} \tag{4.1}$$

for every configuration ω on G. The constant $Z^\xi_{p,q,G}$ is a normalizing constant, referred to as the *partition function*, defined in such a way that the sum over all configurations equals 1. From now on, $\phi^\xi_{p,q,G}$ denotes the measure and the expectation with respect to this measure.

Four boundary conditions play a special role in the study of the random-cluster model.

- The *wired* boundary conditions: they are specified by the fact that all the vertices on the boundary are pairwise wired (the partition is equal to $\{\partial G\}$). The random-cluster measure with wired boundary conditions on G is denoted by $\phi^1_{p,q,G}$.
- The *free* boundary conditions: they are specified by no wiring between vertices on the boundary (the partition is composed of singletons only). The random-cluster measure with free boundary conditions on G is denoted by $\phi^0_{p,q,G}$.
- The *periodic* boundary conditions: for $n \geq 1$, the torus $\mathbb{T}_n$ of size n can be seen as a subgraph of $\mathbb{Z}^2$ with specific boundary conditions as follows. Consider the subgraph of $\mathbb{Z}^2$ induced by the vertex set $\{0,\dots,n\}^2$, with the boundary conditions obtained by wiring $(i,0)$ and (i,n) for every $i \in \{0,\dots,n\}$, and $(0,j)$ and (n,j) for every $j \in \{0,\dots,n\}$. The random-cluster measure on the torus of size n is denoted by $\phi^{\mathrm{per}}_{p,q,n}$.
- The *Dobrushin* boundary conditions, or *domain-wall* boundary conditions: let (Ω,a,b) be a discrete Dobrushin domain. The Dobrushin boundary conditions are defined to be free on ∂_{ab} and

[1] Formally, ω^ξ can be seen as $(\tilde V,\tilde E)$, where $\tilde V$ is the vertex set V_G quotiented by the equivalence relation $x\mathcal{R}y$ if x and y are in the same P_i, and $\tilde E$ is the image of the open edges of ω by the canonical projection from E_{V_G} to $E_{\tilde V}$. We will not really use this formal definition.

wired on ∂_{ba} (in other words, the partition is composed of ∂_{ba} together with singletons). These arcs are referred to as the *free arc* and the *wired arc*, respectively. The measure associated to these boundary conditions will be denoted by $\phi^{a,b}_{p,q,\Omega}$.

Other boundary conditions will come from a configuration outside the graph G (see the next paragraph). For a configuration ξ on $\mathbb{E} \setminus E_G$, the boundary conditions *induced* by ξ are defined by the partition $P_1 \sqcup \cdots \sqcup P_k$, where x and y are in the same P_i if and only if there exists a path in ξ connecting x and y. In general, we identify the boundary conditions induced by ξ with the configuration itself, and we denote the random-cluster measure by $\phi^{\xi}_{p,q,G}$.

4.2 Finite energy and Domain Markov properties

4.2.1 The domain Markov property

The following theorem describes how the influence of the configuration outside a graph F on the measure within F can be encoded using appropriate boundary conditions ξ. This property is the analog of the Dobrushin-Lanford-Ruelle property for Gibbs measures (see [Geo11] or Proposition 7.4 for the Ising case). It will be useful when decoupling events depending on disjoint sets of edges.

Let $\mathcal{F}_E$ be the σ-algebra of events depending on the states of edges in E only.

Theorem 4.2. *Let $p \in [0,1]$, $q > 0$ and ξ some boundary conditions. Fix $F \subset G$ two finite subgraphs of $\mathbb{Z}^2$. For any $\mathcal{F}_{E_F}$-measurable random variable X,*

$$\phi^{\xi}_{p,q,G}(X|\omega(e) = \psi(e), \forall e \in E_G \setminus E_F)(\psi) = \phi^{\psi^{\xi}}_{p,q,F}(X),$$

where $\psi \in \{0,1\}^{E_G \setminus E_F}$ (recall the definition of ψ^{ξ} from the previous section).

From now on, we set $\omega_{|E}$ to denote the restriction of ω to edges in E only.

Proof. Let us deal with the case $F = (V_G, E_G \setminus \{e\})$. Let ω be a configuration on G and set ω^e to be the configuration on G equal to 1 on e and ω elsewhere. We find

$$\begin{aligned}\phi^{\xi}_{p,q,G}(\omega_{|E_F}|\omega(e)=1) &= \frac{\phi^{\xi}_{p,q,G}(\omega^e)}{\phi^{\xi}_{p,q,G}(\omega(e)=1)}\\ &= \frac{\dfrac{p^{o(\omega_{|E_F})+1}(1-p)^{c(\omega_{|E_F})}q^{k((\omega^e)^{\xi})}}{Z^{\xi}_{p,q,G}}}{\dfrac{\sum\limits_{\tilde\omega\in\{0,1\}^{E_G}:\tilde\omega(e)=1} p^{o(\tilde\omega)}(1-p)^{c(\tilde\omega)}q^{k(\tilde\omega^{\xi})}}{Z^{\xi}_{p,q,G}}}\\ &= \frac{p^{o(\omega_{|E_F})+1}(1-p)^{c(\omega_{|E_F})}q^{k(\omega^{\xi^e}_{|E_F})}}{\sum\limits_{\tilde\omega_{|E_F}\in\{0,1\}^{E_F}} p^{o(\tilde\omega_{|E_F})+1}(1-p)^{c(\tilde\omega_{|E_F})}q^{k(\tilde\omega^{\xi^e}_{|E_F})}}\\ &= \phi^{\xi^e}_{p,q,F}(\omega_{|E_F})\end{aligned}$$

where ξ^e is given by the boundary conditions ξ with the two end-points of e wired together. Similarly

$$\phi^{\xi}_{p,q,G}(\omega_{|E_F}|\omega(e)=0)=\phi^{\xi}_{p,q,F}(\omega_{|E_F})$$

and the claim follows easily for F. The result can be deduced for every random variable X by linearity. Now, one can repeat the previous reasoning recursively and the result follows for any subgraph F of G. $\square$

Example. Let $F \subset G$ with $E_G \setminus E_F$ connected. Then if $\psi(e) = 1$ for any $e \notin E_F$, ψ^{ξ} corresponds to wired boundary conditions. Similarly, if $\psi(e) = 0$ for any $e \notin E_F$, ψ^{ξ} corresponds to free boundary conditions. This justifies the notations 0 and 1 for the free and wired boundary conditions.

4.2.2 Finite energy property

The finite energy property roughly yields that the conditional probability for an edge to be open knowing the states of all the other edges is bounded away from 0 and 1 uniformly in $p \in [\varepsilon, 1-\varepsilon]$ (here $\varepsilon \in (0,1/2)$ is fixed) and in the states of all the other edges. This fact extends to any finite family of edges. This property is a useful tool when comparing the probabilities of two configurations differing by only finitely many edges.

Lemma 4.3. *Let $\varepsilon \in (0,1/2)$, $q > 0$ and $F \subset G$ two finite graphs. There exists $c = c(\varepsilon, q, F) > 0$ such that for any $p \in [\varepsilon, 1-\varepsilon]$, any configuration $\psi \in \{0,1\}^{E_F}$ and any boundary conditions ξ,*

$$c \le \phi^{\xi}_{p,q,G}(\omega_{|F} = \psi) \le 1 - c.$$

Of course, the control on $c = c(\varepsilon, q, F)$ deteriorates exponentially fast in the size of F.

Proof. Let us first prove that

$$c < \phi^{\xi}_{p,q,G}(\omega(e) = 1) < 1 - c$$

uniformly on G and $p \in [\varepsilon, 1-\varepsilon]$. In order to see this, observe that conditionally on the states of the edges different from e, the two only boundary conditions on the graph composed solely of the edge e and its end-points are either the free boundary conditions (the two end-points of e are not connected by the configuration outside e), or the wired ones (the two end-points are connected outside e). In the first case, the probability is equal to $p/[p + (1-p)q]$, in the second one, it is equal to p.

The claim follows readily by successive applications of the domain Markov property and by setting $c = \min\{\varepsilon, \varepsilon/[\varepsilon q + (1-\varepsilon)]\}^{|E_G|}$ (the power $|E_G|$ comes from these successive applications). □

Proposition 4.4. *Let $\varepsilon \in (0,1/2)$, $q > 0$ and $F \subset G$ two finite graphs. There exists $c = c(\varepsilon, q, F) > 0$ such that for any event A depending on edges outside F and any $\eta \in \{0,1\}^{E_F}$,*

$$\phi^{\xi}_{p,q,G}(A \cap \{\omega_{|E_F} = \eta\}) \ge c\phi^{\xi}_{p,q,G}(A).$$

Proof. Let $c = c(\varepsilon, q, F) > 0$ given by the previous lemma. We obtain

$$\begin{aligned}
&{\phi_{p,q,G}}^{\xi}(A \cap \{\omega_{|E_F} = \eta\}) \\
&\qquad = \sum_{\psi \in \{0,1\}^{E_G \setminus E_F}} \phi^{\xi}_{p,q,G}(\{\omega_{|E_G \setminus E_F} = \psi\} \cap \{\omega_{|E_F} = \eta\} \cap A) \\
&\qquad = \sum_{\psi \in \{0,1\}^{E_G \setminus E_F}} \phi^{\xi}_{p,q,G}(\{\omega_{|E_G \setminus E_F} = \psi\} \cap A)\phi^{\xi}_{p,q,G}(\omega_{|E_F} = \eta | \omega_{|E_G \setminus E_F} = \psi) \\
&\qquad = \sum_{\psi \in \{0,1\}^{E_G \setminus E_F}} \phi^{\xi}_{p,q,G}(\{\omega_{|E_G \setminus E_F} = \psi\} \cap A)\phi^{\xi^{\psi}}_{p,q,F}(\eta) \\
&\qquad \ge c \sum_{\psi \in \{0,1\}^{E_G \setminus E_F}} \phi^{\xi}_{p,q,G}(\{\omega_{|E_G \setminus E_F} = \psi\} \cap A) = \phi^{\xi}_{p,q,G}(A).
\end{aligned}$$

In the second line, we drop the condition on A in the conditioning since this would be redundant: if ψ is not in A, then the first term equals 0 anyway. □

Remark 4.5. A typical example of a model not satisfying the finite energy property is the uniform measure on spanning trees. Indeed, knowing the whole configuration outside an edge e, it is not necessarily possible for e to be open (for instance if there is a cycle once e is open).

4.3 Planar duality

The planar random-cluster model enjoys a very interesting property called *planar duality.* The next sections are devoted to this notion but before, let us introduce a useful involution on edge-weights. For any p and q, define

$$p^\star = p^\star(p,q) := \frac{(1-p)q}{(1-p)q+p}, \tag{4.2}$$

and the *self-dual point* $p_{sd} = p_{sd}(q)$ as being the unique solution of the equation $p^\star(p_{sd},q) = p_{sd}$, *i.e.*

$$p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{4.3}$$

4.3.1 Planar duality for planar boundary conditions

Let G be a finite graph. We start by discussing planarity for planar boundary conditions on G, i.e. boundary conditions coming from a planar configuration outside G. In this section, let $G^\star$ be the graph with edge-set $E^\star = \{e^\star : e \in E_G\}$ and vertex-set given by the end-points of $E^\star$.

Definition 4.6. For a configuration $\omega \in \{0,1\}^{E_G}$, we define the *dual configuration* $\omega^\star \in \{0,1\}^{E_{G^\star}}$ by the formula

$$\omega^\star(e^\star) = 1 - \omega(e), \quad \forall e \in E_G.$$

A dual-edge $e^\star$ is said to be *dual-open* if $\omega^\star(e^\star) = 1$ and *dual-closed* otherwise. Two sites u and v in $G^\star$ are said to be *dual-connected* if there is a *dual-open path*, *i.e.* a path of open dual-edges between u and v. Two sets U and V are *dual-connected* if there exists a dual-open path connecting $u \in U$ and $v \in V$. These events are denoted by $u \overset{\star}{\longleftrightarrow} v$ and $U \overset{\star}{\longleftrightarrow} V$ respectively. The maximal dual-connected components will be called *dual-clusters.*

Proposition 4.7 (Duality for planar boundary conditions)**.** *Let $\xi \in \{0,1\}^{\mathbb{E}\setminus E_G}$. The dual model of the random-cluster on G with parameters (p,q) and boundary conditions ξ is the random-cluster with parameters $(p^\star,q)$ on $G^\star$ with boundary conditions induced by $\xi^\star$, where $\xi^\star$ is defined by $\xi^\star(e^\star) = 1 - \xi(e)$ for any $e \notin E_G$.*

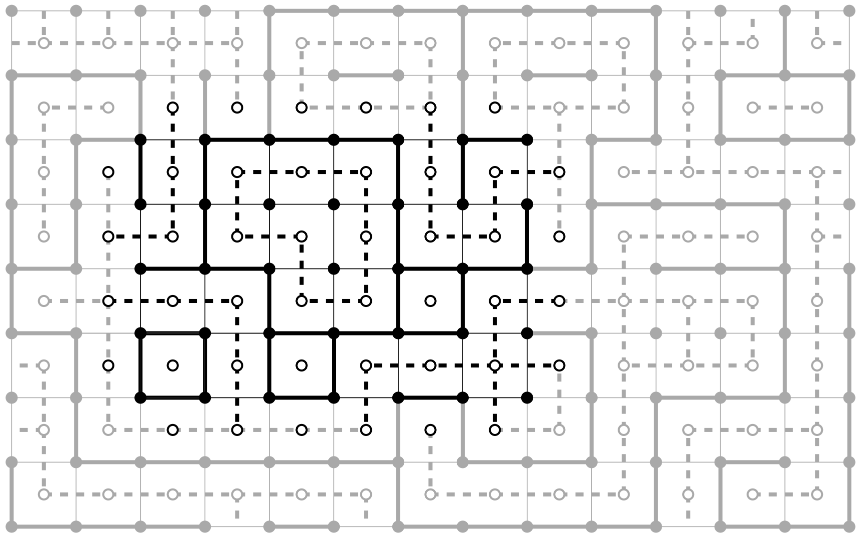


Figure 4.1: A configuration and its dual configuration. The graphs G and $G^\star$ are in black and ξ and $\xi^\star$ are in gray.

Proof. Recall that the graph ω^ξ is obtained from ω by identifying the vertices of ∂G which are connected in ξ. Since $\xi \in \{0,1\}^{\mathbb{E}\setminus E_G}$, the boundary conditions ξ are planar in the sense that ω^ξ is planar. Let $(\omega^\xi)^\star$ be the dual of ω^ξ in the standard graph theoretical sense, meaning that the vertices correspond to faces of ω^ξ, and the edges connect vertices associated to adjacent faces. Then, $(\omega^\xi)^\star$ is equal to $(\omega^\star)^{\xi^\star}$ (in fact, sometimes it possesses an additional isolated vertex but this will be irrelevant here).

Let $f(\omega^\xi)$ be the number of faces in ω^ξ. Using Euler's formula applied to ω^ξ, we find that

$$|E_{\omega^\xi}| \;+\; k(\omega^\xi) \;+\; 1 \;=\; |V_{\omega^\xi}| \;+\; f(\omega^\xi).$$

Recall that $V_{\omega^\xi} = V_{G^\xi}$ does not depend on ω. From the definition of the dual configuration $\omega^\star$ of ω, we have $o(\omega)+o(\omega^\star) = |E_G|$, where $o(\omega^\star)$ is the number of open dual-edges, and therefore $|E_{\omega^\xi}|+o(\omega^\star) = o(\omega)+o(\omega^\star)$ does not depend on ω. Moreover, connected components of $(\omega^\star)^{\xi^\star}$ correspond exactly to faces of ω^ξ, so that $f(\omega^\xi) = k((\omega^\star)^{\xi^\star})$. Overall, there exists a constant C_ξ not depending on ω such that

$$k(\omega^\xi) \quad = \quad C_\xi + k((\omega^\star)^{\xi^\star}) + o(\omega^\star).$$

The probability of $\omega^\star$ is equal to $\phi^\xi_{p,q,G}(\omega)$ which can be rewritten in terms of $\omega^\star$ as follows:

$$\begin{aligned}
\phi^{\xi}_{p,q,G}(\omega) &= \frac{1}{Z^{\xi}_{p,q,G}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^{\xi})} \\
&= \frac{(1-p)^{|E_G|}}{Z^{\xi}_{p,q,G}} [p/(1-p)]^{o(\omega)} q^{k(\omega^{\xi})} \\
&= \frac{(1-p)^{|E_G|}}{Z^{\xi}_{p,q,G}} [p/(1-p)]^{|E_G|-o(\omega^\star)} q^{C_\xi + k((\omega^\star)^{\xi^\star}) + o(\omega^\star)} \\
&= \frac{p^{|E_G|} q^{C_\xi}}{Z^{\xi}_{p,q,G}} [q(1-p)/p]^{o(\omega^\star)} q^{k((\omega^\star)^{\xi^\star})} \\
&= \frac{p^{|E_G|} q^{C_\xi}}{Z^{\xi}_{p,q,G}} [p^\star/(1-p^\star)]^{o(\omega^\star)} q^{k((\omega^\star)^{\xi^\star})} \\
&= \frac{p^{|E_G|} q^{C_\xi}}{(1-p^\star)^{|E_{G^\star}|} Z^{\xi}_{p,q,G}} (p^\star)^{o(\omega^\star)} (1-p^\star)^{c(\omega^\star)} q^{k((\omega^\star)^{\xi^\star})} \\
&= \phi^{\xi^\star}_{p^\star,q,G^\star}(\omega^\star).
\end{aligned}$$

In the third and sixth lines, we used the relation $o(\omega^\star)+o(\omega) = |E_{G^\star}| = |E_G|$. The Euler formula was invoked in the third line, and the relation between p and $p^\star$ in the fifth. □

Let us provide a few examples of dual boundary conditions.

Example 1 (free boundary conditions). Let Ω be a discrete domain. In such case, $\xi = 0$ and therefore $\xi^\star = 1$. We obtain wired boundary conditions on the dual graph $\Omega^\star$ (defined as in the introduction).

Example 2 (wired boundary conditions). Let Ω be a discrete domain. Note that the state of edges between vertices of $\partial\Omega$ is not relevant for wired boundary conditions. Indeed, they are open with probability p and closed with probability $1-p$ (since their end-points are connected anyway), and furthermore the connectivity properties of the configuration are not altered by the states of these edges. For this reason, we will usually assume that these edges are open. In such case ξ can be chosen to be all 1 and therefore $\xi^\star = 0$. The dual of wired boundary conditions is therefore free boundary conditions. With our convention, it is defined on the dual graph of $\Omega \setminus \partial\Omega$, which is slightly smaller than $\Omega^\star$.

Example 3 (Dobrushin boundary conditions). Let (Ω, a, b) be a Dobrushin domain. The Dobrushin boundary conditions are achieved by taking ξ to be 0 everywhere, except on ∂_{ba} for which it is 1. Then, $\xi^\star$ is equal to 0 on $\{e^\star : e \in \partial_{ba}\}$ and 1 everywhere else. Equivalently, the boundary conditions induced by $\xi^\star$ correspond to the boundary conditions induced by the configuration equal to 1 on $\partial^\star_{ab}$ and 0 everywhere else. Therefore the dual of Dobrushin boundary conditions on (Ω, a, b) are Dobrushin boundary conditions on $(\Omega^\star, b^\star, a^\star)$.

Example 4 (mixed boundary conditions). Let Ω be a discrete domain with four marked points a, b, c and d found in counter-clockwise order on its boundary. These points determine arcs ∂_{ab}, ∂_{bc}, ∂_{cd} and ∂_{da} when going around the boundary in the counter-clockwise order. The boundary conditions *mix* are wired on ∂_{ab} and ∂_{cd}, and free elsewhere. Note that ∂_{ab} and ∂_{cd} are not wired together. Then, the dual boundary conditions are wired on $\partial^\star_{bc} \cup \partial^\star_{da}$ (this time the two arcs are wired together), and free elsewhere, where the dual arcs are defined in a similar way to Dobrushin domains.

Remark 4.8. Often, most of the configurations ξ (or equivalently $\xi^\star$) will not be necessary to determine which partition on ∂G is associated to the configuration. In such case, we do not keep track of the whole configuration in $\mathbb{E}$, but only of some subset of the edges. For instance, if any edge outside G with one end-point on ∂G is closed, then the associated boundary conditions are necessarily free.

4.3.2 Duality for periodic boundary conditions

The case of periodic boundary conditions, or equivalently the case of the random-cluster model defined on a torus (with no boundary conditions) is a little more involved: its dual is *not* a random-cluster model because its boundary conditions cannot be achieved by a planar configuration. Yet, the dual model is not very different from a random-cluster model, and that will be enough for our purposes.

In order to state duality in this case, additional notations expressed in terms of the geometry of the torus are required. Let $f(\omega)$ be the number of faces delimited by ω, *i.e.* the number of connected components of the complement of the set of open edges. We also introduce an additional parameter $\delta(\omega)$. Call a cluster of ω a *net* if it contains two non-contractible simple loops of different homotopy classes, and a *cycle* if it is non-contractible but is not a net. Notice that every configuration ω can be of one of three types:

- One of the clusters of ω is a net. Then no other cluster of ω can be a net or a cycle. In that case, let $\delta(\omega) = 2$;

- One of the clusters of ω is a cycle. Then no other cluster can be a net, but other clusters can be cycles as well (in which case all the involved, simple loops are in the same homotopy class). Then let $\delta(\omega) = 1$;

- None of the clusters of ω is a net or a cycle. Let $\delta(\omega) = 0$.

With this additional notation, Euler's formula becomes

$$|V_{\mathbb{T}_n}| + f(\omega) + \delta(\omega) = k(\omega) + 1 + o(\omega). \tag{4.4}$$

Besides, these terms transform in a simple way under duality: $o(\omega)+o(\omega^\star)$ is a constant, $f(\omega) = k(\omega^\star)$ and $\delta(\omega) = 2 - \delta(\omega^\star)$.

Define the *balanced random-cluster model* with weights

$$\tilde{\phi}^{\mathrm{per}}_{p,q,n}(\{\omega\}) = \sqrt{q}^{\,1-\delta(\omega)} \cdot \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}}{\widehat{Z}^{\mathrm{per}}_{p,q,n}},$$

where $\widehat{Z}^{\mathrm{per}}_{p,q,n}$ is a normalizing constant defined in such a way that the sum over all configurations equals 1. The same proof as that of usual duality implies the following result.

Proposition 4.9. *The dual model of the balanced random-cluster on the torus $\mathbb{T}_n$ of size n with parameters (p,q) and periodic boundary conditions is the balanced random-cluster with parameters $(p^\star, q)$ on $\mathbb{T}_n^\star$.*

This means that even though the dual model of the random-cluster model with periodic boundary conditions is not exactly a random-cluster model at the dual parameter, it is absolutely continuous with respect to a self-dual model and the Radon-Nikodym derivative is bounded from above and below by constants depending only on q. More precisely, the partition functions of both models differ by a multiplicative factor at most q, while the denominators involved in the expressions of the probabilities also differ by a multiplicative factor at most q. Overall, the Radon-Nikodym derivative is therefore between $1/q^2$ and q^2.

4.4 Strong positive association when $q \geq 1$

A partial order can be naturally defined on $\{0,1\}^{E_G}$: $\omega \leq \omega'$ if and only if for any $e \in E_G$, $\omega(e) \leq \omega'(e)$. A function $f : \{0,1\}^{E_G} \longrightarrow \mathbb{R}$ is *increasing* if it is increasing for this order. An event A is *increasing* if $\mathbf{1}_A$ is increasing, which is equivalent to the fact that $\omega \in A$ and $\omega \leq \omega'$ implies $\omega' \in A$. (In words, an event is increasing if it is preserved by addition of open edges.) Let us give several examples.

Example 1. $\omega(e)$ is an increasing function.

Example 2. The number of open edges in $E \subset E_G$ is an increasing function.

Example 3. The event that two sets of vertices A and B are connected by an open path is increasing (even if the open paths are constrained to use edges in $E \subset E_G$ only).

Example 4. The event that there exists a cluster of radius larger than n is an increasing event.

The class of increasing events is central in the study of random-cluster models because of the so-called positive association of the model. We will present this link in detail in the next section, but let us first introduce the central notion of stochastic domination.

Definition 4.10. A measure μ_1 *stochastically dominates* μ_2 if for every increasing event A, $\mu_1(A) \geq \mu_2(A)$.

4.4.1 Holley criterion and FKG inequality

Let us start by discussing stochastic ordering and correlation inequalities for general probability measures on $\{0,1\}^{E_G}$. Define $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ by the formulæ

$$(\omega_1 \vee \omega_2)(e) = \max\{\omega_1(e), \omega_2(e)\} \text{ and } (\omega_1 \wedge \omega_2)(e) = \min\{\omega_1(e), \omega_2(e)\}.$$

Theorem 4.11 (Holley inequality [Hol74])**.** *Let μ_1, μ_2 be two positive measures on $\{0,1\}^{E_G}$ such that*

$$\mu_1(\omega_1 \vee \omega_2)\mu_2(\omega_1 \wedge \omega_2) \;\geq\; \mu_1(\omega_1)\mu_2(\omega_2), \quad \omega_1, \omega_2 \in \{0,1\}^{E_G}, \tag{4.5}$$

then μ_1 stochastically dominates μ_2.

We sketch the proof here. For a configuration ω and an edge e, define the configurations ω^e and ω_e as the two configurations coinciding with ω for $f \neq e$, and equal to respectively 1 and 0 at e. Formally,

$$\omega^e(f) = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 1 & \text{if } f = e, \end{cases} \qquad \omega_e(f) = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 0 & \text{if } f = e. \end{cases}$$

(We already used ω^e in the proof of Theorem 4.2.)

Proof. Let us construct a coupling $\mathbb{P}$ with marginals μ_1 and μ_2 in such a way that $\mathbb{P}(\omega_1 \geq \omega_2) = 1$. The result will follow from the fact that

$$\mu_2(A) = \mathbb{P}(\omega_2 \in A) = \mathbb{P}(\omega_2 \in A \text{ and } \omega_1 \geq \omega_2) \leq \mathbb{P}(\omega_1 \in A) = \mu_1(A).$$

In the inequality, we used the fact that A is increasing.

An efficient way of constructing the measure $\mathbb{P}$ is to consider a Markov chain whose stationary measure is $\mathbb{P}$. Since we want the two marginals to be μ_1 and μ_2, we need the induced (marginal) Markov chains to have stationary measures μ_1 and μ_2. This strongly suggests the following Markov chain: consider the infinitesimal generator on

$$\Omega = \Big\{(\omega, \eta) \in \{0,1\}^{E_G} \times \{0,1\}^{E_G} : \omega \geq \eta\Big\}$$

defined by

$$\begin{cases} H(\omega,\eta_e,\omega^e,\eta^e) & = 1, \\ H(\omega^e,\eta,\omega_e,\eta_e) & = \dfrac{\mu_1(\omega_e)}{\mu_1(\omega^e)}, \\ H(\omega^e,\eta^e,\omega^e,\eta_e) & = \dfrac{\mu_2(\eta_e)}{\mu_2(\eta^e)} - \dfrac{\mu_1(\omega_e)}{\mu_1(\omega^e)}. \end{cases}$$

It is easy to check that the continuous time Markov chain with infinitesimal generator is aperiodic and thus possesses a unique stationary measure $\mathbb{P}$. Now, when starting from the configuration (ω,η) with $\omega(e) = 1$ and $\eta(e) = 0$ for any $e \in E_G$, every step preserves the inequality $\omega \geq \eta$ and so $\mathbb{P}(\omega \geq \eta) = 1$. Finally, the Markov chains induced on the first and second coordinates can be checked to have stationary measures μ_1 and μ_2. All these facts together imply the proof readily. □

Theorem 4.11 possesses an elegant simplification: (4.5) does not need to be checked for every configurations ω_1 and ω_2. It is in fact sufficient to check that for any ω and $e \neq f$,

$$\mu_1(\omega^e)\mu_2(\omega_e) \ \geq \ \mu_1(\omega_e)\mu_2(\omega^e) \tag{4.6}$$

$$\text{and} \quad \mu_1(\omega^{ef})\mu_2(\omega_{ef}) \ \geq \ \mu_1(\omega_e^f)\mu_2(\omega_f^e), \tag{4.7}$$

where $\omega_{ef} = (\omega_e)_f$, $\omega_e^f = (\omega_e)^f$ and $\omega^{ef} = (\omega^e)^f$. We refer to [Gri06, Theorem 2.3] for more details on the reduction of the general Holley criterion to this simpler inequalities.

The Holley criterion is particularly suitable to prove the famous Fortuin-Kasteleyn-Ginibre inequality [FKG71] (FKG inequality in short). First proved by Harris in the case of product measures (in this case, it is called Harris inequality), the inequality relates the probability of the intersection of two increasing events to the product of the probabilities.

Theorem 4.12 (FKG lattice condition). *Let G be a finite graph and μ be a positive measure on $\{0,1\}^{E_G}$. If for any configuration ω and $e \neq f \in E_G$*

$$\mu(\omega^{ef})\mu(\omega_{ef}) \ \geq \ \mu(\omega_e^f)\mu(\omega_f^e), \tag{4.8}$$

then for any increasing events A and B,

$$\mu(A \cap B) \ \geq \ \mu(A)\mu(B). \tag{4.9}$$

Proof. Equation (4.9) can be understood as $\mu(\,\cdot\,|B)$ stochastically dominating μ. Let us check Holley criterion (more precisely inequalities (4.6) and (4.7)) for these two measures. Fix ω as well as $e \neq f$. Let us start by the inequality (4.6). We have

$$\mathbf{1}_{\omega^e \in B}\ \mu(\omega^e)\mu(\omega_e) \ \geq \ \mathbf{1}_{\omega_e \in B}\ \mu(\omega_e)\mu(\omega^e)$$

since the indicator function on the left is equal to 1 if the one on the right is equal to 1 (recall that B is increasing). Dividing by $\mu(B)$, we get

$$\mu(\omega^e|B)\mu(\omega_e) \;\geq\; \mu(\omega_e|B)\mu(\omega^e)$$

which is (4.6). We now focus on (4.7). We obtain

$$\mathbf{1}_{\omega^{ef}\in B}\;\mu(\omega^{ef})\mu(\omega_{ef}) \;\geq\; \mathbf{1}_{\omega_e^f\in B}\;\mu(\omega_e^f)\mu(\omega_f^e).$$

We used that $\mathbf{1}_{\omega^{ef}\in B}\geq\mathbf{1}_{\omega_e^f\in B}$ (because B is increasing) and (4.8). Dividing once again by $\mu(B)$ gives us (4.7). □

Remark 4.13. By taking complements, the inequality $\mu(A\cap B)\geq\mu(A)\mu(B)$ holds for decreasing events. Similarly, if A is increasing and B is decreasing, then $\mu(A\cap B)\leq\mu(A)\mu(B)$. The theorem above also implies that $\mu(XY)\geq\mu(X)\mu(Y)$ for any two increasing (resp. decreasing) random variables X,Y. Indeed, by adding constants one may prove this result for positive random variables only. Now, observe that $\{X>t\}$ and $\{Y>s\}$ are increasing for any $s,t>0$. Since $X=\int_0^\infty\mathbf{1}_{\{X>t\}}dt$ and $Y=\int_0^\infty\mathbf{1}_{\{Y>s\}}ds$, the FKG inequality applied in the second line gives

$$\begin{aligned}\mu(XY)&=\int_0^\infty\int_0^\infty\mu(\{X>t\}\cap\{Y>s\})dtds\\&\geq\int_0^\infty\int_0^\infty\mu(X>t)\mu(Y>s)dtds\\&=\Big(\int_0^\infty\mu(X>t)dt\Big)\Big(\int_0^\infty\mu(Y>s)ds\Big)=\mu(X)\mu(Y).\end{aligned}$$

4.4.2 The FKG inequality for random-cluster models

The technology developed in the previous section enables us to prove the following fundamental inequality.

Theorem 4.14 (Fortuin-Kasteleyn-Ginibre inequality [FKG71])**.** *Let $p\in[0,1]$, $q\geq1$, and boundary conditions ξ. For any two increasing events A and B,*

$$\phi^\xi_{p,q,G}(A\cap B)\;\geq\;\phi^\xi_{p,q,G}(A)\phi^\xi_{p,q,G}(B).\tag{4.10}$$

Remark 4.15. Beware of the fact that q is required to be larger than or equal to 1: the result is false when $q<1$. It is in fact natural to conjecture that negative association would hold whenever $q<1$ (uniform spanning trees can be obtained as limits of the random-cluster model with q going to 0 and thus provide one example of negatively correlated random-cluster model with $q<1$). The theorem of negative association is not so understood. We refer to [Pem00] for details on the subject.

Proof. Let us check the condition (4.8). Fix a configuration ω and two edges $e \neq f$. By multiplying by partition functions, we need to prove that

$$p^{o(\omega^{ef})+o(\omega_{ef})}(1-p)^{o(\omega^{ef})+o(\omega_{ef})}q^{k(\omega^{ef})+k(\omega_{ef})}$$
$$\geq p^{o(\omega_e^f)+o(\omega_f^e)}(1-p)^{o(\omega_e^f)+o(\omega_f^e)}q^{k(\omega_e^f)+k(\omega_f^e)}.$$

The terms involving p and $(1-p)$ do not create any difficulty since $o(\omega^{ef}) + o(\omega_{ef}) = o(\omega_e^f) + o(\omega_f^e)$ and $c(\omega^{ef}) + c(\omega_{ef}) = c(\omega_e^f) + c(\omega_f^e)$. Recalling that $q \geq 1$, we only need to check that $k(\omega^{ef}) + k(\omega_{ef}) \geq k(\omega_e^f) + k(\omega_f^e)$. This inequality follows by studying whether both endpoints of f are already connected or not in $\omega_{|E_G \setminus \{e,f\}}$. □

Remark 4.16 (Square-root trick)**.** Let us mention an important implication of the FKG inequality, called the *square-root trick*. Let A and B be two increasing events. Then,

$$\max\left\{\phi^\xi_{p,q,G}(A), \phi^\xi_{p,q,G}(B)\right\} \geq 1 - (1-\phi^\xi_{p,q,G}(A\cup B))^{1/2}. \tag{4.11}$$

This trick will be very useful since (4.11) improves the standard bound

$$\max\left\{\phi^\xi_{p,q,G}(A), \phi^\xi_{p,q,G}(B)\right\} \geq \tfrac{1}{2}\phi^\xi_{p,q,G}(A\cup B).$$

We now turn to two important applications of the FKG inequality.

4.4.3 Stochastic ordering of random-cluster measures

Corollary 4.17 (Comparison in $\boldsymbol{p}$)**.** *Fix boundary conditions ξ and $q \geq 1$. For any $p_1 \leq p_2$, $\phi^\xi_{p_2,q,G}$ stochastically dominates $\phi^\xi_{p_1,q,G}$.*

This corollary legitimates the intuition that the larger p is, the more edges are open.

Proof. An easy computation implies the existence of $K > 0$ such that for every random variable X,

$$\phi^\xi_{p_2,q,G}(X) = \phi^\xi_{p_1,q,G}(XY)/K$$

where

$$Y(\omega) = \left(\frac{p_2/(1-p_2)}{p_1/(1-p_1)}\right)^{o(\omega)}.$$

Plugging $X = 1$, we find $K = \phi^\xi_{p_1,q,G}(Y)$. Now, $Y \geq 0$ is increasing (recall that $p_1 \leq p_2$), so that if X is also assumed to be increasing, the FKG inequality implies

$$\phi^\xi_{p_2,q,G}(X) = \phi^\xi_{p_1,q,G}(XY)/\phi^\xi_{p_1,q,G}(Y) \geq \phi^\xi_{p_1,q,G}(X).$$

□

Remark 4.18. The previous result about stochastic ordering between random-cluster measures can in fact be extended as follows [Gri06, Theorem (3.21)]: if $q_1 \le q_2$ and $\frac{p_1}{q_1(1-p_1)} \le \frac{p_2}{q_2(1-p_2)}$, then $\phi^{\xi}_{p_1,q_1,G}$ is stochastically dominated by $\phi^{\xi}_{p_2,q_2,G}$.

4.4.4 Comparison between boundary conditions

Let ξ and ψ be two boundary conditions. We say that $\xi \le \psi$ if any two vertices wired in ξ are wired in ψ. In other words, the partition corresponding to ψ is coarser than the one corresponding to ξ.

Corollary 4.19 (Comparison between boundary conditions)**.** *Fix $p \in [0,1]$ and $q \ge 1$. For any boundary conditions $\xi \le \psi$ and any increasing event A,*

$$\phi^{\xi}_{p,q,G}(A) \;\le\; \phi^{\psi}_{p,q,G}(A). \tag{4.12}$$

Proof. Consider boundary conditions ψ corresponding to the partition $V_1 \sqcup \cdots \sqcup V_k$ and construct a new graph by adding, for each i, edges between vertices of V_i. Call this new graph G_0 and the set of additional edges E (we have $E = E_{G_0} \smallsetminus E_G$). Now, the domain Markov property implies

$$\begin{aligned}\phi^{\xi}_{p,q,G}(\cdot) &= \phi^{\xi}_{p,q,G_0}(\cdot\,|\omega(e) = 0, \forall e \in E)\\ \phi^{\psi}_{p,q,G}(\cdot) &= \phi^{\xi}_{p,q,G_0}(\cdot\,|\omega(e) = 1, \forall e \in E).\end{aligned}$$

Using the FKG inequality twice, we obtain

$$\phi^{\xi}_{p,q,G}(A) \;\le\; \phi^{\xi}_{p,q,G_0}(A) \;\le\; \phi^{\psi}_{p,q,G}(A)$$

for any increasing event A depending on edges in E_G only. $\square$

Let us take some time to discuss in detail some applications of the comparison between boundary conditions. We start by the easiest observation and we go towards harder and harder corollaries, adding more and more steps in the reasoning. These arguments will be used repeatedly in the book. The easiest corollary is the following.

Corollary 4.20. *Let $p \in [0,1]$, $q \ge 1$ and G a finite graph. For any increasing event A and any boundary conditions ξ,*

$$\phi^{0}_{p,q,G}(A) \;\le\; \phi^{\xi}_{p,q,G}(A) \;\le\; \phi^{1}_{p,q,G}(A). \tag{4.13}$$

We sometimes say that for the stochastic order, the free and the wired boundary conditions are extremal.

Proof. The definition of the ordering implies immediately that $0 \leq \xi \leq 1$. □

Combined with the domain Markov property, the comparison between boundary conditions will allow us to decouple events as follows.

Corollary 4.21. *Let $p \in [0,1]$, $q \geq 1$, a finite graph G and ξ some boundary conditions on G. Let G_1 and G_2 be two subgraphs of G* with disjoint sets of edges. *Then, for any increasing events A and B depending on edges of G_1 and G_2 respectively,*

$$\phi^0_{p,q,G_1}(A)\phi^0_{p,q,G_2}(B) \leq \phi^\xi_{p,q,G}(A \cap B) \leq \phi^1_{p,q,G_1}(A)\phi^1_{p,q,G_2}(B).$$

Proof. We treat the free boundary conditions. The wired boundary conditions are handled similarly.

Let us first look at the conditional expectation $\phi^\xi_{p,q,G}(B|A)$. The event A depends on edges in G_1 only so that it can be seen as a subset $\tilde{A}$ of $\{0,1\}^{E_G \setminus E_{G_2}}$. By partitioning A into events

$$E_\psi = \{\omega(e) = \psi(e), \forall e \notin E_{G_2}\} \subset \{0,1\}^{E_G},$$

where $\psi \in \tilde{A}$ (the event E_ψ corresponds to fixing the configuration outside G_2 to be equal to ψ), we obtain

$$\begin{aligned}\phi^\xi_{p,q,G}(B|A) &= \sum_{\psi \in \tilde{A}} \phi^\xi_{p,q,G}(B|E_\psi)\phi^\xi_{p,q,G}(E_\psi|A)\\ &= \sum_{\psi \in \tilde{A}} \phi^{\xi^\psi}_{p,q,G_2}(B)\phi^\xi_{p,q,G}(E_\psi|A)\\ &\geq \phi^0_{p,q,G_2}(B) \sum_{\psi \in \tilde{A}} \phi^\xi_{p,q,G}(E_\psi|A) = \phi^0_{p,q,G_2}(B).\end{aligned}$$

We used the domain Markov property in the second line, the comparison between boundary conditions in the third, and the fact that A is partitioned into the events E_ψ for $\psi \in \tilde{A}$ in the last equality.

By exchanging the roles of A and B, one gets $\phi^\xi_{p,q,G}(A|B) \geq \phi^0_{p,q,G_1}(A)$. Setting B to be the full state-space $\{0,1\}^{E_G}$ in this inequality gives

$$\phi^\xi_{p,q,G}(A) \geq \phi^0_{p,q,G_1}(A). \tag{4.14}$$

Therefore

$$\phi^\xi_{p,q,G}(A \cap B) = \phi^\xi_{p,q,G}(A)\phi^\xi_{p,q,G}(B|A) \geq \phi^0_{p,q,G_1}(A)\phi^0_{p,q,G_2}(B).$$

□

Example. Let A be an increasing event depending on edges in $F \subset G$ only. By applying the previous corollary (more precisely (4.14)), we find

$$\phi^0_{p,q,F}(A) \le \phi^0_{p,q,G}(A) \quad \text{and} \quad \phi^1_{p,q,F}(A) \ge \phi^1_{p,q,G}(A). \tag{4.15}$$

The use of the Markov property and the comparison between boundary conditions sometimes requires to "push boundary conditions away". We illustrate this reasoning with the following fundamental example (note how intuitive the statement is).

Lemma 4.22. *Let $p \in [0,1]$, $q \ge 1$, $k \le n$ and arbitrary boundary conditions ξ on $\partial\Lambda_n$. For any increasing event A depending on edges in Λ_k only,*

$$\phi^\xi_{p,q,\Lambda_n}(\,A\,|\,\partial\Lambda_k \nleftrightarrow \partial\Lambda_n) \le \phi^0_{p,q,\Lambda_n}(A).$$

Proof. Let $\mathbf{C}$ be the set of dual self-avoiding circuits $\gamma = \{\gamma_0 \sim \gamma_1 \sim \cdots \sim \gamma_m \sim \gamma_0\}$ on $\Lambda_n^\star$ surrounding Λ_k. Define $\overline{\gamma}$ to be the subgraph of Λ_n surrounded by $\gamma \in \mathbf{C}$ [2]. Dual circuits in $\mathbf{C}$ are naturally ordered via the following order relation: γ_1 is "more exterior" than γ_2 if $\overline{\gamma_2}$ is included in $\overline{\gamma_1}$.

On the event $\{\partial\Lambda_k \nleftrightarrow \partial\Lambda_n\}$, define Γ to be the random-variable given by the outermost dual-open circuit in $\mathbf{C}$. Partition $\{\partial\Lambda_k \nleftrightarrow \partial\Lambda_n\}$ into the events $\{\Gamma = \gamma\}$ for $\gamma \in \mathbf{C}$ to obtain

$$\begin{aligned}\phi^\xi_{p,q,\Lambda_n}(\,A\,|\,\partial\Lambda_k \nleftrightarrow \partial\Lambda_n) &= \sum_{\gamma\in\mathbf{C}} \phi^\xi_{p,q,\Lambda_n}(A \cap \{\Gamma = \gamma\}|\,\partial\Lambda_k \nleftrightarrow \partial\Lambda_n)\\ &= \sum_{\gamma\in\mathbf{C}} \phi^\xi_{p,q,\Lambda_n}(A|\Gamma = \gamma)\phi^\xi_{p,q,\Lambda_n}(\Gamma = \gamma|\,\partial\Lambda_k \nleftrightarrow \partial\Lambda_n).\end{aligned}$$

For $\gamma \in \mathbf{C}$, the event $\{\Gamma = \gamma\}$ is measurable with respect to the edges in $E_{\Lambda_n} \setminus E_{\overline{\gamma}}$ only. Indeed, $\Gamma = \gamma$ if all the dual-edges of γ are open in $\omega^\star$ and if any other self-avoiding circuit which is more exterior that γ contains at least one closed dual-edge in $\omega^\star$.

The fact that $\{\Gamma = \gamma\}$ is measurable with respect to edges in $E_{\Lambda_n} \setminus E_{\overline{\gamma}}$ implies that the configuration inside $\overline{\gamma}$ follows the law of a random-cluster model with some boundary conditions which we can identify easily: since conditionally on $\{\Gamma = \gamma\}$, all dual-edges of γ are open in $\omega^\star$, the boundary conditions on the primal graph $\overline{\gamma}$ are free (all edges connecting a vertex of the boundary of $\overline{\gamma}$ to a vertex outside $\overline{\gamma}$ are closed in ω). As a consequence, one may rewrite

$$\phi^\xi_{p,q,\Lambda_n}(A|\Gamma = \gamma) = \phi^0_{p,q,\overline{\gamma}}(A) \le \phi^0_{p,q,\Lambda_n}(A),$$

where the inequality is due to the example following Corollary 4.21 applied to $F = \overline{\gamma}$ and $G = \Lambda_n$.

[2] The graph $\overline{\gamma}$ is composed of the vertices and edges in the finite component of $\mathbb{R}^2 \setminus \gamma$, when γ is considered as a piecewise linear curve in the plane.

Altogether, we find

$$\phi^{\xi}_{p,q,\Lambda_n}(\,A\,|\,\partial\Lambda_k \not\leftrightarrow \partial\Lambda_n) \le \sum_{\gamma\in\mathbf{C}} \phi^0_{p,q,\Lambda_n}(A)\phi^{\xi}_{p,q,\Lambda_n}(\Gamma=\gamma|\,\partial\Lambda_k \not\leftrightarrow \partial\Lambda_n)$$
$$= \phi^0_{p,q,\Lambda_n}(A). \tag{4.16}$$

In the last equality we used the fact that the sets $\{\Gamma=\gamma\}$ for $\gamma\in\mathbf{C}$ partition the set $\{\partial\Lambda_k \not\leftrightarrow \partial\Lambda_n\}$. □

We will be using the strategy of the last proof quite often. Rather than including all the details, we will say that we are conditioning on the outermost closed circuit satisfying a certain condition (we will sometimes do it with the outermost open circuit but it works the same).

In the previous argument, A and $\{\partial\Lambda_k \not\leftrightarrow \partial\Lambda_n\}$ are depending on a disjoint set of edges but we only used the fact that on its intersection with $\{\Gamma=\gamma\}$, the set A depends on edges in $\overline{\gamma}$ only. We will sometimes use the previous argument for events that do not depend on disjoint edges *a priori* but that satisfy this weaker condition instead. The following lemma will both illustrate one important example and be useful several times in the book.

Lemma 4.23. *Let $p\in[0,1]$, $q\ge 1$ and $1\le k\le n$,*

$$\phi^0_{p,q,\Lambda_n}(0\longleftrightarrow\partial\Lambda_k) \le \sum_{m\ge k} 72m^4 \max_{\substack{a\in\{0\}\times[0,m]\\ b\in\{m\}\times[0,m]}} \phi^0_{p,q,[0,m]^2}(a\longleftrightarrow b)\,.$$

The interest of this lemma lies in the fact that on the right-hand side, a and b are on the boundary of a graph with free boundary conditions (these events will be easier to control, see later in the book).

Proof. For $x=(x_1,x_2)$, define $\|x\|_\infty=\max\{|x_1|,|x_2|\}$.

Define $\mathcal{C}$ to be the connected component of the origin. Consider the event that a and b are two vertices in $\mathcal{C}$ maximizing the $\|\cdot\|_\infty$-distance between each other. Since these vertices are at maximal distance, they can be placed on the two opposite sides of a square box Λ in such a way that $\mathcal{C}$ is included in this box. Let $\mathcal{A}_{\max}(a,b,\Lambda)$ the event that a and b are connected in Λ and that their cluster is contained in Λ.

We now wish to estimate the probability of $\mathcal{A}_{\max}(a,b,\Lambda)$. Let $\Lambda^\star$ be the dual discrete domain associated to Λ [3]. Let $\mathbf{C}$ be the set of dual self-avoiding circuits $\gamma=\{\gamma_0\sim\gamma_1\sim\cdots\sim\gamma_m\sim\gamma_0\}$ on $\Lambda^\star$ surrounding a and b. As before, we denote by $\overline{\gamma}$ the interior of γ.

[3] Recall that it is the subgraph of $(\mathbb{Z}^2)^\star$ composed of dual-edges whose end-points correspond to faces touching Λ (they can be included in Λ or simply share an edge with it).

On the event $\mathcal{C}$, there exists $\gamma \in \mathbf{C}$ which is dual-open[4], and a and b are connected in $\overline{\gamma}$. As before, we may condition on the outermost dual-open circuit Γ in $\mathbf{C}$. We deduce as in the last proof that

$$\begin{aligned}\phi^{\xi}_{p,q,\Lambda_n}(a \longleftrightarrow b \text{ in } \overline{\gamma} \,|\, \Gamma = \gamma) &= \phi^{0}_{p,q,\overline{\gamma}}(a \longleftrightarrow b \text{ in } \overline{\gamma}) \\ &\le \phi^{0}_{p,q,\Lambda}(a \longleftrightarrow b \text{ in } \overline{\gamma}) \\ &\le \phi^{0}_{p,q,\Lambda}(a \longleftrightarrow b)\end{aligned}$$

and following the same two lines as in (8.20) (meaning that we partition $\mathcal{A}_{\max}(a,b,\Lambda)$ into the events $\{\Gamma = \gamma\}$), we find

$$\phi^{\xi}_{p,q,\Lambda_n}(\mathcal{A}_{\max}(a,b,\Lambda)) \le \phi^{0}_{p,q,\Lambda}(a \longleftrightarrow b)$$

and therefore

$$\phi^{\xi}_{p,q,\Lambda_n}(\mathcal{A}_{\max}(a,b,\Lambda)) \le \max_{\substack{a \in \{0\}\times[0,m] \\ b \in \{m\}\times[0,m]}} \phi^{0}_{p,q,[0,m]^2}\left(a \longleftrightarrow b\right), \tag{4.17}$$

where $m = \|a - b\|_\infty$.

We may now use the fact that if 0 is connected to distance k, there exist a and b at distance $m \ge k$ of each others and a box Λ having a and b on opposite sides such that $\mathcal{A}_{\max}(a,b,\Lambda)$ occurs. Let us bound the number of choices for a, b and Λ.

For a fixed $m \ge k$, there are $|\Lambda_m| = (2m+1)^2$ choices for a (since a must be at distance smaller or equal to m from the origin). Then a must be on the boundary of Λ and there are therefore $|\partial\Lambda| = 4m$ choices for Λ. The number of choices for b is bounded by $m+1$ (it must be on the opposite sides of Λ). Therefore, for fixed m we can bound the number of possible triples (a,b,Λ) by $4m(2m+1)^2(m+1) \le 72m^4$. We have been very wasteful in the previous reasoning and the bound on this number could be improved greatly but this will be irrelevant for applications.

Overall, (4.17) and a union bound gives

$$\phi^0\left(0 \leftrightarrow \partial\Lambda_k\right) \le \sum_{m\ge k} 72m^4 \max_{\substack{a \in \{0\}\times[0,m] \\ b \in \{m\}\times[0,m]}} \phi^0_{[0,m]^2}\left(a \leftrightarrow b\right).$$

□

[4]Note that this is true even when $\Lambda = \Lambda_n$ since free boundary conditions can be seen as dual-wired boundary conditions on $\Lambda_n^\star$, and that therefore $\partial\Lambda_n^\star$ provides us with a dual self-avoiding circuit in $\mathbf{C}$ which is dual-open. A similar reasoning applies when Λ only shares some sides with Λ_n.

4.5 Infinite-volume measures and phase transition.

4.5.1 Definition of infinite-volume measures

The definition of an infinite-volume random-cluster measure is not direct. Indeed, one cannot count the number of open or closed edges on $(\mathbb{Z}^2, \mathbb{E})$ since they could be (and would be) infinite. We thus define infinite-volume measures indirectly: they are the measures which coincide, when restricted to a finite box, with random-cluster measures in finite volume.

Definition 4.24. Let $p \in [0,1]$ and $q > 0$. A probability measure ϕ on $\{0,1\}^{\mathbb{E}}$ is called an *infinite-volume random-cluster* measure with parameters p and q if for every finite graph G and any configuration $\xi \in \{0,1\}^{\mathbb{E} \setminus E_G}$,

$$\phi(\,\omega_{|E_G} = \eta \,|\, \omega(e) = \xi(e) : \forall e \notin E_G) = \phi^{\xi}_{p,q,G}(\eta) \quad , \quad \forall \eta \in \{0,1\}^{E_G}, \quad (4.18)$$

where ξ are the boundary conditions induced by the configuration ξ.

Remark 4.25. Note that the conditional probability on the left of (4.18) is not properly defined for any ξ, since $\{\omega(e) = \xi(e) : e \notin E_G\}$ has probability 0, but this formula still determines the random variable $\phi(\,\omega_{|E_G} = \eta | \mathcal{F}_{\mathbb{E} \setminus E_G})$ (in particular because it depends on the partition induced by ξ only, and that there are only finitely many such partitions).

Remark 4.26. Many properties of these infinite-volume measures can be deduced from measures in finite volume thanks to (4.18). Often, we will use a result in finite volume and then pass to the limit. When passing to the limit is straightforward, we shall not mention it.

Proving the existence of an infinite-volume measure is not very difficult: one may take a sequence of measures on Λ_n and take a sub-sequential limit (the space of probability measures is compact). Nevertheless, this construction is not very explicit and we prefer the following one. The domain Markov property and the comparison between boundary conditions allow us to construct two interesting infinite-volume measures when $q \geq 1$.

Proposition 4.27. *Let $q \geq 1$. There exist two (possibly equal) infinite-volume random-cluster measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$, called the infinite-volume random-cluster measures with free and wired boundary conditions respectively, such that for any event A depending on a finite number of edges,*

$$\lim_{n\to\infty} \phi^1_{p,q,\Lambda_n}(A) = \phi^1_{p,q}(A) \quad \textit{and} \quad \lim_{n\to\infty} \phi^0_{p,q,\Lambda_n}(A) = \phi^0_{p,q}(A).$$

Proof. We deal with the case of free boundary conditions. Wired boundary conditions are treated similarly. Fix an increasing event A depending on edges in Λ_N only. Applying (4.15) to $F = \Lambda_n$ and $G = \Lambda_{n+1}$, we find that for any $n \geq N$,

$$\phi^0_{p,q,\Lambda_{n+1}}(A) \geq \phi^0_{p,q,\Lambda_n}(A).$$

We deduce that $(\phi^0_{p,q,\Lambda_n}(A))_{n\geq 0}$ is increasing, and therefore converges to a certain value $P(A)$ as n tends to infinity.

Since the ϕ^0_{p,q,Λ_n}-probability of an event B depending on finitely many edges can be written by inclusion-exclusion as a linear combination of the ϕ^0_{p,q,Λ_n}-probability of increasing events, taking the same linear combination defines a natural value $P(B)$ for which

$$\phi^0_{p,q,\Lambda_n}(B) \longrightarrow P(B).$$

The fact that $(\phi^0_{p,q,\Lambda_n})_{n\geq 0}$ are probability measures implies that the function P (which is a priori defined on the set of events depending on finitely many edges) can be extended into a probability measure on $\mathcal{F}_{\mathbb{E}}$. We denote this measure by $\phi^0_{p,q}$. □

The free and wired infinite-volume measures have special properties that we describe now. Let $\tau_x : \{0,1\}^{\mathbb{E}} \to \{0,1\}^{\mathbb{E}}$ defined by

$$\tau_x\omega(\{a,b\}) = \omega(\{a+x, b+x\}) \quad \forall \{a,b\} \in \mathbb{E}.$$

Let $\tau_x A = \{\omega \in \{0,1\}^{\mathbb{E}} : \tau_x\omega \in A\}$. An event A is *translational-invariant* if for any $x \in \mathbb{Z}^2$, $\tau_x A = A$. A measure μ is *invariant under translations* if $\mu(\tau_x A) = \mu(A)$ for any event A.

Theorem 4.28. *Fix $p \in [0,1]$ and $q > 0$. The measures $\phi^1_{p,q}$ and $\phi^0_{p,q}$ are invariant under translations and any translational-invariant event A has probability 0 or 1.*

The second property is called *ergodicity* of the measure.

Proof. Let us treat the case of $\phi^1_{p,q}$. Let A be an increasing event depending on finitely many edges and $x \in \mathbb{Z}^2$ which is a neighbor of the origin (this will imply the result for every $x \in \mathbb{Z}^2$ by successive applications of this result). We have

$$\phi^1_{p,q}(A) = \lim_{n\to\infty} \phi^1_{p,q,\Lambda_n}(A) = \lim_{n\to\infty} \phi^1_{p,q,\tau_x\Lambda_n}(\tau_x A).$$

Now, $\Lambda_{n-1} \subset \tau_x\Lambda_n \subset \Lambda_{n+1}$ and therefore (4.15) implies that

$$\phi^1_{p,q,\Lambda_{n+1}}(\tau_x A) \leq \phi^1_{p,q,\tau_x\Lambda_n}(\tau_x A) \leq \phi^1_{p,q,\Lambda_{n-1}}(\tau_x A)$$

and thus

$$\lim_{n\to\infty} \phi^1_{p,q,\tau_x\Lambda_n}(\tau_x A) = \phi^1_{p,q}(\tau_x A).$$

We conclude that $\phi^1_{p,q}(A) = \phi^1_{p,q}(\tau_x A)$ for any increasing event depending on finitely many edges. Since the increasing events depending on finitely many edges span the σ-algebra of measurable events, we obtain that $\phi^1_{p,q}$ is invariant under translations.

Since any (translation-invariant) events can be approximated by events depending on finitely many edges, the ergodicity follows from the fact[5] that for any events A and B depending on finitely many edges,

$$\lim_{|x|\to\infty} \phi^1_{p,q}(A \cap \tau_x B) = \phi^1_{p,q}(A)\phi^1_{p,q}(B).$$

Observe that by inclusion-exclusion, it is sufficient to prove the equivalent result for A and B increasing and depending on finitely many edges. Let us give ourselves these two increasing events A and B depending on edges in Λ_N only, and $x \in \mathbb{Z}^2$. The FKG inequality implies that

$$\phi^1_{p,q}(A \cap \tau_x B) \geq \phi^1_{p,q}(A)\phi^1_{p,q}(\tau_x B) = \phi^1_{p,q}(A)\phi^1_{p,q}(B).$$

In the other direction, the comparison between boundary conditions implies that for $|x| \geq 2N$,

$$\phi^1_{p,q}(A \cap \tau_x B) \leq \phi^1_{p,q,\Lambda_{|x|/2}}(A)\phi^1_{p,q,\tau_x\Lambda_{|x|/2}}(\tau_x B) = \phi^1_{p,q,\Lambda_{|x|/2}}(A)\phi^1_{p,q,\Lambda_{|x|/2}}(B).$$

In the equality, we used the invariance under translation. The result follows by taking $|x|$ to infinity.

The case of $\phi^0_{p,q}$ follows by taking A and B decreasing instead of increasing. □

The construction of $\phi^1_{p,q}$ and $\phi^0_{p,q}$ can be performed with many other sequences of measures, as explained previously. It could also be possible to see infinite-volume measures existing intrinsically, in the sense that they are not limits of random-cluster measures in finite volume. We conclude this section by discussing uniqueness criteria for these infinite-volume measures.

Proposition 4.29. *Let $p \in [0,1]$ and $q \in (0,\infty)$. If $\phi^1_{p,q} = \phi^0_{p,q}$, then there exists a unique infinite-volume measure with parameters p and q.*

[5] Let us briefly justify this classical fact. Let C be an event which is invariant under translation. Let $\varepsilon > 0$ and A be an event depending on finitely many edges such that $\phi^1_{p,q}(A\Delta C) \leq \varepsilon$. We deduce that

$$\phi^1_{p,q}(C) = \phi^1_{p,q}(C \cap \tau_x C) = \phi^1_{p,q}(A \cap \tau_x A) + O(\varepsilon) \longrightarrow \phi^1_{p,q}(A)^2 + O(\varepsilon) = \phi^1_{p,q}(C)^2 + O(\varepsilon),$$

where the limit means $x \to \infty$. Letting ε tend to 0 gives us that $\phi^1_{p,q}(C) = \phi^1_{p,q}(C)^2$ and therefore $\phi^1_{p,q}(C) \in \{0,1\}$.

Proof. Let $\phi_{p,q}$ be an infinite-volume measure. Equation (4.18) and the comparison between boundary conditions implies that $\phi^0_{p,q,\Lambda_n}(A) \le \phi_{p,q}(A) \le \phi^1_{p,q,\Lambda_n}(A)$ for any $n \ge N$ and any increasing event A depending on edges in Λ_N only. Taking the limit as n tends to infinity, we find

$$\phi^0_{p,q}(A) \le \phi_{p,q}(A) \le \phi^1_{p,q}(A).$$

Now if $\phi^1_{p,q} = \phi^0_{p,q}$, we deduce that $\phi^0_{p,q}(A) = \phi_{p,q}(A) = \phi^1_{p,q}(A)$. Since increasing events depending on finitely many edges generate the σ-algebra, we obtain that $\phi^0_{p,q} = \phi_{p,q} = \phi^1_{p,q}$. □

The following theorem shows that the set of edge-weights for which uniqueness fails is somehow small.

Theorem 4.30. *For $q \ge 1$, the set $\mathcal{D}_q$ of edge-weight p for which uniqueness fails is at most countable.*

The proof is based on the differentiability of the free energy.

Lemma 4.31. *There exists a quantity, called the* free energy*, such that*

$$f(p,q) = \lim_{n\to\infty} \frac{1}{|E_{\Lambda_n}|} \log\left[Z^{\xi}_{p,q,\Lambda_n}\right],$$

where the convergence holds uniformly in the choice of boundary conditions ξ.

Proof. We will simplify the proof and present it only for the subsequence of boxes of size 2^n. The general case follows using the dyadic decomposition of n. Define

$$f^{\xi}_n(p,q) = \frac{1}{|E_{\Lambda_n}|} \log\left[Z^{\xi}_{p,q,\Lambda_n}\right].$$

Now,

$$\begin{aligned} Z^1_{p,q,\Lambda_{2n}} &= \sum_{\omega\in\{0,1\}^{E_{\Lambda_{2n}}}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^1)} \\ &\ge \sum_{\omega\in\{0,1\}^{E_{\Lambda_{2n}}}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k_1(\omega^1)+\dots+k_4(\omega^1)} \\ &= \left(Z^1_{p,q,\Lambda_n}\right)^4, \end{aligned}$$

where $k_1(\omega^1)$, $k_2(\omega^1)$, $k_3(\omega^1)$ and $k_4(\omega^1)$ are the numbers of clusters (counted with wired boundary conditions) in the graphs $\omega_{|\tau_{(-n,-n)}\Lambda_n}$, $\omega_{|\tau_{(-n,n)}\Lambda_n}$, $\omega_{|\tau_{(n,-n)}\Lambda_n}$ and $\omega_{|\tau_{(n,n)}\Lambda_n}$.

Hence, the sequence $(f^1_{2^n}(p,q))_{n\ge0}$ is increasing and converges to a limit denoted by $f(p,q)$.

Now, consider boundary conditions ξ. Since

$$k(\omega^\xi) \le k(\omega^1) \le k(\omega^\xi) + |\partial\Lambda_n|,$$

taking the logarithm and letting n go to infinity implies the convergence to the same limit. □

Proof of Theorem 4.30. We will use the variable π defined by $\pi(p) = \log[p/(1-p)]$. Also set $p_\pi = \frac{e^\pi}{1+e^\pi}$. We set $\tilde f(\pi, q)$ and $\tilde f_n^\xi(\pi, q)$ for $f(p_\pi, q) - \log(1+e^\pi)$ and $f_n^\xi(p_\pi, q) - \log(1+e^\pi)$. When differentiating

$$\tilde f_n^\xi(\pi, q) = \frac{1}{|E_{\Lambda_n}|} \log\Big(\sum_{\omega\in\{0,1\}^{E_{\Lambda_n}}} e^{\pi o(\omega)} q^{k(\omega^\xi)}\Big)$$

in π, we find

$$\partial_\pi \tilde f_n^\xi(\pi, q) = \frac{1}{|E_{\Lambda_n}|} \sum_{e\in\Lambda_n} \phi^\xi_{p_\pi,q,\Lambda_n}(\omega(e) = 1)$$

which is increasing in p. As a consequence, $\tilde f_n^\xi(\pi, q)$ is convex, and therefore its limit $\tilde f(\pi, q)$ also is. This immediately implies that $\tilde f(\pi, q)$ is differentiable except for at most countably many points.

Let us now prove that $\phi^0_{p_\pi,q}(\omega(e) = 1) = \phi^1_{p_\pi,q}(\omega(e) = 1)$ for values of π for which $\tilde f(\pi, q)$ is differentiable. Since $\tilde f_n^1(\pi, q)$ is convex, differentiable in π and increasing to $\tilde f(\pi, q)$, we deduce that

$$\begin{aligned}\lim_{\varepsilon\searrow 0} \frac{\tilde f(\pi+\varepsilon, q) - \tilde f(\pi, q)}{\varepsilon} &= \lim_{n\to\infty} \frac{1}{|E_{\Lambda_{2n}}|} \sum_{e\in E_{\Lambda_{2n}}} \phi^1_{p_\pi,q,\Lambda_{2n}}(\omega(e) = 1)\\ &= \phi^1_{p_\pi,q}(\omega(e) = 1).\end{aligned}$$

The first equality is due to a general fact about convex functions, and the second equality to the invariance under translations and the convergence of $\phi^1_{p_\pi,q,\Lambda_{2n}}$ to $\phi^1_{p_\pi,q}$. Similarly, one may prove that $\tilde f_n^0(\pi, q)$ is decreasing to $\tilde f(\pi, q)$ and therefore

$$\lim_{\varepsilon\searrow 0} \frac{\tilde f(\pi-\varepsilon, q) - \tilde f(\pi, q)}{-\varepsilon} = \phi^0_{p_\pi,q}(\omega(e) = 1).$$

Putting these two facts together, we obtain that at any point of differentiability in π of $\tilde f(\pi, q)$,

$$\phi^1_{p_\pi,q}(\omega(e) = 1) = \phi^0_{p_\pi,q}(\omega(e) = 1). \tag{4.19}$$

We are close to the end of the proof. We need to prove that (4.19) implies that the infinite volume measure is unique at p_π. Proposition 4.29 shows that it is sufficient to prove that $\phi^0_{p_\pi,q} = \phi^1_{p_\pi,q}$. Recall that $\phi^0_{p_\pi,q,\Lambda_n}$ is stochastically dominated by $\phi^1_{p_\pi,q,\Lambda_n}$. In other words, there exists a coupling $\mathbf{P}_n$ with marginals $\phi^0_{p_\pi,q,\Lambda_n}$ and $\phi^1_{p_\pi,q,\Lambda_n}$ such that $\mathbf{P}_n(\omega_1 \le \omega_2) = 1$. By passing to the limit we obtain a coupling $\mathbf{P}$ with marginals $\phi^0_{p_\pi,q}$ and $\phi^1_{p_\pi,q}$. Let us consider an increasing event A depending on a finite set E of edges. We find

$$\begin{aligned}
0 \le \phi^1_{p_\pi,q}(A) - \phi^0_{p_\pi,q}(A) &= \mathbf{P}(\omega_2 \in A \text{ and } \omega_1 \notin A)\\
&\le \mathbf{P}(\exists e \in E : \omega_2(e) = 1 \text{ and } \omega_1(e) = 0)\\
&\le \sum_{e\in E} \mathbf{P}(\omega_2(e) = 1 \text{ and } \omega_1(e) = 0)\\
&= \sum_{e\in E} \phi^1_{p_\pi,q}(\omega(e) = 1) - \phi^0_{p_\pi,q}(\omega(e) = 0) = 0.
\end{aligned}$$

Since $\phi^1_{p_\pi,q}(A) = \phi^0_{p_\pi,q}(A)$ for any increasing event A depending on finitely many edges, we deduce the claim. □

Remark 4.32. It will sometimes be convenient to work in an infinite subgraph G of $\mathbb{Z}^2$. Let ξ be some boundary conditions for G. We define the random-cluster measure in G with boundary conditions ξ on ∂G, wired at infinity as the monotone limit

$$\phi^{1,\xi}_G = \lim \phi^{\xi_n}_{G\cap\Lambda_n}, \tag{4.20}$$

where the boundary conditions ξ_n are wired on the boundary intersecting $\partial\Lambda_n$, and defined by ξ on the rest of the boundary. We define similarly the measure $\phi^{0,\xi}_G$ with free boundary conditions at infinity.

Remark 4.33. One example of infinite set that will come back several times in this book is the strip $\mathbb{S}_n = \mathbb{Z} \times [0, n]$. The finite energy property (Proposition 4.4) implies that for any $p < 1$, there exists $c = c(p) > 0$ such that each vertical segment $\{k\} \times [0, n]$ ($k \in \mathbb{Z}$) is composed of closed edges only with probability larger or equal to c, and this independently of the state of the other edges. We easily deduce that there is no infinite cluster in $\mathbb{S}_n$ almost surely, independently of $p < 1$ and of the boundary conditions. Also, $\phi^{0,\xi}_{\mathbb{S}_n} = \phi^{1,\xi}_{\mathbb{S}_n}$ for any boundary conditions ξ.

4.5.2 Critical point

We are now in a position to discuss the phase transition of the random-cluster model.

Theorem 4.34. *There exists a* critical point $p_c \in [0,1]$ *such that:*

- *For $p < p_c$, any infinite-volume measure has no infinite cluster almost surely.*
- *For $p > p_c$, any infinite-volume measure has an infinite cluster almost surely.*

Proof. Let us define

$$p_c = \inf\{p \in [0,1] : \phi^0_{p,q}(0 \leftrightarrow \infty) > 0\}.$$

Since the event $0 \leftrightarrow \infty$ is increasing, we deduce that $\phi^0_{p,q}(0 \leftrightarrow \infty) > 0$ for any $p > p_c$. Ergodicity implies that

$$\phi^0_{p,q}(\text{there exists an infinite cluster}) = 1.$$

Furthermore, Proposition 4.29 implies that this claim is true for any infinite-volume measure, since $\phi^0_{p,q}$ is the smallest among all of them.

On the other hand, let $p < p_c$. There exists $p < p_0 < p_c$ such that there is a unique infinite-volume measure at p_0 (since the set $\mathcal{D}_q$ is at most countable). We deduce that for any infinite-volume measure $\phi_{p,q}$ and any $x \in \mathbb{Z}^2$

$$\phi_{p,q}(x \leftrightarrow \infty) \le \phi^1_{p,q}(x \leftrightarrow \infty) \le \phi^1_{p_0,q}(x \leftrightarrow \infty) = \phi^0_{p_0,q}(x \leftrightarrow \infty) = 0$$

by uniqueness of the measure at p_0 and $p_0 < p_c$. The claim follows by taking the union over all $x \in \mathbb{Z}^2$. □

Remark 4.35. In order to complete the picture of the existence of a phase transition, we should prove that p_c lies strictly between 0 and 1. The most elementary argument would be an extension of the famous Peierls argument [Pei36]. Since we will compute the critical value explicitly in the next chapter, and since the Peierls argument will be presented in its original context (the Ising model) in Section 7.5.3, we do not spend more time on it now.

We do not resist the pleasure of presenting one of the most beautiful arguments in probability theory, namely the Burton-Keane argument. Let $\ell \in \mathbb{N} \cup \{\infty\}$. Define $\mathcal{E}_\ell$ to be the event that there exist exactly ℓ distinct infinite clusters.

Proposition 4.36 (Uniqueness of the infinite cluster [BK89])**.** *Fix $p \in [0,1]$ and $q \ge 1$. For any $\ell \ge 2$, we have that $\phi^1_{p,q}(\mathcal{E}_\ell) = \phi^0_{p,q}(\mathcal{E}_\ell) = 0$.*

In other words, either there is no infinite cluster almost surely, or there is a *unique* infinite cluster almost surely.

Proof. We present the proof in the case of $\xi = 0$. The case of $\xi = 1$ is treated similarly.

Step 1: proof of $\phi^0_{p,q}[\mathcal{E}_\ell] = 0$ for $2 \le \ell < \infty$. Let $\ell \ge 2$. Let $\mathcal{F}_n$ be the event that the ℓ infinite clusters intersect Λ_n. Fix N large enough that $\phi^0_{p,q}[\mathcal{F}_N] \ge \frac{1}{2}\phi^0_{p,q}[\mathcal{E}_\ell]$. Since $\mathcal{F}_N$ depends on edges outside Λ_N only, the finite-energy property (Proposition 4.4) implies that

$$\phi^0_{p,q}[\mathcal{F}_N \cap \{\omega(e) = 1, \forall e \in \Lambda_N\}] \ge \tfrac{c}{2}\phi^0_{p,q}[\mathcal{E}_\ell].$$

Any configuration in the event on the left contains exactly one infinite cluster since all the vertices in Λ_N are connected. Therefore,

$$\phi^0_{p,q}[\mathcal{E}_1] \ge \tfrac{c}{2}\phi^0_{p,q}[\mathcal{E}_\ell].$$

Ergodicity implies that $\phi^0_{p,q}[\mathcal{E}_\ell]$ and $\phi^0_{p,q}[\mathcal{E}_1]$ are equal to 0 or 1, therefore $\phi^0_{p,q}[\mathcal{E}_\ell] = 0$.

Step 2: proof of $\phi^0_{p,q}[\mathcal{E}_\infty] = 0$. Assume that $\phi^0_{p,q}[\mathcal{E}_\infty] > 0$ and consider $N > 0$ large enough that

$$\phi^0_{p,q}[\text{three distinct infinite clusters intersect the box } \Lambda_N] > 0.$$

The finite-energy property (Proposition 4.4) implies that $\phi^0_{p,q}[\mathrm{CT}_0] > 0$, where CT_0 is the following event:

- all vertices in Λ_N are connected to each other in Λ_N,
- if $\mathcal{C}$ is the cluster of 0, then $\mathcal{C} \cap (\mathbb{Z}^d \smallsetminus \Lambda_N)$ contains at least three distinct infinite connected components.

A vertex $x \in (2N+1)\mathbb{Z}^d$ is called a coarse trifurcation if $\tau_x \mathrm{CT}_0 =: \mathrm{CT}_x$ occurs. By invariance under translation, $\phi^0_{p,q}[\mathrm{CT}_x] = \phi^0_{p,q}[\mathrm{CT}_0]$.

Fix $n \gg N$. The set T of coarse trifurcations in Λ_n has a natural structure of forest $\mathcal{F}$ constructed inductively as follows.

Step 1: At time 0, all the vertices in T are *unexplored.*

Step 2: If there does not exist any unexplored vertex in T left, the algorithm terminates. Otherwise, pick an unexplored vertex $t \in T$ and mark it explored (by this we mean that it is not considered as an unexplored vertex anymore). Go to Step 3.

Step 3: Consider the cluster $\mathcal{C}_t$ of vertices $x \in \mathbb{Z}^d$ connected to t in Λ_n. This cluster decomposes into $k \ge 3$ disjoint connected components of $\mathbb{Z}^d \smallsetminus (x + \Lambda_N)$ denoted by $\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(k)}$. For $i = 1 \dots k$, do the following

- if there exists a vertex $x \in \mathcal{C}^{(i)} \cap \partial\Lambda_n$ and there exists an open path in Λ_n going from x to t and not passing at distance N from a coarse-trifurcation in $\mathcal{C}^{(i)} \cap (T \smallsetminus \{t\})$, then add the vertex x to $\mathcal{F}$ together with the edge $\{t, x\}$.

– if there is no such vertex, then there must be a coarse-trifurcation s connected by an open path not passing at distance N from a trifurcation in $\mathcal{C}^{(i)} \cap (T \setminus \{t,s\})$. If s is not already a vertex of $\mathcal{F}$, add it. Then, add the edge $\{t,s\}$.

Step 4: Go to Step 2.

The graph obtained is a forest (due to the structure of coarse-trifurcations). Each coarse-trifurcation corresponds to a vertex of the forest of degree at least three. Thus, the number of coarse-trifurcations must be smaller than the number of leaves $\mathcal{N}$ of the forest. Taking the expected number of coarse-trifurcations, we find

$$\phi^0_{p,q}[\mathrm{CT}_0]\frac{(2n+1)^2}{(2N+1)^2} \le \phi^0_{p,q}[\mathcal{N}]. \tag{4.21}$$

Yet, leaves are vertices of $\partial\Lambda_n$, thus

$$\phi^0_{p,q}[\mathcal{N}] \le 8n$$

which gives

$$0 < \frac{\phi^0_{p,q}[\mathrm{CT}_0]}{(2N+1)^2} \le \frac{\phi^0_{p,q}[\mathcal{N}]}{(2n+1)^2} \longrightarrow 0 \quad \text{as } n \to \infty.$$

This contradicts $\phi^0_{p,q}[\mathrm{CT}_0] > 0$ and therefore $\phi^0_{p,q}[\mathcal{E}_\infty] > 0$. The claim follows. □

Remark 4.37. The original Burton-Keane argument does not invoke the notion of coarse trifurcation, but simply trifurcation (which would correspond to $N = 0$). There is a tiny difficulty when working with trifurcations (some rewiring construction) that we avoid when working with coarse-trifurcations, at the cost of a slight loss in elegance.

4.5.3 The inequality $p_c \ge p_{sd}$

Overall, the existence of a critical point is not completely direct. Nevertheless, it is now a classical fact which is well understood in many models. The computation of the critical point on the other hand is a much harder task and the existence of a nice formula for $p_c(q)$ is not even obvious in general. For the random-cluster model on the square lattice, it is nevertheless natural to conjecture that the critical point satisfies $p_c = p_{sd}$.

Let us start by a lower bound for the critical value which can be derived using an elegant argument (due to Zhang in the case of percolation) based on the uniqueness of the infinite cluster

Proposition 4.38. *For $q \geq 1$, there exists almost surely no infinite cluster at $p_{sd}(q)$ for the infinite-volume measure with free boundary conditions. As a consequence, $p_{sd}(q) \leq p_c(q)$.*

Proof. Let $\varepsilon \ll 1$. Assume that $\phi^0_{p_{sd},q}(0 \leftrightarrow \infty) > 0$ and choose n large enough that

$$\phi^0_{p_{sd},q}(\Lambda_n \leftrightarrow \infty) > 1 - \varepsilon.$$

The integer n exists since the infinite cluster exists almost surely (therefore the quantity on the left tends to 1 as n tends to infinity).

Let A_{left} (resp. A_{right}, A_{top} and A_{bottom}) be the events that $\{-n\} \times [-n,n]$ (resp. $\{n\}\times[-n,n]$, $[-n,n]\times\{n\}$ and $[-n,n]\times\{-n\}$) are connected to infinity in the complement of Λ_n. By symmetry,

$$\phi^0_{p_{sd},q}(A_{\text{left}} \cup A_{\text{right}}) = \phi^0_{p_{sd},q}(A_{\text{top}} \cup A_{\text{bottom}})$$

and

$$\phi^0_{p_{sd},q}(A_{\text{left}}) = \phi^0_{p_{sd},q}(A_{\text{right}}).$$

We also find that

$$\phi^0_{p_{sd},q}(A_{\text{left}} \cup A_{\text{right}} \cup A_{\text{top}} \cup A_{\text{bottom}}) = \phi^0_{p_{sd},q}(\Lambda_n \leftrightarrow \infty) > 1 - \varepsilon.$$

We can thus invoke the square-root trick (4.11) twice to obtain that

$$\phi^0_{p_{sd},q}(A_{\text{left}}) \geq 1 - \varepsilon^{1/4}.$$

As a consequence,

$$\phi^0_{p_{sd},q}(A_{\text{left}} \cap A_{\text{right}}) \geq 1 - 2\varepsilon^{1/4}.$$

We now use that $p^\star_{sd} = p_{sd}$. Note that the dual measure stochastically dominates the primal one since the dual measure of $\phi^0_{p_{sd},q}$ is $\phi^1_{p_{sd},q}$ [6]. In particular, let $A^\star_{\text{top}}$ and $A^\star_{\text{bottom}}$ be the events that $[-(n+\frac{1}{2}), n+\frac{1}{2}]\times\{n+\frac{1}{2}\}$ and $[-(n-\frac{1}{2}), n+\frac{1}{2}]\times\{-(n+\frac{1}{2})\}$ are dual-connected to infinity using edges outside $E^\star_{\Lambda_n} = \{e^\star : e \in E_{\Lambda_n}\}$. Following the same argument as for the primal model, we find that

$$\phi^1_{p_{sd},q}(A^\star_{\text{top}} \cap A^\star_{\text{bottom}}) \geq 1 - 2\varepsilon^{1/4}.$$

Putting all these facts together, we obtain

$$\phi^0_{p_{sd},q}(A_{\text{left}} \cap A_{\text{right}} \cap A^\star_{\text{top}} \cap A^\star_{\text{bottom}}) > 1 - 4\varepsilon^{1/4}.$$

[6] One can easily see that the duality relation between free and wired boundary conditions extends to infinite volume by looking at measures on Λ_n and letting n go to infinity

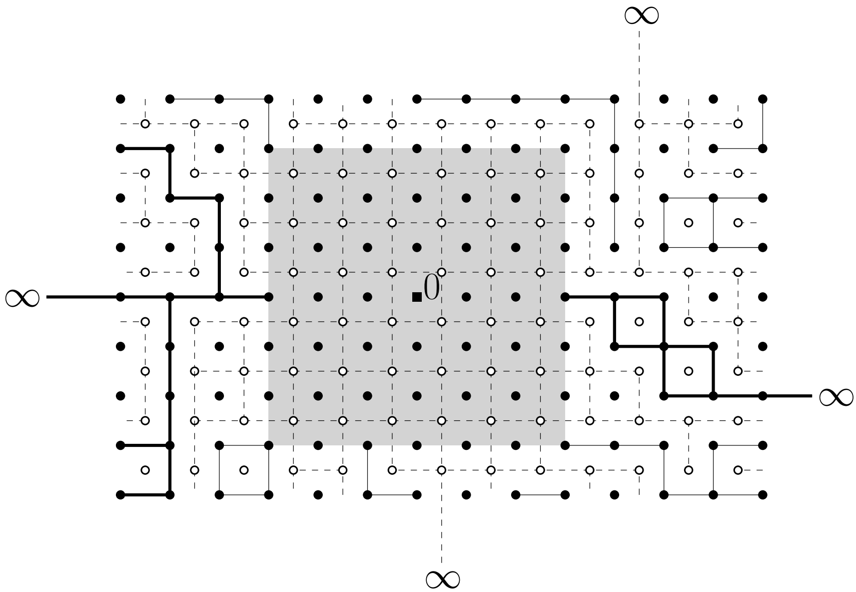


Figure 4.2: In this configuration, two infinite clusters (in bold) coexist.

Now, Let B be the event that every dual-edge in $E^{\star}_{\Lambda_n}$ is open in $\omega^{\star}$. The events B and $A_{\text{left}} \cap A_{\text{right}} \cap A^{*}_{\text{top}} \cap A^{*}_{\text{bottom}}$ depend on disjoint sets. The finite-energy property (Proposition 4.4) implies that

$$\phi^{0}_{p_{sd},q}(B \cap A_{\text{left}} \cap A_{\text{right}} \cap A^{*}_{\text{top}} \cap A^{*}_{\text{bottom}}) > 0,$$

see Fig. 4.2. But this last event is contained in the event that there are two disjoint infinite clusters, which we excluded, thus leading to a contradiction. □

Remark 4.39. The much more difficult inequality $p_c \le p_{sd}$ will be proved in Chapter 5.

Let us conclude this chapter by a useful corollary.

Corollary 4.40. *Fix $q \ge 1$. The unique edge-weight $p \in [0,1]$ for which there can exist distinct infinite-volume measures is $p_{sd}(q)$.*

Remark 4.41. From now on, when $p \ne p_{sd}(q)$, the unique infinite-volume measure with parameters (p,q) is denoted by $\phi_{p,q}$. This measure can be equivalently thought of as $\phi^{0}_{p,q}$ or $\phi^{1}_{p,q}$.

Proof. By Proposition 4.29, we only need to prove that $\phi^0_{p,q} = \phi^1_{p,q}$. Fix $p < p_{sd}$ and an increasing event A depending on a finite number of edges (all included in the box of size say k). For $N \geq n \geq k$, Lemma 4.22 (in fact a trivial modification of it) implies that

$$\phi^1_{p,q,\Lambda_N}(A \cap \{\partial\Lambda_k \not\leftrightarrow \partial\Lambda_n\}) \leq \phi^0_{p,q,\Lambda_n}(A)\phi^1_{p,q,\Lambda_N}(\partial\Lambda_k \not\leftrightarrow \partial\Lambda_n).$$

Letting N tends to infinity, we find that

$$\phi^1_{p,q}(A \cap \{\partial\Lambda_k \not\leftrightarrow \partial\Lambda_n) \leq \phi^0_{p,q,\Lambda_n}(A)\phi^1_{p,q}(\partial\Lambda_k \not\leftrightarrow \partial\Lambda_n).$$

Since $p < p_{sd}(q) \leq p_c(q)$ (by Proposition 4.38), the term on the right tends to $\phi^0_{p,q}(A)$ as n tends to infinity. The left-hand side tends to $\phi^1_{p,q}(A)$. Therefore, we have $\phi^1_{p,q}(A) \leq \phi^0_{p,q}(A)$. Since the other inequality is trivial by stochastic domination, we find that $\phi^1_{p,q}(A) = \phi^0_{p,q}(A)$ for any increasing event A depending on finitely many edges. Since these events generate the whole algebra of measurable events, we obtain that $\phi^1_{p,q} = \phi^0_{p,q}$.

For $p > p_{sd}$, we use that the dual measures of $\phi^1_{p,q}$ and $\phi^0_{p,q}$ are respectively $\phi^0_{p^\star,q}$ and $\phi^1_{p^\star,q}$. Since these two measures are equal, we deduce that $\phi^1_{p,q} = \phi^0_{p,q}$ in this case as well. □

Chapter 5

RSW theory for the random-cluster model with $q \geq 1$

Motivated by the fact that critical exponents of the random-cluster model may be obtained by studying fractal properties of the critical phase (see the introduction for more details), we now focus on the global geometry of big clusters in a large box. Obviously, keeping track of all long open paths in a box would be inefficient, and we are therefore looking for a better way of encoding the information. In the nineties, Aizenman suggested to consider only the information that some subdomains Ω with four points a, b, c and d on their boundary contain "an open path from the arc ab to the arc cd staying in Ω" or not. If one possesses this information for a sufficiently large family of subdomains (for instance, one may look at those which are roughly of the size of the box), one understands the geometry of the big clusters fairly well.

This (vague) discussion motivates our interest in so-called *crossing events*. For simplicity, we will restrict our attention to subdomains which have a rectangular shape (we will treat more general shapes in Chapter 10).

Definition 5.1. A *rectangle* R is a graph of the form $[a,b]\times[c,d]$. For a rectangle $R=[a,b]\times[c,d]$, let $\mathcal{C}_v(R)$ denote the event that there exists an open path from top to bottom in R, i.e. from $[a,b]\times\{c\}$ to $[a,b]\times\{d\}$. Such a path is called a *vertical (open) crossing* of the rectangle.

Similarly, define $\mathcal{C}_h(R)$ to be the event that there exists an open path from left to right in R. Such a path is called an *horizontal (open) crossing* between the left and the right sides.

When the configuration is supercritical ($p > p_c$), one may easily check that most rectangles are crossed vertically and horizontally provided that they are big enough[1] (the infinite cluster will then visit the rectangle with large probability). On the contrary, when the model is subcritical ($p < p_c$), one may prove (it is the object of Theorem 5.18) that the largest cluster in a box of size n is of size $O(\log n)$ with large probability. In particular, the probability that a rectangle of the size of the box is crossed vertically or horizontally is very small (see Remark 5.20). Thus, the phase at $p = p_c$ seems to be of specific interest for crossing events. We will see in this chapter that in such case, the probability of crossing for rectangles remains bounded away from 0 and 1 provided that the aspect ratio of these rectangles remains bounded away from 0 and ∞. Note that in our case, p_c is expected to be p_{sd}.

The most important result of this chapter will be a proof that for $p = p_{sd}$ and $\alpha > 0$, the probability of $\mathcal{C}_h([0, \alpha n] \times [0, n])$ remains bounded away from 0 or 1 uniformly in n. This result extends the Russo-Seymour-Welsh (RSW) theory available for Bernoulli percolation. This theory started with the articles [Rus78] and [SW78] on percolation (also see the more recent approaches of Bollobas and Riordan [BR06a, BR10, BR06c]). Nevertheless, all known approaches are based on independence. In the random-cluster model with $q > 1$, the dependence inherent in the model forces us to develop new arguments. In particular, specific boundary conditions will be easier to handle and we will start by periodic boundary conditions. Later in the chapter, we will treat general boundary conditions.

The chapter is organized as follows. Section 5.1 is devoted to the development of the RSW theory for periodic boundary conditions. In Section 5.2, we present the most important application of this theory: the determination of the critical value $p_c = \sqrt{q}/(1 + \sqrt{q})$. In Section 5.3, we show exponential decay of correlations in the subcritical phase. Section 5.4 presents a more general version of the RSW theory. This improved theory represents a milestone in the theory of the critical random-cluster model for $q \geq 1$ and one important application is directly presented in Section 5.5.

Remark 5.2. The Russo-Seymour-Welsh theory usually refers to a specific aspect of the study of probabilities of crossings: namely the fact that the probability of crossing vertically a rectangle of the form $[0, n] \times [0, \alpha n]$ with $\alpha > 0$ can be expressed in terms of the probability of crossing vertically the rectangle $[0, n] \times [0, \beta n]$ with $\beta < \alpha$ in such a way that if the later remains bounded away from 0 in n, so does the former. In this book, we will allow ourselves some latitude and simply refer to the Russo-Seymour-Welsh theorem as being the fact that crossing probabilities remain bounded away from 0 and 1 provided that α is fixed.

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[1] And also that their aspect ratio is not to small or too big, see Remark 5.20.

5.1 RSW theory for periodic boundary conditions

The following theorem states that, at the self-dual point, the probability of crossing a rectangle horizontally is bounded away from 0 uniformly in the sizes of both the rectangle and the torus provided that the aspect ratio of the rectangle remains constant. The size of the ambient torus is denoted by m.

Theorem 5.3 (Beffara, Duminil-Copin [BDC12a]). *Let $\alpha > 1$ and $q \geq 1$. There exists $c(\alpha) > 0$ such that for every $m > \alpha n > 0$,*

$$c(\alpha) \leq \phi^{\mathrm{per}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,\alpha n] \times [0,n])\big) \leq 1 - c(\alpha). \tag{5.1}$$

By invariance under rotations, we obtain similar bounds for crossings from bottom to top.

The periodic boundary conditions are not helping the existence of a dual-open crossing from top to bottom that would be preventing the existence of an open crossing from left to right, in the sense that the dual model also has periodic boundary conditions[2]. Therefore, it is natural to expect such a result to hold at least for a square shape (this statement will in fact be easy to prove, see Lemma 5.7). The difficult part of the proof will be to extend this result to aspect ratio $\alpha \neq 1$.

Remark 5.4. Note that even for $\alpha = 1$, the above result would not necessarily be true when working with free boundary conditions for example, since the dual model would have wired boundary conditions, and it could be that the existence of a dual-open crossing from top to bottom would be much more likely than the existence of a open crossing from left to right (even if $\alpha \ll 1$). We will discuss this phenomenon in the next chapters.

Theorem 5.3 implies a similar result for the infinite-volume random-cluster measure with wired boundary conditions[3].

Corollary 5.5. *Let $\alpha > 1$ and $q \geq 1$; there exists $c(\alpha) > 0$ such that for every $n \geq 1$,*

$$\phi^{1}_{p_{sd},q}\big[\mathcal{C}_h\big([0,\alpha n] \times [0,n]\big)\big] \geq c(\alpha). \tag{5.2}$$

[2] As least when working with the balanced model, see Section 4.3.2.

[3] Note that in this case also, the dual model has free boundary conditions, and therefore the probability that there exists a dual crossing is a priori smaller than the probability of primal crossings.

Proof. Let $\alpha > 1$ and $m > 2\alpha n > 0$. Using the invariance under translations of $\phi^{\mathrm{per}}_{p_{sd},q,m}$ and comparison between boundary conditions, we have

$$\phi^{1}_{p_{sd},q,\Lambda_{m/2}}\big[\mathcal{C}_h\big([0,\alpha n]\times[0,n]\big)\big] \geq \phi^{\mathrm{p}}_{p_{sd},q,m}\big[\mathcal{C}_h\big([0,\alpha n]\times[0,n]\big)\big] \geq c(\alpha).$$

When m goes to infinity, we find

$$\phi^{1}_{p_{sd},q}\big[\mathcal{C}_h\big([0,\alpha n]\times[0,n]\big)\big] \geq c(\alpha).$$

□

We now focus on the proof of Theorem 5.3 and we work on the torus of size m. For technical reasons, it will be convenient to **rotate the lattice in this torus by $\pi/4$ for the reminder of this section**. In such case, the graph $[0,\alpha n]\times[0,n]$ is then the intersection of the rotated lattice with the rectangle $[0,\alpha n]\times[0,n]$ (when seen as a subset of $\mathbb{R}^2$). The definition of the events $\mathcal{C}_h$ and $\mathcal{C}_v$ is extended to this context (we still go from left to right, and from top to bottom). We will prove the following result.

Proposition 5.6. *Let $q \geq 1$. There exists $c > 0$ such that for every $m > \frac{3}{2}n > 0$,*

$$\phi^{\mathrm{per}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,\tfrac{3}{2}n]\times[0,n])\big) \geq c. \tag{5.3}$$

Let us explain how this result implies Theorem 5.3.

Proof of Theorem 5.3. We use Proposition 5.6. Let us emphasize once again that in the statement and also in this proof, the lattice is rotated by $\pi/4$. Let $R = T([0,n]\times[0,\alpha n])$, where T is the composition of the rotation of angle $\pi/4$ and the translation of vector $(\frac{n}{2},0)$, see Figure 5.1. Define the following rectangles:

$$\begin{aligned} R_j^v &= [j\tfrac{n}{4},(j+1)\tfrac{n}{4}]\times[j\tfrac{n}{4},(j+\tfrac{3}{2})\tfrac{n}{4}],\\ R_j^h &= [j\tfrac{n}{4},(j+\tfrac{3}{2})\tfrac{n}{4}]\times[(j+\tfrac{1}{2})\tfrac{n}{4},(j+\tfrac{3}{2})\tfrac{n}{4}], \end{aligned}$$

for $j \in [0,\lfloor 4\sqrt{2}\alpha\rfloor + 4]$, where $\lfloor x\rfloor$ denotes the integer part of x. If every rectangle R_j^h is crossed horizontally, and every rectangle R_j^v is crossed vertically, then $T([0,\alpha n]\times[0,n])$ is crossed in the long direction. The FKG inequality and Proposition 5.6 provide us with a lower bound on the crossing probability.

Now, the graph $T([0,\alpha n]\times[0,n])$ is isomorphic to the rectangle $[0,\alpha n]\times[0,n]$ when the lattice is not rotated. This concludes the proof. □

The proof of Proposition 5.6 begins with a lemma, which corresponds to the existence of $c(1)$ and is based on the self-duality of the (balanced) random-cluster measures on the torus. This lemma is the starting point for any attempt to obtain bounds on crossing probabilities.

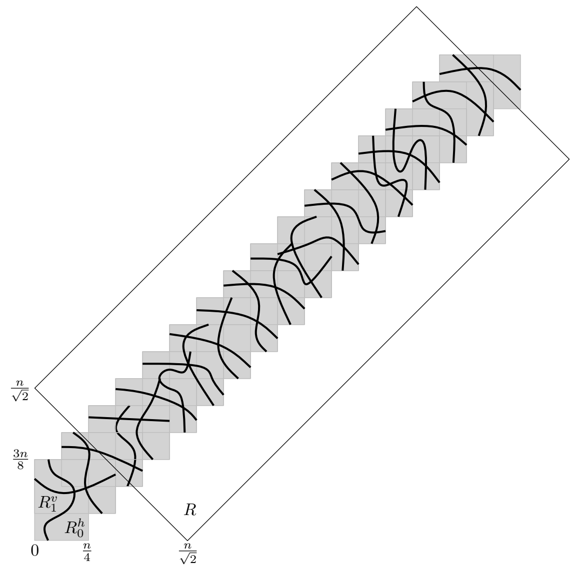

Figure 5.1: A combination of crossings in smaller rectangles creating a crossing of a very long rectangle.

Lemma 5.7. *Let $q \geq 1$, there exists $c(1) = c(1,q) > 0$ such that for every $m > n \geq 1$, $\phi^{\mathrm{per}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,n]^2)\big) \geq c(1)$.*

Proof. We will use the balanced random-cluster measure $\tilde{\phi}^{\mathrm{per}}_{p_{sd},q,m}$ on the torus of size m. Note that the dual of the subgraph of $\mathbb{Z}^2$ induced by $[0,n]^2$ is the subgraph of $(\mathbb{Z}^2)^\star$ induced by $[0,n]^2$ (the latter graph is isomorphic to the former), see Figure 5.2. If there is no open crossing from left to right in $[0,n]^2$, there exists necessarily a dual-open crossing from top to bottom in the dual configuration. Hence, the complement of $\mathcal{C}_h([0,n]^2)$ is the event $\mathcal{C}_v^\star([0,n]^2)$ that there exists a vertical dual-open dual path from top to bottom in $[0,n]^2$ (this dual-open path prevents the existence of an horizontal open crossing), thus yielding

$$\tilde{\phi}^{\mathrm{per}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,n]^2)\big) + \tilde{\phi}^{\mathrm{per}}_{p_{sd},q,m}\big(\mathcal{C}_v^\star([0,n]^2)\big) = 1.$$

Using the duality property for periodic boundary conditions and the symmetry of the lattice, the probability $\tilde{\phi}^{\mathrm{per}}_{p_{sd},q,m}(\mathcal{C}_v^\star([0,n]^2))$ is equal to $\tilde{\phi}^{\mathrm{per}}_{p_{sd},q,m}(\mathcal{C}_h([0,n]^2))$, giving

$$\tilde{\phi}^{\mathrm{per}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,n]^2)\big) = 1/2.$$

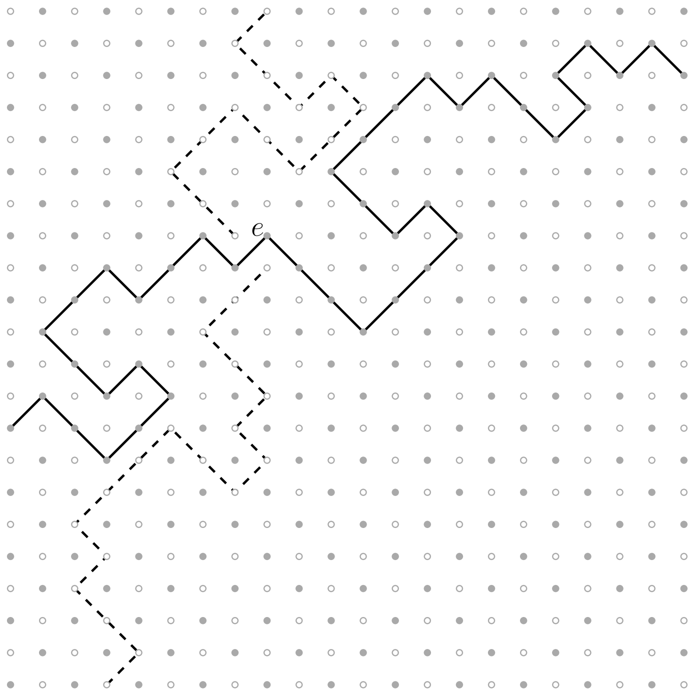

Figure 5.2: The square $[0,n]^2$ and its dual. The event $\mathcal{C}_h([0,n]^2)$ is occurring. If the right edge e is turned to closed, then $\mathcal{C}_v^\star([0,n]^2)$ occurs.

Using the fact that the Radon-Nykodym derivative of the random-cluster model with respect to the balanced random-cluster model is bounded by a constant depending on q only, the result follows. □

The only major difficulty is now to prove that rectangles of aspect ratio α are crossed in the horizontal direction — with probability uniformly bounded away from 0 — for $\alpha = \frac{3}{2}$. There are many ways of proving this in the case of percolation. Nevertheless, these strategies always involve independence in a crucial way. In our case, independence fails, thus a new argument is needed. The main idea is to invoke self-duality in order to force the existence of crossings, even in the case where boundary conditions could look disadvantageous. In order to do that, we introduce the following family of domains, which are in some sense nice symmetric domains.

Define the line $d := \{(x,y) : x = -\sqrt{2}/4\}$. The orthogonal symmetry σ_d with respect to this line maps $e^{i\pi/4}\mathbb{Z}^2$ to $e^{i\pi/4}(\mathbb{Z}^2)^\star$. Let γ_1 and γ_2 be two paths on $e^{i\pi/4}\mathbb{Z}^2$ satisfying the following Hypothesis ($\star$) (see Figure 5.3):

- γ_1 remains on the left of d and γ_2 remains on the right;

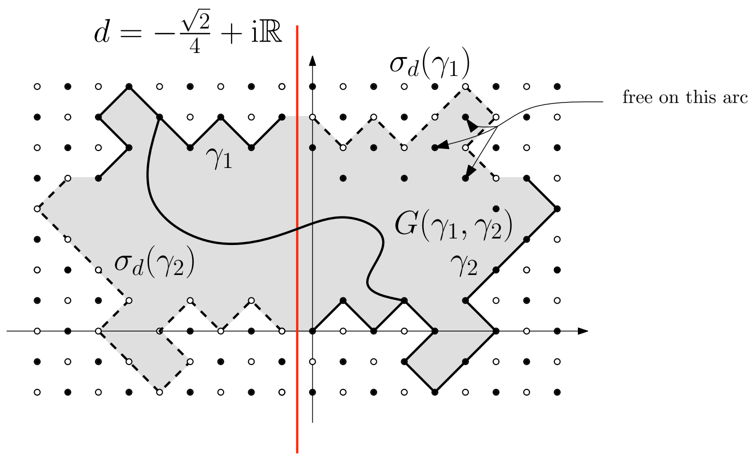

Figure 5.3: Two paths γ_1 and γ_2 satisfying Hypothesis $(\star)$ and the graph $G(\gamma_1, \gamma_2)$.

- γ_2 begins at 0 and γ_1 begins on a vertex of $e^{i\pi/4}\mathbb{Z}^2 \cap d'$, where $d' = \{(x, y) : x = -\sqrt{2}/2\}$;
- γ_1 and $\sigma_d(\gamma_2)$ do not intersect (as curves in the plane);
- γ_1 and $\sigma_d(\gamma_2)$ end at two vertices (one primal and one dual) which are at distance $\sqrt{2}/2$ from each other.

Note that $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$ are paths on $e^{i\pi/4}(\mathbb{Z}^2)^\star$. The definition extends trivially via translation, so that the pair (γ_1, γ_2) is said to satisfy Hypothesis $(\star)$ if one of its translations does.

When following the paths in counter-clockwise order, one can create a circuit by linking by a straight line the end points of γ_1 and $\sigma_d(\gamma_2)$, the start points of $\sigma_d(\gamma_2)$ and γ_2, the end points of γ_2 and $\sigma_d(\gamma_1)$, and the start points of $\sigma_d(\gamma_1)$ and γ_1. Constructed like that, the circuit $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$ surrounds a set of vertices. Define the graph $G(\gamma_1, \gamma_2)$ composed of the vertices that are surrounded by the circuit $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$, and of edges that remain entirely within the circuit (boundary included).

The *mixed boundary conditions* on this graph are wired on γ_1 (all the edges are pairwise connected), wired on γ_2, and free elsewhere. The measure on $G(\gamma_1, \gamma_2)$ with parameters (p_{sd}, q) and mixed boundary conditions is denoted by $\phi_{p_{sd},q,\gamma_1,\gamma_2}$ or more simply ϕ_{γ_1,γ_2}.

Lemma 5.8. *For any pair (γ_1, γ_2) satisfying Hypothesis $(\star)$, the following estimate holds:*

$$\phi_{\gamma_1,\gamma_2}(\gamma_1 \longleftrightarrow \gamma_2) \geq \frac{1}{1+q^2},$$

where $\gamma_1 \leftrightarrow \gamma_2$ means that γ_1 and γ_2 are connected inside $G(\gamma_1, \gamma_2)$.

Proof. On the one hand, if γ_1 and γ_2 are not connected in ω, $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$ must be connected by a dual path in $\omega^\star$ (this event corresponds to $\sigma_d(\gamma_1) \longleftrightarrow \sigma_d(\gamma_2)$ in the dual model). Hence,

$$1 = \phi_{\gamma_1,\gamma_2}(\gamma_1 \longleftrightarrow \gamma_2) + \sigma_d * \phi^\star_{\gamma_1,\gamma_2}(\gamma_1 \longleftrightarrow \gamma_2), \tag{5.4}$$

where $\sigma_d * (\phi^\star_{\gamma_1,\gamma_2})$ denotes the image under σ_d of the dual measure of ϕ_{γ_1,γ_2}. This measure lies on $G(\gamma_1,\gamma_2)$ as well and has parameters (p_{sd}, q).

As explained in Section 4.3.1, the dual boundary conditions of the mixed boundary conditions wired on γ_1 and γ_2, and free elsewhere are wired on $\gamma_1 \cup \gamma_2$ and free elsewhere (observe that we went to the dual model and then used the reflection σ_d). It is very important to notice that the boundary conditions are *not* exactly the mixed one, since γ_1 and γ_2 are wired *together*. Nevertheless, the Radon-Nikodym derivative of $\sigma_d * \phi^\star_{\gamma_1,\gamma_2}$ with respect to ϕ_{γ_1,γ_2} is easy to bound. Indeed, for any configuration ω, the number of clusters can differ only by 1 when counted in $\sigma_d * \phi^\star_{\gamma_1,\gamma_2}$ or ϕ_{γ_1,γ_2} so that the ratio of partition functions belongs to $[1/q, q]$. Therefore, the ratio of probabilities of the configuration ω remains between $1/q^2$ and q^2. This estimate extends to events by summing over all configurations. Therefore,

$$\sigma_d * \phi^\star_{\gamma_1,\gamma_2}(\gamma_1 \longleftrightarrow \gamma_2) \leq q^2 \phi_{\gamma_1,\gamma_2}(\gamma_1 \longleftrightarrow \gamma_2).$$

When plugging this inequality into (5.4), we obtain

$$\phi_{\gamma_1,\gamma_2}(\gamma_1 \longleftrightarrow \gamma_2) + q^2 \phi_{\gamma_1,\gamma_2}(\gamma_1 \longleftrightarrow \gamma_2) \geq 1$$

which implies the claim. □

Remark 5.9. The most important example (on the rotated lattice) of a symmetric domain is the rotated version of a square (it has a $\diamond$ shape). When rotating back the shape and the lattice, we obtain a "standard" square with wired/free/wired/free boundary conditions (i.e. mixed boundary conditions) and we thus obtain the following useful inequality on the square lattice: for every $n \geq 1$,

$$\phi^{\text{mixed}}_{p_c,q,[0,n]^2}\left(\mathcal{C}_h([0,n]^2)\right) \geq \frac{1}{1+q^2}. \tag{5.5}$$

We are now in a position to prove Proposition 5.6.

Proof of Proposition 5.6. The proof goes as follows: we start with creating two paths crossing square boxes, and we then prove that they are connected with good probability.

Step 1: setting of the proof. Consider the rectangle $R = [0, 3n/2] \times [0, n]$ which is the union of the rectangles $R_1 = [0, n] \times [0, n]$ and $R_2 = [n/2, 3n/2] \times [0, n]$, see Figure 5.4. Let A be the event defined by the following conditions:

- R_1 and R_2 are both crossed horizontally (these events have probability at least $c(1)$ to occur, using Lemma 5.7);
- $[n/2, n] \times \{0\}$ is connected inside R_2 to the top side of R_2 (this event has probability greater than $c(1)/2$ to occur using symmetry and Lemma 5.7).

Employing the FKG inequality, we deduce that

$$\phi^{\mathrm{per}}_{p_{sd},q,m}(A) \geq \frac{c(1)^3}{2}. \tag{5.6}$$

When A occurs, define Γ_1 to be the top-most open self-avoiding path crossing R_1 horizontally, and Γ_2 the right-most open self-avoiding path crossing R_2 from $[n/2, n] \times \{0\}$ to the top side. Note that this path is automatically connected by an open path to the right-hand side of R_2 — which is the same as the right-most side of R. In particular, if Γ_1 and Γ_2 are connected, then there exists a horizontal crossing of R. In the following, Γ_1 and Γ_2 are shown to be connected with good probability.

Step 2: the reflection argument. Assume first that $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$, and that they do not intersect. Let x be the vertex at the right end of γ_1 (it is the unique vertex in γ_1 which is on the right side of R_1). We wish to define a set $G_0(\gamma_1, \gamma_2)$ similar to those considered in Lemma 5.8. Apply the following "surgical procedure" (see Figure 5.4 to help visualize):

- First, define the symmetric paths $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$ of γ_1 and γ_2 with respect to the line $d := \{(x, y) : x = n - \sqrt{2}/4\}$.
- Then, parametrize the path $\sigma_d(\gamma_1)$ by the number of steps (along the path) from the starting point $\sigma_d(x)$ and define $\tilde{\gamma}_1 \subset \gamma_1$ so that $\sigma_d(\tilde{\gamma}_1)$ is the part of $\sigma_d(\gamma_1)$ between the beginning of the path and the first time it intersects γ_2. As before, the paths are considered as curves of the plane. Denote the intersection point of the two curves by z. Note that γ_1 and γ_2 do not intersect, which forces $\sigma_d(\gamma_1)$ and γ_2 to do.
- From this, parametrize the path γ_2 by the distance to its "starting point" on $[\frac{n}{2}, n] \times \{0\}$ and set y to be the last visited vertex in $e^{i\frac{\pi}{4}}\mathbb{Z}^2$ before the intersection z. Define $\tilde{\gamma}_2$ to be the part of γ_2 between the last point intersecting $\{n\} \times [0, n]$ before y and y itself.

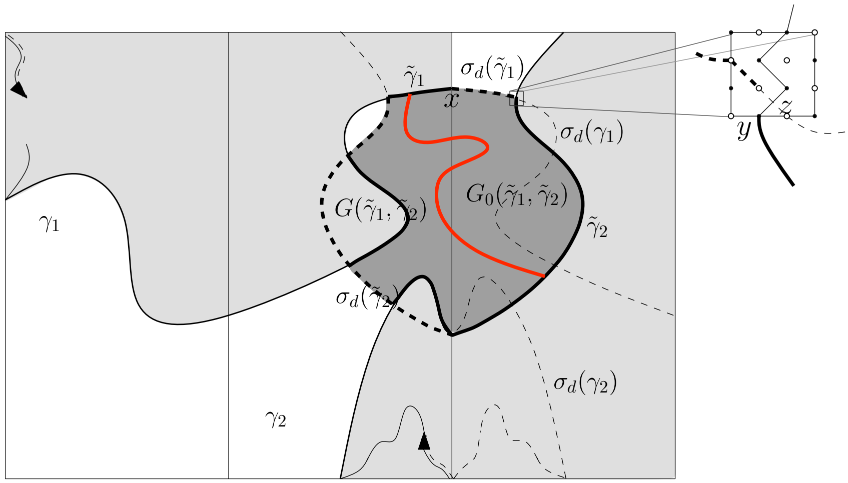

Figure 5.4: The light gray area denotes the edges of R on which the event $\{\Gamma_1 = \gamma_1\} \cap \{\Gamma_2 = \gamma_2\}$ depend. The dark gray is the domain $G_0(\tilde\gamma_1, \tilde\gamma_2)$. All the paths involved in the construction are depicted. Note that dashed curves are "virtual paths" of the dual lattice obtained by the reflection σ_d: they are not necessarily dual open.

- Paths $\tilde\gamma_1$ and $\tilde\gamma_2$ satisfy Hypothesis $(\star)$ so that the graph $G(\tilde\gamma_1, \tilde\gamma_2)$ can be defined.
- Construct a sub-graph $G_0(\tilde\gamma_1, \tilde\gamma_2)$ of $G(\tilde\gamma_1, \tilde\gamma_2)$ as follows: the edges are given by the edges of $e^{i\pi/4}\mathbb{Z}^2$ included in the connected component of $G(\tilde\gamma_1, \tilde\gamma_2) \smallsetminus (\gamma_1 \cup \gamma_2)$ (*i.e.* $G(\tilde\gamma_1, \tilde\gamma_2)$ minus the set $\gamma_1 \cup \gamma_2$) containing d (it is the connected component which contains $x - (0, \varepsilon)$, where $\varepsilon > 0$ is a very small number), and the vertices are given by their endpoints.

Step 3: conditional probability estimate. Still assuming that γ_1 and γ_2 do not intersect, we would like to estimate the probability of γ_1 and γ_2 being connected by a path knowing that $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$. The very important fact here is that $\{\Gamma_1 = \gamma_1\} \cap \{\Gamma_2 = \gamma_2\}$ is measurable in terms of edges above or on γ_1 and on the right of or on γ_2. In particular, the event is measurable in terms of edges outside $G(\tilde\gamma_1, \tilde\gamma_2)$ and edges in γ_1 and γ_2 (This is the same argument as the outermost dual-circuit in the proof of Corollary 4.21). Therefore, the configuration in the domain follows a random-cluster model with specific boundary conditions.

The boundary of $G_0(\tilde\gamma_1, \tilde\gamma_2)$ can be split into several sub-arcs of various types (see Figure 5.4): some are sub-arcs of γ_1 or γ_2, while the others are (adjacent to) sub-arcs of their symmetric images $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$. The conditioning on $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$ ensures that the edges along the sub-arcs of the first type are open; the connections along the others depend on the configuration outside $G_0(\tilde\gamma_1, \tilde\gamma_2)$ in a much more intricate way, but in any case the boundary conditions imposed on the configuration inside $G(\tilde\gamma_1, \tilde\gamma_2)$ are larger than the mixed boundary conditions. Notice, for instance by looking at Fig. 5.4, that *any* boundary conditions dominate the free one and that $\tilde\gamma_1$ and $\tilde\gamma_2$ are two sub-arcs of the first type (they are then wired). Thus, the measure restricted to $G_0(\tilde\gamma_1, \tilde\gamma_2)$ stochastically dominates the restriction of $\phi_{\tilde\gamma_1,\tilde\gamma_2}$ to $G_0(\tilde\gamma_1, \tilde\gamma_2)$.

From these observations, we deduce that for any increasing event B *depending only on edges in* $G_0(\tilde\gamma_1, \tilde\gamma_2)$,

$$\phi^{\mathrm{per}}_{p_{sd},q,m}(B|\Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2) \geq \phi_{\tilde\gamma_1,\tilde\gamma_2}(B). \tag{5.7}$$

In particular, this inequality can be applied to $\{\gamma_1 \leftrightarrow \gamma_2 \text{ in } G_0(\tilde\gamma_1, \tilde\gamma_2)\}$. If $\tilde\gamma_1$ and $\tilde\gamma_2$ are connected in $G(\tilde\gamma_1, \tilde\gamma_2)$, then γ_1 and γ_2 are connected in $G_0(\tilde\gamma_1, \tilde\gamma_2)$. The first event is of $\phi_{\tilde\gamma_1,\tilde\gamma_2}$-probability at least $1/(1+q^2)$ (Lemma 5.8), thus implying

$$\begin{aligned}\phi^{\mathrm{per}}_{p_{sd},q,m}(\gamma_1 \leftrightarrow \gamma_2|\Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2) &\geq \phi_{\tilde\gamma_1,\tilde\gamma_2}(\gamma_1 \leftrightarrow \gamma_2 \text{ in } G_0(\tilde\gamma_1, \tilde\gamma_2)) \\ &\geq \phi_{\tilde\gamma_1,\tilde\gamma_2}(\tilde\gamma_1 \leftrightarrow \tilde\gamma_2) \geq \frac{1}{1+q^2}.\end{aligned} \tag{5.8}$$

Step 5: conclusion of the proof. Note the following obvious fact: if γ_1 and γ_2 intersect, the conditional probability that Γ_1 and Γ_2 intersect, knowing $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$ is equal to 1 — in particular, it is greater than $1/(1+q^2)$. Now,

$$\begin{aligned}\phi^{\mathrm{per}}_{p_{sd},q,m}(\mathcal{C}_h(R)) &\geq \phi^{\mathrm{per}}_{p_{sd},q,m}(\mathcal{C}_h(R) \cap A) \\ &\geq \phi^{\mathrm{per}}_{p_{sd},q,m}(\{\Gamma_1 \leftrightarrow \Gamma_2\} \cap A) \\ &= \phi^{\mathrm{per}}_{p_{sd},q,m}\big(\phi^{\mathrm{per}}_{p_{sd},q,m}(\Gamma_1 \leftrightarrow \Gamma_2|\Gamma_1, \Gamma_2)\mathbf{1}_A\big) \\ &\geq \frac{1}{1+q^2}\phi^{\mathrm{per}}_{p,q,m}(A) \geq \frac{c(1)^3}{2(1+q)^2}\end{aligned}$$

where the first two inequalities are due to inclusion of events, the third one to the definition of conditional expectation, and the fourth and fifth ones, to (5.8) and (5.6) respectively. □

5.2 Application I: critical point of the random-cluster model

5.2.1 Statement of the theorem

The RSW theory for periodic boundary conditions has a very important consequence: it enables us to compute the critical value of the random-cluster model.

Theorem 5.10 (Beffara, Duminil-Copin [BDC12a]). *Let $q \geq 1$. The critical point $p_c = p_c(q)$ for the random-cluster model with cluster-weight q on the square lattice satisfies*

$$p_c = p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

A rigorous derivation of the critical point was previously known in three cases. For $q = 1$, the model is simply bond percolation, proved by Kesten in 1980 [Kes80] to be critical at $p_c(1) = 1/2$. Onsager derived the critical temperature of the Ising model in 1944 [Ons44]. One can actually couple realizations of the Ising and random-cluster models to relate their critical points, see Chapter 7, so that the $q = 2$ case follows from Onsager's result. For modern proofs in that case, see [AKN87] or Chapter 11. Finally, for sufficiently large q, a proof is known based on the fact that the random-cluster model exhibits a first order phase transition; see [LMMS+91, KS82] (the proofs are valid for q larger than 25.72). Let us mention that physicists derived the critical temperature for the Potts models with $q \geq 4$ in 1978, using non-geometric arguments based on analytic properties of the Hamiltonian [HKW78], and that we will present an alternative proof for this special case in Section 6.3.

Now, a few words on the method of proofs. The first ingredient is the RSW theory for periodic boundary conditions. The second ingredient is a collection of sharp threshold theorems, which were originally introduced for product measures.

5.2.2 Sharp threshold for boolean functions

Let us start by presenting the sharp threshold theorems. We wish to understand the behavior of the function $p \mapsto \phi^{\xi}_{p,q,n}(A)$ for an increasing event A (different from $\varnothing$ and $\{0,1\}^{E_{\mathbb{T}_n}}$). This increasing function is equal to 0 at $p = 0$ and to 1 at $p = 1$, and we are interested in the range of p for which its value is between ε and $1 - \varepsilon$ for some positive ε. Under mild conditions on A, the width will be bounded from above in terms of the size of the underlying graph, which is known as a *sharp threshold* behavior. The proof of this fact is based on differential inequalities.

Historically, the general theory of sharp thresholds was first developed by Bourgain, Kahn, Kalai, Katznelson and Linial [BKK+92] (see also [Fri04, FK96, KS06]) in the case of product measures. In lattice models such as percolation, these results have been used (see [BR06a, BR06b]) via a differential equality known as Russo's formula (see [Gri99, Rus81]). Both sharp threshold theory and Russo's formula were later extended to random-cluster measures with $q \geq 1$, see references below. These arguments being not totally standard, we remind the readers of the classical results and refer them to [Gri06] for general statements. Except for Theorem 5.13, the proofs are quite short so that it is natural to include them. The proofs are directly extracted from Grimmett's monograph [Gri06].

Intuitively, the derivative of $\phi^{\xi}_{p,q,G}(A)$ with respect to p is governed by the influence of edges switching from closed to open. The following definition is therefore natural in this setting. The *(conditional) influence* on A of the edge $e \in E_G$, denoted by $I_A(e)$, is defined as

$$I_A(e) := \phi^{\xi}_{p,q,G}(A|\omega(e) = 1) - \phi^{\xi}_{p,q,G}(A|\omega(e) = 0).$$

Proposition 5.11. *Let G be a finite graph, $q \geq 1$ and $\varepsilon > 0$; there exists $c = c(q,\varepsilon) > 0$ such that for any $p \in [\varepsilon, 1-\varepsilon]$, any boundary conditions ξ, and any increasing event A,*

$$\frac{d}{dp}\phi^{\xi}_{p,q,G}(A) \geq c \sum_{e \in E_G} I_A(e).$$

Proof. Let A be an increasing event. The key step is the following inequality which can be obtained by differentiating with respect to p (for details of the computation, see [Gri06, Theorem (2.46)]):

$$\frac{d}{dp}\phi^{\xi}_{p,q,G}(A) = \frac{1}{p(1-p)} \sum_{e \in E_G} \Big[\phi^{\xi}_{p,q,G}\big(\mathbf{1}_A\omega(e)\big) - \phi^{\xi}_{p,q,G}[\omega(e)]\phi^{\xi}_{p,q,G}(A)\Big]. \tag{5.9}$$

A similar differential formula is actually true for any random variable X, but this fact will not be used in the proof. Note that, by definition of $I_A(e)$,

$$\phi^{\xi}_{p,q,G}\big(\mathbf{1}_A\omega(e)\big) - \phi^{\xi}_{p,q,G}(A)\phi^{\xi}_{p,q,G}\big(\omega(e)\big) = I_A(e)\phi^{\xi}_{p,q,G}\big(\omega(e)\big)\big[1 - \phi^{\xi}_{p,q,G}\big(\omega(e)\big)\big]$$

so that (5.9) becomes

$$\begin{aligned}\frac{d}{dp}\phi^{\xi}_{p,q,G}(A) &= \frac{1}{p(1-p)} \sum_{e \in E_G} \phi^{\xi}_{p,q,G}\big(\omega(e)\big)\big[1 - \phi^{\xi}_{p,q,G}\big(\omega(e)\big)\big] I_A(e)\\ &= \sum_{e \in E_G} \frac{\phi^{\xi}_{p,q,G}\big(\omega(e)\big)\big[1 - \phi^{\xi}_{p,q,G}\big(\omega(e)\big)\big]}{p(1-p)} I_A(e) \qquad (5.10)\end{aligned}$$

from which the claim follows since the term

$$\frac{\phi^{\xi}_{p,q,G}\big(\omega(e)\big)\big[1-\phi^{\xi}_{p,q,G}\big(\omega(e)\big)\big]}{p(1-p)} \tag{5.11}$$

is bounded away from 0 uniformly in $p \in [\varepsilon, 1-\varepsilon]$ and $e \in E_G$ when q is fixed thanks to the finite-energy property (Proposition 4.4). □

Remark 5.12. When $q = 1$, (5.10) corresponds to Russo's formula. Indeed, let $\phi_p = \phi^{\xi}_{p,1,G}$. The influence can be rewritten as

$$\begin{aligned} I_A(e) &= \frac{\phi_p(A,\omega(e)=1)}{p} - \frac{\phi_p(A,\omega(e)=0)}{1-p} \\ &= \frac{\phi_p(\omega^e \in A, \omega_e \notin A, \omega(e)=1)}{p} + \frac{\phi_p(\omega_e \in A, \omega(e)=1)}{p} - \frac{\phi_p(\omega_e \in A, \omega(e)=0)}{1-p} \\ &= \phi_p(\omega^e \in A, \omega_e \notin A). \end{aligned}$$

In the last line, we used that $\omega(e)$ is independent of the events $\omega^e \in A$ and $\omega_e \in A$, and the independence of the measure. Furthermore, the inequality is an equality since (5.11) equals 1. The event $\{\omega^e \in A, \omega_e \notin A\}$ is usually called e is pivotal for A. We thus obtain

$$\frac{d}{dp}\phi_p(A) = \sum_{e\in A} \phi_p(e \text{ is pivotal for } A).$$

There has been an extensive study of the largest influence in the case of product measures. It was initiated in [BKK+92]. The following theorem is a special case of the generalization to positively-correlated measures.

Theorem 5.13 (Graham, Grimmett [GG06, GG11])**.** *Let G be a finite graph, $q \geq 1$ and $\varepsilon > 0$ as well as boundary conditions ξ. There exists a constant $c = c(q,\varepsilon) > 0$ such that the following holds. For every $p \in [\varepsilon, 1-\varepsilon]$ and every increasing event A,*

$$\max\big\{I_A(e) : e \in E_G\big\} \geq c\,\phi^{\xi}_{p,q,G}(A)\big(1-\phi^{\xi}_{p,q,G}(A)\big)\frac{\log|E_G|}{|E_G|}.$$

There is a particularly efficient way of using Proposition 5.11 together with Theorem 5.13. In the case of a translation-invariant event on a torus of size n, horizontal (resp. vertical) edges play symmetric roles, so that the influence is the same for all the edges of a given orientation. In particular, Proposition 5.11 together with Theorem 5.13 provide us with the following differential inequality:

Theorem 5.14. *Let $q \geq 1$ and $\varepsilon > 0$. There exists a constant $c = c(q,\varepsilon) > 0$ such that the following holds. For every $p \in [\varepsilon, 1-\varepsilon]$, every $n \geq 1$ and every increasing translation-invariant event A on $\mathbb{T}_n$,*

$$\frac{d}{dp}\phi^{\mathrm{per}}_{p,q,n}(A) \geq c\big(\phi^{\mathrm{per}}_{p,q,n}(A)(1-\phi^{\mathrm{per}}_{p,q,n}(A)\big)\log n.$$

The presence of $\phi^{\mathrm{per}}_{p,q,n}(A)(1-\phi^{\mathrm{per}}_{p,q,n}(A))$ should not be a surprise. When the probability of the event A becomes close to 0 or 1, the lower bound on the derivative cannot be large (since the derivative is small), which one can see since $\phi^{\mathrm{per}}_{p,q,n}(A)$ or $1-\phi^{\mathrm{per}}_{p,q,n}(A)$ is small.

For a non-empty increasing event A, the previous inequality can be integrated between two parameters $p_1 < p_2$ (we recognize the derivative of $\log(x/(1-x))$) to obtain

$$\frac{1-\phi^{\mathrm{per}}_{p_1,q,n}(A)}{\phi^{\mathrm{per}}_{p_1,q,n}(A)} \geq \frac{1-\phi^{\mathrm{per}}_{p_2,q,n}(A)}{\phi^{\mathrm{per}}_{p_2,q,n}(A)} n^{c(p_2-p_1)}.$$

If $\phi^{\mathrm{per}}_{p_1,q,n}(A)$ is assumed to stay bounded away from 0 uniformly in $n \geq 1$, we deduce the existence of $c' > 0$ such that

$$\phi^{\mathrm{per}}_{p_2,q,n}(A) \geq 1 - c' n^{-c(p_2-p_1)}. \tag{5.12}$$

This inequality will be instrumental in the next section. In conclusion, one may keep in mind that *probabilities of nontrivial (meaning different from* $\varnothing$ *and* $\{0,1\}^{E_{\mathbb{T}_n}}$*) symmetric increasing events undergo a sharp threshold when* p *is varied from 0 to 1.*

5.2.3 The proof of Theorem 5.10

We adapt the method developed first by Bollobàs and Riordan [BR06a, BR06b] in the case of Bernoulli percolation. Let us sketch the main steps of the proof first. We first argue that crossing probabilities tend to 1 (as n tends to infinity) when $p > p_{sd}$. In order to see that, consider the translational-invariant event that *some* rectangle of $\mathbb{T}_n$ with width $n/2$ and height $\alpha^2 n$ (with $\alpha \gg 1$) is crossed vertically. At $p = p_{sd}$, the probability of this event is known to be bounded away from 0 uniformly in the size of the torus (thanks to Theorem 5.3). Therefore, Theorem 5.14 can be applied to conclude that the probability goes to 1 when $p > p_{sd}$ (there is also an explicit lower bound on the probability). It is then an easy step to deduce that the probability of crossing vertically a *particular* rectangle with width n and height αn also tends to 1. Note that in order to go from *some* rectangle to a *particular* one, we will need to change the aspect-ratio of rectangles we are considering (see the proof for more details).

Theorem 5.10 is then proved by constructing a path from 0 to infinity when $p > p_{sd}$, which is usually done by combining crossings of rectangles. There is a major difficulty in doing such a construction: one needs to transform estimates in the torus into estimates in the whole plane. One solution is to replace the periodic boundary conditions by wired boundary conditions. The path construction is a little tricky since it must propagate wired boundary conditions through the construction (see Proposition 5.17); it does not follow the standard lines.

We now implement the program sketched above. We start with proving that crossings of long rectangles exist with very high probability when $p > p_{sd}$.

Lemma 5.15. *Let $\alpha > 1$, $q \geq 1$ and $p > p_{sd}$. There exist $\varepsilon_0 = \varepsilon_0(p,q,\alpha) > 0$ and $c_0 = c_0(p,q,\alpha) > 0$ such that for every $n \geq 1$*

$$\phi^{\mathrm{per}}_{p,q,\alpha^2 n}\left(\mathcal{C}_v([0,n]\times[0,\alpha n])\right) \geq 1 - c_0 n^{-\varepsilon_0}. \tag{5.13}$$

Proof. The proof will make it clear that it is sufficient to treat the case of integer α, we therefore assume that α is a positive integer (not equal to 1). Let B be the event that there exists a vertical crossing of a rectangle with dimensions $(n/2, \alpha^2 n)$ in the torus of size $\alpha^2 n$. This event is invariant under translations and satisfies

$$\phi^{\mathrm{per}}_{p_{sd},q,\alpha^2 n}(B) \geq \phi^{\mathrm{per}}_{p_{sd},q,\alpha^2 n}\big(\mathcal{C}_v([0,n/2]\times[0,\alpha^2 n])\big) \geq c(2\alpha^2)$$

uniformly in n. Since B is increasing, Theorem 5.14 (more precisely (5.12)) can be applied to deduce that for any $p > p_{sd}$, there exist $\varepsilon = \varepsilon(p,q,\alpha)$ and $c = c(p,q,\alpha)$ such that

$$\phi^{\mathrm{per}}_{p,q,\alpha^2 n}(B) \geq 1 - cn^{-\varepsilon}. \tag{5.14}$$

If B holds, one of the $2\alpha^3$ rectangles

$$[in/2, in/2+n]\times[j\alpha n,(j+1)\alpha n], \quad (i,j)\in\{0,\dots,2\alpha^2-1\}\times\{0,\dots,\alpha-1\}$$

must be crossed from top to bottom. Denote these events by A_{ij} — they are translates of $\mathcal{C}_v([0,n]\times[0,\alpha n])$. We find

$$\begin{aligned}\phi^{\mathrm{per}}_{p,q,\alpha^2 n}(B) &= 1 - \phi^{\mathrm{per}}_{p,q,\alpha^2 n}(B^c) \leq 1 - \phi^{\mathrm{per}}_{p,q,\alpha^2 n}\Big(\bigcap_{i,j} A^c_{ij}\Big)\\ &\leq 1 - \prod_{i,j}\phi^{\mathrm{per}}_{p,q,\alpha^2 n}(A^c_{ij})\\ &= 1 - \Big[1 - \phi^{\mathrm{per}}_{p,q,\alpha^2 n}\big(\mathcal{C}_v([0,n]\times[0,\alpha n])\big)\Big]^{2\alpha^3}.\end{aligned}$$

The FKG inequality was used in the second line (the reader may recognize the implementation of the "square-root trick" mentioned earlier). Plugging (5.14) into the previous inequality, we deduce

$$\phi^{\mathrm{per}}_{p,q,\alpha^2 n}\left(\mathcal{C}_v\big([0,n]\times[0,\alpha n]\big)\right) \geq 1 - (cn^{-\varepsilon})^{1/(2\alpha^3)}.$$

The claim follows by setting $c_0 := c^{1/(2\alpha)^3}$ and $\varepsilon_0 := \varepsilon/(2\alpha^3)$. □

Let $K > 1$ and $n \geq 1$; define the annulus

$$A^K_n := \Lambda_{K^{n+1}} \smallsetminus \Lambda_{K^n}.$$

An *open circuit* in an annulus is an open path which surrounds the origin. Denote by $\mathcal{A}_n^K$ the event that there exists an open circuit surrounding the origin and contained in A_n^K, together with an open path from this circuit to the boundary of $\Lambda_{K^{n+2}}$, see Figure 5.5. The following lemma shows that the probability of $\mathcal{A}_n^K$ goes to 1, provided that $p > p_{sd}$ and that boundary conditions are wired on $\Lambda_{K^{n+2}}$.

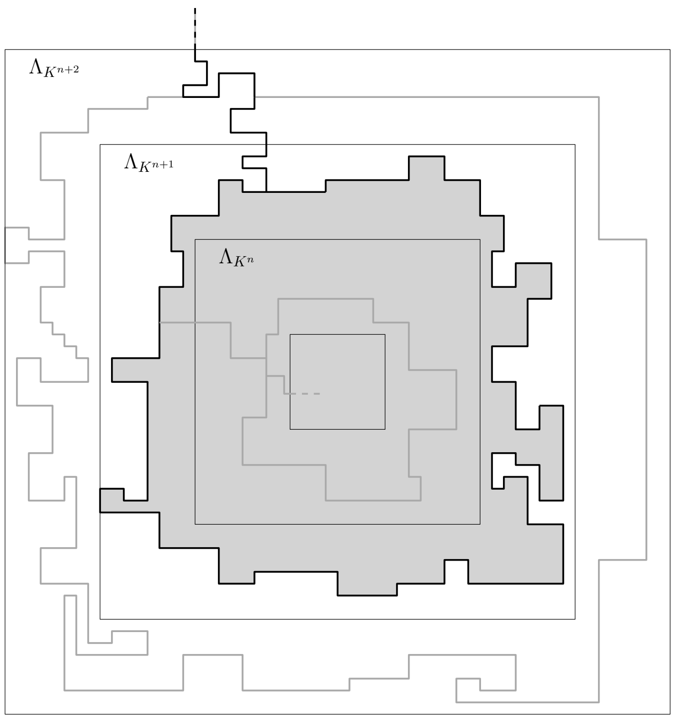

Figure 5.5: The event $\mathcal{A}_n^K$. The gray area is the part of $\mathbb{Z}^2$ delimited by the outermost open self-avoiding circuit in A_n^K. The configuration inside this outermost open self-avoiding circuit (namely the gray area) stochastically dominates a random-cluster configuration with wired boundary conditions on the boundary of the box. The combination of events $\mathcal{A}_n^K$ constructs a path from the origin to infinity.

Lemma 5.16. *Let $K > 1$, $q \geq 1$ and $p > p_{sd}$. There exist $c_1 = c_1(p,q,K) > 0$ and $\varepsilon_1 = \varepsilon_1(p,q,K) > 0$ such that for every $n \geq 1$,*

$$\phi^1_{p,q,K^{n+2}}(\mathcal{A}_n^K) \geq 1 - c_1 e^{-\varepsilon_1 n}.$$

Proof. First, observe that $\mathcal{A}_n^K$ occurs whenever the following events occur simultaneously:

- The following rectangles are crossed vertically:

$$R_1 := [K^n, K^{n+1}] \times [-K^{n+1}, K^{n+1}],$$
$$R_2 := [-K^{n+1}, -K^n] \times [-K^{n+1}, K^{n+1}].$$

- The following rectangles are crossed horizontally:

$$R_3 := [-K^{n+1}, K^{n+1}] \times [K^n, K^{n+1}],$$
$$R_4 := [-K^{n+1}, K^{n+1}] \times [-K^{n+1}, -K^n],$$
$$R_5 := [-K^{n+2}, K^{n+2}] \times [-K^n, K^n].$$

Using the comparison between periodic boundary conditions and wired boundary conditions on $\partial\Lambda_{K^{n+2}}$, the previous lemma implies that the probability of each of these events is greater than $1 - c(K^n)^{-\varepsilon}$ with $c = c_0(p, q, K')$ and $\varepsilon = \varepsilon_0(p, q, K')$, where $K' = \max\{K^2, 2K/(K-1)\}$. Using the FKG inequality, we obtain

$$\phi^1_{p,q,K^{n+2}}(\mathcal{A}_n^K) \geq (1 - c(K^n)^{-\varepsilon})^5.$$

The claim follows by setting $c_1 := 5c$ and $\varepsilon_1 := \varepsilon \log K$. □

We wish to prove that the probability of the intersection of events $\mathcal{A}_n^K$ for $n \geq 0$ is of positive probability when $p > p_{sd}$. So far, we only know that there is an open circuit with very high probability when we consider the random-cluster measure with wired boundary conditions in a slightly larger box. In order to prove the result, assume the existence of a large circuit. Then, we iteratively condition on events $\mathcal{A}_{n-k}^K$, $k \geq 0$. By conditioning "from the outside to the inside", there exists an outermost open self-avoiding circuit in A_{n-k+1}^K that surrounds A_{n-k}^K at every step k. Using comparison between boundary conditions, the measure in A_{n-k}^K stochastically dominates the measure in A_{n-k+1}^K with wired boundary conditions. In other words, we keep track of advantageous boundary conditions. Note that the reasoning, while reminiscent of Kesten's construction of an infinite path for percolation, is not standard.

Let $p > p_{sd}$ and $q \geq 1$. Recall that $\phi_{p,q}$ is the unique infinite-volume measure on $\mathbb{Z}^2$.

Proposition 5.17. *Let $K > 1$, $q \geq 1$ and $p > p_{sd}$. There exist $c, c_1, \varepsilon_1 > 0$ (depending on p, q and K) such that for every $N \geq 1$,*

$$\phi_{p,q}\Big(\bigcap_{n \geq N} \mathcal{A}_n^K\Big) \geq c \prod_{n=N}^{\infty} (1 - c_1 \mathrm{e}^{-\varepsilon_1 n}) > 0.$$

We will only use $N = 1$ in order to prove Theorem 5.10. Nevertheless, the more general statement with arbitrary N will be useful in the next section.

Proof. Let $K > 1$, $q \geq 1$, $p > p_{sd}$ and $N \geq 1$. For every $n \geq 1$, we know that

$$\phi_{p,q}\Big(\bigcap_{k=N}^{n} \mathcal{A}_k^K\Big) = \phi_{p,q}(\mathcal{A}_n^K) \prod_{k=N}^{n-1} \phi_{p,q}(\mathcal{A}_k^K | \mathcal{A}_j^K, k+1 \leq j \leq n). \tag{5.15}$$

Let $N \leq k < n$. We now wish to estimate the probability of $\mathcal{A}_k^K$ conditionally on $\mathcal{A}_j^K$ for $k+1 \leq j \leq n$. In order to do so, we use a "conditioning on the outermost circuit" argument (we refer to Section 4.4.4 for more details on this argument). This time, the circuits are primal instead of dual, and the boundary conditions that we will use to make the comparison are the wired ones. Let us spell the whole argument.

Conditionally on $\mathcal{A}_j^K$, $k+1 \leq j \leq n$, there exists an open self-avoiding circuit in the annulus A_{k+1}^K. Consider the outermost such circuit, denoted by Γ. Conditionally on $\Gamma = \gamma$, the configuration in the inner part $\overline{\gamma}$ of the box $\Lambda_{K^{k+2}}$ has the law of a random-cluster configuration with wired boundary condition. In particular, the conditional probability that there exists a circuit in A_k^K connected to γ is greater than the probability that there exists a circuit in A_k^K connected to the boundary of $\Lambda_{K^{k+2}}$ with wired boundary conditions. Therefore, we obtain that almost surely

$$\begin{aligned}\phi_{p,q}(\mathcal{A}_k^K | \mathcal{A}_j^K, k+1 \leq j \leq n) &= \phi_{p,q}\big(\phi_{p,q}(\mathcal{A}_k^K | \Gamma) | \mathcal{A}_j^K, k+1 \leq j \leq n\big)\\ &\geq \phi_{p,q}\big(\phi^1_{p,q,K^{k+2}}(\mathcal{A}_k^K)\big)\\ &\geq 1 - c_1 \mathrm{e}^{-\varepsilon_1 k},\end{aligned}$$

where Lemma 5.16 was harnessed in the last inequality.

For $p = p_{sd}$, consider the event $\mathcal{A}_n^K$ on $\mathbb{Z}^2$. Thanks to Corollary 5.5, its probability is bounded away from 0 uniformly in n. Since the event is increasing, there exists $c = c(K) > 0$ such that

$$\phi_{p,q}(\mathcal{A}_n^K) = \phi^1_{p,q}(\mathcal{A}_n^K) \geq \phi^1_{p_{sd},q}(\mathcal{A}_n^K) \geq c$$

for any $n \geq N$ and $p > p_{sd}$. Plugging the two estimates into (5.15), we obtain

$$\phi_{p,q}\Big(\bigcap_{k=N}^{n} \mathcal{A}_k^K\Big) \geq c \prod_{k=N}^{n-1} (1 - c_1 \mathrm{e}^{-\varepsilon_1 k}) \geq c \prod_{k=N}^{\infty} (1 - c_1 \mathrm{e}^{-\varepsilon_1 k}).$$

Letting n go to infinity concludes the proof. □

Proof of Theorem 5.10. The bound $p_c \geq p_{sd}$ is provided by Zhang's argument (Proposition 4.38). For $p > p_{sd}$, fix $K > 1$. Applying Proposition 5.17 with $N = 1$, we find

$$\phi_{p,q}(0 \leftrightarrow \infty) \geq c\phi_{p,q}\Big(\bigcap_{n\geq 1} \mathcal{A}_n^K\Big) > 0$$

so that p is supercritical. The constant $c > 0$ is due to the fact that Λ_{K^2} is required to contain open edges only, and therefore $c > 0$ exists using the finite-energy property (Proposition 4.4). Therefore $p \geq p_c$ for every $p > p_{sd}$ and we deduce $p_c \leq p_{sd}$. □

5.3 Application II: exponential decay in the subcritical phase

In this section, we study the subcritical and supercritical phases. In the subcritical phase, the probability for two vertices x and y to be connected by an open path is proved to decay exponentially fast with respect to the distance between x and y. In the supercritical phase, the same behavior holds in the dual model since $p_c = p_{sd}$. This phenomenon is known as a *sharp phase transition.*

Theorem 5.18 (Beffara, Duminil-Copin [BDC12a])**.** *Let $q \geq 1$. For any $p < p_c(q)$, there exists $c = c(p,q) > 0$ such that for any $x, y \in \mathbb{Z}^2$,*

$$\phi_{p,q}(x \leftrightarrow y) \leq e^{-c|x-y|}. \tag{5.16}$$

Remark 5.19. Theorem 5.18 has important consequences. Over the last twenty years, a deep understanding of the subcritical regime was developed under the hypothesis that probabilities of connection between two vertices decay exponentially fast in the distance between them. Unfortunately, this assumption was known only for small p. The result above justifies this assumption in the whole subcritical regime. We refer to [Gri06] for more details on potential applications of Theorem 5.18.

Remark 5.20. Going back for a moment to crossing probabilities, one sees that in the subcritical phase $p < p_c$, the probability of crossing vertically $R = [0,n] \times [0,\alpha n]$ decays exponentially fast in n. In fact, the largest cluster in R can be proved (simply use the union bound) to be smaller than $C \log n$, where $C = C(p)$ depends on α and the constant c of the theorem only. In the reverse direction, the probability of not crossing vertically the rectangle R for $p > p_c$ is exactly the probability of crossing horizontally the dual graph of R in the dual configuration. Since the dual model is subcritical, this probability decays exponentially fast. In other words, the probability of crossing vertically R tends to 1 exponentially fast.

Remark 5.21. The previous result has been extended to more general planar random-cluster models. More precisely, it is proved in [DCM13a] that the phase transition is sharp: correlations decay exponentially fast in the subcritical phase.

The proof runs as follows. Combining the fact that the crossing probabilities go to 0 when $p < p_{sd}$ with a very general differential inequality (see the proposition below), we deduce that the cluster-size at the origin has finite moments of any order. It is then a classical step to deduce exponential decay using a general result on the greedy lattice animals model, see [CGGK93, GK94] for details.

Consider a configuration ω as a vertex of the graph $\{0,1\}^{E_G}$. For $A \subset \{0,1\}^{E_G}$, let $H_A(\omega)$ be the graph distance between the configuration ω and A. This quantity is called the Hamming distance of ω to A. Another way of seeing the Hamming distance is simply to say that it is the minimum number of edges that must be changed on ω in order to be in A. For a set $A \subset \{0,1\}^{E_G}$, the Hamming distance can also be seen as a random variable H_A.

Proposition 5.22. *Let $q \geq 1$ and G be a finite graph. For any random-cluster measure $\phi^\xi_{p,q,G}$ with $p \in (0,1)$ and any increasing event A,*

$$\frac{d}{dp}\phi^\xi_{p,q,G}(A) \geq 4\phi^\xi_{p,q,G}(A)\phi^\xi_{p,q,G}(H_A), \tag{5.17}$$

where $H_A(\omega)$ is the Hamming distance between ω and A.

Proof. Define $|\omega|$ to be the number of open edges in the configuration, i.e. simply the sum over $e \in E_G$ of random variables $\omega(e)$. With this notation, one can rewrite (5.9) as

$$\begin{aligned}\frac{d}{dp}\phi^\xi_{p,q,G}(A) &= \frac{1}{p(1-p)}\Big[\phi^\xi_{p,q,G}(|\omega|\mathbf{1}_A) - \phi^\xi_{p,q,G}(|\omega|)\phi^\xi_{p,q,G}(A)\Big]\\ &= \frac{1}{p(1-p)}\Big[\phi^\xi_{p,q,G}\big((|\omega|+H_A)\mathbf{1}_A\big) - \phi^\xi_{p,q,G}\big(|\omega|+H_A\big)\phi^\xi_{p,q,G}(A)\\ &\qquad\qquad - \phi^\xi_{p,q,G}(H_A\mathbf{1}_A) + \phi^\xi_{p,q,G}(H_A)\phi^\xi_{p,q,G}(A)\Big]\\ &\geq \frac{1}{p(1-p)}\phi^\xi_{p,q,G}(H_A)\phi^\xi_{p,q,G}(A).\end{aligned}$$

To obtain the second line, simply add and subtract the same quantities. In order to go from the second line to the third, remark two things: in the second line, the third term equals 0 (when A occurs, the Hamming distance to A is 0), and the sum of the first two terms is positive thanks to the FKG inequality (indeed, it is easy to check that $|\omega| + H_A$ is increasing). The claim follows since $p(1-p) \leq 1/4$. □

This proposition has an interesting reformulation: integrating the formula (5.17) between p_1 and $p_2 > p_1$, we obtain

$$\phi^{\xi}_{p_1,q,G}(A) \leq \phi^{\xi}_{p_2,q,G}(A)\ \exp\big[-4(p_2 - p_1)\phi^{\xi}_{p_2,q,G}(H_A)\big] \tag{5.18}$$

(we used that H_A is a decreasing random variable). If one can prove that the typical value of H_A is sufficiently large, for instance because A occurs with small probability, then one can obtain good bounds for the probability of A.

Remark 5.23. The inequalities (5.17) and (5.18) hold in infinite-volume provided that A depends on finitely many edges by taking the limit of inequalities in finite volume (we implicitly use the fact that H_A depends on the same edges as A).

Proof of Theorem 5.18. Let x be a vertex of $\mathbb{Z}^2$, and let $\mathcal{C}_x$ be the cluster of x. Its cardinality is denoted by $|\mathcal{C}_x|$. We first prove that $|\mathcal{C}_x|$ has finite moments of any order at $p < p_c$. Then we deduce that the probability of $\{|\mathcal{C}_x| \geq n\}$ decays exponentially fast in n for $p' < p$ by proving that the expected Hamming distance is of order n at p. The proof of this second step is extracted from [Gri06].

Step 1: finite moments for $|\mathcal{C}_x|$. Using the invariance under translations, we may assume without loss of generality that $x = 0$. Let $d > 0$ and $p < p_{sd}$; we wish to prove that

$$\phi_{p,q}(|\mathcal{C}_0|^d) < \infty. \tag{5.19}$$

In order to do so, let $p_1 := (p + p_{sd})/2$ and denote by H_n the Hamming distance to $\{0 \longleftrightarrow \partial\Lambda_n\}$ (recall that $H_n(\omega)$ is the graph distance between ω and $\{0 \longleftrightarrow \partial\Lambda_n\}$ in $\{0,1\}^{\mathbb{E}}$). In this specific case, H_n is simply the minimal number of closed edges that must be crossed in order to go from 0 to $\partial\Lambda_n$. Let

$$K := \exp\left[\frac{p_1 - p}{2d+1}\right] > 1.$$

From Proposition 5.17 applied to the (supercritical) dual model, the probability of $\bigcap_{n>N}(\mathcal{A}_n^K)^\star$ is larger than $c\prod_N^\infty(1 - c_1 e^{-\varepsilon_1 n}) > 0$ (where $(\mathcal{A}_n^K)^\star$ is the occurrence of $\mathcal{A}_n^K$ in the dual model). Hence, there exists $N = N(p_1, q, \alpha)$ sufficiently large such that

$$\phi_{p_1,q}\left(\bigcap_{n\geq N}^{\infty}(\mathcal{A}_n^K)^\star\right) \geq \frac{1}{2}.$$

On this event, H_n is greater than $(\log n/\log K) - N$ since there is at least one closed circuit in each annulus A_k^K with $k \geq N$ (thus increasing the Hamming distance by 1). We obtain

$$\phi_{p_1,q}(H_n) \geq \Big(\frac{\log n}{\log K} - N\Big)\phi_{p_1,q}\Big(\bigcap_{n\geq N}^{\infty}(\mathcal{A}_n^K)^\star\Big) \geq \frac{\log n}{4\log K}$$

for n sufficiently large. Then, since $\{0 \longleftrightarrow \partial\Lambda_n\}$ depends on finitely many edges, (5.18) and Remark 5.23 imply

$$\phi_{p,q}(0 \longleftrightarrow \partial\Lambda_n) \leq \phi_{p_1,q}(0 \longleftrightarrow \partial\Lambda_n)\exp\big[-4(p_1-p)\phi_{p_1,q}(H_n)\big] \leq n^{-(2d+1)} \tag{5.20}$$

for n sufficiently large.

Now, for $x > 0$, let $u(x) = \inf\{k > 0 : |\Lambda_k| \geq x\}$ and observe that $u(x) = \Theta(\sqrt{x})$ as x tends to infinity. Therefore,

$$\begin{aligned}\phi_{p,q}(|\mathcal{C}_0|^d) &= \sum_{n\geq1}\phi_{p,q}\big[|\mathcal{C}_0|^d \geq n\big] = \sum_{n\geq1}\phi_{p,q}\big[|\mathcal{C}_0| \geq n^{1/d}\big]\\ &\leq \sum_{n\geq1}\phi_{p,q}\big[0 \longleftrightarrow \partial\Lambda_{u(n^{1/d})}\big]\\ &\leq 1 + \sum_{n\geq2}u(n^{1/d})^{-(2d+1)} \leq C\sum_{n\geq1}n^{-(1+1/(2d))} < \infty,\end{aligned}$$

where the constant C is universal. The first inequality is due to the fact that $\mathcal{C}_0$ cannot be included in the box of radius $u(n^{1/d}) - 1$ if it has cardinality $n^{1/d}$.

Step 2: exponential decay. Let $p < p_c$. From the first inequality of (5.18) (and Remark 5.23) applied to $p' < p$, it is sufficient to prove that there exists $c > 0$ such that

$$\phi_{p,q}(H_n) \geq cn, \quad \forall n \geq 0$$

in order to prove that $\phi_{p',q}(D_n)$ decays exponentially. It is thus enough to prove that there exists $c > 0$ such that,

$$\phi_{p,q}\Big(\liminf_{n\to\infty}\frac{H_n}{n} > c\Big) = 1.$$

Consider a (not necessarily open) self-avoiding path γ going from the origin to the boundary of the box of size n. The number $T(\gamma)$ of closed edges divided by n along this path can be bounded from below by the following quantity:

$$\frac{1+T(\gamma)}{n} \geq \frac{1+T(\gamma)}{|\gamma|} \geq \frac{1}{|\gamma|}\sum_{z\in\gamma}\frac{1}{|\mathcal{C}_z|} \geq \Big(\frac{1}{|\gamma|}\sum_{z\in\gamma}|\mathcal{C}_z|\Big)^{-1}.$$

We obtained the second inequality by noticing that the number of closed edges in γ is larger than the number of distinct clusters intersecting γ (if $\mathcal{C}$ denotes such a cluster, we have that $1 \geq \sum_{z\in\gamma} |\mathcal{C}|^{-1}\mathbf{1}_{z\in\mathcal{C}}$). The last inequality is due to Jensen's inequality. Since H_n can be rewritten as the infimum of $T(\gamma)$ on paths going from 0 to the boundary of the box, we obtain

$$\frac{1+H_n}{n} \geq \frac{1}{\displaystyle\max_{\gamma:0\leftrightarrow\partial\Lambda_n} \frac{1}{|\gamma|}\sum_{z\in\gamma} |\mathcal{C}_z|}. \tag{5.21}$$

This inequality comes in handy for the following reason: it translates the problem of bounding H_n from below into a last-passage percolation problem. Roughly speaking, the last-passage percolation model is defined as follows. A random variable ω_z, also called *weight*, is associated to every vertex of $\mathbb{Z}^d$ and the goal is to maximize the average of weights $\frac{1}{|\gamma|}\sum_{z\in\gamma}\omega_z$ along a certain family of subsets of $\mathbb{Z}^d$. This type of problems is classical in the case of iid weights on $\mathbb{Z}^d$. In particular, if ω_z has finite moments of sufficiently high order, then the maximal average of weights is bounded uniformly. Here, the subsets on which we maximize the average of weights are the self-avoiding walks from 0 to $\partial\Lambda_n$. The weights are $\omega_z = |\mathcal{C}_z|$ and therefore the distribution of weights has finite moments of any order (since $p < p_c$, this fact follows from Step 1). If the weights would be iid, it would exactly mean that the Hamming distance is linear in n. Unfortunately, the weight distribution is correlated. In order to circumvent this difficulty, we will compare these weights with iid weights.

We proceed in two steps. First, we replace these highly correlated weights by weights which are expressed in terms of iid weights. Let $(\tilde{\mathcal{C}}_z)_{z\in\Lambda_n}$ be a family of independent subsets of $\mathbb{Z}^2$ distributed as $\mathcal{C}_z$. We claim that $(|\mathcal{C}_z|)_{z\in\Lambda_n}$ is stochastically dominated by the family $(M_z)_{z\in\Lambda_n}$ defined as

$$M_z := \sup\big\{|\tilde{\mathcal{C}}_y| : y \in \mathbb{Z}^2 \text{ such that } \tilde{\mathcal{C}}_y \text{ contains } z\big\}.$$

Let $v_1, v_2, \dots$ be a deterministic ordering of $\mathbb{Z}^2$. Given the random family $(\tilde{\mathcal{C}}_z)_{z\in\Lambda_n}$, we shall construct a family $(D_z)_{z\in\Lambda_n}$ having the same joint law as $(\mathcal{C}_z)_{z\in\Lambda_n}$ and satisfying the following condition: for each z, there exists y such that $D_z \subset \tilde{\mathcal{C}}_y$. First, set $D_{v_1} = \tilde{C}_{v_1}$. Given D_{v_1}, D_{v_2}, $\dots$, D_{v_n}, define $E = \bigcup_{i=1}^n D_{v_i}$. If $v_{n+1} \in E$, set $D_{v_{n+1}} = D_{v_j}$ for some j such that $v_{n+1} \in D_{v_j}$. If $v_{n+1} \notin E$, proceed as follows. Let $\partial_e E$ be the set of edges of $\mathbb{Z}^2$ having exactly one end-vertex in E. A (random) subset F of $\tilde{C}_{v_{n+1}}$ may be found in such a way that F has the conditional law of $C_{v_{n+1}}$ given that all edges in $\partial_e E$ are closed; now set $D_{v_{n+1}} = F$. The domain Markov property and the positive association can be used to show that the law of $C_{v_{n+1}}$ depends only on $\partial_e E$, and is stochastically dominated by the law of the cluster in the bulk without any conditioning. The required stochastic domination follows accordingly. In particular, $|\mathcal{C}_z| \le M_z$ and M_z has finite moments. From the previous stochastic domination, we get that almost surely

$$\sup_{\gamma:0\leftrightarrow\partial\Lambda_n} \frac{1}{|\gamma|} \sum_{z\in\gamma} |\mathcal{C}_z| \le \sup_{\gamma:0\leftrightarrow\partial\Lambda_n} \frac{1}{|\gamma|} \sum_{z\in\gamma} M_z.$$

The second step is now to replace M_z by random variables that are really independent. The trick is to enlarge the set of subsets of $\mathbb{Z}^2$ on which the average of weights is maximized. Namely, Lemma 2 of [FN93] can be harnessed to show that

$$\sup_{\gamma:0\leftrightarrow\partial\Lambda_n} \frac{1}{|\gamma|} \sum_{z\in\gamma} M_z \le 2 \sup_{|\Gamma|\ge n} \frac{1}{|\Gamma|} \sum_{z\in\gamma} |\tilde{\mathcal{C}}_z|^2 \tag{5.22}$$

almost surely, where the second supremum is over all finite connected graphs Γ of cardinality larger than n that contain the origin (also called *lattice animals*). Since the $|\tilde{\mathcal{C}}_z|^2$ are now independent and have finite moments of any order at p by Step 1, the main result of [CGGK93, GK94] guarantees that there exists $C > 0$ such that

$$2\phi_{p,q}\Big(\limsup_{n\to\infty} \sup_{|\Gamma|\ge n} \frac{1}{|\Gamma|} \sum_{z\in\gamma} |\tilde{\mathcal{C}}_z|^2 \le C\Big) = 1. \tag{5.23}$$

Putting (5.23) and (5.22) in (5.21) implies that $\liminf H_n/n \ge 1/C$ almost surely, which concludes the proof. □

5.4 Strong RSW theory

5.4.1 Statement

The RSW theory for periodic boundary conditions enables us to compute the critical value of the random-cluster model, yet it provides us with a rather weak understanding of the critical regime. One of the weaknesses of the previous crossing estimates is that they deal with periodic boundary conditions which are somehow balanced between the primal and dual model. It will be important for applications to understand what happens for more general boundary conditions[4]. In this section, we show that the fact that Theorem 5.3 holds for more general boundary conditions is equivalent to several other conditions which are easier to check. *The following theorem is very important, yet we would advise the reader to skip the technical proof in a first reading and to come back to it afterwards.*

[4] In fact, an equivalent of Theorem 5.3 will not be true in general for arbitrary boundary conditions (we will discuss for which value of q it is in Section 6.2).

Theorem 5.24 (Duminil-Copin, Sidoravicius, Tassion [DCST13]). *Let $q \ge 1$. The following assertions are equivalent :*

P1 *(Absence of infinite cluster at criticality)* $\phi^1_{p_c,q}(0 \longleftrightarrow \infty) = 0$.

P2 $\phi^0_{p_c,q} = \phi^1_{p_c,q}$.

P3 *(Infinite susceptibility)* $\chi^0(p_c, q) := \sum_{x \in \mathbb{Z}^2} \phi^0_{p_c,q}(0 \longleftrightarrow x) = \infty$.

P4 *(Sub-exponential decay for free boundary conditions)*

$$\lim_{n\to\infty} \frac{1}{n} \log \phi^0_{p_c,q}(0 \longleftrightarrow \partial\Lambda_n) = 0.$$

P5 *(RSW) Let $\alpha > 0$. There exists $c_1 > 0$ such that for all $n \ge 1$ and any boundary conditions ξ,*

$$c_1 \le \phi^\xi_{p_c,q,[-n,(\alpha+1)n]\times[-n,2n]}(\mathcal{C}_h([0,\alpha n] \times [0,n])) \le 1 - c_1.$$

The previous theorem does not show that these conditions are all satisfied, but that they are equivalent. In fact, whether the conditions are satisfied or not will depend on the value of q, see Section 6.2 for a more detailed discussion.

The previous result was previously known in a few cases:

- *Bernoulli percolation (random-cluster model with $q = 1$).* In such case **P2** is obviously satisfied. Furthermore, Russo [Rus78] proved that **P1**, **P3** and **P4** are all true (and therefore equivalent). Finally, **P5** was proved by Russo [Rus78] and Seymour-Welsh [SW78].
- *Random-cluster model with $q = 2$.* This model is directly related to the Ising model as we will see in Chapter 7. Therefore, all of these properties can be proved to be true using the following results on the Ising model: Onsager proved that the critical Ising measure is unique and that the phase transition is continuous in [Ons44], thus implying **P1** and **P2**. Properties **P3** and **P4** follow from Simon's correlation inequality for the Ising model [Sim80]. Property **P5** was proved in [DCHN11] using a proof specific to the Ising model. Interestingly, in the Ising case each property is derived independently and no direct equivalence was known previously.
- *Random-cluster model with $q \ge 25.72$.* In this case, none of the above properties are satisfied, as proved by using the Pirogov-Sinai technology [LMR86].

Remark 5.25. **P4**$\Rightarrow$**P1** implies that whenever there is an infinite cluster for the wired boundary conditions, correlations decay exponentially fast at criticality for free boundary conditions.

Before presenting the proof of this theorem, let us discuss alternative conditions which could replace the conditions **P1–P5**.

Proposition 5.26. *Let $q \geq 1$. The following properties are equivalent:*

P1 *(Absence of infinite cluster at criticality)* $\phi^1_{p_c,q}(0 \longleftrightarrow \infty) = 0$.

P1' *(Continuous phase transition)* $\lim_{p \searrow p_c} \phi_{p,q}(0 \longleftrightarrow \infty) = 0$.

Note that the (almost sure) absence of an infinite-cluster for $\phi^0_{p_c,q}$ follows from Zhang's argument (Proposition 4.38) but that it does not imply the continuity of the phase transition nor the (almost sure) absence of an infinite-cluster for $\phi^1_{p_c,q}$. In order to prove Proposition 5.26, we start with a simple lemma, which is very useful.

Lemma 5.27. *The weak limits as $p \nearrow p_c$ and $p \searrow p_c$ of $\phi_{p,q}$ are respectively $\phi^0_{p_c,q}$ and $\phi^1_{p_c,q}$. Furthermore, for every increasing event A depending on finitely many edges,*

$$\phi^0_{p_c,q}(A) = \sup_{p<p_c} \phi_{p,q}(A) \quad \textit{and} \quad \phi^1_{p_c,q}(A) = \inf_{p>p_c} \phi_{p,q}(A).$$

Proof. First observe that the supremum and the infimum are in fact limits by monotonicity in p. Now, the second part of the lemma implies the first one, since increasing events depending on finitely many edges generate the whole σ-algebra.

Let us therefore focus on the second part of the statement. Assume that A depends on the state of edges in Λ_k only. We have

$$\begin{aligned}\phi^0_{p_c,q}(A) &= \sup_{n\geq k} \phi^0_{p_c,q,\Lambda_n}(A) = \sup_{n\geq k}\sup_{p<p_c} \phi^0_{p,q,\Lambda_n}(A)\\ &= \sup_{p<p_c}\sup_{n\geq k} \phi^0_{p,q,\Lambda_n}(A) = \sup_{p<p_c} \phi^0_{p,q}(A) = \sup_{p<p_c} \phi_{p,q}(A).\end{aligned}$$

In the first and fourth inequalities, we used the convergence of finite-volume measures to the infinite-volume measure and the comparison between boundary conditions (to show that the limit is in fact a supremum). The third equality is due to the continuity in p in finite volume (everything is even analytic in such case). The last equality follows from the uniqueness of the infinite-volume measure (Corollary 4.40). Similarly,

$$\begin{aligned}\phi^1_{p_c,q}(A) &= \inf_{n\geq k} \phi^1_{p_c,q,\Lambda_n}(A) = \inf_{n\geq k}\inf_{p>p_c} \phi^1_{p_c,q,\Lambda_n}(A)\\ &= \inf_{p>p_c}\inf_{n\geq k} \phi^1_{p,q,\Lambda_n}(A) = \inf_{p>p_c} \phi^1_{p,q}(A) = \inf_{p>p_c} \phi_{p,q}(A).\end{aligned}$$

□

Remark 5.28. The second part of the lemma extends to many events depending on infinitely many edges. We will see examples in the next proofs.

Proof of Proposition 5.26. We only need to check that

$$\lim_{p \searrow p_c} \phi_{p,q}(0 \longleftrightarrow \infty) = \phi^1_{p_c,q}(0 \longleftrightarrow \infty).$$

The previous lemma used in the third equality implies

$$\begin{aligned}\lim_{p \searrow p_c} \phi_{p,q}(0 \longleftrightarrow \infty) &= \inf_{p>p_c} \inf_{n \in \mathbb{N}} \phi_{p,q}(0 \longleftrightarrow \partial\Lambda_n) = \inf_{n \in \mathbb{N}} \inf_{p>p_c} \phi_{p,q}(0 \longleftrightarrow \partial\Lambda_n)\\ &= \inf_{n \in \mathbb{N}} \phi^1_{p_c,q}(0 \longleftrightarrow \partial\Lambda_n) = \phi^1_{p_c,q}(0 \longleftrightarrow \infty).\end{aligned}$$

□

Proposition 5.29. *Let $q \geq 1$. The following properties are equivalent:*

P2 $\phi^0_{p_c,q} = \phi^1_{p_c,q}$.

P2' *The infinite-volume measure at p_c and q is unique.*

This proposition is a direct reformulation of Proposition 4.29.

Remark 5.30. **P1** together with **P3** have an interesting consequence in terms of the order of the phase transition for the so-called Potts model. We do not enter in the details here since we have not introduced this model but let us briefly mention that properties **P1** and **P3** are respectively equivalent to the continuity and the non-differentiability with respect to the magnetic field h of the Potts model free energy at $(\beta = \beta_c, h = 0)$. Therefore, these properties mean that the phase transition of the Potts model is of *second order*.

Let us now turn to **P4** which can be understood in terms of the so-called correlation length.

Lemma 5.31. *Let $q \geq 1$ and $p < p_c$. Then, the quantity*

$$\xi(p,q) = \Big(-\lim_{n \to \infty} \tfrac{1}{n} \log \phi_{p,q}[(0,0) \longleftrightarrow (n,0)]\Big)^{-1}$$

is well-defined and $\xi(p,q) \in (0,\infty)$.

The quantity $\xi(p,q)$ is called the *correlation length* (see Chapter 11 for more details).

Proof. Let $p < p_c$. For every $n, m > 0$, the FKG inequality implies that

$$\begin{aligned}\phi_{p,q}\big[(0,0) \longleftrightarrow (n+m,0)\big] &\ge \phi_{p,q}\big[(0,0) \longleftrightarrow (n,0) \text{ and } (n,0) \longleftrightarrow (n+m,0)\big]\\ &\ge \phi_{p,q}\big[(0,0) \longleftrightarrow (n,0)\big] \cdot \phi_{p,q}\big[(0,0) \longleftrightarrow (m,0)\big].\end{aligned}$$

Fekete's lemma (for a supermultiplicative sequence) implies that $\xi(p,q) \in (0,\infty]$ is well defined. Furthermore,

$$\phi_{p,q}\big[(0,0) \longleftrightarrow (n,0)\big] \;\le\; \exp[-n/\xi(p,q)]. \tag{5.24}$$

Finally, Theorem 5.18 forces $\xi(p,q) < \infty$. □

Proposition 5.32. *Let $q \ge 1$. The following properties are equivalent:*

P4 *(sub-exponential decay for free boundary conditions)*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \phi^0_{p_c,q}\,(0 \longleftrightarrow \partial\Lambda_n) = 0.$$

P4' *(vanishing mass-gap) $\xi(p,q)$ tends to $+\infty$ as $p \nearrow p_c(q)$.*

Proof. Let us first assume that **P4** is not satisfied. In such case, there exists $c > 0$ such that

$$\begin{aligned}\lim_{n\to\infty} \tfrac{1}{n} \log \phi^0_{p,q}\,((0,0) \longleftrightarrow (n,0)) &\le \lim_{n\to\infty} \tfrac{1}{n} \log \phi^0_{p,q}\,(0 \longleftrightarrow \partial\Lambda_n)\\ &\le \lim_{n\to\infty} \tfrac{1}{n} \log \phi^0_{p_c,q}\,(0 \longleftrightarrow \partial\Lambda_n) =: -c < 0.\end{aligned}$$

In particular, $\xi(p,q) \le \frac{1}{c}$ for any $p < p_c$ and **P4'** is not satisfied as well.

Let us now assume that **P4'** is not satisfied and that $\xi(p,q) \le M$ for every $p < p_c$. In such case,

$$\begin{aligned}\phi^0_{p_c,q}\big[(0,0) \longleftrightarrow (n,0)\big] &= \sup_{k\ge n} \phi^0_{p_c,q}\big[(0,0) \longleftrightarrow (n,0) \text{ in } \Lambda_k\big]\\ &= \sup_{k\ge n} \sup_{p<p_c} \phi_{p,q}\big[(0,0) \longleftrightarrow (n,0) \text{ in } \Lambda_k\big]\\ &= \sup_{p<p_c} \sup_{k\ge n} \phi_{p,q}\big[(0,0) \longleftrightarrow (n,0) \text{ in } \Lambda_k\big]\\ &= \sup_{p<p_c} \phi_{p,q}\big[(0,0) \longleftrightarrow (n,0)\big]\\ &\le \sup_{p<p_c} \mathrm{e}^{-n/\xi(p,q)} \le \mathrm{e}^{-n/M}.\end{aligned}$$

In the second line we used Lemma 5.27 and in the last (5.24).

Now, take $x \in \partial\Lambda_n$ and assume without loss of generality that the first coordinate of x is equal to n. The FKG inequality and an orthogonal reflection with respect to $d = \{(x_1, x_2) : x_1 = n\}$ imply that

$$\begin{aligned}\phi^0_{p_c,q}[0 \longleftrightarrow x]^2 &= \phi^0_{p_c,q}[0 \longleftrightarrow x]\phi^0_{p_c,q}[x \longleftrightarrow (2n,0)] \\ &\leq \phi^0_{p_c,q}[0 \longleftrightarrow (2n,0)] \leq \mathrm{e}^{-2n/M}, \end{aligned}\tag{5.25}$$

from which we deduce that

$$\phi^0_{p_c,q}[0 \longleftrightarrow \partial\Lambda_n] \leq |\partial\Lambda_n|\mathrm{e}^{-n/M}.$$

As a consequence, **P4'** is not satisfied. □

The properties **P1**–**P4** (and their equivalent formulations) are classical definitions describing continuous phase transitions and are believed to be equivalent for many natural models, even though it is a priori unclear how this can be proved in a robust way. Now that we have an interpretation for properties **P1**–**P4**, let us explain why Property **P5** is of particular interest: it provides an equivalent to Theorem 5.3 uniform in boundary conditions. The uniformity with respect to boundary conditions is crucial for applications, especially when trying to decouple events, see e.g. Section 5.5.

Before diving into the proof, let us mention two equivalent formulations of **P5**. For $z \in \mathbb{R}^2$, define $\mathcal{A}_n(z)$ to be the event that there exists an open circuit (i.e. an open path $v_0 \sim v_1 \sim \cdots \sim v_k \sim v_0$) in the annulus $z + (\Lambda_{2n} \setminus \Lambda_n)$ surrounding z. Also define $\mathcal{A}_n = \mathcal{A}_n(0)$.

Proposition 5.33. *The following propositions are equivalent;*

P5 *For any $\alpha > 0$, there exists $c_1 = c_1(\alpha) > 0$ such that for all $n \geq 2$ and for all boundary conditions ξ on the boundary of $[-n, (\alpha+1)n] \times [-n, 2n]$, we have*

$$c_1 \leq \phi^\xi_{p_c,q,[-n,(\alpha+1)n]\times[-n,2n]}\left(\mathcal{C}_h([0,\alpha n] \times [0,n])\right) \leq 1 - c_1.$$

P5' *There exists $c_2 > 0$ such that for all $n \geq 2$,*

$$\phi^0_{p_c,q,\Lambda_{2n}\setminus\Lambda_n}\left(\mathcal{A}_n\right) \geq c_2.$$

P5" *For any $R \geq 2$, there exists $c_3 = c_3(R) > 0$ such that for all $n \geq 2$,*

$$\phi^0_{p_c,q,\Lambda_{Rn}}\left(\mathcal{A}_n\right) \geq c_3.$$

Proof. The proof of **P5'⇒P5"** is obvious by comparison between boundary conditions. In order to prove **P5⇒P5'**, consider the four rectangles

$$
\begin{aligned}
R_1 &:= [4n/3, 5n/3] \times [-5n/3, 5n/3],\\
R_2 &:= [-5n/3, -4n/3] \times [-5n/3, 5n/3],\\
R_3 &:= [-5n/3, 5n/3] \times [4n/3, 5n/3],\\
R_4 &:= [-5n/3, 5n/3] \times [-5n/3, -4n/3].
\end{aligned}
$$

If the intersection of $\mathcal{C}_v(R_1)$, $\mathcal{C}_v(R_2)$, $\mathcal{C}_h(R_3)$ and $\mathcal{C}_h(R_4)$ occurs, then $\mathcal{A}_n$ occurs. In particular, the FKG inequality and the comparison between boundary conditions implies that c_2 can be chosen to be equal to $c_1(10)^4$.

Let us now turn to the proof of **P5"**$\Rightarrow$**P5**. We start by the lower bound. Fix some $R \geq 2$ as in **P5"** and the corresponding $c_3 > 0$. Let $\alpha > 0$. For $n \geq 4R$, the intersection of the events $\mathcal{A}_{n/(2R)}[(j\lfloor \frac{n}{R} \rfloor, \frac{n}{2})]$ for $j = 0, \dots, \lceil R\alpha \rceil$ is included in $\mathcal{C}_h([0, \alpha n] \times [0, n])$. The FKG inequality implies

$$\phi^0_{p_c, q, [-n, (\alpha+1)n] \times [-n, 2n]} (\mathcal{C}_h([0, \alpha n] \times [0, n])) \geq c_3^{1 + R\lceil \alpha \rceil}.$$

By comparison between boundary conditions, we obtain the lower bound for every ξ.

The upper bound may be obtained from this lower bound as follows. By comparison between boundary conditions once again, it is sufficient to prove the bound for the wired boundary conditions. In such case, the complement of $\mathcal{C}_h([0, \alpha n] \times [0, n])$ is the event that the rectangle $[\frac{1}{2}, \alpha n - \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ is crossed from top to bottom by a dual-open dual-path. Since the dual of the wired boundary conditions are the free ones, the boundary conditions for the dual measure are free. We can now harness **P5"** for the dual model to construct a dual-open dual-path from top to bottom with probability bounded away from 0. This finishes the proof. □

Remark 5.34. The restriction on boundary conditions being at distance n from the rectangle can be relaxed in the following way: if **P5** holds, then for any $\alpha > 0$ and $\varepsilon > 0$, there exists $c = c(\alpha, \varepsilon) > 0$ such that for every $n \geq 1$,

$$c \leq \phi^{\xi}_{p_c, q, [-\varepsilon n, (\alpha+\varepsilon)n] \times [-\varepsilon n, (1+\varepsilon)n]} (\mathcal{C}_h([0, \alpha n] \times [0, n])) \leq 1 - c.$$

(Simply follow the same proof as above to obtain this result.) It is natural to ask why boundary conditions are fixed at distance εn of the rectangle $[0, \alpha n] \times [0, n]$ and not simply on the boundary. It may in fact be the case that **P5** holds but that crossing probabilities of rectangles $[0, \alpha n] \times [0, n]$ with free boundary conditions on their boundary converge to zero as n tends to infinity. Such a phenomenon does not occur for $1 \leq q < 4$ as shown in [DCST13] (for such values of q, the crossing probabilities on rectangles with free boundary conditions directly on the boundary are bounded away from 0 uniformly in n provided that the aspect ratio remains bounded away from 0 and 1) but is expected to occur for $q = 4$. In conclusion, we will always work with boundary conditions at "macroscopic distance" from the boundary.

5.4.2 Proof of Theorem 5.24

We now dive into the proof. *Once again, we advise to skip this part during the first reading.* In this section, $q \geq 1$ is fixed and $p = p_c(q)$. In order to lighten the notation, we drop the reference to p and q and simply write ϕ_G^ξ instead of $\phi_{p_c(q),q,G}^\xi$. Note that ϕ^1 is the infinite-volume measure with wired boundary conditions at criticality. We will also use the following notation: the event that A and B are connected by an open path included in C will be denoted by $A \overset{C}{\longleftrightarrow} B$.

Preliminaries: easy implications

In order to isolate the hard part of the proof, let us start by checking the four "simple" implications **P1⇒P2**, **P2⇒P3**, **P3⇒P4** and **P5⇒P1**.

Property **P1** *implies* **P2***:* The proof of this fact follows from the proof of Corollary 4.40 since the only assumption used there was $\phi^1(0 \leftrightarrow \infty) = 0$.

Property **P2** *implies* **P3***:* If **P2** holds,

$$\begin{aligned}(2n+1)\,\phi^0(0 \longleftrightarrow \partial\Lambda_n) &= (2n+1)\,\phi^1(0 \longleftrightarrow \partial\Lambda_n)\\ &\geq \sum_{x \in \{0\}\times[-n,n]} \phi^1(x \longleftrightarrow (x + \partial\Lambda_n))\\ &\geq \phi^1(\mathcal{C}_v([-n,n]\times[0,n])) \geq c,\end{aligned}$$

where $c > 0$ is a constant independent of n. The first equality is due to the uniqueness of the infinite-volume measure given by **P2** and the second inequality by Corollary 5.5. This leads to

$$\sum_{x \in \partial\Lambda_n} \phi^0(0 \longleftrightarrow x) \geq \phi^0(0 \longleftrightarrow \partial\Lambda_n) \geq \frac{c}{2n+1}.$$

As a consequence, $\sum_{x \in \mathbb{Z}^2} \phi^0(0 \longleftrightarrow x) = \infty$ and **P3** holds true.

Property **P3** *implies* **P4***:* Assume that **P4** does not hold. In this proof, we recall the dependency in p and q when $p \neq p_c(q)$. In such case, there exists $c > 0$ such that

$$-\lim_{n\to\infty} \frac{1}{n} \log[\phi_{p,q}(0 \longleftrightarrow (n,0))] \geq \lim_{n\to\infty} -\frac{1}{n} \log[\phi_{p,q}(0 \longleftrightarrow \partial\Lambda_n)] > c$$

for any $p < p_c$. Thus, (5.24) implies that $\phi_{p,q}(0 \longleftrightarrow (n,0)) \le e^{-cn}$ uniformly in n and $p < p_c$. Lemma 5.27 thus leads to $\phi^0(0 \longleftrightarrow (n,0)) \le e^{-cn}$ for every $n \ge 1$. Now, (5.25) implies that for every $x = (x_1, x_2) \in \mathbb{Z}^2$,

$$\phi^0(0 \longleftrightarrow x) \le \sqrt{\phi^0(0 \longleftrightarrow (2\|x\|_\infty, 0))} \le e^{-c\|x\|_\infty},$$

where $\|x\|_\infty = \max\{|x_1|, |x_2|\}$. Summing over every $x \in \mathbb{Z}^2$ gives

$$\sum_{x \in \mathbb{Z}^2} \phi^0(0 \leftrightarrow x) < \infty$$

and thus **P3** does not hold.

Property **P5** *implies* **P1***:* Recall that **P5** implies **P5'**. We now prove a slightly stronger result which obviously implies **P1** and will be useful later in the proof.

Lemma 5.35. *Property* **P5'** *implies that there exists* $\varepsilon > 0$ *such that for any* $n \ge 1$,

$$\phi^1(0 \leftrightarrow \partial\Lambda_n) \le n^{-\varepsilon}.$$

Proof. Let k be such that $2^k \le n < 2^{k+1}$. Also define the annuli $A_j = \Lambda_{2^j} \smallsetminus \Lambda_{2^{j-1}-1}$ for $j \ge 1$. We have

$$\begin{aligned}\phi^1(0 \longleftrightarrow \partial\Lambda_n) &\le \prod_{j=1}^{k} \phi^1\left(\partial\Lambda_{2^{j-1}} \xleftrightarrow{A_j} \partial\Lambda_{2^j} \,\Big|\, \bigcap_{i>j} \{\partial\Lambda_{2^{i-1}} \xleftrightarrow{A_i} \partial\Lambda_{2^i}\}\right)\\ &\le \prod_{j=1}^{k} \phi^1_{A_j}\left(\partial\Lambda_{2^{j-1}} \xleftrightarrow{A_j} \partial\Lambda_{2^j}\right).\end{aligned}$$

In the second line, we used the fact that the event upon which we condition depends only on edges outside of Λ_{2^j} together with the comparison between boundary conditions (Corollary 4.21).

Now, the complement of $\{\partial\Lambda_{2^{j-1}} \xleftrightarrow{A_j} \partial\Lambda_{2^j}\}$ is the event that there exists a dual-open circuit in $A_j^\star$ surrounding the origin. Property **P5'** implies[5] that this dual-open circuit exists with probability larger than or equal to $c > 0$ independently of $n \ge 1$. This implies that

$$\phi^1(0 \longleftrightarrow \partial\Lambda_n) \le \prod_{j=1}^{k}(1-c) = (1-c)^k \le (1-c)^{\log n/\log 2}.$$

The proof follows by setting $\varepsilon = -\frac{\log(1-c)}{\log 2}$. □

[5]There is a slight technical issue here: the annulus $A_j^\star$ is not really of the form $\Lambda_{2n} \smallsetminus \Lambda_n$ for some n. Nevertheless, it contains a translation of $\Lambda_{2n} \smallsetminus \Lambda_{n+1}$ for $n = 2^{j-1} - 1$. Now, the open circuit constructed in the proof of **P5⇒P5'** is contained in $\Lambda_{2n} \smallsetminus \Lambda_{n+1}$ and therefore there exists an open circuit in $\Lambda_{2n} \smallsetminus \Lambda_{n+1}$ surrounding the origin with probability bounded away from 0.

Remark 5.36. The proof of the previous lemma illustrates the need for *bounds which are uniform with respect to boundary conditions.* Indeed, it could be the case that the ϕ^1-probability of an open path from the inner to the outer sides of A_j is bounded away from 1, but conditioning on the existence of paths in each annulus A_i (for $i < j$) could favor open edges drastically, and imply that the probability of the event under consideration is close to 1.

The only remaining implication to prove is **P4** implies **P5**. Recall from Proposition 5.33 that **P5** is equivalent to **P5"** and we therefore rather choose to prove that **P4** implies **P5"** when $R = 8$. The proof follows two steps. First, we prove that either **P5"** holds or $\phi^0(0 \leftrightarrow \partial\Lambda_n)$ tends to 0 stretched-exponentially fast. We then prove that in the second case, the speed of convergence is actually exponential.

Proposition 5.37. *Exactly one of these two cases occurs:*

1. $\inf_{n\geq 1} \phi^0_{\Lambda_{8n}}(\mathcal{A}_n) > 0.$

2. *There exists $\alpha > 0$ such that for any $n \geq 1$,*

$$\phi^0(0 \longleftrightarrow \partial\Lambda_n) \leq \exp(-n^\alpha).$$

First, consider the strip $G = \mathbb{Z} \times [-n, 3n]$, and the boundary conditions ξ defined to be wired on $\mathbb{R} \times \{3n\}$, and free on $\mathbb{R} \times \{-n\}$. Recall that in this case, $\phi^{1,\xi}_G$ and $\phi^{0,\xi}_G$ are equal, and we thus write ϕ_{strip} for this measure.

Lemma 5.38. *For all $k \geq 1$, there exists a constant $c = c(k) > 0$ such that, for all $n \geq 1$,*

$$\phi_{\text{strip}}(\mathcal{C}_h([-kn, kn] \times [0, 2n])) \geq c. \tag{5.26}$$

Proof. Fix $n, k \geq 1$. We will assume that n is divisible by 9 (one may adapt the argument for general values of n). By duality, the complement of $\mathcal{C}_h([-kn, kn] \times [0, 2n])$ is the event $\mathcal{C}^\star_v(R^\star)$ that there exits a vertical dual-open dual crossing in $R^\star := [-kn + \frac{1}{2}, kn - \frac{1}{2}] \times [-\frac{1}{2}, 2n + \frac{1}{2}]$. Therefore, either (5.26) is true for $c = 1/2$, or

$$\phi_{\text{strip}}(\mathcal{C}^\star_v(R^\star)) \geq 1/2.$$

We assume that we are in this second situation for the rest of the proof.

The dual of the measure on the strip with free boundary conditions on the bottom and wired on the top is the measure on the strip with free boundary conditions on the top and wired on the bottom. This measure is the image of ϕ_{strip} under the orthogonal reflection with respect to the horizontal line $\mathbb{R}\times\{n-\frac{1}{4}\}$ composed with a translation by the vector $(\frac{1}{2},0)$. We thus obtain that

$$\begin{aligned}\phi_{\text{strip}}\left(\mathcal{C}_v([-kn,kn]\times[0,2n])\right) &\geq \phi_{\text{strip}}\left(\mathcal{C}_v([-kn,kn-1]\times[-1,2n])\right)\\ &= \phi_{\text{strip}}\left(\mathcal{C}_v^\star(R^\star)\right) \geq 1/2.\end{aligned}$$

Partitioning the segment $[-kn,kn]$ into the union of $18k$ segments of length $\lambda := n/9$ (note that λ is an integer), the union bound gives us

$$\phi_{\text{strip}}\left(I \longleftrightarrow \mathbb{R}\times\{2n\}\right) \geq \frac{1}{36k} =: c_1, \tag{5.27}$$

where $I = [4\lambda,5\lambda]\times\{0\}$. For future reference, let us also introduce the segment $J = [6\lambda,7\lambda]\times\{0\}$.

Define the rectangle $R = [0,9\lambda]\times[0,2n]$. When the event estimated in equation (5.27) is realized, there exists an open path in R connecting I to the union of the top, left and right boundaries of R. Using the reflection with respect to the vertical line $\{\frac{n}{2}\}\times\mathbb{R}$, we find that at least one of the two following inequalities occurs:

Case 1: $\phi_{\text{strip}}\left(I \overset{R}{\longleftrightarrow} [0,n]\times\{2n\}\right) \geq c_1/3.$

Case 2: $\phi_{\text{strip}}\left(I \overset{R}{\longleftrightarrow} \{0\}\times[0,2n]\right) \geq c_1/3.$

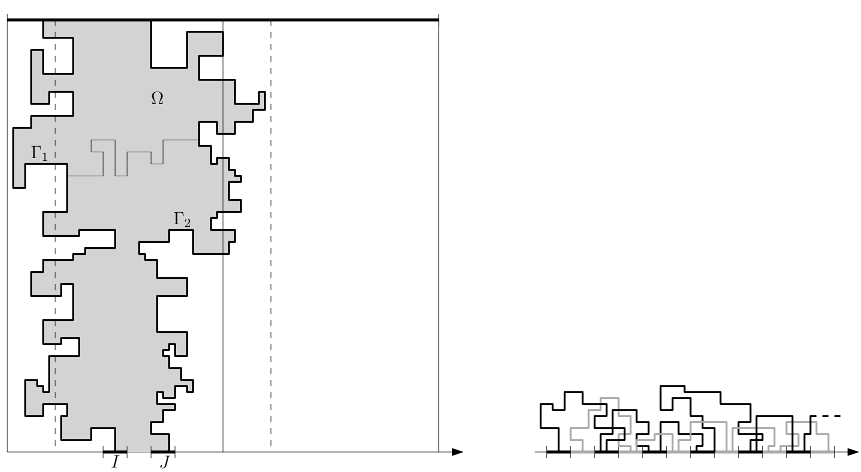

Figure 5.6: The construction in Case 1 with the two paths Γ_1 and Γ_2 and the domain Ω between the two paths. On the right, a combination of paths creating a long path from left to right.

Proof of (5.26) *in Case 1:* Consider the event that there exist

(i) an open path from I to the top of $[0,2n]^2$ contained in $[0,2n]^2$,
(ii) an open path from J to the top of $[0,2n]^2$ contained in $[0,2n]^2$,
(iii) an open path connecting these two paths in $[0,2n]^2$.

Each path in (i) and (ii) exists with probability larger than $c_1/3$ (since R and $(2\lambda,0)+R$ are included in $[0,2n]^2$). Furthermore, let Γ_1 be the left-most path satisfying (i) and Γ_2 the right-most path satisfying (ii); see Fig. 5.6. The subgraph of $[0,2n]^2$ between Γ_1 and Γ_2 is denoted by Ω. Conditioning on Γ_1 and Γ_2, the boundary conditions on Ω are wired on Γ_1 and Γ_2, and dominate the free boundary conditions on the rest of $\partial\Omega$. Following an argument close to those described in Section 4.4.4, we deduce that boundary conditions on Ω stochastically dominate boundary conditions induced by wired boundary conditions on the left and right sides of the box $[0,2n]^2$, and free on the top and bottom sides. As a consequence of (5.5), conditionally on Γ_1 and Γ_2, there exists an open path in Ω connecting Γ_1 to Γ_2 with probability larger than $1/(1+q^2)$. In conclusion,

$$\phi_{\text{strip}}\left(I \xleftrightarrow{[0,2n]^2} J\right) \geq \phi_{\text{strip}}\left((\text{i}),\ (\text{ii}) \text{ and } (\text{iii}) \text{ occur}\right) \geq \left(\frac{c_1}{3}\right)^2 \times \frac{1}{(1+q^2)}. \tag{5.28}$$

For $x = j\lambda$, where $j \in \{-9k-5,\dots,9k-6\}$, define the translate of the event considered in (5.28):

$$A_x := (x+I) \xleftrightarrow{x+[0,2n]^2} (x+J).$$

If A_x occurs for every such x, we obtain an open crossing from left to right in $[-kn,kn]\times[0,2n]$. The FKG inequality implies that this happens with probability larger than $\left(\frac{c_1^2}{9(1+q^2)}\right)^{18k}$.

Proof of (5.26) *in Case 2:* Define the rectangle $R' = [4\lambda, 9\lambda]\times[0,2n]$. Note that in Case 2, J is connected to one side of $[2\lambda,11\lambda]\times[0,2n]$ with probability bounded from below by $c_1/3$, hence the same is true for R' (since $[2\lambda,11\lambda]\times[0,2n]$ is wider than R'). Consider the event that there exist

(i) an open path from I to the right side of R contained in R,
(ii) an open path from J to the left side of R' contained in R',
(iii) an open path connecting these two paths in $[0,2n]^2$.

The first path occurs with probability larger than $c_1/3$, and the second one with probability larger than $c_1/6$ (there exists a path to one of the sides with probability at least $c_1/3$, and therefore by symmetry in R' to the left side with probability larger than $c_1/6$). By the FKG inequality, the event that both (i) and (ii) occur has probability larger than $c_1^2/18$. We now wish to prove that conditionally on (i) and (ii) occurring, the event (iii) occurs with good probability.

Define the segments $K(y,z) = \{4\lambda\} \times [y,z]$ for $y \le z \le \infty$. They are all subsegments of the vertical line of first coordinate equal to 4λ.

Consider the right-most open path Γ_1 satisfying (ii). It intersects the segment $K(0,2n)$ at a unique point with second coordinate denoted by y. Also consider the left-most open path $\tilde{\Gamma}_2$ satisfying (i). Either Γ_1 and $\tilde{\Gamma}_2$ intersect, or they do not. In the first case, we are already done since (iii) automatically occurs. In the second, we consider the subpath Γ_2 of $\tilde{\Gamma}_2$ from I to the first intersection with $K(y,2n)$ (this intersection must exist since $\tilde{\Gamma}_2$ goes to the right side of R'). Let us now show that Γ_1 and Γ_2 are connected with good probability. Note the similarity with the construction in Proposition 5.6 with symmetric domains, except that the lattice is not rotated here. The proof is therefore slightly more technical and we choose to isolate it from the rest of the argument.

Claim: There exists $c_2 > 0$ such that for any possible realizations γ_1 and γ_2 of Γ_1 and Γ_2,

$$\phi_{\text{strip}}\left(\gamma_1 \overset{R}{\longleftrightarrow} \gamma_2 \,\middle|\, \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2\right) \ge c_2. \tag{5.29}$$

Proof of the Claim. Fig. 5.7 should be very helpful in order to follow this proof. Construct the subgraph Ω "between γ_1 and γ_2" formally delimited by:

- the arc γ_2,
- the segment $[0,n] \times \{0\}$,
- the arc γ_1,
- the segment $K(y+1,2n)$ *excluded* (the vertices on this segment are not part of the domain).

We wish to compare Ω (left of Fig. 5.7) to a reference domain D (center of Fig. 5.7) defined as the upper half-plane minus the edges intersecting $\{4\lambda - \frac{1}{2}\} \times (y,\infty)$. Define the boundary conditions *mix* on D by:

- wired boundary conditions on $K(y,\infty)$ and $A := (-\infty, 4\lambda] \times \{0\}$;
- wired boundary conditions at infinity (See Remark 4.32 for more details on what is meant by wired boundary conditions at infinity);
- free boundary conditions elsewhere.

The boundary conditions on Ω inherited by the conditioning $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$ dominate wired on γ_1 and γ_2, and free elsewhere. Thus, we deduce that

$$\phi_{\text{strip}}\left(\gamma_1 \overset{\Omega}{\longleftrightarrow} \gamma_2 \,\middle|\, \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2\right) \ge \phi_D^{\text{mix}}\left(K(y,\infty) \overset{D}{\longleftrightarrow} A\right). \tag{5.30}$$

As mentioned above, the domain D is not exactly a symmetric domain but it is still very close to being one. Consider the domain $\tilde{D}$ (see on the right of Fig. 5.7) obtained from D by the reflection with respect to the vertical line $d = \{(4\lambda - \frac{1}{4}, y) : y \in \mathbb{R}\}$ and a translation by $(\frac{1}{2}, \frac{1}{2})$. Let $B = (-\infty, 4\lambda - 1] \times \{0\}$. Define the boundary conditions *mix* on $\tilde{D}$ as

- wired boundary conditions on $K(y+1,\infty) \cup B$ (it is very important that the two arcs are wired together);
- free boundary conditions at infinity;
- free boundary conditions elsewhere.

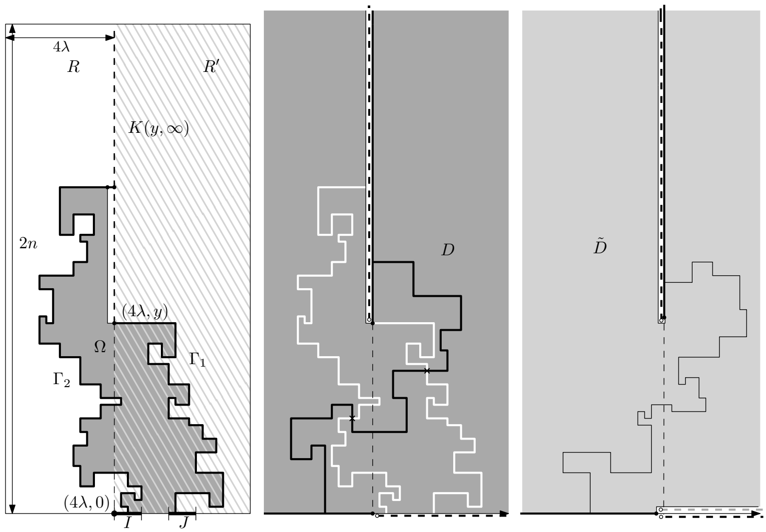

Figure 5.7: **Left.** The domain Ω. We depicted the part of the domain with free boundary conditions by putting dual wired boundary conditions on the associated dual arcs. The wired boundary conditions are depicted in bold. The rectangles R and R' are also specified (R' is in dashed). **Center.** The domain D. We depicted the domain Ω in white. The existence of an open path between $K(y,\infty)$ and A implies the existence of an open path between γ_1 and γ_2 in D (between the two crossings). **Right.** The domain $\tilde{D}$ with one path from $K(y+1,\infty)$ to B. The pre image of this path by the reflection mapping D onto $\tilde{D}$ is a dual-path in D preventing the existence of an open path from $K(y,\infty)$ to A.

Using duality, we find that

$$\phi_D^{\mathrm{mix}}\Big(K(y,\infty) \overset{D}{\not\longleftrightarrow} A\Big) = \phi_{\tilde{D}}^{\mathrm{mix}}\Big(K(y+1,\infty) \overset{\tilde{D}}{\longleftrightarrow} B\Big)$$

and thus

$$\phi_D^{\mathrm{mix}}\Big(K(y,\infty) \overset{D}{\longleftrightarrow} A\Big) + \phi_{\tilde{D}}^{\mathrm{mix}}\Big(K(y+1,\infty) \overset{\tilde{D}}{\longleftrightarrow} B\Big) = 1. \tag{5.31}$$

Define the *mix′* boundary conditions on D as wired boundary conditions on $K(y+1,\infty)\cup B$ (the two arcs are once again wired together) and free elsewhere (they correspond to the boundary conditions *mix* on $\tilde D$). The comparison between boundary conditions and the fact that $\tilde D \subset D$ lead to

$$\phi_{\tilde D}^{\text{mix}}\left(K(y+1,\infty) \overset{\tilde D}{\longleftrightarrow} B\right) \le \phi_D^{\text{mix}'}\left(K(y,\infty) \overset{D}{\longleftrightarrow} A\right). \tag{5.32}$$

The boundary conditions for the probability on the right can be compared to the boundary conditions *mix*. First, one may wire the vertices $(4\lambda, y)$ and $(4\lambda, y+1)$ together, and the vertices $(4\lambda-1, 0)$ and $(4\lambda, 0)$ together, which increases the probability of an open path between $K(y,\infty)$ and A. Second, one may unwire the arcs B and $K(y{+}1,\infty)$, paying a multiplicative cost of q^2. Using the previous inequality and the comparison between the boundary conditions described in this paragraph, we deduce

$$\phi_D^{\text{mix}'}\left(K(y,\infty) \overset{D}{\longleftrightarrow} A\right) \le q^2 \phi_D^{\text{mix}}\left(K(y,\infty) \overset{D}{\longleftrightarrow} A\right).$$

Putting this inequality in (5.32) and then in (5.31), and finally using (5.30), we find that

$$\phi_{\text{strip}}\left(\gamma_1 \overset{\Omega}{\longleftrightarrow} \gamma_2 \,\middle|\, \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2\right) \ge \phi_D^{\text{mix}}\left(K(y,\infty) \overset{D}{\longleftrightarrow} A\right) \ge \frac{1}{1+q^2}.$$

◇

It follows from (5.29) and the probabilities of (i) and (ii) that

$$\phi_{\text{strip}}\left(I \overset{R}{\longleftrightarrow} J\right) \ge \frac{1}{1+q^2} \times \frac{c_1^2}{18}.$$

Here again, $18k$ translations of the event above guarantee the occurrence of an open crossing from left to right in $[-kn, kn] \times [0, 2n]$. This occurs with probability larger than $(\frac{c_1^2}{18(1+q^2)})^{18k}$ thanks to the FKG inequality again. □

In the next lemma, we consider horizontal crossings in rectangular shaped Dobrushin domains with free boundary conditions on the bottom and wired elsewhere.

Lemma 5.39. *For all $k > 0$ and $\ell \ge 4/3$, there exists a constant $c = c(k,\ell) > 0$ such that for all $n > 0$,*

$$\phi_D^{a,b}\left(\mathcal{C}_h\left([-kn, kn] \times [0, n]\right)\right) \ge c \tag{5.33}$$

with $D = [-kn, kn] \times [0, \ell n]$, and $\phi_D^{a,b}$ is the random-cluster measure with Dobrushin boundary conditions on (D, a, b) where $a = (-kn, 0)$ and $b = (kn, 0)$.

Proof. For $\ell = 4/3$, the result follows directly from Lemma 5.38 since the boundary conditions on $\phi_D^{a,b}$ stochastically dominate wired boundary conditions on the top of the strip $\mathbb{Z} \times [0, \frac{4n}{3}]$, and free on the bottom, and therefore there exists an horizontal crossing of the rectangle $[-kn, kn] \times [\frac{n}{3}, n]$ with probability bounded away from 0.

Now assume that the result holds for ℓ and let us prove it for $\ell + 1/3$. By comparison between boundary conditions in $[-kn, kn] \times [\frac{n}{3}, \ell n + \frac{n}{3}]$, we know that

$$\phi_D^{a,b}\left(\mathcal{C}_h([-kn, kn] \times [\tfrac{n}{3}, \tfrac{4n}{3}])\right) \geq c(k, \ell).$$

Conditioning on the highest such crossing, the boundary conditions below this crossing stochastically dominate the Dobrushin boundary conditions in $[-kn, kn] \times [0, \frac{4n}{3}]$ with $a = (-kn, 0)$ and $b = (kn, 0)$. An application of the case $\ell = \frac{4}{3}$ enables us to set $c(k, \ell + \frac{1}{3}) = c(k, \ell)c(k, \frac{4}{3})$.

The proof follows from the fact that the probability in (5.33) is decreasing in ℓ. □

Lemma 5.40. *There exists a constant $C < \infty$ such that, for all $n \geq 1$,*

$$\phi^0_{\Lambda_{56n}}(\mathcal{A}_{7n}) \leq C\, \phi^0_{\Lambda_{8n}}(\mathcal{A}_n)^2. \tag{5.34}$$

Proof. Define $z_\pm = (\pm 5n, 0)$. If $\mathcal{A}_{7n}$ occurs, the boundary conditions on Λ_{7n} stochastically dominate the wired boundary conditions on Λ_{56n} due to the existence of the open circuit in $\Lambda_{14n} \setminus \Lambda_{7n}$ (simply apply the same proof as for free boundary conditions in **P1⇒P2** above). The use of the RSW theorem from Section 5.1 (Corollary 5.5) thus implies the existence of a constant $c_1 > 0$ such that, for all n,

$$\phi^0_{\Lambda_{56n}}[\mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-) | \mathcal{A}_{7n}] \geq \phi^1_{\Lambda_{56n}}[\mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-)] \geq c_1. \tag{5.35}$$

It directly implies that for all n,

$$\phi^0_{\Lambda_{56n}}[\mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-)] \geq c_1 \phi^0_{\Lambda_{56n}}[\mathcal{A}_{7n}]. \tag{5.36}$$

Now, examine the domain $D = [-56n, 56n] \times [2n, 56n]$ and define $a = (-56n, 2n)$ and $b = (56n, 2n)$. Under $\phi^0_{56n}[\,\cdot\,|\mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-)]$, the boundary conditions on D are stochastically dominated by wired boundary conditions on the bottom and free boundary conditions on the other sides. Let $\mathcal{C}_h^\star(R_+^\star)$ be the event that

$$R_-^\star := [-56n - \tfrac{1}{2}, 56n + \tfrac{1}{2}] \times [2n + \tfrac{1}{2}, 3n - \tfrac{1}{2}]$$

contains a dual-open dual-path from left to right. As a consequence, Lemma 5.39 applied[6] to $k = 57$ and $\ell = 55$ implies that

$$\phi^0_{\Lambda_{56n}}\big[\mathcal{C}^\star_h(R^\star_+)\,\big|\mathcal{A}_n(z_+)\cap\mathcal{A}_n(z_-)\big] \geq \phi^{a,b}_D\Big(\mathcal{C}^\star_h(R^\star_+)\Big) \geq c_2 \tag{5.37}$$

for some universal constant $c_2 > 0$ independent of n. Similarly,

$$\phi^0_{\Lambda_{56n}}\Big(\mathcal{C}^\star_h(R^\star_-)\,\Big|\mathcal{A}_n(z_+)\cap\mathcal{A}_n(z_-)\Big) \geq c_2, \tag{5.38}$$

where $\mathcal{C}^\star_h(R^\star_-)$ is the event that

$$R^\star_+ := [-56n - \tfrac{1}{2}, 56n + \tfrac{1}{2}] \times [-3n + \tfrac{1}{2}, 2n - \tfrac{1}{2}]$$

contains a dual-open dual-path from left to right. Define the event $\mathcal{B}_n$, illustrated on Fig. 5.8, which is the intersection of the events $\mathcal{A}_n(z_+)$, $\mathcal{A}_n(z_-)$, $\mathcal{C}^\star_h(R^\star_+)$, and $\mathcal{C}^\star_h(R^\star_-)$. Equations (5.36), (5.37) and (5.38) lead to the estimate

$$\phi^0_{\Lambda_{56n}}(\mathcal{B}_n) \geq c_3\phi^0_{\Lambda_{56n}}(\mathcal{A}_{7n}), \tag{5.39}$$

where $c_3 > 0$ is a positive constant independent of n.

Assume $\mathcal{B}_n$ occurs and define Γ_1 to be the top-most horizontal dual-crossing of $R^\star_+$ and Γ_2 to be the lowest horizontal dual-crossing of $R^\star_-$. Note that these paths are dual paths. Let Ω be the set of vertices in $[-3n, 3n]^2$ below Γ_1 and above Γ_2. Exactly as in the proof of Lemma 5.38, when conditioning on Γ_1, Γ_2 and everything outside Ω, the boundary conditions inside Ω are dual-wired on Γ_1 and Γ_2, and dominated by wired elsewhere. The dual measure inside Ω therefore dominates the restriction to Ω of the dual measure on $[-3n + \frac{1}{2}, 3n - \frac{1}{2}]^2$ with dual-wired boundary conditions on $\{\pm(3n - \frac{1}{2})\} \times [-3n + \frac{1}{2}, 3n - \frac{1}{2}]$ and dual-free boundary conditions on $[-3n + \frac{1}{2}, 3n - \frac{1}{2}] \times \{\pm(3n - \frac{1}{2})\}$. Using (5.5), we find

$$\phi^0_{\Lambda_{56n}}\big(\mathcal{C}_n\,\big|\,\mathcal{B}_n\big) \geq \frac{1}{1+q^2},$$

where $\mathcal{C}_n = \{\Gamma_1 \overset{\star}{\longleftrightarrow} \Gamma_2 \text{ in } [-3n, 3n]^2\}$ (here in $[-3n, 3n]^2$ means in the subgraph of $(\mathbb{Z}^2)^\star$ induced by dual-vertices in $[-3n, 3n]^2$ when seen as a subset of $\mathbb{R}^2$). Similar inequalities hold for the events

$$\mathcal{D}_n = \{\Gamma_1 \overset{\star}{\longleftrightarrow} \Gamma_2 \text{ in } [-13n, -7n] \times [-3n, 3n]\},$$
$$\mathcal{E}_n = \{\Gamma_1 \overset{\star}{\longleftrightarrow} \Gamma_2 \text{ in } [7n, 13n] \times [-3n, 3n]\}.$$

[6] Morally, one may think that taking $k = 56$ and $\ell = 54$ is sufficient. Nevertheless, the rectangle $R^\star_+$ is of length $112n + 1$ and width $n - 1$ and not $112n$ and n, thus explaining why it is necessary to pick $k = 57$ and $\ell = 55$.

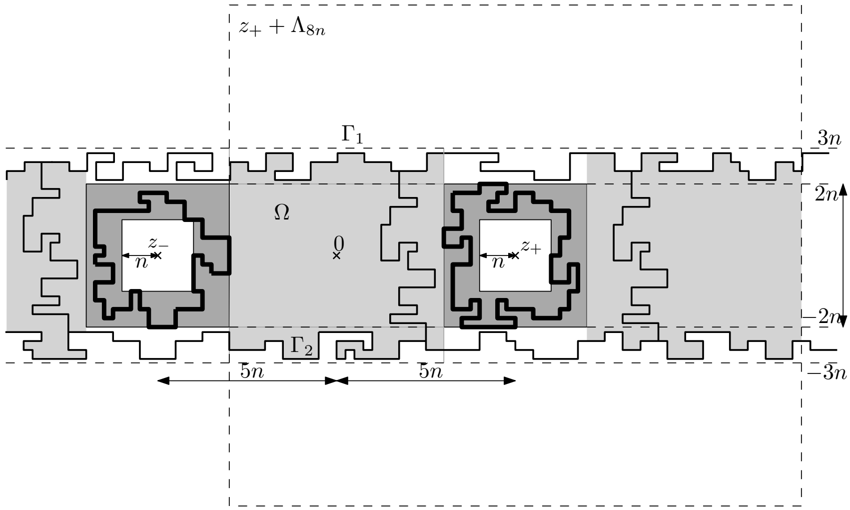

Figure 5.8: Primal open crossings are in bold, dual-open are in plain. The events $\mathcal{A}_n(z_+)$, $\mathcal{A}_n(z_-)$ and the existence of the dual horizontal crossings of $R_+^\star$ and $R_-^\star$ form $\mathcal{B}_n$. Conditionally on $\mathcal{B}_n$, Γ_1 and Γ_2 are connected in Ω by a dual-open path with probability larger than $1/(1+q^2)$.

The FKG inequality thus implies

$$\phi^0_{\Lambda_{56n}}\left(\mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{E}_n \,\middle|\, \mathcal{B}_n\right) \geq \frac{1}{(1+q^2)^3} \tag{5.40}$$

which, together with (5.39), leads to

$$\phi^0_{\Lambda_{56n}}\left(\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{E}_n\right) \geq \frac{c_3}{(1+q^2)^3}\phi^0_{\Lambda_{56n}}\left(\mathcal{A}_{7n}\right). \tag{5.41}$$

The event estimated in (5.41) implies in particular the existence of dual circuits in $z_+ + \Lambda_{8n}$ and $z_- + \Lambda_{8n}$ disconnecting $z_+ + \Lambda_{2n}$ from $z_- + \Lambda_{2n}$. Writing $\mathcal{F}_n$ for the event that such dual circuits exist and using the comparison between boundary conditions one last time (more precisely a "conditioning on the outermost dual-circuit" argument), we obtain

$$\begin{aligned}
\phi^0_{\Lambda_{8n}}\left(\mathcal{A}_n\right)^2 &= \phi^0_{z_-+\Lambda_{8n}}\left(\mathcal{A}_n(z_-)\right)\phi^0_{z_++\Lambda_{8n}}\left(\mathcal{A}_n(z_+)\right)\\
&\geq \phi^0_{\Lambda_{56n}}\left(\mathcal{A}_n(z_-)\,|\,\mathcal{A}_n(z_+)\cap\mathcal{F}_n\right)\phi^0_{\Lambda_{56n}}\left(\mathcal{A}_n(z_+)\,|\,\mathcal{F}_n\right)\phi^0_{\Lambda_{56n}}\left(\mathcal{F}_n\right)\\
&= \phi^0_{\Lambda_{56n}}\left(\mathcal{A}_n(z_-)\cap\mathcal{A}_n(z_+)\cap\mathcal{F}_n\right)\\
&\geq \frac{c_3}{(1+q^2)^3}\phi^0_{\Lambda_{56n}}\left(\mathcal{A}_{7n}\right).
\end{aligned}$$

This inequality implies the claim. □

Proof of Proposition 5.37. Obviously the cases 1 and 2 cannot occur simultaneously. Suppose that the first case does not occur and let us prove that the second does.

For all $n \geq 1$, set $u_n = C\phi^0_{\Lambda_{8n}}(\mathcal{A}_n)$, where C is defined as in Lemma 5.40. With this notation, Lemma 5.40 implies that $u_{7n} \leq u_n^2$ for any $n \geq 1$ and therefore

$$u_{7^k n_0} \leq u_{n_0}^{2^k} \tag{5.42}$$

for any positive $k \geq 0$ and $n_0 \geq 1$. Now, if $\liminf_{n\to\infty} \phi^0_{\Lambda_{8n}}(\mathcal{A}_n) = 0$, then we may pick n_0 such that $u_{n_0} < 1$. By (5.42), there exists $c_1 > 0$ such that for all n of the form $n = 7^k n_0$,

$$u_n \leq \exp\left(-c_1 n^{\log 2/\log 7}\right). \tag{5.43}$$

Fix $n = 7^k n_0$ and consider $\frac{n}{7} \leq m < n$. The FKG inequality and the comparison between boundary conditions imply that

$$\begin{aligned}
\phi^0_{[0,m]^2}((0,p) \longleftrightarrow (m,\ell)) &\leq \left(\phi^0_{[-m,m]\times[0,m]}((-m,\ell) \longleftrightarrow (m,\ell))\right)^{1/2} \\
&\leq \left(\phi^0_{\Lambda_{8n}}(\mathcal{C}_h([-2n,2n]\times[0,m]))\right)^{1/14} \\
&\leq \left(\phi^0_{\Lambda_{8n}}(\mathcal{A}_n)\right)^{1/56} \leq \exp\left(-c_2 n^{\log 2/\log 7}\right).
\end{aligned}$$

In the first inequality, we used that if $(0,p) \longleftrightarrow (m,\ell)$ and $(-m,\ell) \longleftrightarrow (0,p)$, then $(-m,\ell) \longleftrightarrow (m,\ell)$. In the second inequality, we have used that if $(x,\ell) \longleftrightarrow (x+2m,\ell)$ occur for $x = 2mj$ with $j \in \{-7,\dots,7\}$, then $\mathcal{C}_h([-2n,2n]\times[0,m])$ occurs. Finally in the third inequality we combined four crossings as in the proof of **P5⇒P5'**. Lemma 4.23 implies the claim. □

Theorem 5.24 follows directly from Proposition 5.37 and the following proposition:

Proposition 5.41. *If there exists $\alpha > 0$ such that for all $n \geq 1$,*

$$\phi^0(0 \longleftrightarrow \partial\Lambda_n) \leq \exp(-n^\alpha),$$

then there exists $c > 0$ such that for all $n \geq 1$,

$$\phi^0(0 \longleftrightarrow \partial\Lambda_n) \leq \exp(-cn).$$

We start by a lemma. Let $n \geq 1$ and $\theta \in [-1,1]$. Define the *tilted strip* in direction θ:

$$S(n,\theta) = \{(x,y) \in \mathbb{Z}^2 : 0 \leq y - \theta x \leq n\}.$$

Write $\phi_{S(n,\theta)}^{-\infty,\infty}$ for the random-cluster measure on the tilted strip $S(n,\theta)$ with wired boundary conditions on the top side and free on the bottom side[7].

We will also consider a truncated version of the tilted strip $S(n,\theta)$. For $m \geq 0$, consider the *truncated tilted strip*

$$S(n,m,\theta) = S(n,\theta) \cap \Lambda_m.$$

We will always assume that $\theta m \in \mathbb{N}$ and we will always see $S(n,m,\theta)$ as a Dobrushin domain with $a = (-m,-\theta m)$ and $b = (m,\theta m)$. Write $\phi_{S(n,m,\theta)}^{a,b}$ for the random-cluster measure with Dobrushin boundary conditions on $(S(n,m,\theta),a,b)$, i.e. free on the bottom side and wired on the other sides.

For simplicity, we will call the bottom side of the strip or the truncated strip the *free arc*, and the rest of the boundary the *wired arc*.

Lemma 5.42. *For all $m \geq n \geq 1$ and $\theta \in [-1,1]$,*

$$\phi_{S(n,m,\theta)}^{a,b}(0 \longleftrightarrow \text{wired arc}) \geq \frac{1}{5m^2n^2}. \tag{5.44}$$

Proof. Fix $n \geq 0$ and $\theta \in [-1,1]$. Let us work in the strip $S(2n,\theta)$. From now on, we drop the dependence in n and θ and write for instance $S = S(2n,\theta)$ and $S(m) = S(n,m,\theta)$. Beware that there is a slightly confusing notation here: the height of the strip is $2n$ while the one of the truncated strip is n.

For $x \in S$, define the translate $S_x(m) := x + S(m)$ of $S(m)$. We extend the definition of wired and free arcs to this context. Let $\mathcal{A}(x)$ be the event that x is connected to the wired arc of $S_x(m)$ and every open path from a vertex $y \notin S_x(m)$ to x intersects the wired arc (of $S_x(m)$). In other words, no open path starting from x "exits" $S_x(m)$ through the free arc (i.e. the bottom side).

We consider the random function $F : \mathbb{N} \longrightarrow [0,2n]$ defined by

$$F(k) := \min\{\ell : (k,\ell) \text{ is connected to the top side of } S\} - \theta k. \tag{5.45}$$

Recall that $\theta m \in \mathbb{N}$. Therefore, F can take only the $2nm+1$ following values:

$$\{0, \tfrac{1}{m}, \dots, \tfrac{2nm-1}{m}, 2n\}.$$

On the event $\{F(0) \leq n\}$, there must exist $k \in \{-nm^2,\dots,nm^2\}$ such that $F(k) \leq n$ and $F(k') \geq F(k)$ for every $|k'-k| \leq m$. Otherwise, if there is no such k, then there exists a sequence $0 = k_0,\dots,k_{nm}$ with $|k_{i+1}-k_i| \leq m$ and $0 < F(k_{i+1}) < F(k_i)$. But this provides $nm+1$ distinct values for F, all smaller or equal to n and strictly larger than 0, which is contradictory.

[7] The boundary conditions at infinity are irrelevant since the tilted strip is essentially a one-dimensional graph (this is the same argument as for ϕ_{strip} before).

Now, for k satisfying $F(k) \le n$ and $F(k') \ge F(k)$ for every $|k'-k| \le m$, the event $\mathcal{A}((k,F(k)))$ is realized. In conclusion, if $F(0) \le n$, then there exists $x \in S(n,nm^2,\theta)$ such that $\mathcal{A}(x)$ is realized and the union bound shows the existence of x in the bottom half $S(n,\theta)$ of S such that

$$\phi_S^{-\infty,\infty}(\mathcal{A}(x)) \ge \frac{\phi_S^{-\infty,\infty}(F(0)\le n)}{|S(n,nm^2,\theta)|} = \frac{\phi_S^{-\infty,\infty}(F(0)\le n)}{n(2nm^2+1)}.$$

Consider the interface between the open cluster connected to the top side of the box and the dual-open cluster dual-connected to the bottom side. By duality, this interface intersects $\{0\}\times[0,n]$ with probability larger or equal to 1/2. Thus, $\phi_S^{-\infty,\infty}(F(0)\le n) \ge \frac{1}{2}$ and therefore

$$\phi_S^{-\infty,\infty}(\mathcal{A}(x)) \ge \frac{1}{5n^2m^2}.$$

In order to conclude, we simply need to prove that

$$\phi_{S(m)}^{a,b}(0 \longleftrightarrow \text{wired arc of } S(m)) \ge \phi_S^{-\infty,\infty}(\mathcal{A}(x)). \tag{5.46}$$

First, observe that since x is contained in the bottom half $S(n,\theta)$ of S, the set $S_x(m)$ is entirely included in S. Second, since there is no open path containing x and exiting $S_x(m)$ by the free arc, there exists a lowest dual-open path in $S_x(m)$, denoted by $\Gamma^\star$, preventing the existence of such a path, see Fig. 5.9. Let Ω be the set of vertices of $S_x(m)$ above $\Gamma^\star$. The law of the random-cluster on Ω is dominated by the law of $\omega_{|\Omega}$, where ω is sampled according to a random-cluster model on $S_x(m)$ with Dobrushin boundary conditions. If $\mathcal{A}(x)$ occurs, then conditionally on $\Gamma^\star$, x is connected to the wired arc of $S_x(m)$ by an open path contained in Ω. Thus,

$$\begin{aligned}\phi_S^{-\infty,\infty}(x \longleftrightarrow \text{wired arc of } S_x(m)|\Gamma^\star) &\le \phi_{S_x(m)}^{a,b}\Big(x \stackrel{\Omega}{\longleftrightarrow} \text{wired arc of } S_x(m)\Big)\\ &\le \phi_{S_x(m)}^{a,b}(x \longleftrightarrow \text{wired arc of } S_x(m))\\ &= \phi_{S(m)}^{a,b}(0 \longleftrightarrow \text{wired arc of } S(m)).\end{aligned}$$

We omitted a few lines to get the first inequality since we applied such a reasoning already several times in this chapter. The equality follows from invariance under translations. Since the previous bound is uniform in the possible realizations of $\Gamma^\star$, we deduce (5.46) and the result follows readily. □

The next lemma will be used recursively in the proof of Proposition 5.41.

Lemma 5.43. *Assume that there exists $\alpha > 0$ such that for all $n \ge 1$,*

$$\phi^0(0 \longleftrightarrow \partial\Lambda_n) \le \exp(-n^\alpha).$$

Then for $\varepsilon > 0$ small enough, there exists a constant $C < \infty$ such that for any $n \geq 1$, any $u \in \{-n\} \times [-n, n]$ and any $v \in \{n\} \times [-n, n]$,

$$\phi^0_{\Lambda_n}(u \leftrightarrow v) \leq e^{Cn^\varepsilon} \phi^0(0 \leftrightarrow \partial\Lambda_n)^2 + Cn^6 \sum_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell=2n}} \phi^0(0 \leftrightarrow \partial\Lambda_k)\,\phi^0(0 \leftrightarrow \partial\Lambda_\ell).$$

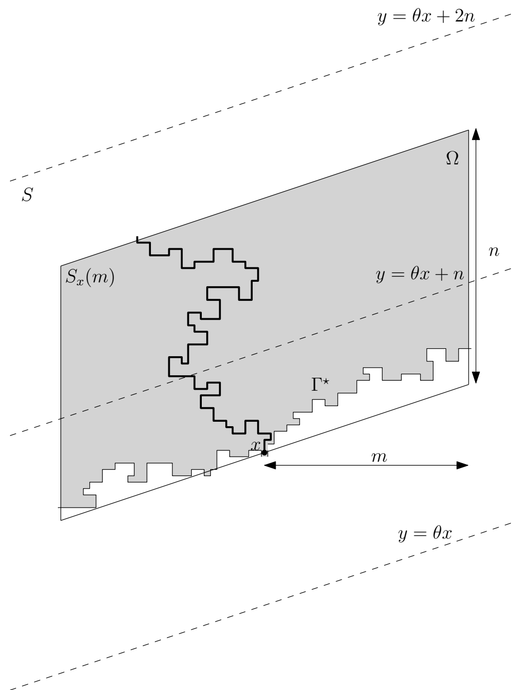

Figure 5.9: The event $\mathcal{A}(x)$ and the lowest dual-path $\Gamma^\star$.

Proof. Fix $\varepsilon > 0$. Let us translate the box Λ_n in such a way that $u = -v$; the new box is denoted by $\tilde{\Lambda}_n$. Let us also introduce $\tilde{\Lambda}_n^\star$ to be the dual graph of the Dobrushin domain $\tilde{\Lambda}_n$ (recall that it is the subgraph of $(\mathbb{Z}^2)^\star$ induced by faces *touching* $\tilde{\Lambda}_n$). Define the set

$$D = \left\{z \in \tilde{\Lambda}_n : d(z, [u, v]) < n^\varepsilon\right\}$$

(here $[u, v]$ denotes the segment between u and v and the distance is the Euclidean distance between a point and a set). As illustrated in Fig. 5.10, we consider the sets D_- and D_+ of points $z \in \tilde{\Lambda}_n$ lying respectively below and above D. On $\{u \xleftrightarrow{\tilde{\Lambda}_n} v\}$, define Γ^- and Γ^+ to be respectively the lowest and highest open (non-necessarily self-avoiding) paths connecting u

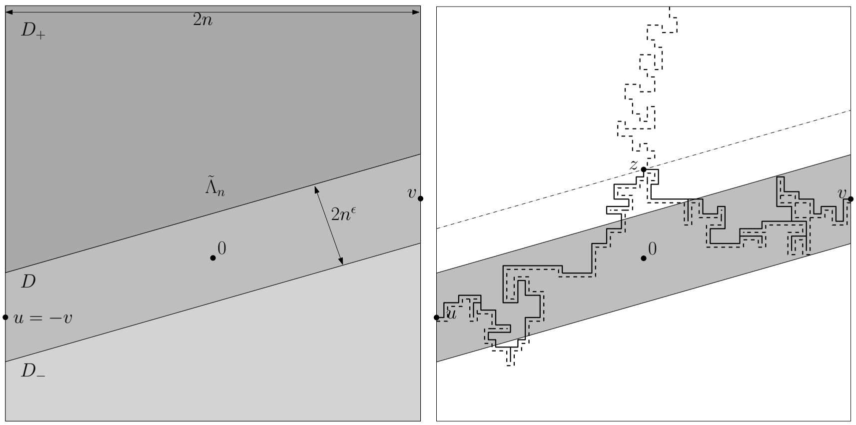

Figure 5.10: **Left.** The regions D, D_- and D_+. Note that 0 is not necessarily at the center of $\tilde\Lambda_n$. **Right.** The situation before closing the edges surrounding z when $\mathcal{G}_n(z)$ and $\{z+(\frac12,\frac12)\overset{\star}{\longleftrightarrow}\partial\tilde\Lambda_n^\star\}$ occurs. The dual-open paths are depicted in dash lines.

to v. The event $\{u\overset{\tilde\Lambda_n}{\longleftrightarrow}v\}$ is included in the union of the following three sub-events:

$$\mathcal{E}_- = \{u\overset{\tilde\Lambda_n}{\longleftrightarrow}v\}\cap\{\Gamma^-\cap D_+\neq\varnothing\}, \tag{5.47}$$

$$\mathcal{E}_+ = \{u\overset{\tilde\Lambda_n}{\longleftrightarrow}v\}\cap\{\Gamma^+\cap D_-\neq\varnothing\}, \tag{5.48}$$

$$\mathcal{E} = \{u\overset{\tilde\Lambda_n}{\longleftrightarrow}v\}\cap\{\Gamma^+\subset D_+\cup D\}\cap\{\Gamma^-\subset D_-\cup D\}. \tag{5.49}$$

In the rest of the proof, we will bound separately $\phi^0_{\tilde\Lambda_n}(\mathcal{E}_-)$ (and therefore $\phi^0_{\tilde\Lambda_n}(\mathcal{E}_+)$ by symmetry) and $\phi^0_{\tilde\Lambda_n}(\mathcal{E})$, hence the two terms on the right-hand side of the inequality in the statement.

Estimation of $\phi^0_{\tilde\Lambda_n}(\mathcal{E}_-)$. For $z\in D_+\cap\mathbb{Z}^2$, let $\mathcal{G}_n(z)$ be the event that:

- u is connected to v in $\tilde\Lambda_n$,
- $z\in\Gamma^-$ and $\mathrm{d}(z,[u,v]) = \max_{z'\in\Gamma^-\cap D_+}\mathrm{d}(z',[u,v])$.

Note that

$$\mathcal{E}_- = \bigcup_{z\in D_+}\mathcal{G}_n(z). \tag{5.50}$$

Conditionally on Γ^-, what is above Γ^- follows a random-cluster measure with wired boundary conditions on Γ^- and free on $\partial\tilde\Lambda_n$. Thus, by comparison between boundary conditions and Lemma 5.42 (with $m=n$

and $\theta = \frac{v_2 - u_2}{v_1 - u_1}$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$), we find that

$$\phi^0_{\tilde{\Lambda}_n}\left(z + (1/2, 1/2) \overset{\star}{\longleftrightarrow} \partial\tilde{\Lambda}^\star_n \,\middle|\, \mathcal{G}_n(z)\right) \geq \frac{1}{5(2n)^4}. \tag{5.51}$$

When both $\mathcal{G}_n(z)$ and $\{z + (1/2, 1/2) \overset{\star}{\longleftrightarrow} \partial\tilde{\Lambda}^\star_n\}$ occur, closing the four dual edges surrounding the vertex z disconnects Γ^- into two paths separated by dual-open circuits (see Fig. 5.10). The respective $\|\cdot\|_\infty$-end-to-end distances ℓ and k of these paths satisfy $k + \ell \geq 2n - 2$.

Using the comparison between boundary conditions once-again, we find

$$\begin{aligned}
\phi^0_{\tilde{\Lambda}_n}(\mathcal{G}_n(z)) &\leq 5(2n)^4\, \phi^0_{\tilde{\Lambda}_n}\left(\mathcal{G}_n(z) \cap \{z + (1/2, 1/2) \overset{\star}{\longleftrightarrow} \partial\tilde{\Lambda}^\star_n\}\right) && (5.52)\\
&\leq \frac{80}{c^4} n^4 \sum_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell = 2n-2}} \phi^0(u \leftrightarrow u + \partial\Lambda_k)\, \phi^0(v \leftrightarrow v + \partial\Lambda_\ell). && (5.53)
\end{aligned}$$

The finite-energy property (Proposition 4.4) is used in the second line to close the edges around z. Summing over all possible $z \in D_+$ gives

$$\phi^0_{\tilde{\Lambda}_n}(\mathcal{E}_-) \leq C_1 n^6 \sum_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell = 2n-2}} \phi^0(0 \leftrightarrow \partial\Lambda_k)\, \phi^0(0 \leftrightarrow \partial\Lambda_\ell).$$

The finite-energy property (Proposition 4.4) once again implies that

$$\phi^0(0 \leftrightarrow \partial\Lambda_{r+1}) \geq c\phi^0(0 \leftrightarrow \partial\Lambda_r)$$

for any $r \geq 0$ and thus

$$\phi^0_{\tilde{\Lambda}_n}(\mathcal{E}_-) \leq C_2 n^6 \sum_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell = 2n}} \phi^0(0 \leftrightarrow \partial\Lambda_k)\, \phi^0(0 \leftrightarrow \partial\Lambda_\ell). \tag{5.54}$$

Estimation of $\phi^0_{\tilde{\Lambda}_n}(\mathcal{E})$. First, we wish to justify that conditionally on the occurrence of $\mathcal{E}$, there exists an open path between u and v which is staying in D with probability close to 1. To see this, remark that any open path between u and v must lie in the region Ω between Γ^- and Γ^+ (see Fig. 5.11). Furthermore, conditioning on Γ^+ and Γ^-, the boundary conditions on Ω are wired. In particular, the configuration in Ω stochastically dominates the restriction to Ω of a configuration $\tilde{\omega}$ sampled according to a random-cluster measure with wired boundary conditions at infinity. Since Γ^+ and Γ^- are already open, u and v are connected in D if there exists an open path in $\tilde{\omega}$ from left to right in D. The complement of this event is included in the event that a dual-vertex of $D^\star$ is dual-connected to distance n^ε of itself in $\tilde{\omega}$. The ϕ^1-probability of this event can thus be bounded by $|D|\exp(-n^{\alpha\varepsilon})$ thanks to the assumption made on connection probabilities. We deduce

$$\phi^0_{\tilde{\Lambda}_n}\left(u \overset{D}{\longleftrightarrow} v\right) \geq (1 - |D|\exp(-n^{\alpha\varepsilon}))\phi^0_{\tilde{\Lambda}_n}(\mathcal{E}). \tag{5.55}$$

Now, consider the set of edges E of D intersecting the line $\{\frac{1}{2}\} \times \mathbb{R}$. Also define w_- and w_+ to be respectively the highest point of $D_-^\star$ and the lowest point of $D_+^\star$ with first coordinate equal to $\frac{1}{2}$. Let $\mathcal{F}$ be the event that

- all the edges of E are closed,
- w_- and w_+ are dual-connected to $\partial\tilde{\Lambda}_n^\star$ in $D_-^\star$ and $D_+^\star$ respectively.

Consider the event $u \overset{D}{\longleftrightarrow} v$ and modify the configuration by closing all edges in E. The finite-energy property (Proposition 4.4) implies that

$$\begin{aligned}\phi^0_{\tilde{\Lambda}_n}&\left(\mathcal{F} \cap \{u \leftrightarrow u + \partial\Lambda_n\} \cap \{v \leftrightarrow v + \partial\Lambda_{n-1}\}\right) \\ &\geq \phi^0_{\tilde{\Lambda}_n}\left(u \overset{D}{\longleftrightarrow} v\right) \times c^{2\sqrt{2}n^\varepsilon} \times \left(\frac{1}{5(2n)^4}\right)^2,\end{aligned} \tag{5.56}$$

where the term $c^{2\sqrt{2}n^\varepsilon}$ is a uniform lower bound for the probability that all edges in E are closed (a simple computation involving the Euclidean distance shows that there are less than $2\sqrt{2}n^\varepsilon$ edges in E), and $[5(2n)^4)]^{-2}$ comes from the estimate

$$\phi^0_{\tilde{\Lambda}_n}\left(w_- \overset{\star}{\longleftrightarrow} \partial\tilde{\Lambda}_n^\star \text{ in } D_-^\star \,\middle|\, u \overset{D}{\longleftrightarrow} v\right) \geq \frac{1}{5(2n)^4} \quad \text{and}$$
$$\phi^0_{\tilde{\Lambda}_n}\left(w_+ \overset{\star}{\longleftrightarrow} \partial\tilde{\Lambda}_n^\star \text{ in } D_+^\star \,\middle|\, u \overset{D}{\longleftrightarrow} v\right) \geq \frac{1}{5(2n)^4}$$

implied by Lemma 5.42. Overall, (5.55) and (5.56) give us

$$\phi^0_{\tilde{\Lambda}_n}(\mathcal{E}) \leq e^{Cn^\varepsilon} \phi^0_{\tilde{\Lambda}_n}\left(\mathcal{F} \cap \{u \leftrightarrow u + \partial\Lambda_n\} \cap \{v \leftrightarrow v + \partial\Lambda_{n-1}\}\right). \tag{5.57}$$

We are almost done (we only need to bound the right-hand side of the previous inequality by a constant times $\phi^0(0 \leftrightarrow \partial\Lambda_n)^2$). In order to do so, note that the event $\mathcal{F}$ forces the existence of a dual path disconnecting the cluster of u and the cluster of v (see Fig. 5.11). Conditioning on the cluster of u and its boundary, the boundary conditions in what remains are stochastically dominated by free boundary conditions at infinity, and we deduce (by the the same strategy that we already used several times) that

$$\begin{aligned}\phi^0_{\tilde{\Lambda}_n}&\left(\mathcal{F} \cap \{u \leftrightarrow u + \partial\Lambda_n\} \cap \{v \leftrightarrow v + \partial\Lambda_{n-1}\}\right) \\ &\leq \phi^0(0 \leftrightarrow \partial\Lambda_n)\,\phi^0(0 \leftrightarrow \partial\Lambda_{n-1}) \leq \frac{1}{c}\phi^0(0 \leftrightarrow \partial\Lambda_n)^2,\end{aligned}$$

where once again we used insertion tolerance in the last inequality. The claim follows readily from this inequality and (5.57). $\square$

Remark 5.44. The previous lemma implies that $\phi^0_{[0,2n]^2}(u \leftrightarrow v)$ is bounded by the right hand-side of (5.47) for any u and v on two opposite

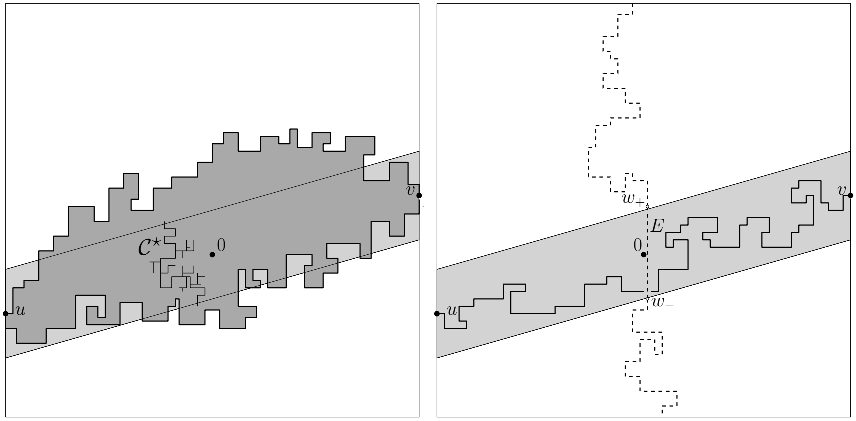

Figure 5.11: **Left.** The domain Ω between Γ_- and Γ_+. Inside, a dual cluster $\mathcal{C}^\star$ preventing the existence of an open path from u to v in D. Since we assumed that connection probabilities decay as a stretched exponential, this cluster exists with very small probability. **Right.** Splitting the open path from u to v in two pieces.

sides of $[0,2n]^2$. Let us argue that $\phi^0_{[0,2n-1]^2}(u' \leftrightarrow v')$ is also bounded by a universal constant C times the right-hand side of (5.47) uniformly on u' and v' on opposite sides of $[0,2n-1]^2$. Indeed, the comparison between boundary conditions shows that

$$\phi^0_{[0,2n-1]^2}(u' \leftrightarrow v') \leq \phi^0_{[0,2n]^2}\big(u' \leftrightarrow v' \text{ in } [0,2n-1]^2\big).$$

Now let u and v be two neighbors of u' and v' on opposite sides of $[0,2n]^2$. The finite-energy property (Proposition 4.4) implies that

$$\phi^0_{[0,2n-1]^2}(u' \leftrightarrow v') \leq C\phi^0_{[0,2n]^2}(u \leftrightarrow v)$$

and we may apply the previous lemma.

Proof of Proposition 5.41. Assume that there exists $\alpha > 0$ such that

$$\phi^0\left(0 \longleftrightarrow \partial\Lambda_n\right) \leq \exp(-n^\alpha) \tag{5.58}$$

for any $n \geq 0$. Fix $\varepsilon < \beta < \alpha$ to be chosen later. Set

$$q_n = e^{n^\beta}\phi^0\left(0 \longleftrightarrow \partial\Lambda_n\right).$$

Lemma 4.23 applied to $2n$ and Lemma 5.43 (more precisely Remark 5.44) imply that there exists $C_3 > 0$ such that

$$\begin{aligned}
q_{2n} &\le e^{(2n)^\beta} \sum_{m\ge n} C_3 m^4 \Big(e^{Cm^\varepsilon} \phi^0(0\leftrightarrow \partial\Lambda_m)^2 \\
&\qquad\qquad + Cm^6 \sum_{\substack{k,\ell\ge m^\varepsilon\\ k+\ell=2m}} \phi^0(0\leftrightarrow\partial\Lambda_k)\phi^0(0\leftrightarrow\partial\Lambda_\ell)\Big)\\
&\le e^{(2n)^\beta} \sum_{m\ge n} C_3 m^4 \Big(e^{Cm^\varepsilon} e^{-2m^\beta} q_m^2 + Cm^6 \sum_{\substack{k,\ell\ge m^\varepsilon\\ k+\ell=2m}} e^{-(k^\beta+\ell^\beta)} q_k q_\ell\Big)\\
&\le \Big(\max_{\substack{k,\ell\ge n^\varepsilon\\ k+\ell\ge 2n}} q_k q_\ell\Big) e^{(2n)^\beta} \sum_{m\ge n} C_3 m^4 \Big(e^{Cm^\varepsilon} e^{-2m^\beta} + Cm^6 \sum_{\substack{k,\ell\ge m^\varepsilon\\ k+\ell=2m}} e^{-(k^\beta+\ell^\beta)}\Big)\\
&\le C_4 \max_{\substack{k,\ell\ge n^\varepsilon\\ k+\ell\ge 2n}} q_k q_\ell,
\end{aligned}$$

where $C_4 < \infty$ is a constant independent of n. The constant C was introduced in Lemma 5.43. The existence of C_4 follows from a simple computation using $\varepsilon < \beta$ and the fact that $\beta < 1$ and $k,\ell \ge n^\varepsilon$ imply

$$e^{-(k^\beta+\ell^\beta)} \le e^{-(k+\ell)^\beta} e^{-c_2 n^{\varepsilon\beta}}$$

for some constant $c_2 > 0$ [8].

Let us now come back to the proof. The finite-energy property implies the existence of $c_3 \in (0,\infty)$ such that $c_3 q_k \le q_{k+1} \le q_k/c_3$ for any $k \ge 0$. Using this fact, the previous inequality immediately extends to odd integers and there exists $C_5 < \infty$ such that

$$q_n \le C_5 \max_{\substack{k,\ell\ge n^\varepsilon\\ k+\ell\ge n}} q_k q_\ell.$$

Since we do not know a priori that (q_n) is decreasing unfortunately, we need to include the following technical trick. Set $Q_n = C_5 \max\{q_m : m \ge n\}$. For this definition, we still get

$$Q_n \le \max_{\substack{k,\ell\ge n^\varepsilon\\ k+\ell\ge n}} Q_k Q_\ell.$$

We are now in a position to conclude. The assumption implies that (Q_n) tends to zero. Pick n_0 such that $Q_n < 1$ for $n \ge n_0^\varepsilon$. Since $(Q_n)_{n\ge0}$ is

[8] Let us make a small remark before proceeding forward with the proof. It was crucial to keep the division in the inequality of Lemma 5.43 between a term $k = \ell = m$ with a stretched exponential penalty $C_3 m^4 e^{Cm^\varepsilon}$, and the general term $k+\ell = 2m$, for which we have only a polynomial penalty $C_3 m^4 C m^6$. If we would have replaced the polynomial bound by a stretched exponential one for every k and ℓ, the values of k or ℓ close to n^ε would have created difficulties since the correction would have been of the order of the largest of the two terms.

decreasing, the maximum of $Q_k Q_\ell$ is not reached for $k \geq n$ or $\ell \geq n$ and we obtain that for $n \geq n_0$,

$$Q_n \leq \max_{\substack{n>k,\ell\geq n^\varepsilon \\ k+\ell\geq n}} Q_k Q_\ell.$$

We can now proceed by induction to prove that for $n \geq n_0$,

$$Q_n \leq \exp(-c_4 n) \quad \text{where} \quad c_4 := \max_{n_0^\varepsilon \leq n \leq n_0} -\tfrac{1}{n}\log(Q_n) > 0.$$

We therefore conclude that

$$\phi^0\left(0 \longleftrightarrow \partial\Lambda_n\right) \leq \exp(n^\beta) Q_n \leq \exp(n^\beta) \cdot \frac{1}{C_5} \exp(-c_4 n).$$

□

5.5 Applications of the strong RSW theory to spatial mixing

The bound **P5** on crossing probabilities enables us to study the spatial mixing properties at criticality. One may decouple events which are depending on edges in different areas of the space, and therefore compensate for the lack of independence. The next theorem illustrates this fact. It will be used in many occasions in the reminder of this book.

Theorem 5.45 (Polynomial ratio weak mixing under condition **P5**). *Fix $q \geq 1$ such that Property* **P5** *of Theorem 5.24 is satisfied. There exists $\alpha > 0$ such that for any $2k \leq n$ and for any event A depending only on edges in Λ_k,*

$$\left|\phi^\xi_{p_c,q,\Lambda_n}(A) - \phi^\psi_{p_c,q,\Lambda_n}(A)\right| \leq \left(\frac{k}{n}\right)^\alpha \phi^\xi_{p_c,q,\Lambda_n}(A) \tag{5.59}$$

uniformly in boundary conditions ξ and ψ.

Together with the Domain Markov property, this result implies the following inequality for $2k \leq m \leq n$ (with n possibly infinite),

$$\left|\phi^\xi_{p_c,q,\Lambda_n}(A \cap B) - \phi^\xi_{p_c,q,\Lambda_n}(A)\,\phi^\xi_{p_c,q,\Lambda_n}(B)\right| \leq \left(\frac{k}{m}\right)^\alpha \phi^\xi_{p_c,q,\Lambda_n}(A)\,\phi^\xi_{p_c,q,\Lambda_n}(B),$$

where the boundary conditions ξ are arbitrary, A is an event depending on edges of Λ_k only, and B is an event depending on edges of $\Lambda_n \setminus \Lambda_m$.

Remark 5.46. For $p \neq p_c(q)$, estimates of this type (with an exponential speed of convergence instead of polynomial) can be established by using the rate of spatial decay for the influence of a single vertex [Ale98]. At criticality, the correlation between distant events does not boil down to correlations between points and a finer argument must be harnessed. Crossing-probability estimates which are *uniform in boundary conditions* are the key in order to prove such results.

Remark 5.47. We will see several specific applications of this theorem in the next chapters. The most striking consequence is the fact that the dependence on boundary conditions can be forgotten as long as the boundary conditions are sufficiently distant from the set of edges determining whether the events under consideration are satisfied or not. For instance, it allows us to state several theorems in infinite volume, keeping in mind that most of these results possess natural counterparts in finite volume by using the fact that

$$c\phi_{p_c,q}(A) \leq \phi^{\xi}_{p_c,q,\Lambda_{2n}}(A) \leq C\phi_{p_c,q}(A)$$

for any event A depending on edges in Λ_n only, and any boundary conditions ξ on $\partial\Lambda_{2n}$ (the constants c and C are universal).

The proof of Theorem 5.45 is based on the following lemma.

Lemma 5.48. *Let $k \leq n$ and ξ arbitrary boundary conditions on $\partial\Lambda_n$. There exist two couplings $\mathbf{P}$ and $\mathbf{Q}$ on configurations (ω_ξ, ω_1) with the following properties:*

- *ω_ξ and ω_1 have respective laws $\phi^{\xi}_{p_c,q,\Lambda_n}$ and $\phi^{1}_{p_c,q,\Lambda_n}$.*
- *$\mathbf{P}$-almost surely, if $\omega_1^\star$ contains a dual-open dual-circuit in $\Lambda_n^\star \setminus \Lambda_k^\star$ and $\Gamma^\star$ is the outermost such circuit, then $\Gamma^\star$ is also closed in ω_ξ, and furthermore ω_1 and ω_ξ coincide inside $\Gamma^\star$.*
- *$\mathbf{Q}$-almost surely, if ω_ξ contains an open circuit in $\Lambda_n \setminus \Lambda_k$ and $\tilde{\Gamma}$ is the outermost such circuit, then $\tilde{\Gamma}$ is also open in ω_1, and furthermore ω_1 and ω_ξ coincide inside $\tilde{\Gamma}$.*

Proof. Holley's criterion provides us with a monotonic coupling between $\phi^{\xi}_{p_c,q,\Lambda_n}$ and $\phi^{1}_{p_c,q,\Lambda_n}$. Nevertheless, this coupling does not necessarily satisfy the required property. Let us construct another coupling via an explicit construction. We start by explaining how to sample $\phi^{\xi}_{p_c,q,\Lambda_n}$. The Domain Markov property enables us to construct a configuration as follows. Consider uniform random variables U_e on $[0,1]$ for every edge e. Choose an edge e_1 and declare it open if U_{e_1} is smaller or equal to $\phi^{\xi}_{p_c,q,\Lambda_n}(\omega(e_1)=1)$. Choose another edge e_2 and set it to open if

$U_{e_2} \leq \phi^{\xi}_{p_c,q,\Lambda_n}(\omega(e_2) = 1|\omega(e_1))$. We iterate this procedure for every edge. Also note that we can stop the procedure after a certain number of edges and sample the rest of the edges according to the right conditional law. The domain Markov property guarantees that the measure thus obtained is $\phi^{\xi}_{p_c,q,\Lambda_n}$. Note that the choice of the next edge can be random, as long as it depends only on the state of edges discovered so far.

Of course, the previous construction is useless for one measure, but it becomes interesting if we consider two measures: one may sample both configurations based on the same random variables U_e with a specific way of choosing the next edges. Let us now describe the way we are choosing the edges:

- *Construction of* $\mathbf{P}$: After t steps, the edge $e_{t+1} \in E_{\Lambda_n} \setminus E_{\Lambda_k}$ is chosen in such a way that it has one end-point connected to $\partial\Lambda_n$ by an open path in ω_1, until it is not possible anymore. Then sample all remaining edges at once according to the correct conditional law. If there is a closed circuit surrounding Λ_k in ω_1, then there was a time t such that at time $t+1$, no undiscovered edges had an end-point which was connected to the boundary in ω_1. Since this procedure guarantees that $\omega_1 \geq \omega_\xi$, no such edges were connected to the boundary in ω_ξ as well. Therefore, the configuration sampled inside the remaining domain is a random-cluster model with free boundary conditions in both cases. In particular, both configurations coincide in Λ_k.

- *Construction of* $\mathbf{Q}$: After t steps, the edge $e_{t+1} \in E_{\Lambda_n} \setminus E_{\Lambda_k}$ is chosen in such a way that one end-point of $e^{\star}_{t+1}$ is dual-connected to $\partial\Lambda^{\star}_n$ by a dual-open path in $\omega^{\star}_{\xi}$, until it is not possible anymore. Then sample all remaining edges according to the correct conditional law. If there is an open circuit surrounding Λ_k in ω_ξ, then there was a first time t such that the open circuit was discovered at time t. Once again, $\omega_1 \geq \omega_\xi$ and this circuit is also open in ω_1. Then, the configuration inside the connected component of Λ_k in $\Lambda_n \setminus \{e_1, e_2, \ldots, e_t\}$ will be sampled according to a random-cluster configuration with wired boundary conditions. In particular, both configurations coincide in Λ_k.

□

Proof of Theorem 5.45. It is clearly sufficient to prove that there exists $\alpha > 0$ such that

$$\left|\phi^{\xi}_{p_c,q,\Lambda_n}(A) - \phi^{1}_{p_c,q,\Lambda_n}(A)\right| \leq \left(\frac{k}{n}\right)^{\alpha} \phi^{\xi}_{p_c,q,\Lambda_n}(A)$$

for any event A depending on edges in Λ_k. Let E be the event that there exists a dual-open dual-circuit in $\omega_\xi^\star$ included in $\Lambda_n^\star \setminus \Lambda_k^\star$. We deduce

$$\begin{aligned}\phi^\xi_{p_c,q,\Lambda_n}(A) &\geq \phi^\xi_{p_c,q,\Lambda_n}(A\cap E) = \mathbf{Q}[\omega_\xi \in A\cap E] \geq \mathbf{Q}[\omega_1 \in A\cap E]\\ &= \phi^1_{p_c,q,\Lambda_n}(A\cap E) \geq (1-(k/n)^\alpha)\phi^1_{p_c,q,\Lambda_n}(A)\end{aligned}$$

where in the third inequality, we used the fact that if ω_1 belongs to $A\cap E$, then ω_1 and therefore ω_ξ belong to E. Since ω_1 and ω_ξ coincide in Λ_k, then $\omega_\xi \in A$. The existence of α in the last inequality follows exactly as in the proof of Lemma 5.35 from Property **P5'** applied in concentric annuli $\Lambda_{k2^{i+1}} \setminus \Lambda_{k2^i}$ with $0 \leq i \leq \log_2(n/k)$.

Reciprocally, if F denotes the event that there is an open circuit in $\Lambda_n \setminus \Lambda_k$, we find

$$\begin{aligned}\phi^1_{p_c,q,\Lambda_n}(A) &\geq \phi^1_{p_c,q,\Lambda_n}(A\cap F) = \mathbf{P}[\omega_1 \in A\cap F] \geq \mathbf{P}[\omega_\xi \in A\cap F]\\ &= \phi^\xi_{p_c,q,\Lambda_n}(A\cap F) \geq (1-(k/n)^\alpha)\phi^\xi_{p_c,q,\Lambda_n}(A)\end{aligned}$$

where once again, we used in the third inequality that if $\omega_\xi \in A\cap F$, then ω_1 is in F, and since ω_1 and ω_ξ then coincide on Λ_k, we get that $\omega_1 \in A$. The last inequality is due to **P5'** once again. $\square$

Chapter 6

Parafermions in the random-cluster model

Critical random-cluster models exhibit a rich variety of behaviours depending on the value of q. Exact computations can be performed (see [Bax89]), and despite the fact that they do not lead to fully rigorous mathematical proofs, they do provide insight and conjectures on the behavior of these models at and near criticality. For instance, the random-cluster phase transition is conjectured to be of *first order* for $q > 4$ and *second order* for $q < 4$. In the latter case, the so-called *scaling limit* is expected to be *conformally invariant* (see the second part of the manuscript for additional details). Despite Baxter's computations, very little of the behavior of the model is rigorously understood. In particular, the question of the order of the phase transition is far from being solved.

In order to understand the phase transition in random-cluster models, we introduce the so-called *parafermionic observables* and we study them in depth. The observable was first introduced in [FK80] before being rediscovered in [RC06, Smi06]. For random-cluster models with parameter $q \in [0,4]$, these observables are vertex operators that are (anti)-holomorphic parafermions of fractional spin $\sigma \in [0,1]$.

6.1 The parafermionic observable

The parafermionic observable is defined in terms of the Temperley-Lieb representation of the random-cluster model (also called the fully packed loop $O(n)$-model on the square lattice) in Dobrushin domains with Dobrushin boundary conditions.

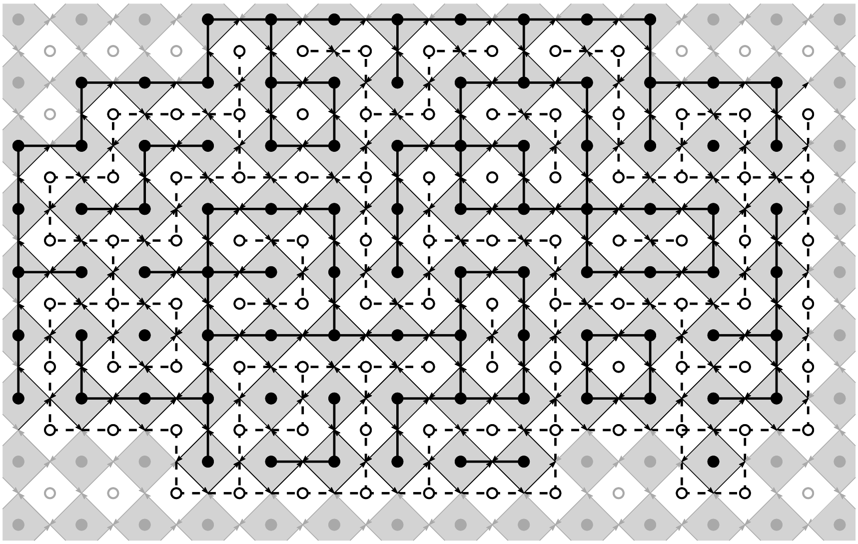

Figure 6.1: The configuration ω with its dual configuration $\omega^\star$.

6.1.1 Loop representation of the random-cluster model

We start by defining the loop-configuration associated to a percolation-configuration. Fix a Dobrushin domain (Ω, a, b) and consider a configuration $\omega \in \{0,1\}^{E_\Omega}$ together with its dual-configuration $\omega^\star \in \{0,1\}^{E_{\Omega^\star}}$. As suggested in the section on planar duality (Section 4.3.1), we define the Dobrushin boundary conditions by taking the edges (between endpoints) of ∂_{ba} to be open, and the dual-edges of $\partial^\star_{ab}$ to be dual-open.

Through every vertex of the medial graph $\Omega^\diamond$ of Ω passes either an open bond of Ω or a dual open bond of $\Omega^\star$. Draw self-avoiding loops on $\Omega^\diamond$ as follows: a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open bond through this vertex, see Fig. 6.2. The loop representation contains loops together with a self-avoiding path going from $a^\diamond$ to $b^\diamond$, see Fig. 6.2. This curve is called the *exploration path* and is denoted by γ.

Remark 6.1. The loops correspond to the interfaces separating clusters from dual clusters, and the exploration path corresponds to the interface between the cluster connected to ∂_{ba} and the dual cluster connected to $\partial^\star_{ab}$.

The loop representation of the random-cluster model is a well-known representation. It allows to map the random-cluster model to a so-called

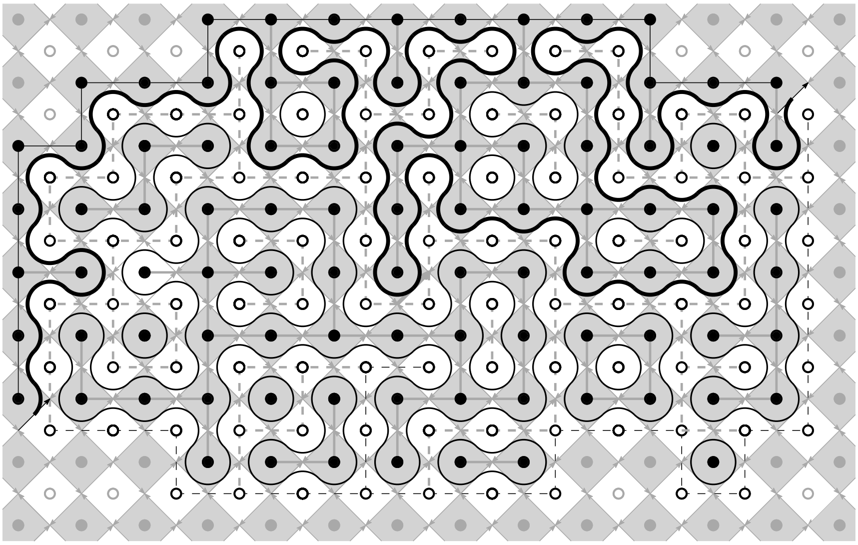

Figure 6.2: The loop representation associated to the primal and dual configurations in the previous picture. The exploration path is drawn in bold.

ice-type model (we refer to [Bax89, Chapter 10] for more details on the subject). We will not use this fact here, but simply the fact that the probability of a configuration is conveniently rewritten in terms of the number of loops.

Fix

$$x = x(p,q) := \frac{p}{\sqrt{q}(1-p)}.$$

Proposition 6.2. *Let (Ω, a, b) be a Dobrushin domain. Let $p \in (0,1)$ and $q > 0$. For any configuration ω,*

$$\phi_{p,q,\Omega}^{a,b}(\omega) = \frac{x^{o(\omega)}\sqrt{q}^{\ell(\omega)}}{Z(\Omega,a,b,p,q)}, \tag{6.1}$$

where $\ell(\omega)$ is the number of loops in the loop configuration associated to ω and $Z(\Omega,a,b,p,q)$ is a normalizing constant.

In particular, when $p = p_{sd} = p_c$, we obtain that $x = 1$ and the probability of a configuration is expressed in terms of the number of loops only.

Proof. Let $\omega^\star$ be the dual configuration of ω. The dual of $\phi^{a,b}_{p,q,\Omega}$ is $\phi^{b^\star,a^\star}_{p^\star,q,\Omega^\star}$. Recall that

$$\phi^{a,b}_{p,q,\Omega}(\omega) = \frac{1}{Z}\Big(\frac{p}{1-p}\Big)^{o(\omega)} q^{k(\omega)} \text{ and } \phi^{a^\star,b^\star}_{p^\star,q^\star,\Omega^\star}(\omega^\star) = \frac{1}{Z^\star}\Big(\frac{p^\star}{1-p^\star}\Big)^{o(\omega^\star)} q^{k(\omega^\star)},$$

where Z and $Z^\star$ are normalizing constants and $k(\omega)$ and $k(\omega^\star)$ denote the number of clusters with Dobrushin boundary conditions. We thus obtain

$$\begin{aligned}
\phi^{a,b}_{p,q,\Omega}(\omega)^2 &= \phi^{a,b}_{p,q,\Omega}(\omega)\cdot\phi^{b^\star,a^\star}_{p^\star,q,\Omega^\star}(\omega^\star)\\
&= \frac{1}{ZZ^\star}\Big(\frac{p}{1-p}\Big)^{o(\omega)} q^{k(\omega)}\Big(\frac{p^\star}{1-p^\star}\Big)^{o(\omega^\star)} q^{k(\omega^\star)}\\
&= \frac{1}{ZZ^\star}\Big(\frac{p^\star}{1-p^\star}\Big)^{o(\omega^\star)+o(\omega)}\left(\frac{p(1-p^\star)}{(1-p)p^\star}\right)^{o(\omega)} q^{k(\omega)+k(\omega^\star)}\\
&= \frac{1}{\tilde Z^2}\left(\frac{p(1-p^\star)}{(1-p)p^\star}\right)^{o(\omega)} q^{k(\omega)+k(\omega^\star)-2}
\end{aligned}$$

where we have set

$$\tilde Z^2 = \frac{ZZ^\star}{q^2[p^\star/(1-p^\star)]^{o(\omega)+o(\omega^\star)}}.$$

Note that $\tilde Z$ does not depend on the configuration since $o(\omega)+o(\omega^\star) = |E_\Omega|$. Now, the definition of $p^\star$ gives that

$$\frac{pp^\star}{(1-p)(1-p^\star)} = q$$

which implies

$$\frac{p}{1-p}\,\frac{1-p^\star}{p^\star} = \frac{p}{1-p}\,\frac{p}{(1-p)q} = x^2.$$

We can also check that $\ell(\omega) = k(\omega)+k(\omega^\star)-2$. Altogether, this implies

$$\phi^{a,b}_{p,q,\Omega}(\omega)^2 = \frac{1}{\tilde Z}\,x^{2o(\omega)}q^{\ell(\omega)}.$$

□

6.1.2 Definition of the parafermionic observable

We are now in a position to define the parafermionic observable.

Definition 6.3. The *winding* $\mathrm{W}_\Gamma(e,e')$ of a curve Γ (on the medial lattice) between two medial-edges e and e' of the medial graph is the total signed rotation in radians that the curve makes from the mid-point of the edge e to that of the edge e' (see Fig. 6.3). By convention, if Γ does not go through e', we set $\mathrm{W}_\Gamma(e,e') = 0$.

For a curve drawn on the medial lattice, the winding can be computed in a very simple way: it corresponds to the number of $\frac{\pi}{2}$-turns on the left minus the number of $\frac{\pi}{2}$-turns on the right times $\pi/2$.

Definition 6.4 (Smirnov [Smi10])**.** Consider a Dobrushin domain (Ω,a,b). The *edge parafermionic observable* $F = F(p,q,\Omega,a,b)$ is defined for any medial edge $e \in E_{\Omega^\diamond}$ by

$$F(e) \;:=\; \phi^{a,b}_{p,q,\Omega}[\mathrm{e}^{\mathrm{i}\sigma \mathrm{W}_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}],$$

where γ is the exploration path and σ is given by the relation $\sin(\sigma\pi/2) = \sqrt{q}/2$.

Remark 6.5. Note that σ is real for $q \le 4$, and belongs to $1 + i\mathbb{R}$ for $q > 4$. For $q \in [0,4]$, σ has the physical interpretation of a spin, which is fractional in general, hence the name parafermionic (fermions have half-integer spin while bosons have integer spin, there are no particles with fractional spin, but the use of such fractional spins at a theoretical level has been very fruitful in physics). For $q > 4$, σ is not real anymore and does not have any physical interpretation. Also note that for $q = 2$, $\sigma = 1/2$ corresponds to the spin of a fermion. For this reason, we will speak of the *fermionic observable* in this special case.

Remark 6.6. We will sometimes work with the *vertex parafermionic observable* which is defined as follows. For a medial-vertex $v \in V_{\Omega^\diamond} \smallsetminus \partial V_{\Omega^\diamond}$ (v has four neighboring medial-vertices in $\Omega^\diamond$), set

$$F(v) := \frac{1}{2}\sum_{u\sim v} F([uv]).$$

6.1.3 Contour integrals of the edge-parafermionic observable

The parafermionic observable satisfies a very specific property at criticality regarding contour integrals.

Let (Ω,a,b) be a Dobrushin domain. One may define a dual $(\Omega^\diamond)^\star$ of $\Omega^\diamond$ in the following way: the vertex set of $(\Omega^\diamond)^\star$ is $V_\Omega \cup V_{\Omega^\star}$ and the edges of the dual connect nearest vertices together. We extend the definitions available for other graphs to this context.

Definition 6.7. A *discrete contour* $\mathcal{C}$ is a finite sequence $z_0 \sim z_1 \sim \cdots \sim z_n = z_0$ in $(\Omega^\diamond)^\star$ such that the path $(z_0, \dots, z_n)$ is edge-avoiding. The discrete contour integral of the parafermionic observable F along $\mathcal{C}$ is defined by

$$\oint_{\mathcal{C}} F(z)dz := \sum_{i=0}^{n-1} (z_{i+1} - z_i) F\left([z_i z_{i+1}]^\star\right),$$

where the z_i are considered as complex numbers and $[z_i z_{i+1}]^\star$ denotes the edge of $\Omega^\diamond$ intersecting $[z_i z_{i+1}]$ in its center.

Theorem 6.8 (Vanishing contour integrals)**.** *Let* (Ω, a, b) *be a Dobrushin domain,* $q > 0$ *and* $p = p_c$*. For any discrete contour* $\mathcal{C}$ *of* (Ω, a, b)*,*

$$\oint_{\mathcal{C}} F(z)dz = 0.$$

Remark 6.9. The fact that discrete contour integrals vanish seems to be close to a well-known property of holomorphic functions: their contour integrals vanish. Nevertheless, one should be slightly careful when drawing such a parallel: the edge-observable is defined on edges, and should rather be understood as the discretization of a form than as the discretization of a function (the function is the vertex-observable itself). As a form, the fact that these discrete contour integrals vanish should be interpreted as the discretization of the property of being closed.

The following lemma will be important for the proof of Theorem 6.8.

Lemma 6.10. *Let* (Ω, a, b) *be a Dobrushin domain,* $p \in [0,1]$ *and* $q > 0$*. Consider* $v \in V_{G^\diamond}$ *with four incident medial edges* A*,* B*,* C *and* D *indexed in counter-clockwise order in such a way that* A *and* C *are pointing towards* v *(the two others are pointing away). Then,*

$$F(A) - F(C) \;=\; i\mathrm{e}^{\mathrm{i}\alpha}[F(B) - F(D)], \tag{6.2}$$

where $\alpha = \alpha(p,q) \in [0, 2\pi)$ *is given by the relation* $\mathrm{e}^{\mathrm{i}\alpha(p)} := \dfrac{\mathrm{e}^{\mathrm{i}\sigma\pi/2} + ix(p)}{\mathrm{e}^{\mathrm{i}\sigma\pi/2}x(p) + i}$.

When $p = p_c$, $\alpha = 0$ and the relation (6.2) can be understood as the fact that the discrete contour integral along the small square surrounding v is zero. The previous theorem thus follows by summing the relation (6.2) over vertices of $\Omega^\diamond$ enclosed by $\mathcal{C}$ (in other words faces of $(\Omega^\diamond)^\star$ surrounded by $\mathcal{C}$). We use the fact that $\mathcal{C}$ does not surround any boundary point of $\Omega^\diamond$. This follows from the fact that $\Omega^\diamond$ can be seen as a simply connected domain of $\mathbb{R}^2$ as explained in Chapter 3. In particular, its complement is a connected graph.

Proof. Assume that $v \in V_{\Omega^\diamond}$ corresponds to a vertical edge of Ω. The case of an horizontal edge can be treated in a similar fashion.

Let s be the involution (on the space of configurations) switching the state open or closed of the edge of Ω associated to v. Let e be an edge of $\Omega^\diamond$ and let

$$e_\omega \quad := \quad \phi^{a,b}_{p,q,\Omega}(\omega)\, \mathrm{e}^{\mathrm{i}\sigma \mathrm{W}_{\gamma(\omega)}(e,e_b)} 1_{e\in\gamma(\omega)}$$

be the contribution of the configuration ω to $F(e)$. With this notation, $F(e) = \sum_\omega e_\omega$. Since s is an involution, the following relation holds:

$$F(e) \;=\; \tfrac{1}{2}\sum_\omega \left[e_\omega + e_{s(\omega)}\right].$$

To prove (6.2), it is thus sufficient to show that

$$A_\omega + A_{s(\omega)} - C_\omega - C_{s(\omega)} \;=\; \mathrm{e}^{\mathrm{i}\alpha(p)} i\left[B_\omega + B_{s(\omega)} - D_\omega - D_{s(\omega)}\right] \qquad (6.3)$$

for any configuration ω.

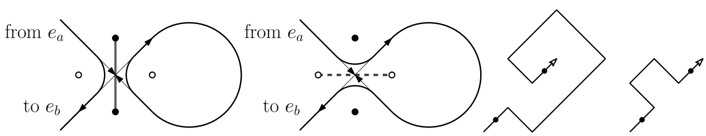

Figure 6.3: **Left.** The neighborhood of v for two associated configurations ω and $s(\omega)$. **Right.** Three examples for the winding: it is respectively equal to 2π, 0 and 0.

There are three possible cases:

Case 1. No edge incident to v belongs to $\gamma(\omega)$. Then, none of these edges is incident $\gamma(s(\omega))$ either. For any e incident to v, e_ω and $e_{s(\omega)}$ equal 0 and (6.3) trivially holds.

Case 2. Two edges incident to v belong to $\gamma(\omega)$, see Fig. 6.3. Since γ and the medial lattice possess a natural orientation, $\gamma(\omega)$ enters through either A or C and leaves through B or D. Assume that $\gamma(\omega)$ enters through the edge A and leaves through the edge D (*i.e.* that the primal edge corresponding to v is open). It is then possible to compute the contributions for ω and $s(\omega)$ of all the edges adjacent to v in terms of A_ω. Indeed,

- Since $s(\omega)$ has one less open edge and one less loop, we find

$$\begin{aligned}\phi^{a,b}_{p,q,\Omega}(s(\omega)) &= \frac{1}{Z} x^{o(s(\omega))} \sqrt{q}^{\ell(s(\omega))} = \frac{1}{Z} x^{o(\omega)-1} \sqrt{q}^{\ell(\omega)-1} \\ &= \frac{1}{x\sqrt{q}} \phi^{a,b}_{p,q,\Omega}(\omega).\end{aligned}$$

- Windings of the curve can be expressed using the winding at A. For instance, $W_{\gamma(\omega)}(B, e_b) = W_{\gamma(\omega)}(A, e_b) - \pi/2$.

The other cases are treated similarly. The contributions are given in the following table.

configuration	A	C	B	D
ω	A_ω	0	0	$e^{i\sigma\pi/2} A_\omega$
$s(\omega)$	$\frac{A_\omega}{x\sqrt{q}}$	$e^{i\sigma\pi} \frac{A_\omega}{x\sqrt{q}}$	$e^{-i\sigma\pi/2} \frac{A_\omega}{x\sqrt{q}}$	$e^{i\sigma\pi/2} \frac{A_\omega}{x\sqrt{q}}$

Using the identity $e^{i\sigma\pi/2} - e^{-i\sigma\pi/2} = i\sqrt{q}$, we deduce (6.3) by summing (with the right weight) the contributions of all the edges incident to v.

Case 3. The four edges incident to v belong to $\gamma(\omega)$. Then only two of these edges belong to $\gamma(s(\omega))$ and the computation is similar to Case 2 with $s(\omega)$ instead of ω.

In conclusion, (6.3) is always satisfied and the claim is proved. □

6.1.4 Boundary values

On the boundary, the complex argument of the observable can be computed explicitly using the fact that the curve cannot wind around a boundary point without exiting the domain.

Contrarily to ∂_{ab} and $\partial_{ab}^\star$ which are sets of vertices and dual-vertices respectively, we remind the reader that $\partial_{ab}^\diamond$ and $\partial_{ba}^\diamond$ are seen as sets of oriented medial-edges, and therefore as curves in the plane. In particular, one may consider their winding.

Lemma 6.11 (Boundary conditions)**.** *Let (Ω, a, b) be a Dobrushin domain, $p \in [0,1]$ and $q > 0$.*

- *For $x \in \partial_{ab}$ and $e \in \partial_{ab}^\diamond$ bordering x,*

$$F(e) = \exp[i\sigma W_{\partial_{ab}^\diamond}(e, e_b)] \cdot \phi_{p,q,\Omega}^{a,b}(x \longleftrightarrow \partial_{ba}).$$

- *For $u \in \partial_{ba}^\star$ and $e \in \partial_{ba}^\diamond$ bordering u,*

$$F(e) = \exp[i\sigma W_{\partial_{ba}^\diamond}(e, e_b)] \cdot \phi_{p,q,\Omega}^{a,b}\left(u \overset{\star}{\longleftrightarrow} \partial_{ab}^\star\right).$$

Proof. We prove the result for $x \in \partial_{ab}$. The proof for $u \in \partial^\star_{ba}$ follows the same lines. Since $\gamma(\omega)$ is the interface between the open cluster connected to ∂_{ba} and the dual open cluster connected to $\partial^\star_{ab}$, x is connected to ∂_{ba} if and only if e is on the exploration path. Therefore,

$$\phi^{a,b}_{p,q,\Omega}\left(x \longleftrightarrow \partial_{ba}\right) = \phi^{a,b}_{p,q,\Omega}\left(e \in \gamma\right).$$

The edge e being on the boundary, the winding of the curve is deterministic and equal to $W_{\partial^\diamond_{ab}}(e, e_b)$, we find

$$\begin{aligned} F(e) &= \phi^{a,b}_{p,q,\Omega}\left(\mathrm{e}^{\mathrm{i}\sigma W_{\partial^\diamond_{ab}}(e,e_b)} 1_{e\in\gamma}\right) \\ &= \mathrm{e}^{\mathrm{i}\sigma W_{\partial^\diamond_{ab}}(e,e_b)} \phi^{a,b}_{p,q,\Omega}\left(e \in \gamma\right) \\ &= \mathrm{e}^{\mathrm{i}\sigma W_{\partial^\diamond_{ab}}(e,e_b)} \phi^{a,b}_{p,q,\Omega}\left(x \longleftrightarrow \partial_{ba}\right). \end{aligned}$$

□

6.1.5 Interpretation as a Boundary Value Problem

Let us now make a small detour and a big leap of faith. We wish to study the possible scaling limits of these observables at criticality. Fix $0 < q < 4$, $p = p_c(q)$ and a simply connected domain Ω with two points on its boundary.

Let F_δ be the *vertex* parafermionic observable (see Remark 6.6) inside $(\Omega_\delta, a_\delta, b_\delta)$, where $(\Omega_\delta, a_\delta, b_\delta)$ is a sequence of Dobrushin domains approximating (Ω, a, b). *Assume* that properly renormalized, F_δ converges when δ tends to zero to a continuous function f.

If the discrete contour integrals of the vertex-observable F_δ would vanish, it would imply that the contour integrals of f also vanish, thus implying by Morera's theorem that the function f is holomorphic. Unfortunately, we do not know this result for the vertex-observable, but only for the edge-observable. Therefore, this only implies that a certain discrete form is "closed" [1]. Nevertheless, evidences suggest that the curl of the form given by the edge-observable also vanishes in the scaling limit. *It is therefore natural to conjecture that f is holomorphic.*

Now, the previous section shows that for any $e \in \partial^\diamond_{ab} \cup \partial^\diamond_{ba}$, $F_\delta(e)$ is collinear with $\eta\nu(e)^{-\sigma}$, where:

[1] One may think that interpreting the edge-observable F_δ as a discrete form rather than a function is an unnecessary obstacle, and that F_δ can be thought of as a function on a different graph. This is indeed true: F_δ is a function on the medial graph of $\Omega^\diamond$. Nevertheless, the function thus obtained does not converge to a continuous function in the scaling limit (for instance for $q = 2$, the complex argument of the edge-observable is not the same for edges with different orientations, see Chapter 9), and therefore this interpretation does not provide us with an alternative way of dealing with the problem.

- η is a complex number of modulus 1 which depends on the domain only. This term is not important in the interpretation;
- $\nu(e)^{-\sigma}$ equals
$$\exp\big(-i\sigma W_{\partial^\diamond_{ab}}(e,e_b)\big)$$
if $e \in \partial^\diamond_{ab}$. The same definition holds for e on the path $\partial^\diamond_{ba}$, with $\partial^\diamond_{ba}$ replacing $\partial^\diamond_{ab}$ in the previous formula. The vector $\nu(e)^{-\sigma}$ can be interpreted as a discretization of the tangent vector to the boundary to the power $-\sigma$ (recall that $\sigma \in [0,1]$ for $0 < q < 4$).

(At the discrete level, we defined the vertex-observable only inside the domain but one may in fact extend the definition fairly naturally to boundary medial-vertices as well. We will discuss this question for the fermionic observable in Chapter 9.)

In the continuum, the previous boundary conditions at the discrete level suggest that the limit f "should" satisfy the following boundary conditions:

$$\mathrm{Im}(f\overline{\eta}\nu^\sigma) = 0 \text{ on } \partial\Omega,$$

where ν is the tangent vector, and be solution of the following Boundary Value Problem (BVP)

$$f \text{ holomorphic on } \Omega, \quad \text{continuous on } \overline{\Omega} \quad \text{and} \quad \mathrm{Im}(f\overline{\eta}\nu^\sigma) = 0 \text{ on } \partial\Omega.$$

These BVPs are called Riemann-Hilbert BVP. In general, the previous discussion seems hard to justify rigorously. Nevertheless, we will see in the second part of this book that it is exactly what happens for $q = 2$.

A natural way of trying to prove that F_δ, properly renormalized, converges to the solution of the Riemann-Hilbert BVP would be to prove that F_δ is solution of a discretization of the Riemann-Hilbert BVP, and the fact that the integrals along discrete contours of the observable vanish (or equivalently that the equations (6.2) hold) seems to be a good starting point to try to implement this strategy. Unfortunately, a simple counting argument shows that the number of unknown variables determining F_δ is $|E_{\Omega^\diamond_\delta}|$ while the number of relations provided by (6.2) is only $|V_{\Omega^\diamond_\delta}|$. Therefore, the relations do not determine the observable completely (for instance, (6.2) is not sufficient to compute the observable using the boundary values only). In particular, F_δ is not necessarily *the unique solution* of the right discrete Riemann-Hilbert BVP.

We sometimes say that the equations given by (6.2) correspond to only half of the discrete Cauchy-Riemann equations (see Chapter 8), and that the other half becomes satisfied in the scaling limit. In the next chapters, we will see that for $q = 2$, one can extract more information from the observables. It becomes determined by the equations given by (6.2) and one may prove that F_δ is indeed the unique solution of a discrete Riemann-Hilbert BVP.

6.1.6 Average behavior of the observable on the boundary

As explained in the previous section, the observable is not determined by the relations provided by (6.2) and the boundary conditions, and therefore its behavior is unlikely to be understood completely using (6.2) only. Nevertheless, a small miracle occurs and the average behavior on the boundary of the domain can be understood quite well by using the fact that the integral along the boundary is equal to zero. Let us describe this now.

For $x \in \partial_{ab}$, define $N(x)$ to be the number of neighboring vertices of x which are *not* in Ω. We also set

$$W(x) := \frac{1}{N(x)+1} \sum_e W_{\partial^\diamond_{ab}}(e, e_b),$$

where the sum runs over medial-edges $e \in \partial^\diamond_{ab}$ bordering the face corresponding to x. Note that there are $N(x)+1$ such medial-edges. This quantity can thus be interpreted as the average winding on adjacent medial-edges of $\partial^\diamond_{ab} \cup \partial^\diamond_{ba}$. In some sense, it is a discrete version of the tangent vector to the boundary.

For $u \in \partial^\star_{ba}$, define $N(u)$ to be the number of neighboring dual-vertices of u which are not in Ω. The quantity $W(u)$ is defined as before, with $\partial^\diamond_{ba}$ replacing $\partial^\diamond_{ab}$.

Corollary 6.12. *Let (Ω, a, b) be a Dobrushin domain, $q > 0$ and $p = p_c$. Then*

$$\sum_{x \in \partial_{ab}} \delta_x \, \phi^{a,b}_{p_c,q,\Omega}(x \leftrightarrow \partial_{ba}) - \sum_{u \in \partial^\star_{ba}} \delta_u \, \phi^{a,b}_{p_c,q,\Omega}\left(u \overset{\star}{\longleftrightarrow} \partial^\star_{ab}\right)$$
$$= 1 - \exp[\mathrm{i}(\sigma-1) W_{\partial^\diamond_{ab}}(e_a, e_b)], \qquad (6.4)$$

where $\delta_x := \sin\left[(1-\sigma)\frac{\pi}{4} N(x)\right] \exp\left[\mathrm{i}(\sigma-1) W(x)\right]$.

While the notations may be confusing, this statement will be extremely useful when using the observable for $q \neq 2$. The proof can be summarized as follows: we integrate the edge parafermionic observable along the outermost contour. Since this contour goes along the boundary, we may factorize the winding terms using Lemma 6.11. We finish the proof by grouping the medial-edges depending on which vertex of the boundary they border.

Proof. Consider the set E of medial-edges of $E_{\Omega^\diamond}$ having only one end-point in $\Omega^\diamond \smallsetminus \partial\Omega^\diamond$. Let E_{i} be the set of medial-edges of E that are pointing towards a vertex of $\Omega^\diamond \setminus \partial\Omega^\diamond$. Similarly, define $E_{\rm o}$ to be the set of medial-edges of E that are pointing away from a vertex of $\Omega^\diamond \setminus \partial\Omega^\diamond$.

Now, consider the circuit $\mathcal{C}$ passing through the edges of $E_{\rm i}$ and $E_{\rm o}$ orthogonally. Observe that this circuit goes around $\Omega^\diamond$. From Theorem 6.8, we have

$$\oint_{\mathcal{C}} F(z)dz = 0.$$

This can be rewritten as

$$\sum_{e\in E_{\rm i}} {\rm e}^{-{\rm i}W(e,e_b)}F(e) - \sum_{e\in E_{\rm o}} {\rm e}^{-{\rm i}W(e,e_b)}F(e) \;=\; 0.$$

Above, $W(e,e_b)$ denotes $W_{\partial^\diamond_{ab}}(e,e_b)$ or $W_{\partial^\diamond_{ba}}(e,e_b)$ depending on whether $e \in \partial^\diamond_{ab}$ or $e \in \partial^\diamond_{ba}$.

Each vertex of ∂_{ab} (resp. dual-vertex of $\partial^\star_{ba}$) is bordered by exactly one medial-edge in $E_{\rm i} \setminus \{e_a\}$ and one medial-edge in $E_{\rm o} \setminus \{e_b\}$. We denote this first medial-edge by $e_{\rm i}(x)$ and the second by $e_{\rm o}(x)$ (similarly $e_{\rm i}(u)$ and $e_{\rm o}(u)$). The expression of F on the boundary (Lemma 6.11) thus implies that

$$\begin{aligned}
&1 - {\rm e}^{{\rm i}(\sigma-1){\rm W}({\rm e_a},{\rm e_b})}\\
&+ \sum_{x\in\partial_{ab}} \Big({\rm e}^{{\rm i}(\sigma-1)W(e_{\rm o}(x),e_b)} - {\rm e}^{{\rm i}(\sigma-1)W(e_{\rm i}(x),e_b)}\Big)\phi^{a,b}_{p_c,q,\Omega}(x \longleftrightarrow \partial_{ba})\\
&= \sum_{u\in\partial^\star_{ba}} \Big({\rm e}^{{\rm i}(\sigma-1)W(e_{\rm o}(u),e_b)} - {\rm e}^{{\rm i}(\sigma-1)W(e_{\rm i}(u),e_b)}\Big)\phi^{a,b}_{p_c,q,\Omega}\Big(u \overset{\star}{\longleftrightarrow} \partial^\star_{ab}\Big). \qquad (6.5)
\end{aligned}$$

We now wish to interpret terms in the parentheses. We do it for $x \in \partial_{ab}$ only ($u \in \partial^\star_{ba}$ can be handled similarly). First,

$$\begin{aligned}
&{\rm e}^{{\rm i}(\sigma-1)W(e_{\rm o}(x),e_b)} - {\rm e}^{{\rm i}(\sigma-1)W(e_{\rm i}(x),e_b)} =\\
&\sin\Big[(1-\sigma)\tfrac{W(e_{\rm o}(x),e_b)-W(e_{\rm i}(x),e_b))}{2}\Big]\exp\Big[{\rm i}(\sigma-1)\tfrac{W(e_{\rm o}(x),e_b)+W(e_{\rm i}(x),e_b)}{2}\Big].
\end{aligned}$$

By construction, $W(e_{\rm o}(x),e_b) - W(e_{\rm i}(x),e_b) = \frac{\pi}{2}N(x)$. Furthermore, one may check by dividing between the cases where $N(x) = 2$, 3 or 4, that

$$\frac{W(e_{\rm o}(x),e_b) + W(e_{\rm i}(x),e_b)}{2} = W(x).$$

This concludes the proof. □

6.1.7 The special case $q = 4$

The case $q = 4$ is in-between the regime of $\sigma \in [0,1]$ (namely $q \in (0,4]$) and $\sigma \in 1 + i\mathbb{R}$ ($q \geq 4$). In particular, the parafermionic observable that we defined previously is degenerated and does not contain any information for $q = 4$: one may easily see that when $\sigma = 1$, (6.2) merely says that the exploration path is a curve, meaning that the number of entries to a vertex is equal to the number of exits. It is therefore necessary to introduce a slightly different observable.

Definition 6.13. Consider a Dobrushin domain (Ω, a, b) and $p \in [0,1]$. The *edge parafermionic observable* $\widehat{F} = \widehat{F}(p, 4, \Omega, a, b)$ for any medial edge $e \in E_{\Omega^\diamond}$ is defined by

$$\widehat{F}(e) \;:=\; \phi_{p,q,\Omega}^{a,b}[\mathrm{W}_\gamma(e, e_b)\mathrm{e}^{\mathrm{i}\mathrm{W}_\gamma(\mathrm{e},\mathrm{e}_\mathrm{b})}\mathbf{1}_{e\in\gamma}]$$

where γ is the exploration path.

Exactly as before, this parafermionic observable satisfies the following properties (we do not necessarily assume that we are at criticality here).

Theorem 6.14. *Fix $q = 4$, $p \in [0,1]$, and let (Ω, a, b) be a Dobrushin domain.*

- *Let $v \in V_{G^\diamond}$ with four incident medial edges A, B, C and D indexed in counter-clockwise order in such a way that A and C are pointing towards v (the two others are pointing away). Then,*

$$\widehat{F}(A) - \widehat{F}(C) \;=\; i[\widehat{F}(B) - \widehat{F}(D)] \;+\; i\frac{\pi}{2}\frac{1-x}{1+x}\big[F(B) - F(D)\big]. \tag{6.6}$$

- *For $y \in \partial_{ab}$ and $e \in \partial_{ab}^\diamond$ bordering y,*

$$\widehat{F}(e) \;=\; W_{\partial_{ab}^\diamond}(e, e_b)\,\mathrm{e}^{\mathrm{i}W_{\partial_{\mathrm{ab}}^\diamond}(\mathrm{e},\mathrm{e}_\mathrm{b})}\;\phi_{p,q,\Omega}^{a,b}\,(y \leftrightarrow \partial_{ba})\,,$$

  and for $u \in \partial_{ba}^\star$ and $e \in \partial_{ba}^\diamond$ bordering u,

$$\widehat{F}(e) \;=\; W_{\partial_{ba}^\diamond}(e, e_b)\,\mathrm{e}^{\mathrm{i}W_{\partial_{\mathrm{ba}}^\diamond}(\mathrm{e},\mathrm{e}_\mathrm{b})}\;\phi_{p,q,\Omega}^{a,b}\left(u \overset{\star}{\longleftrightarrow} \partial_{ab}^\star\right).$$

- *Let $p = p_c$, for any discrete contour $\mathcal{C}$ of (Ω, a, b),*

$$\oint_{\mathcal{C}} \widehat{F}(z)dz = 0.$$

- *Let $p = p_c$,*

$$\sum_{x\in\partial_{ab}} \delta_x\,\phi_{p_c,q,\Omega}^{a,b}\,(x \leftrightarrow \partial_{ba}) - \sum_{u\in\partial_{ba}^\star} \delta_u\,\phi_{p_c,q,\Omega}^{a,b}\left(u \overset{\star}{\longleftrightarrow} \partial_{ab}^\star\right) = W_{\partial_{ab}^\diamond}(e_a, e_b), \tag{6.7}$$

  where $\delta_x := \frac{\pi}{2}N(x)$.

Proof. We only prove (6.6). The other claims are obtained by exactly the same proofs as in the previous sections.

For this proof only, we will consider different values of $q < 4$ and let q tends to 4. For this reason, σ can be equal to 1 or to the solution $\sigma = \sigma(q)$ of $\sin(\sigma\pi/2) = \sqrt{q}/2$. Fix $x > 0$. Denote the observable at $(p(q), q) := (\frac{x\sqrt{q}}{1+x\sqrt{q}}, q)$ with spin $\sigma \in \{1, \sigma(q)\}$ by $F_{q,\sigma}$. Note that the choice of $p = p(q)$ is made in such a way that x is fixed.

For any $q < 4$,

$$\begin{aligned} F_{q,\sigma(q)}(A) - F_{q,\sigma(q)}(C) &= i\mathrm{e}^{\mathrm{i}\alpha(p)}[F_{q,\sigma(q)}(B) - F_{q,\sigma(q)}(D)] \\ F_{q,1}(A) - F_{q,1}(C) &= i[F_{q,1}(B) - F_{q,1}(D)], \end{aligned}$$

where $\alpha(q)$ was defined in Lemma 6.10. The first relation is due to the case $q \neq 4$, and the second follows readily from the fact that γ is a curve (it simply asserts that a curve entering through A or C exits through B or D). Now, since $\sigma(q)$ and $\alpha(q)$ tend to 1 and 0 as $q \nearrow 4$, one may develop the first equation in $\sigma - 1$. The first order cancels thanks to the second equation, and the second order gives the claim. □

6.2 Second order phase transition

In this section, we show that the conditions **P1**–**P5** of Theorem 5.24 are all satisfied when $1 \le q \le 4$. It answers partially Baxter's conjecture [Bax73] that **P1**–**P5** if and only if $q \le 4$. The following theorems are extracted from [DC12, DCST13]. The article [DCST13] also contains a partial result for $q > 4$. We refer to the article itself for more details.

6.2.1 Statement of the theorem

Theorem 6.15 (Duminil-Copin [DC12])**.** *Let $1 \le q \le 4$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log \phi^0_{p_c,q}(0 \longleftrightarrow \partial\Lambda_n) = 0.$$

Theorem 6.15 together with Theorem 5.24 implies the following corollary.

Corollary 6.16 (Duminil-Copin, Sidoravicius, Tassion [DCST13])**.** *Let $1 \le q \le 4$. The following assertions are satisfied:*

P1 $\phi^1_{p_c,q}(0 \longleftrightarrow \infty) = 0$.

P2 $\phi^0_{p_c,q} = \phi^1_{p_c,q}$.

P3 $\chi^0(p_c,q) := \sum_{x\in\mathbb{Z}^2} \phi^0_{p_c,q}(0 \longleftrightarrow x) = \infty.$

P4 $\lim_{n\to\infty} \frac{1}{n} \log \phi^0_{p_c,q}(0 \longleftrightarrow \partial\Lambda_n) = 0.$

P5 *Let $\rho > 0$. There exists $c > 0$ such that for all $n \geq 2$ and any boundary conditions ξ,*

$$c \leq \phi^{\xi}_{p_c,q,[-n,(\rho+1)n]\times[-n,2n]}(\mathcal{C}_h([0,\rho n]\times[0,n])) \leq 1-c.$$

Note that Theorem 6.15 implies the weakest property among **P1**–**P5** (namely **P4**). This justifies the energy spent in the previous chapter to prove Theorem 5.24.

6.2.2 Proof of Theorem 6.15

As for the proof of Theorem 5.24, the reader may skip the proof in a first reading. The original proof of Theorem 6.15 can be found in [DC12]. However, we choose to present the streamlined proof of [DCST13]. In this section, we fix $1 \leq q < 4$ and $p = p_c(q)$. The case $q = 4$ follows the same proof with the modified parafermionic observable. We therefore drop them from the notations.

Set C_n to be the slit domain obtained by removing from Λ_n the edges between the vertices of $\{(0,k) : 0 \leq k \leq n\}$. We define Dobrushin boundary conditions on C_n to be wired on $\{(0,k) : 0 \leq k \leq n\}$ and free elsewhere. For simplicity, we now refer to $\{(0,k) : 0 \leq k \leq n\}$ as the *wired arc*. The measure on C_n with these boundary conditions is denoted $\phi^{\mathrm{dobr}}_{C_n}$. Equivalently, one may obtain $\phi^{\mathrm{dobr}}_{C_n}$ by taking $\phi^0_{\Lambda_n}(\,\cdot\,|\omega(e) = 1 : \text{for all } e \text{ in wired arc})$ and we therefore think of C_n as the box Λ_n with free boundary conditions and $\{(0,k) : 0 \leq k \leq n\}$ wired; see Fig. 6.5.

Lemma 6.17. *There exists $c > 0$ such that for any $n \geq 1$,*

$$\phi^{\mathrm{dobr}}_{C_n}((0,-n) \longleftrightarrow \text{wired arc}) \geq \frac{c}{n^{16}}.$$

The main estimate used in the proof of this lemma is provided by Corollary 6.12 applied in a well-chosen domain. Then, we work a little to compare boundary conditions in this domain to Dobrushin boundary conditions in C_n. To exploit the whole power of Corollary 6.12, we will consider a domain which is non-planar. Namely, let us introduce the following graph $\mathbb{U}$ (see Fig. 6.4): the vertices are given by $\mathbb{Z}^3$ and the edges by

- $[(x_1,x_2,x_3),(x_1,x_2+1,x_3)]$ for every $x_1,x_2,x_3 \in \mathbb{Z}$,

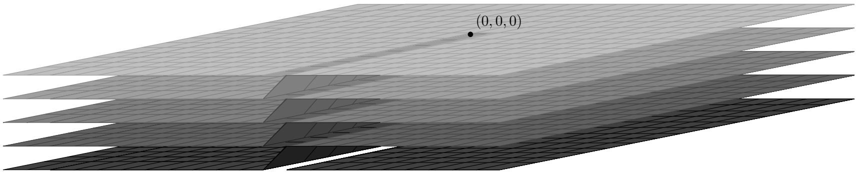

Figure 6.4: The graph $\mathbb{U}$.

- $[(x_1,x_2,x_3),(x_1+1,x_2,x_3)]$ for every $x_1,x_2,x_3\in\mathbb{Z}$ such that $x_1\neq 0$ or such that $x_1=0$ and $x_2\geq 0$,
- $[(0,x_2,x_3),(1,x_2,x_3+1)]$ for every $x_2<0$ and $x_3\in\mathbb{Z}$.

This graph is the universal cover of $\mathbb{Z}^2\smallsetminus F$, where F is the face centered at $(-\frac{1}{2},-\frac{1}{2})$. It can also be seen at $\mathbb{Z}^2$ with a branching point at $(-\frac{1}{2},-\frac{1}{2})$. All definitions of dual and medial graphs extend to this context.

Proof. For $n\geq 1$, define

$$U_n := \left\{(x_1,x_2,x_3)\in\mathbb{U} : |x_1|,|x_2|\leq n \text{ and } |x_3|\leq n^5\right\}.$$

We wish to apply Corollary 6.12 to $(U_n,0,0)$. Even if the domain is non-planar, the proof works exactly in the same way and we get

$$\sum_{x\in\partial U_n}\delta_x\,\phi^0_{U_n}(x\leftrightarrow 0) = 1-\mathrm{e}^{\mathrm{i}(\sigma-1)\frac{3\pi}{2}}.$$

To obtain this equality, we used that:

- $\partial_{ab}=\partial U_n$ and $\partial^\star_{ba}=\varnothing$;
- $W(e_a,e_b)=\frac{3\pi}{2}$;
- The Dobrushin boundary conditions with $a=b=0$ are simply free boundary conditions.

Since $|\delta_x|\leq 1$, we immediately get that

$$\sum_{x\in\partial U_n}\phi^0_{U_n}(0\leftrightarrow x)\geq c_1 \tag{6.8}$$

for some constant $c_1=c_1(q)>0$ independent of n.

We now wish to bootstrap this estimate to an estimate on C_n. Let us start by proving the following claim (observe that $|x_3|<n^5$ in the statement).

Claim: There exists $c_2>0$ (independent of n) such that there exists $x=(x_1,x_2,x_3)\in\partial U_n$ with $|x_3|<n^5$ and

$$\phi^0_{U_n}(0\leftrightarrow x)\geq\frac{c_2}{n^6}.$$

We will prove this fact by showing that vertices x with $|x_3| = n^5$ have very small probability of being connected to the origin and cannot therefore account for much in (6.8).

Proof of the Claim. Let $R_0^\star$ be the dual graph of the subgraph of $\mathbb{U}$ with vertex set $R_0 := [-n,n] \times [0,n] \times \{0\}$, i.e. the graph with edge set $\{e^\star : e \in E_{R_0}\}$ and vertex set given by the end-points of these edges. Note that uniformly in the state of edges outside R_0, the boundary conditions in R_0 are stochastically dominated by wired boundary conditions on the "bottom side" $[-n,n]\times\{0\}\times\{0\}$ of R_0, and free elsewhere. Passing to the dual model, we deduce that uniformly in the state of edges outside R_0,

$$\begin{aligned}\phi^0_{U_n}\Big((-\tfrac12,-\tfrac12,0) \overset{\star}{\longleftrightarrow} \partial U_n^\star \text{ in } R_0^\star \Big| \text{edges outside } R_0\Big)&\\ \ge \phi^{\text{dobr}}_{R_0^\star}\Big((-\tfrac12,-\tfrac12,0) \overset{\star}{\longleftrightarrow} \partial U_n^\star \text{ in } R_0^\star\Big)&,\end{aligned}$$

where Dobrushin boundary conditions on $R_0^\star$ are dual-free on the bottom and dual-wired everywhere else. Lemma 5.42 (with $m = n$ and $\theta = 0$) thus implies that

$$\phi^0_{U_n}\Big((-\tfrac12,-\tfrac12,0) \overset{\star}{\longleftrightarrow} \partial U_n^\star \text{ in } R_0^\star \Big| \text{edges outside } R_0\Big) \ge \frac{1}{5n^4}. \tag{6.9}$$

The same is also true for $R_k^\star = (0,0,k) + R_0^\star$ with $|k| \le n^5$.

If a vertex $x = (x_1,x_2,x_3) \in \partial U_n$ with $x_3 = n^5$ is connected to $(0,0,0)$, then none of the dual vertices $(-\frac12,-\frac12,k)$ are dual connected to ∂U_n in $R_k^\star$, for $0 < k < x_3$ (the symmetric holds for $x_3 = -n^5$). Equation (6.9) applied $|x_3| - 1$ times implies that

$$\phi^0_{U_n}(0 \longleftrightarrow x) \;\le\; \Big(1 - \frac{1}{5n^4}\Big)^{|x_3|-1}.$$

The probability is therefore exponentially small when $|x_3| = n^5$. Together with (6.8), the previous inequality implies that for n large enough,

$$\sum_{x\in\partial U_n : |x_3|<n^5} \phi^0_{U_n}(0 \leftrightarrow x) \ge \frac{c_1}{2}.$$

The claim follows directly from the union bound, provided that c_2 is chosen small enough. $\diamond$

Fix x given by the claim and rotate and translate vertically[2] U_n in such a way that $x = (x_1,-n,0)$ for some $0 \le x_1 \le n$. Consider C_n as a subgraph of U_n. The boundary conditions on C_n induced by the free boundary

[2]Seen as a graph, U_n is invariant by rotation by $\pi/2$ since the line where x_3 "increases" is invisible from inside U_n.

conditions on U_n are dominated by the Dobrushin boundary conditions on C_n defined above. Furthermore, the existence of an open path from x to the origin implies the existence of a path from x to the wired arc in C_n. Thus, the claim implies that

$$\phi_{C_n}^{\mathrm{dobr}}(x \leftrightarrow \text{wired arc}) \;\geq\; \frac{c_2}{n^6}. \tag{6.10}$$

To conclude the proof, we need to obtain a lower bound for the probability that the vertex $(0,-n,0)$ itself is connected to the wired arc. We use once again a "conditioning on the right-most and left-most paths type argument". Since we now work on a sub-domain of $\mathbb{Z}^2$, we drop the third coordinate from the notation.

We may assume that $x_1 \geq 0$ and that the two vertices $x = (x_1,-n)$ and $(-x_1,-n)$ are connected to the wired arc. The FKG inequality implies that this occurs with probability $(\frac{c_2}{n^6})^2$. Consider the right-most open path from $(x_1,-n)$ to the wired arc, and the left-most open path from $(-x_1,-n)$ to the wired arc. Let S be the part of C_n between these two paths, see Fig. 6.5. The boundary conditions in S dominate the free boundary conditions on the bottom of C_n, and wired elsewhere. Lemma 5.42 (applied to $2n$, $m = n$ and $\theta = 0$) thus implies[3] that $(0,-n)$ is connected to the wired arc with probability larger than $\frac{1}{20n^4}$. The claim follows by choosing c small enough. $\square$

We are now in a position to prove Theorem 6.15.

Let ∂_n be the set of vertices at distance $\frac{n}{16}$ of the vertex $(0,-n)$ in C_n. The reasoning is similar to the proof of Lemma 5.40 except that instead of isolating primal circuits around z_- and z_+ from each others, we will isolate the primal path from $(0,-n)$ to ∂_n from the wired arc.

Proof of Theorem 6.15. Let $R^\star_{\mathrm{right}} := \big[\frac{n}{16}, \frac{5n}{16}\big] \times [-n,n]$ and $R^\star_{\mathrm{left}} := \big[-\frac{5n}{16}, -\frac{n}{16}\big] \times [-n,n]$. Define the three events $\mathcal{E} = \{(0,-n) \longleftrightarrow \partial_n\}$, $\mathcal{F}_{\mathrm{right}}$ and $\mathcal{F}_{\mathrm{left}}$ that there exists a dual-open dual-path from bottom to top in $R^\star_{\mathrm{right}}$ and $R^\star_{\mathrm{left}}$ respectively.

Let $\mathcal{C}$ be the event that there exists a dual-open dual-path in the square

$$R^\star := \Big[-\frac{5n}{16}+\frac{1}{2}, \frac{5n}{16}-\frac{1}{2}\Big] \times \Big[-\frac{3n}{4}+\frac{1}{2}, -\frac{n}{8}-\frac{1}{2}\Big],$$

connecting a dual open path crossing $R^\star_{\mathrm{left}}$ from top to bottom to a dual open path crossing $R^\star_{\mathrm{right}}$ from top to bottom.

[3] We use a comparison between boundary conditions. The reasoning is the same as usual: we compare boundary conditions on S with the boundary conditions induced by boundary conditions on Λ_n with free boundary conditions on the bottom and wired boundary conditions on the other sides.

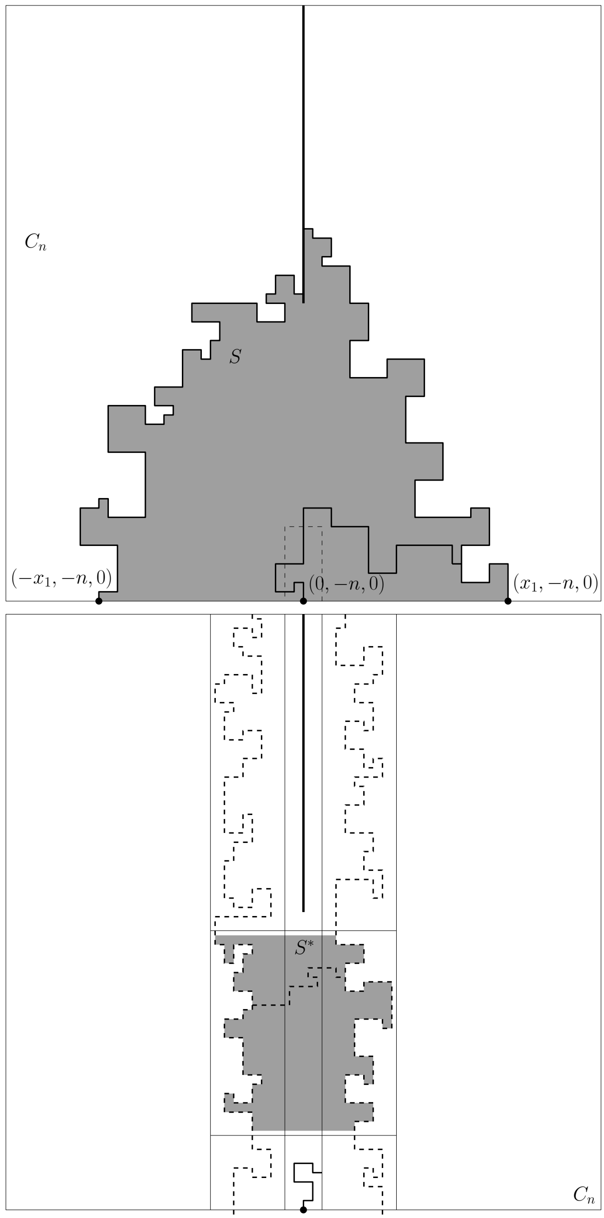

Figure 6.5: **Top.** The two paths connecting the wired arc to $(x_1, -n, 0)$ and $(-x_1, -n, 0)$ (or simply $(x_1, -n)$ and $(-x_1, -n)$) and the area S between them. **Bottom.** The two dual-open paths in the long rectangles $R^{\star}_{\text{right}}$ and $R^{\star}_{\text{left}}$.

Conditioning on $\mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} \cap \mathcal{C}$, boundary conditions on R_n are dominated by free boundary conditions in the plane. Therefore

$$\phi^0_{\mathbb{Z}^2}\left(0 \leftrightarrow \partial\Lambda_{\frac{n}{16}}\right) \geq \phi^{\text{dobr}}_{C_n}\left(\mathcal{E}|\mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} \cap \mathcal{C}\right) \geq \phi^{\text{dobr}}_{C_n}\left(\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} \cap \mathcal{C}\right).$$

We now prove a lower bound on the term on the right:

$$\begin{aligned}&\phi^{\text{dobr}}_{C_n}\left(\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} \cap \mathcal{C}\right)\\ &= \phi^{\text{dobr}}_{C_n}(\mathcal{E}) \cdot \phi^{\text{dobr}}_{C_n}\left(\mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}|\mathcal{E}\right) \cdot \phi^{\text{dobr}}_{C_n}\left(\mathcal{C}|\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}\right).\end{aligned}$$

First, Lemma 6.17 implies that $\phi^{\text{dobr}}_{C_n}(\mathcal{E}) \geq \frac{c}{n^{16}}$.

Second, conditioned on everything on the left of $\{\frac{n}{16}\}\times[-n,n]$, boundary conditions on $[\frac{n}{16},n]\times[-n,n]$ are dominated by wired boundary conditions on the left side and free elsewhere. In particular, boundary conditions for the dual model stochastically dominate free boundary conditions on the left side and wired elsewhere. Lemma 5.39 implies that $\phi^{\text{dobr}}_{C_n}\left(\mathcal{F}_{\text{right}}|\mathcal{E}\right) \geq c_2$ and the same lower bound holds true for $\phi^{\text{dobr}}_{C_n}\left(\mathcal{F}_{\text{left}}|\mathcal{F}_{\text{right}} \cap \mathcal{E}\right)$. We obtain

$$\phi^{\text{dobr}}_{C_n}\left(\mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}|\mathcal{E}\right) \geq c_2^2.$$

Third, we turn to $\phi^{\text{dobr}}_{C_n}\left(\mathcal{C}|\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}\right)$. Let $S^\star$ be the area of the dual graph in $R^\star$ between the right-most dual open path from top to bottom in $R^\star_{\text{right}}$, and the left-most dual open path crossing from top to bottom in $R^\star_{\text{left}}$, see Fig. 6.5. The boundary conditions for the dual model on $S^\star$ dominate dual-free boundary conditions on top and bottom, and dual-wired elsewhere. The domain Markov property and the comparison between boundary conditions imply that boundary conditions for the dual model on $S^\star$ dominate dual-free boundary conditions on top and bottom sides of $R^\star$, and dual-wired on the two other sides. Therefore, the probability of having a dual open path in $S^\star$ crossing from left to right is larger than $1/(1+q^2)$ thanks to (5.5). In particular,

$$\phi^{\text{dobr}}_{C_n}\left(\mathcal{C}|\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}\right) \geq \frac{1}{1+q^2}.$$

Putting everything together, we find that

$$\phi^0\left(0 \leftrightarrow \partial\Lambda_{n/16}\right) \geq \frac{c}{n^{16}} \cdot c_2^2 \cdot \frac{1}{1+q^2}$$

and indeed

$$\lim_{n\to\infty} -\frac{1}{n}\log\left(\phi^0\left(0 \leftrightarrow \partial\Lambda_n\right)\right) = 0.$$

□

6.3 Alternative computation of p_c when $q \geq 4$

We conclude this chapter by providing an alternative derivation of the critical point when $q > 4$. While this result also follows from the previous chapter, the technique gives (a little) more information on the critical phase and is probably more robust.

Theorem 6.18 (Beffara, Duminil-Copin, Smirnov [BDCS12]). *Let $q > 4$. The critical point $p_c = p_c(q)$ for the random-cluster model with parameter q on the square lattice satisfies*

$$p_c = p_{sd} = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

As mentioned above, when $q > 4$, the spin variable σ involved in the definition of the parafermionic observable becomes non-real, therefore it does not have an immediate physical interpretation. However, this allows us to write better estimates even in the absence of an interpretation in terms of holomorphic functions and relate our observables to the connectivity properties of the model. For $p \neq \frac{\sqrt{q}}{1+\sqrt{q}}$, the observable is proved to behave like massive harmonic functions and to decay exponentially fast with respect to the distance to the boundary of the domain. Translated into connectivity properties, this implies a slightly stronger theorem than Theorem 6.18, namely the sharpness of the phase transition for $q \geq 4$. It would be nice to obtain a similar computation of p_c based on the observable for $1 \leq q < 4$ or even more so for $q < 4$.

For $x = (x_1, x_2) \in \mathbb{R}^2$, define $\|x\|_\infty = \max\{|x_1|, |x_2|\}$.

Theorem 6.19. *Let $q > 4$. For every $p < p_{sd}$, there exists $c = c(p,q) > 0$ such that for any $u \in \mathbb{Z}^2$,*

$$\phi^0_{p,q}(0 \leftrightarrow u) \leq \mathrm{e}^{-c\|u\|_\infty}.$$

Once we have the exponential decay of correlations for $p < p_{sd}$, Theorem 6.18 follows right away. Let us explain this quickly.

Proof of Theorem 6.18 assuming Theorem 6.19. Fix $q > 4$ and let $p < p_{sd}$. Theorem 6.19 and the Borel-Cantelli lemma imply that 0 is not connected to infinity almost surely and therefore $p < p_c$. We deduce that $p_c \geq p_{sd}$ [4].

Let us now prove that $p_{sd} \geq p_c$. For a dual vertex $y \in (\mathbb{Z}^2)^\star$, let $A(y)$ be the event that there exists a dual-open circuit surrounding the origin

[4] Note that we could have used Zhang's argument (Proposition 4.38) but once we have exponential decay, this approach is quicker.

and passing through y. Observe that on $A(y)$, y is dual-connected to a dual-vertex z at distance $\|y\|_\infty$ of y.

For $p > p_{sd}$, the dual measure is a random-cluster measure with $p^\star < p_{sd}$. Therefore

$$\phi^1_{p,q}(A(y)) \le 8|y| \exp\big[-c(p^\star,q)\|y\|_\infty\big]$$

for any $y \in (\mathbb{Z}^2)^\star$. The Borel-Cantelli lemma implies that almost surely there are finitely many dual-circuits around the origin. As a consequence, there is an infinite cluster almost surely and $p \ge p_c$. We deduce $p_c \le p_{sd}$. □

Remark 6.20. The comparison between boundary conditions enables us to extend the previous theorems to $q = 4$. Indeed, for every $p < \frac{\sqrt{q}}{1+\sqrt{q}}$, there exists (p',q) with $q > 4$ and $p' < \frac{\sqrt{q}}{1+\sqrt{q}}$ such that the random-cluster measure $\phi^0_{p',q}$ stochastically dominates the random-cluster measure $\phi^0_{p,4}$ (Remark 4.18). It follows from this stochastic domination that for any $u \in \mathbb{Z}^2$,

$$\phi^0_{p,4}(0 \longleftrightarrow u) \le \phi^0_{p',q}(0 \longleftrightarrow u) \le \exp\big[-c(p',q)\|u\|_\infty\big].$$

6.3.1 Proof of Theorem 6.19

Let (Ω,a,b) be a Dobrushin domain, $p \in [0,1]$ and $q > 4$. With the notations of Definition 6.4, we define the function $\tilde F$ by

$$\tilde F(e) = \phi^{a,b}_{p,q,\Omega}\left(\mathrm{e}^{\mathrm{i}(\sigma-1)\mathrm{W}_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}\right). \tag{6.11}$$

Observe that $\mathrm{i}(\sigma-1) \in \mathbb{R}$ whenever $q \ge 4$. Furthermore, (6.2) translates into

$$\tilde F(A) + \tilde F(C) - \Lambda(p)\tilde F(B) - \Lambda(p)\tilde F(D) = 0, \tag{6.12}$$

around any medial-vertex $v \in V_{\Omega^\diamond}$ with four incident medial-edges A, B, C and D indexed in counter-clockwise order in such a way that A and C are pointing towards v (the two others are pointing away). Above,

$$\Lambda(p) = \frac{\mathrm{e}^{\mathrm{i}(\sigma-1)\pi/2} + x(p)}{\mathrm{e}^{\mathrm{i}(\sigma-1)\pi/2}x(p) + 1} \in \mathbb{R}_+.$$

Note that $\Lambda(p) = 1$ if and only if $p = \frac{\sqrt{q}}{1+\sqrt{q}}$.

For a set $E \subset E_{\Omega^\diamond}$, $\partial_{\mathrm{e}} E$ denotes the set of medial-edges sharing exactly one end-point with a medial-edge in E (in particular they are not in E). Let E_{int} be the set of medial-edges having two end-points in $\Omega^\diamond \setminus \partial\Omega^\diamond$.

Proposition 6.21. *Fix $q > 4$ and $p < p_{sd}$. Let $(\Omega^\diamond, a^\diamond, b^\diamond)$ be a Dobrushin domain. There exists $C_1 = C_1(p,q) < \infty$ such that for any $E \subset E_{\mathrm{int}}$,*

$$\sum_{e\in E} \tilde F(e) \le C_1 \sum_{e\in\partial_{\mathrm{e}} E} \tilde F(e).$$

Proof. Since $E \subset E_{\text{int}}$, the end-points of medial-edges in E are in $\Omega^\diamond \setminus \partial\Omega^\diamond$. We may therefore sum (6.12) over all end-points of edges in E. It provides a weighted sum of $\tilde{F}(e)$ (with coefficients denoted by $c(e)$) identical to zero:

$$\sum_{e \in E} c(e)\tilde{F}(e) + \sum_{e \in \partial_{\mathrm{e}} E} c(e)\tilde{F}(e) = 0. \tag{6.13}$$

For an edge $e \in E$, $\tilde{F}(e)$ appears in two identities (6.12), corresponding to its two end-points. Since the edge e is pointing towards one of its end-points and away from the other one, the coefficients will be -1 and $\Lambda(p)$. Thus $\tilde{F}(e)$ for $e \in E$ will enter the sum with a coefficient $c(e) = 1 - \Lambda(p) \neq 0$. For an edge $e \in \partial_{\mathrm{e}} E$, $\tilde{F}(e)$ will appear in exactly one identity (corresponding to the end-point shared with an edge of E). The coefficient will be $-\Lambda(p)$ or 1, depending on the orientation of e with respect to this end-point. In any case, $\tilde{F}(e)$ will enter the sum with a coefficient $c(e)$ which is bounded. Since $\tilde{F}(e)$ is positive for every $e \in E_{\Omega^\diamond}$, the proposition follows immediately by setting

$$C_1 = C_1(p,q) := \frac{\max\{1, \Lambda(p)\}}{|\Lambda(p) - 1|}.$$

□

For $n, m \geq 0$, consider the graph $R(m,n) = [0,n] \times [-m,m]$.

Lemma 6.22. *Fix $q > 4$ and $p \leq p_{sd}$. There exists $C_2 = C_2(q) < \infty$ such that for any $n, m \geq 0$,*

$$\sum_{e \in \partial_{\mathrm{e}} E_{\text{int}}} F(e) \leq C_2,$$

where $\tilde{F}$ is defined in the Dobrushin domain $(R(m,n), 0, 0)$.

Proof. Let us start by recalling that Dobrushin boundary conditions on $(R(m,n), 0, 0)$ coincide with free boundary conditions.

Every edge $e \in \partial_{\mathrm{e}} E_{\text{int}}$ is on the boundary of $R(m,n)^\diamond$. Lemma 6.11 thus implies that

$$\tilde{F}(e) = \mathrm{e}^{\mathrm{i}(\sigma-1)W(e,e_b)} \phi^0_{p,q,R(m,n)}[x \longleftrightarrow 0],$$

where x is the vertex of $\partial R(m,n)$ bordered by e. Since $-\frac{\pi}{2} \leq W(e,e_b) \leq 2\pi$, we find that

$$\sum_{e \in \partial_{\mathrm{e}} E_{\text{int}}} \tilde{F}(e) \leq C_1 \sum_{x \in \partial R(m,n)} \phi^0_{p,q,R(m,n)}[x \longleftrightarrow 0].$$

We wish to prove that the right hand side of the previous inequality is bounded uniformly in m and n. We may now focus on $p = p_{sd}$ by monotonicity. We use Proposition 6.12 again to obtain:

$$\sum_{x\in\partial R(m,n)} \delta_x \, \phi^0_{p,q,R(m,n)}[x \longleftrightarrow 0] = 1 - \exp\Big[\mathrm{i}(\sigma-1)\frac{3\pi}{2}\Big],$$

where we used that $\partial_{ab} = \partial R(m,n)$, $\partial^\star_{ba} = \varnothing$ and $W(e_a, e_b) = \frac{3\pi}{2}$.

Above, δ_x is

$$\delta_x := \sin\big[(1-\sigma)\tfrac{\pi}{4}N(x)\big]\exp\big[\mathrm{i}(\sigma-1)W(x)\big]$$

which is positive since $\sigma \in 1 + i\mathbb{R}$. Furthermore, we once again have $-\frac{\pi}{2} \le W(x) \le \frac{3\pi}{2}$, and therefore δ_x is bounded away from 0 and ∞ uniformly in m and n. The claim follows readily. □

We are now in a position to prove our key corollary. For a graph Ω, let us introduce the following graphs constructed recursively. Let $\Omega^{(0)} := \Omega$ and $\Omega^{(k)} := \Omega^{(k-1)} \setminus \partial\Omega^{(k-1)}$ for any $k \ge 1$. They can be seen as successive "peelings" of Ω, each step consisting in removing the boundary of the existing graph. Let $E_k := E_{\mathrm{int}}[\Omega^{(k)}]$.

Corollary 6.23. *Let $q > 4$ and $p < p_{sd}$. There exist $C_1, C_2 > 0$ such that for any $n, m \ge 0$,*

$$\sum_{e\in E_k} \tilde F(e) \le C_1 C_2 \left(\frac{C_1}{1+C_1}\right)^k,$$

where $\tilde F$ is defined in the Dobrushin domain $(R(m,n),0,0)$.

Proof. Proposition 6.21 can be applied to $\Omega^(k)$ (with $a = b$ be any point on $\partial\Omega^{(k)}$) to give

$$\sum_{e\in E_k} \tilde F(e) \le \frac{C_1}{1+C_1} \sum_{e\in E_k \cup \partial_{\mathrm{e}} E_k} \tilde F(e).$$

Since $E_k \cup \partial_{\mathrm{e}} E_k \subset E_{k-1}$ and $\tilde F(e) \ge 0$, this implies

$$\sum_{e\in E_k} \tilde F(e) \le \frac{C_1}{1+C_1} \sum_{e\in E_{k-1}} \tilde F(e).$$

Using the previous bound iteratively, and Proposition 6.21 one last time (in the second inequality), we find

$$\sum_{e\in E_k} \tilde F(e) \le \left(\frac{C_1}{1+C_1}\right)^k \sum_{e\in E_{\mathrm{int}}} \tilde F(e) \le C_1 \left(\frac{C_1}{1+C_1}\right)^k \sum_{e\in \partial_{\mathrm{e}} E_{\mathrm{int}}} \tilde F(e).$$

The claim follows by bounding the sum on the right-hand side by C_2 using Lemma 6.22. □

Define the strip $\mathbb{S}_n := [0,n] \times \mathbb{Z}$ of with n, and the half-plane $\mathbb{HP} = \mathbb{N} \times \mathbb{Z}$.

Lemma 6.24. *There exists $c_1 = c_1(p,q) > 0$ such that for any $v = (v_1, v_2)$ with $v_1 \in \mathbb{N}$,*

$$\phi^0_{p,q,\mathbb{HP}}(0 \longleftrightarrow v) \le \exp[-c_1 v_1].$$

Proof. Let m and n in such a way that $v \in R(m,n)$. Apply Corollary 6.23 to $R(m,n)$ and k to find (we use the notation of the corollary)

$$\sum_{e \in E_k} \tilde{F}(e) \le C_1 C_2 \left(\frac{C_1}{1+C_1}\right)^k.$$

Consider the orthogonal symmetry τ with respect to $d = \{x = (x_1, x_2) : x_2 = 0\}$. Let γ be the exploration path in $(R(m,n), 0, 0)$ (i.e. starting from 0). By symmetry, every configuration ω such that $e \in \gamma$ is mapped by τ to a configuration for which $\tau(e) \in \gamma$. Furthermore, the winding is transformed from $W_\gamma(e, e_b)$ to $\frac{3\pi}{2} - W_{\tau(\gamma)}(\tau(e), e_b)$. This gives that

$$\begin{aligned}\tilde{F}(e) + \tilde{F}(\tau(e)) &= \phi^0_{p,q,R(m,n)}\left[\left(\mathrm{e}^{\mathrm{i}(\sigma-1)W(e,e_b)} + \mathrm{e}^{\mathrm{i}(\sigma-1)(\frac{3\pi}{2} - W(e,e_b))}\right)\mathbf{1}_{e \in \gamma}\right] \\ &\ge 2\phi^0_{p,q,R(m,n)}(e \in \gamma).\end{aligned}$$

Let $\mathcal{C}$ be the cluster of the origin in $R(m,n)$. Define $\partial_{\mathrm{ext}}\mathcal{C}$ to be the exterior boundary of $\mathcal{C}$, i.e. the set of vertices of $\mathcal{C}$ connected by a path in $\mathbb{Z}^2 \setminus \mathcal{C}$ to the boundary of $R(m,n)$. The exploration path γ is a loop from 0 to 0 going around $\mathcal{C}$. The exterior boundary $\partial_{\mathrm{ext}}\mathcal{C}$ lies directly on its exterior. Let $d_\infty(x,F)$ denote the distance between x and the set F, i.e. $\min\{\|x-y\|_\infty : y \in F\}$. We find

$$\begin{aligned}\phi^0_{p,q,R(m,n)}\Big(\exists u \in \partial_{\mathrm{ext}}\mathcal{C} : d_\infty(u, \partial R(m,n)) \ge k\Big) &\le \sum_{e \in E_k} \phi^0_{p,q,R(m,n)}(e \in \gamma) \\ &\le C_1 C_2 \left(\frac{C_1}{1+C_1}\right)^k.\end{aligned}$$

Letting m go to infinity and using the uniform bound above,

$$\phi^0_{p,q,\mathbb{S}_n}\Big(\exists u \in \partial_{\mathrm{ext}}\mathcal{C} : d_\infty(u, \partial\mathbb{S}_n) \ge k\Big) \le C_1 C_2 \left(\frac{C_1}{1+C_1}\right)^k.$$

Since there is no infinite cluster in $\mathbb{S}_n$ almost surely (see Remark 4.33), the cluster $\mathcal{C}$ intersects $\{u \in \mathbb{S}_n : d_\infty(u, \partial\mathbb{S}_n) \ge k\}$ if and only if $\partial_{\mathrm{ext}}\mathcal{C}$ intersects $\{u \in \mathbb{S}_n : d_\infty(u, \partial\mathbb{S}_n) \ge k\}$. Hence, for $v = (v_1, v_2)$ with $v_1 \ge 0$,

$$\begin{aligned}\phi^0_{p,q,\mathbb{S}_n}(0 \longleftrightarrow v) &\le \phi^0_{p,q,\mathbb{S}_n}(\exists u \in \mathcal{C} : d_\infty(u, \partial\mathbb{S}_n) \ge v_1) \\ &= \phi^0_{p,q,\mathbb{S}_n}(\exists u \in \partial_{\mathrm{ext}}\mathcal{C} : d_\infty(u, \partial\mathbb{S}_n) \ge v_1) \\ &\le C_1 C_2 \left(\frac{C_1}{1+C_1}\right)^{v_1}.\end{aligned}$$

The proof follows by letting n go to infinity and then by choosing $c_1 = c_1(p,q) > 0$ small enough. □

The previous lemma implies that for any $n \geq 0$ and $x \in \partial\Lambda_n$,

$$\phi^0_{p,q,\Lambda_n}(0 \longleftrightarrow x) \leq \exp[-c_1 n].$$

Indeed, rotate the graph Λ_n in such a way that $x = (n, x_2)$. The comparison between boundary conditions in $(n,0) - \mathbb{HP}$ leads to

$$\phi^0_{p,q,\Lambda_n}(0 \longleftrightarrow x) \leq \phi^0_{p,q,(n,0)-\mathbb{HP}}(0 \longleftrightarrow x) \leq \exp[-c_1 n]$$

thanks to Lemma 6.24. Now, Lemma 4.23 implies that for any k,

$$\begin{aligned}\phi^0_{p,q,\Lambda_n}(0 \longleftrightarrow \partial\Lambda_k) &\leq \sum_{m\geq k} 64 m^4 \max_{\substack{a\in\{0\}\times[0,m]\\ b\in\{m\}\times[0,m]}} \phi^0_{p,q,[0,m]^2}(a \longleftrightarrow b)\\ &\leq \sum_{m\geq k} 64 m^4 \exp(-c_1 m) \leq \exp(-c_2 k),\end{aligned} \tag{6.14}$$

where $c_2 > 0$ is chosen small enough but independent of k. In the second inequality, we used the comparison between boundary conditions. Theorem 6.19 thus follows by letting n tend to infinity.

Part III

Ising and FK-Ising models

Chapter 7

Two-dimensional Ising model

We now enter a new part of this book devoted to the Ising model. As presented in the introduction, this model is one of the most famous models of statistical physics. In this chapter, we study the basic properties of this model.

Define the exterior boundary of a graph G by

$$\partial_e G = \{x \notin V_G : \exists y \in V_G \text{ s.t. } x \sim y\}.$$

7.1 Definition of the Ising model

7.1.1 Formal definition on the square lattice

The Ising model can be defined on any graph. However, we will once more restrict ourselves to the square lattice. Let G be a finite subgraph of $\mathbb{Z}^2$, the Ising model with *free boundary conditions* is a random assignment $\sigma \in \{-1,1\}^{V_G}$ of spins $\sigma_x \in \{-1,+1\}$ (or simply $-/+$), where σ_x denotes the spin at the vertex x. The *Hamiltonian* of the Ising model is defined by

$$H_G^{\mathrm{f}}(\sigma) := -\sum_{[xy]\in E_G} \sigma_x\sigma_y.$$

The *partition function* of the model is

$$Z_{\beta,G}^{\mathrm{f}} = \sum_{\sigma\in\{-1,1\}^{V_G}} \exp\left[-\beta H_G^{\mathrm{f}}(\sigma)\right], \tag{7.1}$$

where β is the *inverse temperature* of the model. The Ising measure is the Boltzmann measure with Hamiltonian H_G^{f}. More precisely, the probability of a configuration σ is equal to

$$\mu^{\mathrm{f}}_{\beta,G}(\sigma) = \frac{1}{Z^{\mathrm{f}}_{\beta,G}} \exp\left[-\beta H^{\mathrm{f}}_G(\sigma)\right]. \tag{7.2}$$

Below, $\mu^{\mathrm{f}}_{\beta,G}$ will also denote the expectation with respect to $\mu^{\mathrm{f}}_{\beta,G}$.

Remark 7.1. In order to get the same definition as in the first chapter, the constant β should be understood as $1/T$, where T is the temperature. For this reason, we will speak of high-temperature regime when β is small, and low-temperature regime when β is large.

Remark 7.2. More generally, the Ising model may be defined by the Hamiltonian

$$H^{\mathrm{f}}_G(\sigma) := -\sum_{x,y\in V_G} J_{x,y}\sigma_x\sigma_y,$$

where $(J_{x,y})_{x,y\in\mathbb{Z}^2}$ is a set of *coupling constants*, where $(J_{x,y})$ is invariant under translations, i.e. that $J_{x,y} = J_{0,y-x}$ for any x and y. If $J_{x,y} \geq 0$ for any x and y, the model is called *ferromagnetic*. If $J_{x,y} = J$ if $x \sim y$ and 0 otherwise, the model is called *nearest neighbor*, otherwise it is called *long-range*. In this book, we focus on the nearest neighbor ferromagnetic Ising model and we fix $J = 1$ [1].

For a graph G and $\tau \in \{-1,+1\}^{\mathbb{Z}^2}$, one may also define the Ising model with τ boundary conditions by the Hamiltonian

$$H^{\tau}_G(\sigma) = \begin{cases} -\displaystyle\sum_{x\sim y \text{ and } \{x,y\}\cap V_G\neq\emptyset} \sigma_x\sigma_y & \text{if } \sigma_x = \tau_x, \forall x \notin V_G, \\ \infty & \text{otherwise.} \end{cases} \tag{7.3}$$

Note that in this case the state space of spin configurations is formally $\{-1,+1\}^{\mathbb{Z}^2}$. However, since any configuration not corresponding to τ outside of G has probability 0, the space is in direct correspondence with $\{-1,+1\}^{V_G}$. For this reason, we now consider this later set to be the state space.

Remark 7.3. The boundary conditions τ may be defined as an element of $\{-1,+1\}^{\partial_{\mathrm{e}}G}$ instead since other values outside V_G are not relevant. The approach consisting in considering $\tau \in \{-1,1\}^{\mathbb{Z}^2}$ is sometimes more convenient, in particular when taking the limit (as n tends to infinity) of $\mu^{\tau}_{\beta,\Lambda_n}$ for a fixed τ. Furthermore, this choice is more consistent with the general theory of so-called Gibbs measures and long-range models. Nevertheless, we will sometimes allow ourself some latitude and simply specify the boundary conditions on $\partial_{\mathrm{e}}G$.

[1] Multiplying all the coupling constants by the same multiplicative constant boils down to changing β.

Similarly to the random-cluster case, several boundary conditions will be of particular importance in our study:

- free boundary conditions: defined above.
- $+$ and $-$ boundary conditions: the measure with boundary conditions $\tau_x = +1$ for all $x \in \mathbb{Z}^2$ (resp. $\tau_x = -1$ for all $x \in \mathbb{Z}^2$) is denoted by $\mu^+_{\beta,G}$ (resp. $\mu^-_{\beta,G}$),
- *Dobrushin* (or *domain-wall*) boundary conditions: assume G is a discrete domain. Assume that $\partial_e G$ can be divided into two $\star$-connected paths ∂_- and ∂_+ when going around it (x and y are $\star$-neighbors if $\|x-y\|_\infty = 1$ and each vertex has eight neighbors). The Dobrushin boundary conditions are defined to be $-$ on ∂_- and $+$ on ∂_+. The Dobrushin boundary conditions force the existence of macroscopic interfaces in the model between the $-$ cluster connected to ∂_- and the $+$ cluster connected to ∂_+.

7.2 General properties

7.2.1 Dobrushin-Lanford-Ruelle property

Similarly to the random-cluster case, the Ising model satisfies a strong form of domain Markov property, called the DLR conditions; see [Geo11]. Roughly speaking, this condition asserts that the Ising measure conditioned on the configuration outside of a graph F is equal to the Ising measure with boundary conditions inherited from the conditioning. In particular, the Ising measure only keeps memory of the nearest neighbors.

Proposition 7.4. *Let $F \subset G$ two finite subgraphs of $\mathbb{Z}^2$. Let $\tau \in \{-1,1\}^{\mathbb{Z}^2}$ and $\beta > 0$. Then*

$$\mu^\tau_{\beta,G}\big(\sigma_{|V_F} = \eta \,\big|\, \sigma_x = \tau_x : \forall x \in V_G \setminus V_F\big) \;=\; \mu^\tau_{\beta,F}(\eta) \quad , \quad \forall \eta \in \{-1,1\}^{V_F}.$$

Proof. The proof of this statement is left as an exercise. It is very similar to the proof of the Markov property for random-cluster models. □

7.2.2 Positive association of the Ising model

The set $\{-1,1\}^{V_G}$ is equipped with a partial order: $\sigma \le \sigma'$ if for any $x \in V_G$, $\sigma_x \le \sigma'_x$. A random variable X is increasing if $\sigma \le \sigma'$ implies $X(\sigma) \le X(\sigma')$. An event A is increasing if $\mathbf{1}_A$ is increasing. It is equivalent to the fact that A is stable by the action of switching some spins from -1 to 1.

Theorem 7.5 (FKG inequality). *Let G be a finite graph, τ be boundary conditions and $\beta > 0$. For any two increasing events A and B,*

$$\mu^{\tau}_{\beta,G}(A \cap B) \ \geq \ \mu^{\tau}_{\beta,G}(A)\mu^{\tau}_{\beta,G}(B).$$

Proof. In this proof only, we set $\sigma(x)$ instead of σ_x. We use the FKG lattice condition (4.8) once again, but applied to vertices this time. Let σ be a configuration, and e and f two vertices. Set σ^{ef}, σ_{ef}, σ^{e}_{f} and σ^{f}_{e} to be the configurations agreeing with σ away from e and f, and such that $(\sigma(e), \sigma(f)) = (1,1)$, $(-1,-1)$, $(1,-1)$ and $(-1,1)$ respectively. The following inequality would imply criterion (4.8) immediately:

$$H^{\tau}_{G}(\sigma^{ef}) + H^{\tau}_{G}(\sigma_{ef}) \leq H^{\tau}_{G}(\sigma^{f}_{e}) + H^{\tau}_{G}(\sigma^{e}_{f}), \tag{7.4}$$

Let us prove this inequality now. When e and f are not adjacent, the two sides of (7.4) are equal. When e and f are adjacent, we see that the left-hand term of (7.4) corresponds to configurations with $\sigma(e) = \sigma(f)$, while the right-hand term corresponds to configurations with $\sigma(e) \neq \sigma(f)$. In particular, the left-hand side is indeed smaller than the right-hand one. □

As before, the FKG inequality implies the following comparison between boundary conditions.

Theorem 7.6. *Let G be a finite graph and $\beta > 0$. For boundary conditions $\tau_1 \leq \tau_2$ and an increasing event A,*

$$\mu^{\tau_1}_{\beta,G}(A) \leq \mu^{\tau_2}_{\beta,G}(A). \tag{7.5}$$

Exactly as in the case of measures on $\{0,1\}^{E_G}$, we say that $\mu^{\tau_2}_{\beta,G}$ *stochastically dominates* $\mu^{\tau_1}_{\beta,G}$ if $\mu^{\tau_1}_{\beta,G}(A) \leq \mu^{\tau_2}_{\beta,G}(A)$ for any increasing event A. In this language, the + boundary conditions are the largest ones in the sense of stochastic ordering, while − are the smallest.

Proof. We prove that $H^{\tau_2}_{G}(\sigma^{ef}) + H^{\tau_1}_{G}(\sigma_{ef}) \leq H^{\tau_1}_{G}(\sigma^{f}_{e}) + H^{\tau_2}_{G}(\sigma^{e}_{f})$ following the same reasoning as above. We leave the proof as an exercise. □

7.3 FK-Ising model and Edwards-Sokal coupling

The Ising model can be coupled to the random-cluster model with cluster-weight $q = 2$. For this reason, the $q = 2$ random-cluster model will from now on be called *FK-Ising.* We now present this coupling, called the Edwards-Sokal coupling, along with some consequences for the Ising model.

Let G be a finite graph and let ω be a configuration of open and closed edges on G. A spin configuration σ can be constructed on the graph G as follows:

- Assign independently to each cluster of ω a "cluster-spin" 1 or -1 with probability 1/2;
- Define the spin σ_x of a vertex to be equal to the cluster-spin of its cluster.

Note that all vertices in the same cluster of ω receive the same spin, but that clusters of pluses (or minuses) in σ can be much bigger than the original clusters of ω.

Proposition 7.7 (Edwards-Sokal coupling [ES88]). *Let $p \in (0,1)$ and G a finite graph. If the configuration ω is distributed according to a random-cluster measure with parameters $(p,2)$ and free boundary conditions, then the spin configuration σ is distributed according to an Ising measure with inverse temperature $\beta = -\frac{1}{2}\ln(1-p)$ and free boundary conditions.*

Proof. Consider a finite graph G, and let $p \in (0,1)$. Consider a measure P on pairs (ω,σ), where ω is a random-cluster configuration with free boundary conditions and σ is the corresponding spin-configuration constructed as explained above. Then, for (ω,σ), we have:

$$P[(\omega,\sigma)] = \frac{1}{Z^0_{p,2,G}} p^{o(\omega)}(1-p)^{c(\omega)}2^{k(\omega)} \cdot 2^{-k(\omega)} = \frac{1}{Z^0_{p,2,G}} p^{o(\omega)}(1-p)^{c(\omega)}.$$

Now, we construct another measure $\tilde{P}$ on pairs of percolation and spin configurations as follows. Let $\tilde{\sigma}$ be a spin-configuration distributed according to an Ising model with inverse temperature β satisfying $e^{-2\beta} = 1-p$ and free boundary conditions. We deduce $\tilde{\omega}$ from $\tilde{\sigma}$ by closing all edges between neighboring vertices with different spins, and by independently opening edges between neighboring vertices with same spins with probability p. Then, for any $(\tilde{\omega},\tilde{\sigma})$,

$$\tilde{P}[(\tilde{\omega},\tilde{\sigma})] = \frac{e^{-2\beta r(\tilde{\sigma})}p^{o(\tilde{\omega})}(1-p)^{|E_G|-o(\tilde{\omega})-r(\tilde{\sigma})}}{Z} = \frac{p^{o(\tilde{\omega})}(1-p)^{c(\tilde{\omega})}}{Z}$$

where $r(\tilde{\sigma})$ is the number of edges between vertices with different spins, and Z is a normalizing constant.

To prove that the two measures are the same, we therefore need to check that the two previous measures are in fact defined on the same set of "admissible" pairs of configurations (this would imply that $Z = Z^0_{p,2,G}$ and that the probabilities of each configuration is the same). But the two spaces of configurations are obviously the same since if σ has been obtained from ω, then ω can be obtained from σ via the second procedure described above, and the same is true in the reverse direction for $\tilde{\omega}$ and $\tilde{\sigma}$.

This implies that $P = \tilde{P}$ and the marginals of P are the random-cluster model with parameters $(p, 2)$ and the Ising model at inverse-temperature β, which is the claim. □

The coupling gives a randomized procedure to obtain an Ising configuration from an FK-Ising configuration (it suffices to assign random spins). The proof of Proposition 7.7 also provides a randomized procedure to obtain an FK-Ising configuration from an Ising configuration. This observation was used by Swendsen-Wang to create a fast algorithm to sample the Ising model [SW87].

If one considers wired boundary conditions for the random-cluster, the Edwards-Sokal coupling provides us with an Ising configuration with +1 boundary conditions if we assign cluster-spins uniformly to each clusters, except for the clusters touching the boundary that automatically receive a cluster-spin +1. This extends to other boundary conditions.

Proposition 7.8 (Edwards-Sokal coupling [ES88])**.** *Let $p \in (0,1)$ and G a finite graph. If the configuration ω is distributed according to a random-cluster measure with parameters $(p,2)$ and wired boundary conditions, then the spin configuration σ is distributed according to an Ising measure with inverse temperature $\beta = -\frac{1}{2}\ln(1-p)$ and + boundary conditions.*

The Edwards-Sokal coupling provides us with a dictionary between correlation properties of the Ising model and connectivity properties of the FK-Ising percolation[2]. Indeed, two vertices which are connected in the random-cluster configuration must have the same spin, while vertices which are not have independent spins. Let us give two examples of applications of this fact (we will see many more applications of this coupling in the next sections).

Proposition 7.9. *Fix $p \in (0,1)$, G a finite graph and $\beta = -\frac{1}{2}\ln(1-p)$. For any $x, y \in G$,*

$$\begin{aligned}\mu^{+}_{\beta,G}[\sigma_x] &= \phi^{1}_{p,2,G}(x \leftrightarrow \partial G),\\ \mu^{\mathrm{f}}_{\beta,G}[\sigma_x\sigma_y] &= \phi^{0}_{p,2,G}(x \longleftrightarrow y).\end{aligned}$$

Proof. Let us treat the first case. Let P be the coupling between $\mu^{+}_{\beta,G}$ and $\phi^{1}_{p,q,G}$ and E the expectation with respect to P. We find

$$\mu^{+}_{\beta,G}[\sigma_x] = E[\sigma_x \mathbf{1}_{\{x \not\leftrightarrow \partial G\}}] + E[\sigma_x \mathbf{1}_{\{x \leftrightarrow \partial G\}}] = P(x \leftrightarrow \partial G) = \phi^{1}_{p,2}(0 \leftrightarrow \partial G).$$

[2]The FK-Ising model is called a *geometric representation* of the Ising model due to the fact that correlations are encoded in terms of geometric properties of percolation configurations.

In the second equality, we used that conditionally on $x \not\leftrightarrow \partial G$, the average spin is 0, while conditionally on $x \leftrightarrow \partial G$, it is 1. The second case can be treated similarly. □

7.4 Planar Gibbs measures and phase transition

Let us start by defining infinite-volume measures (also called *Gibbs measures*). See [vEFS93, Section 2.3.2] for details on Gibbs measures.

Definition 7.10 (Gibbs measure)**.** Let $\beta > 0$. A measure μ on $\{-1,1\}^{\mathbb{Z}^2}$ is called an Ising Gibbs measure at inverse-temperature β if for any finite graph G and any configuration τ on $\mathbb{Z}^2$,

$$\mu(\,\sigma_{|V_G} = \eta \,|\, \sigma_x = \tau_x : \forall x \notin V_G) = \mu^\tau_{\beta,G}(\eta) \quad , \quad \forall \eta \in \{-1,1\}^{V_G}.$$

Remark 7.11. Note that the previous conditioning is not really defined since the probability of $\sigma_x = \tau_x$ for any $x \notin V_G$ is a priori zero but there is no difficulty in making sense of this degenerated conditioning (for instance, one may condition on $\sigma_x = \tau_x$ for any $x \in \partial_{\mathrm{e}} G$ only).

The domain Markov property and the comparison between boundary conditions allow us to construct Gibbs measures.

Proposition 7.12. *There exist three Gibbs measures μ^+_β, μ^-_β and μ^{f}_β, called the Gibbs measures with $+$, $-$ and free boundary conditions respectively, such that for any event A depending on a finite number of vertices,*

$$\begin{aligned}
\lim_{n\to\infty} \mu^+_{\beta,\Lambda_n}(A) &= \mu^+_\beta(A) \ , \\
\lim_{n\to\infty} \mu^-_{\beta,\Lambda_n}(A) &= \mu^-_\beta(A) \ , \\
\lim_{n\to\infty} \mu^{\mathrm{f}}_{\beta,\Lambda_n}(A) &= \mu^{\mathrm{f}}_\beta(A) \ .
\end{aligned}$$

Proof. The proof may be performed using the comparison between boundary conditions, but we prefer to use the Edwards-Sokal coupling since it illustrates the fact that the coupling extends to the infinite-volume context. Precisely, define the measure μ^{f}_β obtained by assigning spins uniformly to the clusters of $\phi^0_{p,2}$, where $\beta = -\frac{1}{2}\log(1-p)$. It is then straightforward to deduce the convergence of $\mu^{\mathrm{f}}_{\beta,\Lambda_n}$ to μ^{f}_β from the convergence of $\phi^0_{p,2,\Lambda_n}$ to $\phi^0_{p,2}$ and the fact that conditionally on the random-cluster configuration ω, the cluster-spin configuration follows a product measure on clusters of ω.

Similarly, μ_β^+ and μ_β^- can be constructed by considering $\phi_{p,2}^1$ and assigning spins uniformly except that the infinite cluster (if it exists) receives spin +1 in the first case, and −1 in the second one. □

The measures μ_β^+, μ_β^- and μ_β^{f} are *invariant under translations.* Furthermore, μ_β^+ and μ_β^- are *ergodic* but μ_β^{f} is not necessarily, as can be seen from the fact that in the Edwards-Sokal coupling in infinite volume, the infinite cluster could receive a spin 1 or a spin −1. In fact, μ_β^{f} can be proved to be equal to $\frac{1}{2}(\mu_\beta^+ + \mu_\beta^-)$, and it is therefore ergodic only if $\mu_\beta^+ = \mu_\beta^-$. Let us also mention that μ_β^- and μ_β^+ are extremal for the stochastic ordering between measures on $\{-1,+1\}^{\mathbb{Z}^2}$.

The Ising model in infinite-volume exhibits a phase transition at some critical inverse temperature β_c, above which a spontaneous magnetization appears.

Theorem 7.13. *There exists $\beta_c \in (0,\infty)$ such that:*
- *for any $\beta < \beta_c$, $\mu_\beta^+[\sigma_0] = 0$,*
- *for any $\beta > \beta_c$, $\mu_\beta^+[\sigma_0] > 0$.*

Furthermore, $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$.

Proof of Theorem 7.13. Proposition 7.9 immediately implies that $\beta_c = -\frac{1}{2}\ln[1-p_c(2)]$ by passing to the infinite-volume. Then, Theorem 5.10 applied to $q = 2$ concludes the proof. □

The problem of identifying the critical value of the Ising model is more than fifty years old. Summarizing, Kramers and Wannier identified (without proof) the critical temperature where a phase transition occurs, separating an ordered from a disordered phase, using planar duality [KW41a, KW41b]. Kaufman and Onsager [Ons44, KO50] computed the free energy of the model, paving the way to an analytic derivation of its critical temperature. Later, Aizenman, Barsky and Fernández [ABF87] found a computation of the critical temperature based on differential inequalities. Here, we use yet another method since we invoke the determination of the critical value of the FK-Ising model provided in Chapter 5.

Remark 7.14. One may use the (very useful) Griffiths-Kelly-Sherman inequality [Gri67, KS68] to prove the existence of β_c. This inequality asserts that for any graph G, any $\beta > 0$ and any two sets A, B of vertices of G,

$$\mu_{\beta,G}^+[\sigma_A] \geq 0,$$
$$\mu_{\beta,G}^+[\sigma_A\sigma_B] \geq \mu_{\beta,G}^+[\sigma_A]\mu_{\beta,G}^+[\sigma_B],$$

where $\sigma_A = \prod_{v\in A}\sigma_v$. The second inequality may be used to prove that the derivative with respect to β of $\mu^+_{\beta,G}[\sigma_0]$ is positive:

$$\frac{d}{d\beta}\mu^+_{\beta,G}(\sigma_0) = \sum_{x\sim y}\mu^+_{\beta,G}(\sigma_0\sigma_x\sigma_y) - \mu^+_{\beta,G}(\sigma_0)\mu^+_{\beta,G}(\sigma_x\sigma_y) \geq 0.$$

Observe that, similarly to the random-cluster model, one could construct (a priori) many Gibbs measures and their classification is thus non trivial. Even though we will not use these facts in the future, let us describe planar Gibbs measures, starting from the high-temperature regime, then the critical regime and then the low-temperature regime.

Proposition 7.15. *When $\beta < \beta_c$, there is a unique infinite-volume measure.*

Proof. Let p such that $\beta = -\frac{1}{2}\log(1-p)$. There is no infinite-cluster for $\phi^1_{p,2}$ and therefore μ^+_β and μ^-_β are constructed from $\phi^1_{p,2}$ in the same way. We deduce that $\mu^+_\beta = \mu^-_\beta$ and by a proof similar to the one of Proposition 4.29, this implies that there exists a unique infinite-volume measure. □

For general models the classification at criticality is a priori much more difficult than in the high-temperature regime. For the Ising model, this is not the case and it turns out that there exists a unique Gibbs measure at criticality. The result goes back to Onsager [Ons44] (we also proved it in this book: simply use the Edwards-Sokal coupling together with Property **P2** of Corollary 6.16). We do not resist the temptation to present a beautiful elementary proof due to W. Werner [Wer09].

Proposition 7.16. *On $\mathbb{Z}^2$, there exists a unique Gibbs measure at β_c.*

Let us outline the proof first. We play with the Edwards-Sokal coupling between Ising and FK-Ising models. We wish to prove that $\mu^+_{\beta_c} = \mu^-_{\beta_c}$. In order to do so, we first use Zhang's argument (Proposition 4.38) to show that there is no infinite cluster at criticality almost surely. The core of the proof will be to prove that the measure $\mu^{\mathrm{f}}_{\beta_c}$ is ergodic. Since it is symmetric with respect to flipping all spins, it implies that $\mu^{\mathrm{f}}_{\beta_c}$ does not possess any infinite-clusters of pluses or minuses. In particular, there exists $\mu^{\mathrm{f}}_{\beta_c}$-almost surely an infinite number of + and − circuits (more precisely $\star$-connected circuits). This last fact can then be used to show that $\mu^+_{\beta_c} \leq \mu^{\mathrm{f}}_{\beta_c} \leq \mu^-_{\beta_c}$.

Proof. (sketch) Let us prove that $\mu^+_{\beta_c} = \mu^-_{\beta_c}$. In such case, the comparison between the boundary conditions implies that the critical Gibbs measure is unique. First, there is no infinite cluster for $\phi^0_{p_c,2}$ thanks to Proposition 4.38. The core of the proof consists in proving that this property implies that $\mu^{\mathrm{f}}_{\beta_c}$ is ergodic.

Claim: the measure $\mu^{\rm f}_{\beta_c}$ is ergodic.

Proof of the Claim. Let A and B be two events depending on two sets of vertices V and W. We consider the coupling P between $\mu^{\rm f}_{\beta_c}$ and $\phi^0_{p_c,2}$. Let x with $|x|\ge 2n$. Assume that V and W are included in Λ_n. We find

$$\begin{aligned}\mu^{\rm f}_{\beta_c}(A\cap\tau_x B) &= P(A\cap\tau_x B)\\ &= E\big(P(A\cap\tau_x B|\Gamma_n)\mathbf{1}_{\{\Gamma_n \text{ exists}\}}\big)\\ &\quad + P\big(A\cap\tau_x B\cap\{\Gamma_n \text{ does not exist}\}\big),\end{aligned}$$

where Γ_n is the union of the two outer-most dual-circuits in $\Lambda_{|x|/2}\smallsetminus\Lambda_n$ and $\tau_x(\Lambda_{|x|/2}\smallsetminus\Lambda_n)$ respectively. Since $\phi^0_{p_c,2}(0\leftrightarrow\infty)=0$, the second term tends to zero as x and then n tend to infinity. The conditional probability inside the first term is given by

$$P(A\cap\tau_x B|\Gamma_n) = \mu^{\rm f}_{\beta_c,O_n}(A)\mu^{\rm f}_{\beta_c,X_n}(B),$$

where O_n and X_n are the connected component of 0 and x in $\mathbb{Z}^2\smallsetminus\Gamma_n$. The convergence of $\mu^{\rm f}_{\beta_c,O_n}$ and $\mu^{\rm f}_{\beta_c,X_n}$ to $\mu^{\rm f}_{\beta_c}$ thus implies that the first term (and therefore the sum of the two) satisfies

$$\lim_{|x|\to\infty}\mu^{\rm f}_{\beta_c}(A\cap\tau_x B) = \mu^{\rm f}_{\beta_c}(A)\mu^{\rm f}_{\beta_c}(B).$$

$\diamond$

The symmetry of $\mu^{\rm f}_{\beta_c}$ under flips of all spins $-1/+1$ together with its ergodicity implies

$$\mu^{\rm f}_{\beta_c}(\exists \text{ an infinite-cluster of } +) = \mu^{\rm f}_{\beta_c}(\exists \text{ an infinite-cluster of } -)\in\{0,1\}$$

and therefore both are equal to zero.

Say that two vertices $x=(x_1,x_2)$ and $y=(y_1,y_2)$ are $\star$-neighbors if $|x_1-y_1|\le 1$ and $|x_2-y_2|\le 1$. A $\star$-circuit is a path $v_0,\dots,v_n=v_0$ such that v_i and v_{i+1} are $\star$-neighbors for every $0\le i<n$. Since there is no infinite cluster, there exists an infinite number of disjoint pluses and minuses $\star$-circuits surrounding the origin. We can now apply the same strategy as in the proof of Corollary 4.40. One may for instance use a "conditioning on the outer-most $\star$-connected circuit of pluses argument" to prove that $\mu^+_{\beta_c}\le\mu^{\rm f}_{\beta_c}$. Doing the same with minuses, we can prove that $\mu^{\rm f}_{\beta_c}\le\mu^-_{\beta_c}$. In conclusion, $\mu^+_{\beta_c}\le\mu^-_{\beta_c}$ and therefore $\mu^+_{\beta_c}=\mu^-_{\beta_c}$ (we already mentioned that $\mu^-_{\beta_c}\le\mu^+_{\beta_c}$). $\square$

The classification when $\beta>\beta_c$ is more interesting and more difficult. The space of infinite-volume measures is an interval with two extremal measures μ^+_β and μ^-_β.

Theorem 7.17 (Aizenman, Higuchi [Aiz80, Hig81]). *Fix $\beta > \beta_c$. The set of Gibbs measures with inverse-temperature β is given by*

$$\left\{\lambda\mu_\beta^+ + (1-\lambda)\mu_\beta^- : \lambda \in [0,1]\right\}.$$

Remark 7.18. This result is no longer true in higher dimensions, as shown by [Dob72] (also see [vB75]). Indeed, consider boxes with + boundary conditions on the upper half-space and − boundary conditions on the lower half-space. These boundary conditions imply the existence of a surface between + and −. In dimensions 3 and higher and at very high β, this surface does not fluctuate much and it is possible to prove that the infinite measure constructed by nested sequences of such boxes contains a hyper-surface passing through the origin with positive probability. This rules out the possibility of the measure being translationally invariant in the vertical direction. Since Gibbs measures with + or − boundary conditions are invariant under translations, this measure is not a linear combination of them. In 2D, the corresponding construction (+ on the upper half-plane and − on the lower half-plane) does not lead to the same contradiction. Indeed, in such case, the interface can be proved to have Gaussian fluctuations [CIV03]. As a consequence, it passes through the origin with probability tending to 0 as n tends to infinity.

By studying interfaces in more detail, Coquille and Velenik provided a new proof of the Aizenman-Higuchi result [CV12]. This result was extended to so-called Potts models in [CDCIV12] (see the antepenultimate chapter for a definition of Potts models).

7.5 High and low temperature expansions and Kramers-Wannier duality

7.5.1 High temperature expansion

The *high temperature expansion* of the Ising model is a graphical representation introduced by van der Waerden [vdW41]. It is based on the following identity (which is true since $\sigma_x\sigma_y \in \{-1,+1\}$):

$$e^{\beta\sigma_x\sigma_y} = \cosh(\beta) + \sigma_x\sigma_y \sinh(\beta) = \cosh(\beta)\left[1 + \tanh(\beta)\sigma_x\sigma_y\right]. \qquad (7.6)$$

For $A \subset V_G$, let $\mathcal{E}_G(A)$ be the set of subgraphs ω of G such that:

- every vertex $v \in V_G \setminus A$ is the end-point of an even number of edges of ω,
- every vertex $v \in A$ is the end-point of an odd number of edges of ω.

Note that if $|A|$ is odd, $\mathcal{E}_G(A)$ is empty. The set $\mathcal{E}_G(\varnothing) =: \mathcal{E}_G$ is simply the set of even subgraphs of G. For $\omega \in \mathcal{E}_G(A)$, we set $|\omega|$ for the number of edges in ω.

Recall that $\sigma_A = \prod_{x\in A}\sigma_x$.

Proposition 7.19. *Let G be a finite graph, $\beta > 0$, and $A \subset V_G$. We find*

$$Z^{\mathrm{f}}_{\beta,G} = 2^{|V_G|}\cosh(\beta)^{|E_G|}\sum_{\omega\in\mathcal{E}_G}\tanh(\beta)^{|\omega|} \tag{7.7}$$

and

$$\mu^{\mathrm{f}}_{\beta,G}[\sigma_A] = \frac{\displaystyle\sum_{\omega\in\mathcal{E}_G(A)}\tanh(\beta)^{|\omega|}}{\displaystyle\sum_{\omega\in\mathcal{E}_G}\tanh(\beta)^{|\omega|}}, \tag{7.8}$$

Proof. Let us start with the partition function. We know

$$\begin{aligned}
Z^{\mathrm{f}}_{\beta,G} &= \sum_{\sigma\in\{-1,1\}^{V_G}}\prod_{\{x,y\}\in E_G} e^{\beta\sigma_x\sigma_y}\\
&= \cosh(\beta)^{|E_G|}\sum_{\sigma\in\{-1,1\}^{V_G}}\prod_{\{x,y\}\in E_G}[1+\tanh(\beta)\sigma_x\sigma_y]\\
&= \cosh(\beta)^{|E_G|}\sum_{\sigma\in\{-1,1\}^{V_G}}\sum_{\omega\subset E_G}\tanh(\beta)^{|\omega|}\prod_{e=\{x,y\}\in\omega}\sigma_x\sigma_y\\
&= \cosh(\beta)^{|E_G|}\sum_{\omega\subset E_G}\tanh(\beta)^{|\omega|}\sum_{\sigma\in\{-1,1\}^{V_G}}\prod_{e=\{x,y\}\in\omega}\sigma_x\sigma_y
\end{aligned}$$

where we used (7.6) in the second equality. Notice that

$$\sum_{\sigma\in\{-1,1\}^{V_G}}\prod_{e=\{x,y\}\in\omega}\sigma_x\sigma_y = \begin{cases}2^{|V_G|} & \text{if } \omega\in\mathcal{E}_G\\ 0 & \text{otherwise}\end{cases}$$

and the formula for the partition function follows.

Let us now treat the second case. It is sufficient to prove that

$$\sum_{\sigma\in\{-1,1\}^{V_G}}\sigma_A e^{-\beta H^{\mathrm{f}}_G(\sigma)} = 2^{|V_G|}\cosh(\beta)^{|E_G|}\sum_{\omega\in\mathcal{E}_G(A)}\tanh(\beta)^{|\omega|}.$$

The first lines of the computation for the partition function are exactly the same, and we end up with:

$$\begin{aligned}
&\sum_{\sigma\in\{-1,1\}^{V_G}}\sigma_A e^{-\beta H^{\mathrm{f}}_G(\sigma)}\\
&\qquad= \cosh(\beta)^{|E_G|}\sum_{\omega\subset E_G}\tanh(\beta)^{|\omega|}\sum_{\sigma\in\{-1,1\}^{V_G}}\sigma_A\prod_{e=\{x,y\}\in\omega}\sigma_x\sigma_y.
\end{aligned}$$

Notice that

$$\sum_{\sigma\in\{-1,1\}^{V_G}}\sigma_A\prod_{e=\{x,y\}\in\omega}\sigma_x\sigma_y = \begin{cases}2^{|V_G|} & \text{if } \omega\in\mathcal{E}_G(A)\\ 0 & \text{otherwise}\end{cases}$$

and the formula follows. □

Remark 7.20. The same can be done for any boundary conditions. Let us mention the case of + boundary conditions. Let δ be an additional vertex not in G, sometimes called the *ghost vertex*, and connect every vertex of ∂G to δ to obtain the graph G_δ. Then,

$$Z^+_{\beta,G} = 2^{|V_G|}\cosh(\beta)^{|E_{G_\delta}|}\sum_{\omega\in\mathcal{E}_{G_\delta}}\tanh(\beta)^{|\omega|}$$

(beware of the fact that it is 2 to the power $|V_G|$ and not $|V_{G_\delta}|$) and

$$\mu^+_{\beta,G}[\sigma_A] = \frac{\displaystyle\sum_{\omega\in\mathcal{E}_{G_\delta}(A)\cup\mathcal{E}_{G_\delta}(A\cup\{\delta\})}\tanh(\beta)^{|\omega|}}{\displaystyle\sum_{\omega\in\mathcal{E}_{G_\delta}}\tanh(\beta)^{|\omega|}}. \tag{7.9}$$

Note that in the second formula, either $\mathcal{E}_{G_\delta}(A)$ or $\mathcal{E}_{G_\delta}(A\cup\{\delta\})$ is empty depending whether $|A|$ is odd or even. This representation using the ghost vertex is also very useful when working with an external field.

Remark 7.21. Several other representations of the Ising model have been introduced over the years. For instance, a representation exploiting the expansion

$$e^{\beta\sigma_x\sigma_y} = \sum_{k=0}^{\infty}\frac{(\beta\sigma_x\sigma_y)^k}{k!}$$

instead of (7.6) was proposed by Aizenman in [Aiz82]. This representation, called the *random current representation*, has been used to study the Ising model above 4 dimensions. It was also the main ingredient in the proof of [ABF87]. Recently, it was used to prove that the transition of the three dimensional Ising model is continuous [ADCS13].

7.5.2 The low temperature expansion

The *low temperature expansion* of the Ising model is a graphical representation on the dual lattice. The representation consists in drawing the contours (living on $(\mathbb{Z}^2)^\star$) between clusters of spin 1 and clusters of spin -1. More precisely, let $\omega(\sigma)\in\{0,1\}^{E_{G^\star}}$ be the family of contours associated to σ defined for every dual-edge $e^\star$ by

$$\omega(\sigma)_{e^\star} = \begin{cases}1 & \text{if } \sigma_x \neq \sigma_y \text{ (here } e=[xy]),\\ 0 & \text{otherwise.}\end{cases}$$

Observe that the construction is bijective[3], one may reconstruct the spins (once we know boundary conditions) from the contour configuration.

[3]More precisely, it is one-to-one for any boundary conditions τ, but it is two-to-one for free boundary conditions due to the symmetry $+1/-1$.

For a contour configuration ω, we set $|\omega|$ for the number of dual-edges in ω (we sometimes speak of the *total length of the contours*).

The probability of a contour configuration can be easily expressed in terms of the total lengths of the contours. Indeed, recall that the probability of a configuration σ is proportional to $e^{-\beta H_G^\tau(\sigma)}$. Now, $H_G^\tau(\sigma) = 2n(\sigma) - |E_G|$, where $n(\sigma)$ is the number of pairs of neighboring vertices with different spins. By construction, each edge e whose end-points have different spins in σ is in direct correspondence with an edge $e^\star$ in $\omega(\sigma)$, thus $n(\sigma) = |\omega(\sigma)|$. In conclusion, the probability of a contour configuration ω is simply proportional to $e^{-2\beta|\omega|}$.

Example 1 (+ boundary conditions). In this case, the contours use edges of $G^\star$ only, where here the dual graph contains dual vertices corresponding to faces of $\mathbb{Z}^2$ adjacent to G (for discrete domains, this definition corresponds to the definition of the dual graph introduced in Chapter 3). Furthermore, a family of contours is an even subgraph of $G^\star$. Let $\mathcal{E}_{G^\star}$ be the set of even subgraphs of $G^\star$. We obtain easily

$$Z_{\beta,G}^{+} = e^{\beta|E_{G^\star}|}\sum_{\omega\in\mathcal{E}_{G^\star}} e^{-2\beta|\omega|} \quad\text{and}\quad \mu_{\beta,G}^{+}(\sigma) = \frac{e^{-2\beta|\omega(\sigma)|}}{\sum_{\omega\in\mathcal{E}_{G^\star}} e^{-2\beta|\omega|}}. \tag{7.10}$$

Example 2 (Dobrushin boundary conditions). Assume that spins are -1 on ∂_-, and $+1$ on ∂_+ (recall the definition of ∂_- and ∂_+ from the beginning of this chapter). Then, families of contours are composed of loops together with one interface running between the two edges of $(\mathbb{Z}^2)^\star$ separating ∂_- from ∂_+. If one adds the two endpoints denoted u and v of these two edges, we simply obtain that the set of families of contours is $\mathcal{E}_{G^\star\cup\{u,v\}}(\{u,v\})$, where the notation extends the notation of the previous section to the dual lattice. Furthermore,

$$\mu_{\beta,G}^{\mathrm{dobr}}(\sigma) = \frac{e^{-2\beta|\omega(\sigma)|}}{\sum_{\omega\in\mathcal{E}_{G^\star\cup\{u,v\}}(\{u,v\})} e^{-2\beta|\omega|}} \tag{7.11}$$

and

$$Z_{\beta,G}^{\mathrm{dobr}} = e^{\beta|E_{G^\star}|}\sum_{\omega\in\mathcal{E}_{G^\star\cup\{u,v\}}(\{u,v\})} \left(e^{-2\beta}\right)^{|\omega|}. \tag{7.12}$$

Example 3. For general boundary conditions, the contours live on the edges of $G^\star$, with possibly additional dual-edges of $(\mathbb{Z}^2)^\star$ corresponding to edges where the boundary conditions switch from 1 to -1 or vice-versa. In other words, the contours leave on $\{e^\star : e \in E_{G\cup\partial_e G}\}$.

Remark 7.22. We are now in a position to expose the Kramers-Wannier argumentation [KW41a, KW41b]. It is based on the fact that the high and low temperature expansions are clearly related to each other, with the drawback that they live on different graphs.

If $\beta^\star$ is set to satisfy $\tanh(\beta^\star) = e^{-2\beta}$, then for every graph G, (7.7) and (7.10) give that

$$2^{|V_{G^\star}|}\cosh(\beta^\star)^{|E_{G^\star}|}Z^+_{\beta,G} = e^{\beta|E^\star_G|}Z^{\mathrm{f}}_{\beta^\star,G^\star}. \tag{7.13}$$

Physicists expect that the critical point corresponds exactly to the unique point for which the so-called *free energy* is not analytic in β. The free energy is defined by the formula

$$f(\beta) = \lim_{n\to\infty}\frac{1}{|V_{\Lambda_n}|}\log[Z^+_{\beta,\Lambda_n}] = \lim_{n\to\infty}\frac{1}{|V_{\Lambda_n}|}\log[Z^{\mathrm{f}}_{\beta,\Lambda_n}].$$

The fact that f is well-defined follows from the same sub-multiplicative arguments as in the proof of Lemma 4.31 and the fact that the two limits are equal is due to the fact that for

$$\mathrm{e}^{-2\beta|\partial\Lambda_n|} \le \left|\frac{Z^+_{\beta,\Lambda}(\sigma)}{Z^{\mathrm{f}}_{\beta,\Lambda_n}(\sigma)}\right| \le \mathrm{e}^{2\beta|\partial\Lambda_n|}$$

(only interactions on the boundary change).

Now, (7.13) and $|E_{\Lambda^\star_n}| \approx 2|V_{\Lambda_n}|$ imply that

$$f(\beta^\star) = f(\beta) + \log 2 + 2\log\cosh(\beta^\star) - 2\beta.$$

If $\beta_c^\star \ne \beta_c$, there would be at least two such singularities at β_c and $\beta_c^\star$. Thus, β_c should be equal to $\beta_c^\star$, which implies $\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$. Of course, the mathematical justification that there exists a unique singularity requires some work (which was absent from the original work in [KW41a, KW41b]).

7.5.3 Peierls argument

Since we are almost there anyway, let us mention Peierls argument, which rigorously proves that $\beta_c \in (0,\infty)$. It harnesses the low and high temperature expansions and is of great historical significance. Interestingly, this argument has been generalized to many models, including the random-cluster model. In particular, the (omitted) direct proof that the critical value of the random-cluster model $p_c(q)$ is not equal to 0 or 1 (Theorem 4.34) follows a similar argument.

Proposition 7.23 (Peierls argument [Pei36])**.** *The critical inverse temperature β_c on the square lattice is strictly positive and finite.*

Proof. Let us prove that β_c is finite. We wish to estimate $\mu^+_{\beta,G}[\sigma_0]$ when β is very large. Since

$$\mu^+_{\beta,G}[\sigma_0] = 1 - 2\mu^+_{\beta,G}[\sigma_0 = -1],$$

it is sufficient to show that $\mu^+_{\beta,G}[\sigma_0 = -1] < 1/2$ uniformly in the graph G. The observation is that $\{\sigma_0 = -1\}$ is included in the event that there exists a circuit in the low-temperature expansion surrounding 0. Thus, if $\mathcal{EL}_{G^\star}$ denotes the subset of configurations in $\mathcal{E}_{G^\star}$ containing one self-avoiding loop surrounding 0 and $\mathcal{L}_{G^\star}$ the set of self-avoiding circuits on $G^\star$ surrounding 0, the low-temperature expansion gives that

$$\mu^+_{\beta,G}[\sigma_0 = -1] \le \frac{\sum_{\omega\in\mathcal{EL}_{G^\star}} e^{-2\beta|\omega|}}{\sum_{\omega\in\mathcal{E}_{G^\star}} e^{-2\beta|\omega|}} \le \sum_{\gamma\in\mathcal{L}_{G^\star}} e^{-2\beta|\gamma|} \le \sum_{n=1}^{\infty} n4^n e^{-2\beta n} < 1/2$$

for β large enough. We used the fact that, when removing a prescribed loop, the weighted sum over all possible even graphs avoiding this loop is smaller than the one without this constraint and therefore

$$\sum_{\omega\in\mathcal{EL}_{G^\star}} e^{-2\beta|\omega|} \le \Big(\sum_{\gamma\in\mathcal{L}_{G^\star}} e^{-2\beta|\gamma|}\Big)\Big(\sum_{\omega\in\mathcal{E}_{G^\star}} e^{-2\beta|\omega|}\Big)$$

and in the third inequality the fact that the number of paths of length n surrounding the origin is smaller than $n4^n$.

The inequality $0 < \beta_c$ can be obtained using the high-temperature expansion instead of the low-temperature one. Indeed, the second formula of (7.9) applied to $A = \{0\}$ implies that

$$\mu^+_{\beta,G}[\sigma_0] = \frac{\sum_{\omega\in\mathcal{E}_{G_\delta}(\{0,\delta\})} \tanh(\beta)^{|\omega|}}{\sum_{\omega\in\mathcal{E}_{G_\delta}} \tanh(\beta)^{|\omega|}} \le \sum_{\gamma\in\mathrm{SAW}(0,\delta)} \tanh(\beta)^{|\omega|} \le \sum_{n=d}^{\infty} [4\tanh(\beta)]^n,$$

where d is the distance from the origin to ∂G and $\mathrm{SAW}(0,\delta)$ is the set of self-avoiding walks on G from the origin to the ghost vertex δ. When β is small enough, this last term decays exponentially fast as d tends to infinity and the proof is finished.

Note that we used a reasoning similar to what is above: any configuration in $\mathcal{E}_{G_\delta}(\{0,\delta\})$ is the union of a self-avoiding walk from the origin to δ plus an even subgraph in $\mathcal{E}_{G_\delta}$, and when removing the walk, the weighted sum over all possible even graphs avoiding this walk is smaller than without this constraint. Altogether, this justifies that indeed

$$\sum_{\omega\in\mathcal{E}_{G_\delta}(\{0,\delta\})}\tanh(\beta)^{|\omega|}\le\Big(\sum_{\gamma\in\mathrm{SAW}(0,\delta)}\tanh(\beta)^{|\omega|}\Big)\Big(\sum_{\omega\in\mathcal{E}_{G_\delta}}\tanh(\beta)^{|\omega|}\Big).$$

□

Remark 7.24. Observe that we used the low-temperature expansion to prove that $\mu_\beta^+(\sigma_0)>0$ for low temperatures, and the high-temperature to prove that $\mu_\beta^+(\sigma_0)=0$ for high temperatures. This justifies the denomination of low and high temperature expansions, even though these expansions may be used for any inverse-temperature (for instance we will use this fact at criticality in the following section).

7.5.4 Spin fermionic observable in discrete domains with two marked points

Let $\Omega^\diamond$ be a medial discrete domain and let $u^\diamond$ and $v^\diamond$ be two medial-vertices of $\Omega^\diamond$. Let $\Omega\cup\{u^\diamond,v^\diamond\}$ be the graph obtained from the primal graph Ω by adding the vertices $u^\diamond$ and $v^\diamond$ and replacing the original edges passing through these medial-vertices by mid-edges emanating from $u^\diamond$ and $v^\diamond$; see Fig. 7.1. Define the following subset of configurations on this new graph:

$$\widehat{\mathcal{E}_\Omega}(u^\diamond,v^\diamond)=\mathcal{E}_{\Omega\cup\{u^\diamond,v^\diamond\}}(\{u^\diamond\}\Delta\{v^\diamond\}),$$

where Δ denotes the symmetric difference. While the definition of $\widehat{\mathcal{E}_\Omega}(u^\diamond,v^\diamond)$ involves the graph $\Omega\cup\{u^\diamond,v^\diamond\}$, we will use the vocabulary of the original graph Ω by saying that the edges incident to $u^\diamond$ and $v^\diamond$ are *half-edges*.

Remark 7.25. The configurations in $\widehat{\mathcal{E}_\Omega}(u^\diamond,v^\diamond)$ can be expressed fairly simply in terms of the high temperature expansion on Ω. They contain two half-edges (possibly joining to make a full edge if $u^\diamond=v^\diamond$) that one may remove to obtain configurations entering into the framework of the high-temperature expansion. In particular, the configurations are composed of edge-avoiding loops together with a non-self-crossing path from $u^\diamond$ to $v^\diamond$.

Let $|\omega|$ be the total length of a configuration $\omega\in\widehat{\mathcal{E}_\Omega}(u^\diamond,v^\diamond)$. It is equal to the number of edges in the configuration when removing the two half-edges starting from $u^\diamond$ and $v^\diamond$ plus 1. In other words, it is simply the total length of the contours (since half-edges count for 1/2 instead of 1).

The *winding* $\mathrm{W}_\Gamma(u^\diamond,v^\diamond)$ of a curve Γ between two medial vertices $u^\diamond$ and $v^\diamond$ of the medial graph is the total signed rotation in radians that the curve makes from $u^\diamond$ to $v^\diamond$. With these notations, we can define the spin-Ising fermionic observable.

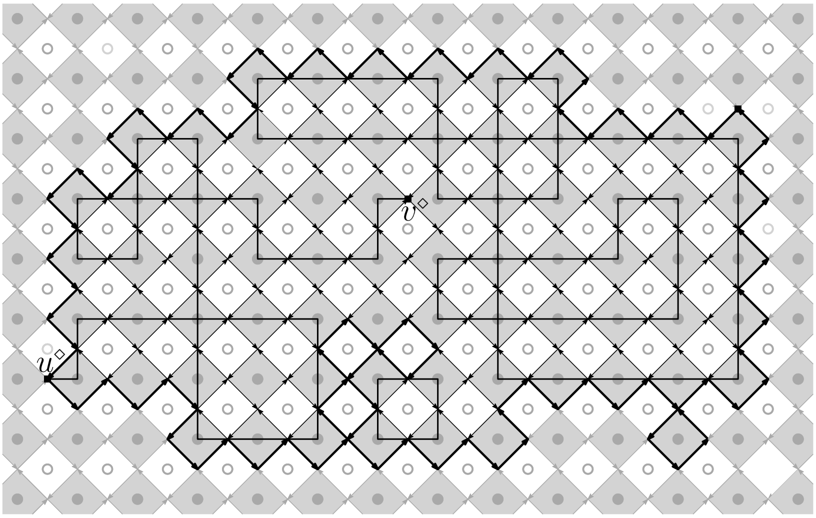

Figure 7.1: An example of a collection of contours in $\widehat{\mathcal{E}}_{\Omega^\diamond}(u^\diamond, v^\diamond)$ on a simply connected domain.

Definition 7.26 (Smirnov [Smi06])**.** Let $\Omega^\diamond$ be a discrete domain and let $u^\diamond$ and $v^\diamond$ be two distinct medial vertices on $\partial\Omega^\diamond$. The spin-Ising fermionic observable at a third medial vertex $z^\diamond$ is defined by

$$F_{\Omega^\diamond,u^\diamond,v^\diamond}(z^\diamond) = \frac{\sum\limits_{\omega\in\widehat{\mathcal{E}}_\Omega(u^\diamond,z^\diamond)} e^{-\frac{i}{2}W_{\gamma(\omega)}(u^\diamond,z^\diamond)}(\sqrt{2}-1)^{|\omega|}}{\sum\limits_{\omega\in\widehat{\mathcal{E}}_\Omega(u^\diamond,v^\diamond)} e^{-\frac{i}{2}W_{\gamma(\omega)}(u^\diamond,v^\diamond)}(\sqrt{2}-1)^{|\omega|}},$$

where $\gamma(\omega)$ is any non-self-crossing path in ω going from $u^\diamond$ to $z^\diamond$ (or from $u^\diamond$ to $v^\diamond$).

Remark 7.27. While there may be several choices for the path $\gamma(\omega)$, the quantity $e^{-\frac{i}{2}W_{\gamma(\omega)}(u^\diamond,v^\diamond)}$ does not depend on the choice of a non-self-crossing path.

Remark 7.28. Fermionic observables in the Ising model are older than parafermionic observables for the random-cluster model. They essentially go back to Kaufman-Onsager. They also appeared in several works from Sato, Miwa, Jimbo [SMJ78, SMJ79a, SMJ79b, SMJ79c, SMJ80]. Nevertheless, the study of Boundary Value Problem, which will eventually lead to conformal invariance, was only performed recently by Smirnov, see Chapter 9.

Remark 7.29. Since the weights of edges are critical (recall that $\sqrt{2}-1 = e^{-2\beta_c}$), the Kramers-Wannier duality has a enlightening interpretation here. The high-temperature expansion can be thought of as the low-temperature expansion of an Ising model on the dual graph with specific boundary conditions. Let us expand on this. Assume for a moment that $u^\diamond$ and $v^\diamond$ are on the boundary of $\Omega^\diamond$ and extend configurations in $\widehat{\mathcal{E}}_\Omega(u^\diamond, v^\diamond)$ by adding the two missing half-edges adjacent to $u^\diamond$ and $v^\diamond$ (they are connecting $u^\diamond$ and $v^\diamond$ to the end-points u and v of the primal edges associated to them that are not in Ω). Applying Kramers-Wannier duality, the denominator thus corresponds exactly to the low-temperature expansion of an Ising model on the dual graph with Dobrushin boundary conditions. This will be crucial in order to connect the behavior of the spin fermionic observable to the geometry of interfaces (see Fig. 7.2).

Remark 7.30. Let us mention that the numerator of the observable has also an interpretation using directly the high-temperature expansion. In fact, it can be shown that it corresponds to the high-temperature expansion of the partition function of an Ising model with a disorder operator at $z^\diamond$. More precisely, this operator introduces a monodromy at $z^\diamond$: every time one turns around $z^\diamond$, the spins are reversed. Equivalently, it boils down to reversing the correlation constants J to $-J$ along an arbitrary simple curve from $z^\diamond$ to the boundary of the domain.

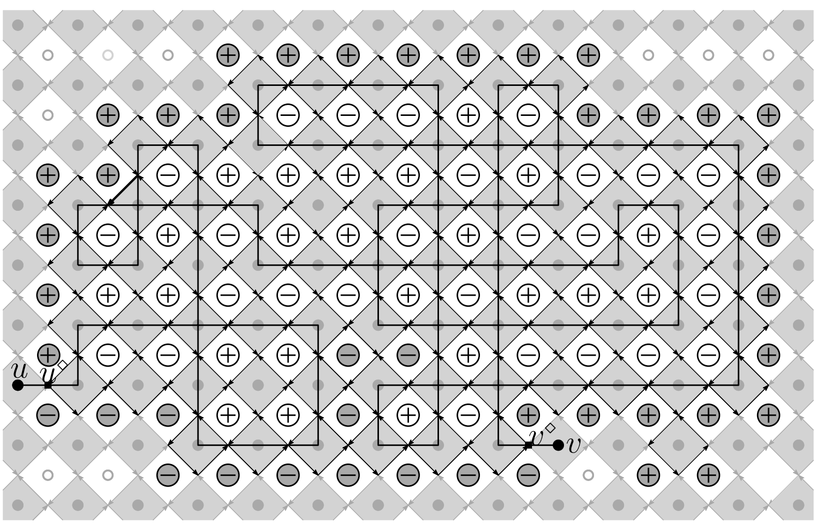

Figure 7.2: The Ising configuration on $\Omega^\star$ corresponding to configurations in $\widehat{\mathcal{E}}_\Omega(u^\diamond, v^\diamond)$. We also depicted the vertices u and v.

Chapter 8

Discrete complex analysis on graphs

Complex analysis is the study of harmonic and holomorphic functions in complex domains. In this chapter, we shall discuss how to discretize harmonic and holomorphic functions, and what are the properties of these discretizations (it is inspired by [DCS12a]).

Since we are aiming at a description of discrete structures that converge to continuum ones, we will work with discrete approximations of the square lattice. We use the notations and terminology introduced in Chapter 3. In particular, Ω, $\Omega^\star$ and $\Omega^\diamond$ always denote discrete domains or Dobrushin domains except otherwise mentioned. Since we discuss complex analysis, we will also work with complex coordinates and variables.

This presentation of discrete holomorphicity will be somewhat biased: it will mostly focus on the relation between discrete harmonicity/holomorphic maps and discrete Boundary Value Problems (BVPs). Let us explain the reasons behind this deliberate choice. In Chapter 6, we saw that parafermionic observables possess properties that are shared by holomorphic maps. Furthermore, the boundary conditions of these parafermionic observables suggest that their scaling limit is the solution of Riemann-Hilbert BVPs:

$$f \text{ is holomorphic on } \Omega, \text{ continuous on } \overline{\Omega}, \text{ and } \mathrm{Im}(f\nu^\sigma) = 0 \text{ on } \partial\Omega,$$

where ν is the tangent vector on the boundary. For general values of q, proving that the parafermionic observables converge in the scaling limit seems to be an Herculean task. Nevertheless, discrete complex analysis offers us a framework and a strategy to justify such a convergence: if a parafermionic observable could be proved to be the solution of a relevant discretization of a Riemann-Hilbert BVP, then its convergence in the scaling limit could follow from abstract results on so-called discrete harmonic/holomorphic maps. This fact alone is a compelling motivation for studying discrete complex analysis from the point of view of BVPs.

We will start with a discussion of discrete harmonic functions and their relation to the Dirichlet problem, which is the easiest discrete BVP to make sense of and to solve. Then, we will define discrete holomorphic functions. We will study these objects very succinctly for the following reason: the notion of discrete holomorphicity will not be strong enough to enable us to study discrete Riemann-Hilbert BVPs. We will therefore quickly move to a notion of discrete holomorphicity introduced by Smirnov, called s-holomorphicity, which enables us to treat a special case of Riemann-Hilbert BVP.

This chapter must be understood as a toolbox for what will follow. In the next chapter, we will show that the strategy outlined above works in a special case : the fermionic observable (for $q = 2$ random-cluster model or for the Ising model) will be proved to be s-holomorphic and the theory developed in this chapter will be harnessed to prove convergence of these two observables in the scaling limit.

In this chapter, we identify a graph and its set of vertices (for instance Ω_δ will mean V_{Ω_δ}).

8.1 Discrete harmonic functions and discrete Dirichlet BVP

We refer to [Law05] for a deeper or more broader study on discrete harmonic functions and their link to random walks.

8.1.1 Definition and connection with random walks

In this section, Ω_δ denotes a discrete domain or a Dobrushin domain. Consider the operator Δ_δ defined as follows. For $f : \Omega_\delta \to \mathbb{R}$ and x in $\Omega_\delta \setminus \partial\Omega_\delta$, set

$$\Delta_\delta f(x) := \frac{1}{4} \sum_{y\sim x} \big[f(y) - f(x)\big].$$

Definition 8.1. A function $h : \Omega_\delta \to \mathbb{R}$ is *discrete harmonic* (resp. *discrete superharmonic*, *discrete subharmonic*) at x if $\Delta_\delta h(x) = 0$ (resp. ≤ 0, ≥ 0).

8.1.2 The discrete Dirichlet BVP for harmonic functions

For a function $g : \partial\Omega_\delta \to \mathbb{C}$, a (harmonic) solution of the *Dirichlet Boundary Value Problem* on Ω_δ with boundary conditions g is given by a function $h : \Omega_\delta \to \mathbb{C}$ which is discrete harmonic on $\Omega_\delta \setminus \partial\Omega_\delta$ and equal to g on $\partial\Omega_\delta$.

Theorem 8.2. *Consider a discrete domain Ω_δ and a function $g : \partial\Omega_\delta \to \mathbb{R}$. There exists a unique solution to the discrete Dirichlet BVP on Ω_δ with boundary conditions g.*

In order to prove this theorem, we will need the two following important facts.

Lemma 8.3 (maximum principle)**.** *Let $h : \Omega_\delta \to \mathbb{R}$ be discrete harmonic on $\Omega_\delta \setminus \partial\Omega_\delta$. Then,*

$$\max\{h(x) : x \in \Omega_\delta\} = \max\{h(x) : x \in \partial\Omega_\delta\}.$$

Proof. Let $m = \max\{h(x) : x \in \Omega_\delta\}$ and let $U = \{x \in \Omega_\delta : f(x) = m\}$. Let $x \in U \setminus \partial\Omega_\delta$. Then $m = h(x) = \frac{1}{4}\sum_{y\sim x} h(y)$ and therefore $h(y) = m$ for any neighbor y of x. This observation implies that $U \cap \partial\Omega_\delta \neq \varnothing$, which is the claim. □

Consider the simple random walk (X_n) on $\delta\mathbb{Z}^2$, i.e. the Markov process on vertices defined by jumping at each time step on one of the nearest neighbors with equal probability. For a graph Ω_δ, let τ be the hitting time of $\partial\Omega_\delta$.

Lemma 8.4 (connection with random walks)**.** *A function $h : \Omega_\delta \to \mathbb{R}$ is discrete harmonic on $\Omega_\delta \setminus \partial\Omega_\delta$ if and only if for any $x \in \Omega_\delta \setminus \partial\Omega_\delta$, $h(X_{n\wedge\tau})$ is a martingale for the simple random walk starting from x.*

Proof. Let $x \in \Omega_\delta \setminus \partial\Omega_\delta$ and (X_n) be the simple random walk starting from x, then $\mathbb{E}[h(X_1)] = h(x)$ is equivalent to $\Delta_\delta h(x) = 0$. This implies the claim readily. □

Proof of Theorem 8.2. Let us start with the uniqueness. Consider two solutions h_1 and h_2 of the BVP. Then, $h_1 - h_2$ is a solution of a discrete Dirichlet BVP with boundary conditions 0. The maximum principle applied to $h_1 - h_2$ and $h_2 - h_1$ implies that $h_1 = h_2$. Let us now turn to the existence. Consider the function

$$f(x) = \mathbb{E}_x[g(X_\tau)],$$

where under $\mathbb{E}_x$, (X_n) is a simple random walk starting at x, and τ is still the hitting time of $\partial\Omega_\delta$. By definition, this function is equal to g on the boundary. Since it can easily be seen to be harmonic on $\Omega_\delta \setminus \partial\Omega_\delta$, we obtain a solution. □

Example 1. A discrete harmonic function is the solution of the Dirichlet BVP with $g = f_{|\partial\Omega_\delta}$.

Example 2. The *discrete harmonic measure* of $y \in \partial\Omega_\delta$ is the solution of the Dirichlet BVP with $g(x) = 1$ if $x = y$ and 0 otherwise. Equivalently, $H_{\Omega_\delta}(x,y)$ is the probability that a simple random walk starting from x reaches $\partial\Omega_\delta$ at y.

Let us mention the following formula involving the discrete harmonic measure.

Proposition 8.5. *For any function $h : \Omega_\delta \to \mathbb{R}$ harmonic on $\Omega_\delta \setminus \partial\Omega_\delta$,*

$$h = \sum_{y \in \partial\Omega_\delta} h(y) H_{\Omega_\delta}(\cdot, y).$$

Proof. Note that

$$h - \sum_{y \in \partial\Omega_\delta} h(y) H_{\Omega_\delta}(\cdot, y)$$

is harmonic in $\Omega_\delta \setminus \partial\Omega_\delta$. Since it vanishes on $\partial\Omega_\delta$, the uniqueness of the Dirichlet BVP implies that it is equal to 0 everywhere, hence the claim. □

8.1.3 Derivative estimates and compactness criteria

For general functions, a control on the gradient provides regularity estimates on the function itself. It is a well-known fact that harmonic functions satisfy the reverse property: controlling the function allows us to control the gradient. The following lemma shows that the same is true for discrete harmonic functions. Recall that $d(x, F) = \inf\{|x - y|, y \in F\}$.

Proposition 8.6. *There exists $C > 0$ such that, for any discrete harmonic function $h : \Omega_\delta \to \mathbb{R}$ and any two neighboring vertices x and y in Ω_δ,*

$$|h(x) - h(y)| \leq C\delta \frac{\sup\{|h(z)| : z \in \Omega_\delta\}}{d(x, \delta\mathbb{Z}^2 \setminus \Omega_\delta)}. \tag{8.1}$$

Proof. Let $x,y \in \Omega_\delta$. Let $2r = d(x,\delta\mathbb{Z}^2 \setminus \Omega_\delta) > 0$, so that $U_\delta = (x+[-r,r]^2)\cap\delta\mathbb{Z}^2$ is included in Ω_δ. Lemma 8.4 implies that for any two neighboring vertices x and y of Ω_δ,

$$h(x)-h(y) = \mathbf{E}[h(X_\tau)-h(Y_{\tau'})], \tag{8.2}$$

where (X_n) and (Y_n) are two simple random walks starting respectively at x and y, and τ and τ' are the hitting times of ∂U_δ. Note that we have some flexibility on the choice of the coupling $\mathbf{P}$ (only the marginals are determined). Consider the following coupling of (X_n) and (Y_n): (X_n) is a simple random walk and (Y_n) is constructed as follows,

- if $X_1 = y$, then $Y_n = X_{n+1}$ for $n \geq 0$,
- if $X_1 \neq y$, then $Y_n = \sigma(X_{n+1})$, where σ is the orthogonal symmetry with respect to the perpendicular bisector ℓ of the segment $[X_1 y]$ until X_{n+1} reaches ℓ. As soon as it does, set $Y_n = X_{n+1}$ for all subsequent steps.

It is easy to check that (Y_n) is also a simple random walk starting at y. Moreover, we have

$$\big|h(x)-h(y)\big| \leq \mathbf{E}\big[|h(X_\tau)-h(Y_{\tau'})|\mathbf{1}_{X_\tau\neq Y_{\tau'}}\big] \leq 2\Big(\sup_{z\in\partial U_\delta}|h(z)|\Big)\,\mathbf{P}(X_\tau \neq Y_{\tau'})$$

Using the definition of the coupling, the probability on the right is known: it is equal to the probability that (X_n) does not touch ℓ before exiting U_δ. Since U_δ is of radius r/δ for the graph distance, the gambler ruin implies that the probability on the right-hand side is smaller than $\frac{c_1}{r}\delta$ (with $c_1 < \infty$ being a universal constant independent of δ). We deduce that

$$\big|h(x)-h(y)\big| \leq 2\left(\sup_{z\in\partial U_\delta}|h(z)|\right)\frac{c_1}{r}\delta \;\leq\; 2\left(\sup_{z\in\Omega_\delta}|h(z)|\right)\frac{c_1}{r}\delta.$$

□

The following proposition will be very useful. Recall that functions on Ω_δ are implicitly extended to the faces of Ω_δ (we denote the union of faces by $\overline{\Omega}_\delta$).

Proposition 8.7. *A family $(h_\delta)_{\delta>0}$ of discrete harmonic functions on the graphs Ω_δ is precompact for the uniform topology on compact subsets of Ω if* one *of the following properties holds:*

(1) $(h_\delta)_{\delta>0}$ is uniformly bounded on any compact subset of Ω,

(2) for any compact subset K of Ω, there exists $M = M(K) > 0$ such that for any $\delta > 0$

$$\delta^2 \sum_{x\in K_\delta} |h_\delta(x)|^2 \leq M.$$

The first condition corresponds to be bounded in the L^∞-norm, the second in the L^2-norm.

Proof. Let us prove that the proposition holds under the first hypothesis and then that the second hypothesis implies the first one. Let K be a compact subset of Ω. We now assume that $\delta_0 > 0$ is such that $K \subset \overline{\Omega}_{\delta_0}$. We are faced with a family of continuous maps $h_\delta : K \longrightarrow \mathbb{C}$ indexed by $\delta < \delta_0$. Let $2r = d(K, \Omega^c) > 0$.

Condition (1) We aim to apply the Arzelà-Ascoli theorem. It is sufficient to prove that functions h_δ are uniformly Lipschitz since by assumption they are uniformly bounded on any compact subset of Ω and therefore on K. Proposition 8.6 (or more precisely the before last displayed inequality in its proof) shows that $|h_\delta(x) - h_\delta(y)| \le C_K \delta$ for any two neighbors $x, y \in K_\delta$, where

$$C_K = C \frac{\sup_{\delta>0} \sup\left\{|h_\delta(z)| : z \in \Omega_\delta \text{ with } d(z, K) \le r\right\}}{2r},$$

implying that $|h_\delta(x) - h_\delta(y)| \le 2C_K|x - y|$ for any $x, y \in K_\delta$ (not necessarily neighbors). The Arzelá-Ascoli theorem concludes the proof.

Condition (2) Assume now that the second hypothesis holds, and let us prove that $(h_\delta)_{\delta>0}$ is bounded on K. Consider $x \in K_\delta$. Now,

$$\frac{r}{2\delta} \min\left\{\delta^2 \sum_{y \in \partial\Lambda_k^\delta} |h_\delta(y)|^2 \; : \; \tfrac{r}{2\delta} \le k \le \tfrac{r}{\delta}\right\} \le \delta^2 \sum_{y \in U_\delta} |h_\delta(y)|^2,$$

where $\Lambda_k^\delta = x + \delta\Lambda_k$ is the rescaled version of the box of size k centered around x. Using the second hypothesis, the right-hand side is bounded and there exists $k := k(x)$ such that $\frac{r}{2\delta} \le k \le \frac{r}{\delta}$ and

$$\delta \sum_{y \in \partial\Lambda_k^\delta} |h_\delta(y)|^2 \le 2M/r,$$

where $M < \infty$ is provided by the assumption. Proposition 8.5 implies

$$h_\delta(x) = \sum_{y \in \partial\Lambda_k^\delta} h_\delta(y) H_{\Lambda_k^\delta}(x, y) \tag{8.3}$$

for every $x \in U_{\delta k}$. Using the Cauchy-Schwarz inequality, we find

$$\begin{aligned} h_\delta(x)^2 &= \left(\sum_{y \in \partial\Lambda_k^\delta} h_\delta(y) H_{\Lambda_k^\delta}(x, y)\right)^2 \\ &\le \left(\delta \cdot \sum_{y \in \partial\Lambda_k^\delta} |h_\delta(y)|^2\right)\left(\frac{1}{\delta} \cdot \sum_{y \in \partial\Lambda_k^\delta} H_{\Lambda_k^\delta}(x, y)^2\right) \le 2M/r \cdot C^2 \end{aligned}$$

where C is a uniform constant. The last inequality used the fact that $H_{\Lambda_k^\delta}(x, y) \le C\delta$ for some $C = C(r) > 0$, which is a very easy estimate that one may obtain using random walks[1]. □

[1] Observe that the harmonic measure is smaller than the harmonic measure in the

8.1.4 Convergence to the continuum Dirichlet BVP

Discrete harmonic functions on square lattices of smaller and smaller mesh size were studied in a number of papers in the early twentieth century (see e.g. [PW23, Bou26, Lus26]), culminating in the seminal work of Courant, Friedrichs and Lewy [CFL28]. In this article, solutions to the Dirichlet problem for a discretization of an elliptic operator were shown to converge to the solution of the analogous continuous problem as the mesh of the lattice tends to zero. We discuss this result here.

Let us now turn to the convergence result of [CFL28]. We start by a lemma.

Lemma 8.8. *Let Ω be a domain of the plane and (Ω_δ) be a sequence of discrete approximation converging in the Carathéodory sense to Ω. Let $(h_\delta)_{\delta>0}$ be a family of discrete harmonic functions on Ω_δ converging uniformly on any compact subset of Ω to a function h. Then, h is harmonic in Ω.*

Proof. Let (h_δ) be a sequence of discrete harmonic functions on Ω_δ converging to h. Via Propositions 8.6 and 8.7, $(\frac{1}{\delta}[h_\delta(\cdot+\delta)-h_\delta])_{\delta>0}$ is precompact and we may extract sub-sequential limits. Note that the limiting object is continuous. Since $\partial_x h$ is the only possible sub-sequential limit[2], $(\frac{1}{\delta}[h_\delta(\cdot+\delta)-h_\delta])_{\delta>0}$ converges. Similarly, one can prove convergence of discrete derivatives of any order. In particular, $\frac{1}{\delta^2}\Delta_\delta h_\delta$ converges to $[\partial_{xx}h+\partial_{yy}h]=\Delta h$. Since the first term is equal to 0, the second also vanishes and h is harmonic. $\square$

We state the result of [CFL28] in the specific context that will be useful in this book. Namely, we consider a Dirichlet BVP with possible singularities at two points on the boundary of a simply connected domain.

Theorem 8.9. *Let Ω be a discrete domain with two points a and b on its boundary. Let f be a bounded continuous function on $\partial\Omega\setminus\{a,b\}$. We consider a sequence of Dobrushin domains $(\Omega_\delta,a_\delta,b_\delta)$ converging to (Ω,a,b) in the Carathéodory sense. Let $f_\delta:\partial\Omega_\delta\to\mathbb{R}$ be a sequence of uniformly bounded functions converging uniformly away from a and b*

strip $[-\delta k,\delta k]\times(\delta\mathbb{Z})$. Now in this strip, one may check that the probability of hitting the top side at y is equal to the probability that $X_N=y$ for a simple random walk X in $\delta\mathbb{Z}$, where N is the sum of M iid geometric random variables of mean $1/2$, where M is distributed as the first hitting time of $\{-k,k\}$ for an independent random walk on $\mathbb{Z}$. It is well-known that M/k^2 tends to the random-variable with density $f(t):=\frac{2}{\sqrt{2\pi}}t^{-3/2}\exp(-1/t^2)$, and the result thus follows easily from the local central-limit theorem. The details are left as an exercise.

[2] To see that, integrate it between z and $z+(\varepsilon,0)$ and let ε tend to 0.

to f. Let h_δ be the unique discrete harmonic function on Ω_δ such that $(h_\delta)_{|\partial\Omega_\delta} = f_\delta$. Then

$$h_\delta \longrightarrow h \quad \text{when } \delta \to 0$$

uniformly on compact subsets of Ω, where $h : \overline{\Omega} \setminus \{a,b\} \to \mathbb{R}$ is the unique continuous function which is harmonic on Ω and equal to f on $\partial\Omega \setminus \{a,b\}$.

Proof. Since $(f_\delta)_{\delta>0}$ is uniformly bounded by some constant M, the minimum and maximum principles imply that $(h_\delta)_{\delta>0}$ is bounded by M. Therefore, the family (h_δ) is precompact (Proposition 8.7). Let $\tilde{h}$ be a subsequential limit. Necessarily, $\tilde{h}$ is harmonic inside the domain (Lemma 8.8) and bounded. To prove that $\tilde{h} = h$, it suffices to show that $\tilde{h}$ can be continuously extended to the boundary by f.

Let $x \in \partial\Omega \setminus \{a,b\}$ and $\varepsilon > 0$. There exists $R > 0$ such that for δ small enough,

$$|f_\delta(x') - f_\delta(x)| < \varepsilon \qquad \text{for every } x' \in \partial\Omega_\delta \cap Q(x,R),$$

where $Q(x,R) = x + [-R,R]^2$. For $r < R$ and $y \in Q(x,r)$, we have

$$|h_\delta(y) - f_\delta(x)| \;=\; \mathbf{E}[f_\delta(X_\tau) - f_\delta(x)]$$

for X a random walk starting at y, and τ its hitting time of the boundary. Decomposing between walks exiting the domain inside $Q(x,R)$ and others, we find

$$|h_\delta(y) - f_\delta(x)| \;\le\; \varepsilon \;+\; 2M\mathbf{P}[X_\tau \notin Q(x,R)].$$

Lemma 8.10 below guarantees that $\mathbf{P}[X_\tau \notin Q(x,R)] \le (r/R)^\alpha$ for some independent constant $\alpha > 0$. Taking $r = R(\varepsilon/2M)^{1/\alpha}$ and letting δ go to 0, we obtain $|\tilde{h}(y) - f(x)| \le 2\varepsilon$ for every $y \in Q(x,r)$. □

Lemma 8.10 (weak Beurling's estimate)**.** *There exists $\alpha > 0$ such that for any $0 < r < \frac{1}{2}$ and any curve γ inside $\mathbb{D} := \{z : |z| < 1\}$ from $\{z : |z| = 1\}$ to $\{z : |z| = r\}$, the probability that a random walk on $\mathbb{D}_\delta$ starting at 0 exits $\mathbb{D}_\delta$ without crossing γ is smaller than r^α uniformly in $\delta > 0$.*

Proof. There exists $c > 0$ such that for any annulus $A_x := \{z : x \le |z| \le 2x\}$, with $r \le x \le \frac{1}{2}$, the random walk trajectory has a probability larger than $c > 0$ of closing a loop around the origin while crossing this annulus. In this case, the trajectory necessarily intersects γ. Since the random walk trajectory must cross roughly $\log_2 r$ annuli of the form $A_{2^{-n}}$, and that at each step it has a probability at least $c > 0$ of closing a circuit, the result follows with $\alpha = -\log_2[1-c]$. □

8.1.5 Discrete Green functions

This paragraph concludes the section by mentioning the important example of discrete Green functions. While slightly technical, the following propositions will be useful to the study of s-holomorphic maps. *The proof may be skipped during a first reading.*

For $y \in \Omega_\delta \setminus \partial\Omega_\delta$, let $G_{\Omega_\delta}(\cdot, y)$ be the *discrete Green function* in the domain Ω_δ with singularity at y, *i.e.* the unique function on Ω_δ such that

- its Laplacian on $\Omega_\delta \setminus \partial\Omega_\delta$ equals 0 except at y, where it equals 1,
- $G_{\Omega_\delta}(\cdot, y)$ vanishes on the boundary $\partial\Omega_\delta$.

The quantity $-G_{\Omega_\delta}(x, y)$ is the expected number of visits at x of a random walk started at y and stopped at the first time it reaches the boundary. Equivalently, it is also the number of visits at y of a random walk started at x stopped at the first time it reaches the boundary.

Green functions are very convenient, in particular because of the Riesz representation formula.

Proposition 8.11 (Riesz representation formula)**.** *Let $f : \Omega_\delta \to \mathbb{C}$ be a (non-necessarily harmonic) function vanishing on $\partial\Omega_\delta$. We have*

$$f = \sum_{y \in \Omega_\delta \setminus \partial\Omega_\delta} \Delta_\delta f(y) G_{\Omega_\delta}(\cdot, y).$$

Proof. Note that $f - \sum_{y \in \Omega_\delta \setminus \partial\Omega_\delta} \Delta_\delta f(y) G_{\Omega_\delta}(\cdot, y)$ is harmonic and vanishes on the boundary. Hence, it equals 0 everywhere. □

Finally, a regularity estimate on discrete Green functions will be needed. This proposition is slightly technical. In the following, $aQ_\delta = [-a, a]^2 \cap \delta\mathbb{Z}^2$ and $\nabla f(x) = (f(x+\delta) - f(x), f(x+i\delta) - f(x))$.

Proposition 8.12. *There exists $C > 0$ such that for any $\delta > 0$ and $y \in 9Q_\delta$,*

$$\sum_{x \in Q_\delta} |\nabla_x G_{9Q_\delta}(x, y)| \le C\delta \sum_{x \in Q_\delta} G_{9Q_\delta}(x, y).$$

Proof. In the proof, $C_1, \ldots, C_6$ denote universal constants. We divide the proof into two cases depending whether $y \in 9Q_\delta \setminus 3Q_\delta$ or $y \in 3Q_\delta$.

Case 1: $y \in 9Q_\delta \setminus 3Q_\delta$. Using random walks, one can easily show that there exists $C_1 > 0$ such that

$$\frac{1}{C_1} G_{9Q_\delta}(x, y) \le G_{9Q_\delta}(x', y) \le C_1 G_{9Q_\delta}(x, y)$$

for every $x, x' \in 2Q_\delta$ (we leave this as an exercise[3]). Using Proposition 8.6, we deduce

$$\sum_{x\in Q_\delta} |\nabla_x G_{9Q_\delta}(x,y)| \le \sum_{x\in Q_\delta} C_2\delta \max_{x\in 2Q_\delta} G_{9Q_\delta}(x,y) \le C_1C_2\delta \sum_{x\in Q_\delta} G_{9Q_\delta}(x,y)$$

which is the claim for $y \in 9Q_\delta \smallsetminus 3Q_\delta$.

Case 2: $y \in 3Q_\delta$. We know that the random walk spends an expected time of C_3/δ^2 in the box $3Q_\delta$ before exiting it. Using the fact that $G_{9Q_\delta}(x,y)$ is the number of visits of x for a random walk starting at y (and stopped on the boundary) and summing over x, we deduce

$$\sum_{x\in Q_\delta} G_{9Q_\delta}(x,y) \ge C_3/\delta^2.$$

Therefore, it suffices to prove that $\sum_{x\in Q_\delta} |\nabla G_{9Q_\delta}(x,y)| \le C_4/\delta$. Let $G_{\delta\mathbb{Z}^2}(x,\cdot)$ be the Green function in the whole plane, *i.e.* the function with Laplacian equal to 1 for $y = x$ and 0 otherwise, normalized so that $G_{\delta\mathbb{Z}^2}(x,x) = 0$, and with sub-linear growth at infinity. This function has been widely studied. In particular, it was proved in [MW40] that

$$G_{\delta\mathbb{Z}^2}(x,y) = \frac{1}{\pi}\ln\left(\frac{|x-y|}{\delta}\right) + C_5 + o\left(\frac{\delta}{|x-y|}\right).$$

Now, $G_{\delta\mathbb{Z}^2}(\cdot,y) - G_{9Q_\delta}(\cdot,y) - \frac{1}{\pi}\ln\left(\frac{1}{\delta}\right)$ is harmonic in $9Q_\delta$. Furthermore, the boundary conditions (on $\partial 9Q_\delta$) on both $G_{9Q_\delta}(\cdot,y)$ and $G_{\delta\mathbb{Z}^2}(\cdot,y) - \frac{1}{\pi}\ln\left(\frac{1}{\delta}\right)$

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[3] This is a special application of the discrete Harnack's principle. Let us sketch a proof of this fact based on the random walk itself. Set $2 < \lambda < 3$. Let H_δ be the union of the top and bottom sides of λQ_δ. Similarly, define V_δ to be the union of the left and right sides of λQ_δ. Also, denote by S the rectangle with same height as λQ and same width as $3Q$, which has the same center as Q. Recall that since $G_{9Q_\delta}(\cdot,y)$ is harmonic on λQ_δ, we get

$$G_{9Q_\delta}(x,y) = \sum_{z\in H_\delta} G_{9Q_\delta}(z,y)\mathrm{H}_{\lambda Q_\delta}(x,z).$$

Without loss of generality, one may assume that $L := \sum_{z\in V_\delta} G_{9Q_\delta}(z,y) \ge \sum_{z\in H_\delta} G_{9Q_\delta}(z,y)$. Now, we previously showed that $\mathrm{H}_{\lambda Q_\delta}(x,z) \le C\delta$ and therefore

$$G_{9Q_\delta}(x,y) \le C\delta \sum_{z\in\partial(\lambda Q_\delta)} G_{9Q_\delta}(z,y) \le 2C\delta L.$$

In the other direction, use the harmonicity in S_δ. Since $G_{9Q_\delta}(z,y)$ is positive for every z, we find that

$$G_{9Q_\delta}(x,y) \ge \sum_{z\in H_\delta} G_{9Q_\delta}(z,y)\mathrm{H}_{S_\delta}(x,z).$$

A simple computation involving random walks (similar to the argument leading to $\mathrm{H}_{\lambda Q_\delta}(x,z) \le C\delta$) shows that there exists $c > 0$ such that for any $x \in 2Q_\delta$ and any $z \in H_\delta$, $\mathrm{H}_{S_\delta}(x,z) \ge c\delta$. We deduce that $G_{9Q_\delta}(x,y) \ge c\delta L$. In conclusion, we always have $c\delta L \le G_{9Q_\delta}(x,y) \le C\delta L$ and the claim follows with $C_1 = C/c$.

are bounded (the first one is 0, the second is tending to C_5 as δ tends to 0). Therefore, Proposition 8.6 implies

$$\sum_{x\in Q_\delta} \left|\nabla_x (G_{\delta\mathbb{Z}^2}(x,y) - G_{9Q_\delta}(x,y))\right| \le C_6\delta\cdot 1/\delta^2 = C_6/\delta.$$

(We used the fact that the gradients in x of $G_{\delta\mathbb{Z}^2}(x,y) - G_{9Q_\delta}(x,y)$ and $G_{\delta\mathbb{Z}^2}(x,y) - G_{9Q_\delta}(x,y) - \frac{1}{\pi}\log(\frac{1}{\delta})$ are obviously the same.) Moreover, the asymptotic of $G_{\delta\mathbb{Z}^2}(\cdot,y)$ leads to

$$\sum_{x\in Q_\delta} \left|\nabla_x G_{\delta\mathbb{Z}^2}(x,y)\right| \le C_7/\delta.$$

Summing the two inequalities, the result follows readily. □

8.2 Discrete holomorphic functions

Historically, discrete holomorphic functions appeared implicitly in Kirchhoff's work [Kir47] in which a graph is modeled as an electric network. Besides the original work of Kirchhoff, one of the first notable applications of discrete holomorphic functions is perhaps the famous article [BSST40] of Brooks, Smith, Stone and Tutte, where discrete holomorphic functions were used to construct tilings of rectangles by squares. We now define discrete holomorphic functions in a more modern fashion.

8.2.1 Isaacs's definition of discrete holomorphic functions

Discrete holomorphic functions distinctively appeared for the first time in the papers [Isa41, Isa52] of Isaacs, where he proposed two definitions[4]. Both definitions ask for a discrete version of the Cauchy-Riemann equations $\partial_{i\alpha}F = i\partial_\alpha F$ or equivalently that the $\bar{z}$-derivative is 0. We will be working with Isaacs's second definition (although the theories based on both definitions are almost the same). The definition involves the following discretization of the $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$ operator. For convenience, we will consider discrete holomorphic functions on the medial lattice (more precisely on the medial lattice of a discrete domain or a Dobrushin domain). For a complex valued function f on $\Omega_\delta^\diamond$, and for $x \in \Omega_\delta \cup \Omega_\delta^\star$, define

$$\bar\partial_\delta f(x) \quad = \quad \tfrac{1}{2}\left[f(E) - f(W)\right] + \tfrac{i}{2}\left[f(N) - f(S)\right]$$

[4] Isaacs called such functions "mono-diffric" functions.

where S, E, N and W denote the four vertices of $\Omega_\delta^\diamond$ adjacent to the medial vertex x indexed in the obvious way (N, E, S and W stand for cardinal directions).

Definition 8.13. A function $f : \Omega_\delta^\diamond \to \mathbb{C}$ is *discrete holomorphic* if $\bar\partial_\delta f(x) = 0$ for every $x \in \Omega_\delta \cup \Omega_\delta^\star$. The equation $\bar\partial_\delta f(x) = 0$ is called the *discrete Cauchy-Riemann equation* at x.

Remark 8.14. In Kirchhoff's work, every edge of the graph Ω_δ is seen as bearing a unit resistor and for $u \sim v$, $F(u,v)$ is the current from u to v. The first and the second Kirchhoff's laws of electricity can be restated as

- the sum of currents flowing from a vertex is zero $\sum_{v\sim u} F(u,v) = 0$,
- the sum of the currents around a closed contour $\gamma_0 \sim \gamma_1 \sim \cdots \sim \gamma_k = \gamma_0$ is zero: $\sum_{i=1}^k F(\gamma_{i-1},\gamma_i) = 0$.

For a second, let us consider an orientation of the lattice $\delta\mathbb{Z}^2$ isomorphic to the one of the medial lattice (namely counterclockwise around black faces when $\delta\mathbb{Z}^2$ is colored in a chessboard way). For a medial vertex x associated to an oriented edge e, define $f(x) = eF(e)$, where e is seen as a complex number. The function f is then discrete holomorphic on $\Omega_\delta^\diamond$.

The theory of discrete holomorphic functions starts pretty much like the usual complex analysis.

Proposition 8.15. *Discrete holomorphic functions $f, g : \Omega_\delta^\diamond \to \mathbb{C}$ satisfy the following properties:*

- *$\lambda f + \mu g$ is discrete holomorphic for any $\lambda, \mu \in \mathbb{C}$.*
- *$Re(f)$ and $Im(f)$ are harmonic functions for the appropriate modification of the discrete Laplacian.*
- *Discrete contour integrals vanish.*
- *If the family (f_δ) of discrete holomorphic functions on Ω_δ converge uniformly on every compact subset of Ω to f, then f is holomorphic.*

Proof. The first claim is obvious. Let us now turn to the second. Let $v \in \Omega_\delta^\diamond$. Let nw, ne, se and sw be the four nearest neighbors of v in $(\delta\mathbb{Z}^2)^\diamond$ and n, e, s and w the next nearest neighbors at distance δ of v (the previous indexation refers once again to cardinal directions). Assume that these eight medial vertices are in $\Omega_\delta^\diamond$. Then, the Cauchy-Riemann equation applied to the four faces bordered by v gives that

$$\begin{aligned}\tilde{\Delta} f(v) := &\frac{1}{4}\Big(f(n) - f(v) + [f(e) - f(v)] + [f(s) - f(v)] + [f(w) - f(v)]\Big)\\ &= \frac{i}{4}\Big(f(ne) - f(nw) + [f(se) - f(ne)] + [f(sw) - f(se)]\\ &\quad + [f(nw) - f(sw)]\Big)\\ &= 0.\end{aligned}$$

Therefore, f is $\tilde{\Delta}$-harmonic (the operator $\tilde{\Delta}$ is a modified Laplacian on $\Omega_\delta^\diamond$).

For the third property, we first recall the definition of discrete contours and discrete observables. It is slightly different from the definition in Section 6.1.3 since we are dealing with functions on vertices rather than edges. Let $\mathcal{C}$ be a self-avoiding polygon $z_0 \sim z_1 \sim \cdots \sim z_n = z_0$ on Ω_δ (or $\Omega_\delta^\star$). We then define

$$\oint_{\mathcal{C}} f(z)dz := \sum_{i=0}^{n-1} f\Big(\frac{z_i + z_{i+1}}{2}\Big)(z_{i+1} - z_i). \tag{8.4}$$

Observe that there are two types of contours (primal and dual). The Cauchy-Riemann equation $\bar{\partial}_\delta f(x)$ can be thought of as the fact that the integral along the discrete contour formed by the four medial vertices around the face x equals 0. Exactly as in Chapter 6, the fact that $\Omega_\delta^\diamond$ is simply connected implies that any discrete contour vanishes. The last property is a trivial application of Morera theorem: f is continuous and its contour integrals vanish. □

Remark 8.16. We only need the Cauchy-Riemann equations for every $x \in \Omega_\delta$ to obtain that the integrals along dual discrete contours vanish, or for every $x \in \Omega_\delta^\star$ to obtain this result for primal discrete contours.

Remark 8.17. Unfortunately, the product of two discrete holomorphic functions is no longer discrete holomorphic in general: while restrictions of 1, z, and z^2 to the square lattice are discrete holomorphic, the higher powers are not. This makes the theory of discrete holomorphic functions significantly harder than the usual complex analysis, since one cannot transpose proofs from continuum to discrete in a straightforward way.

8.2.2 Discrete Dirichlet and Neumann BVP for discrete holomorphic maps

In this section, we prefer working with $\Omega_\delta \cup \Omega_\delta^\star$ instead of $(\delta\mathbb{Z}^2)^\diamond$. Note that $\delta\mathbb{Z}^2 \cup (\delta\mathbb{Z}^2)^\star$ is a translate of $(\delta\mathbb{Z}^2)^\diamond$ and therefore one may generalize the definition of discrete holomorphicity to this graph in an obvious way.

A function $f : \Omega_\delta \cup \Omega_\delta^\star \to \mathbb{C}$ is a (discrete holomorphic) solution to the *Dirichlet BVP* on $\Omega_\delta \cup \Omega_\delta^\star$ with boundary conditions $g : \partial\Omega_\delta \to \mathbb{C}$ if f is discrete holomorphic and $f = g$ on $\partial\Omega_\delta$. Note that we fix boundary conditions on $\partial\Omega_\delta$ only.

Proposition 8.18. *Let Ω_δ be a discrete domain. For any $g : \partial\Omega_\delta \to \mathbb{C}$, there exists a solution f to the Dirichlet BVP on $\Omega_\delta \cup \Omega_\delta^\star$. Furthermore, the solution is unique up to the addition of a constant to $f_{|\Omega_\delta^\star}$.*

Proof. Let $H_1, H_2 : \Omega_\delta \to \mathbb{R}$ be the harmonic maps on $\Omega_\delta \smallsetminus \partial\Omega_\delta$ which are the harmonic solutions of the Dirichlet BVP with boundary condition $\mathrm{Re}(g)$ and $\mathrm{Im}(g)$. We set $f_{|\Omega_\delta} = H_1 + iH_2$. Let $v \in \Omega_\delta^\star$ and $c \in \mathbb{C}$ [5]. Set $f(v) = c$. For $w \sim v$, there is one value λ so that the Cauchy-Riemann equation is satisfied around $\frac{1}{2}(v+w)$. We set $f(w) = \lambda$. Iterating this procedure, one can construct f on any dual-vertex of $\Omega_\delta^\star$ which can be connected by a sequence of dual edges to v. Since Ω_δ is a discrete domain, $\Omega_\delta^\star$ is connected and we therefore constructed $f_{|\Omega_\delta^\star}$ everywhere. □

Remark 8.19. When g is real valued, $f_{|\Omega_\delta}$ takes its values in $\mathbb{R}$. Furthermore, if $c \in i\mathbb{R}$, then $f_{|\Omega_\delta^\star}$ takes its values in $i\mathbb{R}$. In such cases, $f_{|\Omega_\delta}$ and $f_{|\Omega_\delta^\star}$ are discrete versions of two harmonic conjugates in the continuum. For this reason, we will often restrict our attention to boundary conditions g which are real valued. More generally, discrete holomorphic maps are often decomposed into their real and imaginary parts, leaving respectively on Ω_δ and $\Omega_\delta^\star$.

Remark 8.20. Interestingly, when boundary conditions are constant on a portion of $\partial\Omega_\delta$, then the "dual boundary conditions" on $\partial\Omega_\delta^\star$ are Neumann, in the sense that the discrete normal derivative of the function on $\Omega_\delta^\star$ is zero. This duality between Dirichlet and Neumann BVP is very useful in the continuum, and can be exploited in the discrete world as well. We will not focus on this here, since we aim for the solution of more complicated BVPs.

8.3 Riemann-Hilbert BVP and s–holomorphic functions

We are now trying to solve a particular form of so-called Riemann-Hilbert BVPs. These continuum problems can be stated as follows: look for a continuous function $f : \overline{\Omega} \to \mathbb{C}$ such that

$$f \text{ is holomorphic on } \Omega \text{ and } \mathrm{Im}\big[f\nu^{1/2}\big] = 0 \text{ on } \partial\Omega,$$

[5]The existence of the additive constant comes from the freedom in the choice of $c \in \mathbb{C}$.

where ν is the tangent to $\partial\Omega$ viewed as a unit complex number (in order to define properly this tangent for rough domains, simply use conformal invariance to map the domain to a smooth domain). Also observe that ν and $-\nu$ do not play symmetric roles here. Therefore, it will be important to specify the direction along which the tangent vector is considered. In general, we may also introduce complex singularities on the boundary or inside the domain.

In this book, we restrict our attention to the following three BVPs:

BVP₁ In a simply connected domain Ω with two points a and b on its boundary,

- the tangent vectors are oriented from a to b on the boundary arcs ∂_{ab} and ∂_{ba},
- f has a singularity at a and b,
- $\mathrm{Im}(\int_x^y f^2) = 1$, where $x \in \partial_{ab}$ and $y \in \partial_{ba}$ are two other fixed points[6].

BVP₂ In a simply connected domain Ω with two points u and v on its boundary,

- the tangent vectors are oriented counterclockwise along the boundary,
- f has a singularity at $u \in \partial\Omega$,
- $f(v) = 1$.

BVP₃ In a simply connected domain Ω with a point x inside the domain,

- the tangent vectors are oriented counterclockwise on the boundary,
- f has a complex singularity at $x \in \Omega$ with complex residue 1.

Remark 8.21. In the continuum, the solutions to these problems are not hard to find. Let us illustrate this fact with the example of a solution f to **BVP**$_1$. The method for finding the solution is relevant for what will be next. The function $H = \mathrm{Im}(\int^z f^2)$ is the imaginary part of a holomorphic map on Ω, and it is therefore harmonic. Now, on the boundary, f^2 is collinear to the complex conjugate of the tangent vector, and therefore H is constant on ∂_{ab} and ∂_{ba}. Since $\mathrm{Im}(\int_x^y f^2) = H(y) - H(x) = 1$, we obtain that if $H = 0$ on ∂_{ab}, then $H = 1$ on ∂_{ba}. Therefore, $f = \sqrt{\phi'}$, where ϕ is any holomorphic map on Ω with imaginary part H. Interestingly, ϕ can be easily checked to be given by a conformal map from Ω to the strip $\mathbb{R} \times [0,1]$, sending a to $-\infty$ and b to ∞.

As explained in the previous sections, there are difficulties when dealing with the square of a discrete holomorphic function, and this will make

[6] The choice of x and y is irrelevant here.

the study of discrete versions of these Riemann-Hilbert BVP much more intricate (since it seems that introducing the primitive of the square of the solution is convenient). In order to overcome this difficulty, we introduce s-holomorphic functions (for *spin*-holomorphic). This notion was developed in [Smi10, CS11, CS12].

8.3.1 Definition of s-holomorphic functions

In this section, s-holomorphic functions are defined on vertices of the medial graph of a discrete domain or a Dobrushin domain. For any edge e of the medial lattice (the edge e being oriented, it can be thought of as a complex number), the real line passing through the origin and $\sqrt{e}$ is denoted by $\ell(e)$ (the choice of the square root is irrelevant since we will be looking at projections on lines only). The different lines associated with medial edges on $(\delta\mathbb{Z}^2)^\diamond$ are $e^{i\pi/8}\mathbb{R}$, $e^{-i\pi/8}\mathbb{R}$, $e^{-i3\pi/8}\mathbb{R}$ and $e^{-i5\pi/8}\mathbb{R}$, see Fig. 8.1. For a line ℓ, define

$$P_\ell(x) = \alpha \mathrm{Re}(\overline{\alpha} x) = \tfrac{1}{2}\big(x + \alpha^2 \overline{x}\big),$$

where α is any unit vector collinear to ℓ.

Definition 8.22 (Smirnov)**.** A function $f : \Omega_\delta^\diamond \to \mathbb{C}$ is *s-holomorphic* if for any edge $e = [xy]$ of $\Omega_\delta^\diamond$, we have

$$P_{\ell(e)}[f(x)] = P_{\ell(e)}[f(y)].$$

Remark 8.23. The definition differs slightly from the definition in [Smi10] where the lattice was rotated by $\pi/4$.

Let us first confirm that the notion of s-holomorphicity is stronger than the notion of discrete holomorphicity.

Proposition 8.24. *Any s-holomorphic function $f : \Omega_\delta^\diamond \to \mathbb{C}$ is discrete holomorphic on $\Omega_\delta^\diamond$.*

Proof. Let $f : \Omega_\delta^\diamond \to \mathbb{C}$ be a s-holomorphic function. Let v be a vertex of $\delta\mathbb{Z}^2 \cup (\delta\mathbb{Z}^2)^\star$ corresponding to a face of $\Omega_\delta^\diamond$. Assume that $v \in \delta\mathbb{Z}^2$, the case $v \in (\delta\mathbb{Z}^2)^\star$ is similar. We wish to show that $\bar\partial_\delta f(v) = 0$. Let N, W, S and E be the four medial-vertices around v as illustrated in Fig. 8.1, and let us write one relation provided by the s-holomorphicity, for instance

$$P_{e^{-i\pi/8}\mathbb{R}}[f(E)] \;=\; P_{e^{-i\pi/8}\mathbb{R}}[f(S)].$$

Expressed in terms of f and its complex conjugate $\bar f$ only, the previous equality becomes

$$f(E) + e^{-i\pi/4}\overline{f(E)} \;=\; f(S) + e^{-i\pi/4}\overline{f(S)}.$$

Doing the same with the three other relations, we find

$$\begin{array}{rcl} f(S)+ie^{-i\pi/4}\overline{f(S)} & = & f(W)+ie^{-i\pi/4}\overline{f(W)}, \\ f(W)-e^{-i\pi/4}\overline{f(W)} & = & f(N)-e^{-i\pi/4}\overline{f(N)}, \\ f(N)-ie^{-i\pi/4}\overline{f(N)} & = & f(E)-ie^{-i\pi/4}\overline{f(E)}. \end{array}$$

Multiplying the second identity by $-i$, the third by -1, the fourth by i, and then summing the four identities, we obtain

$$0=(1-i)\left[f(E)-f(W)+if(N)-if(S)\right]=2(1-i)\bar{\partial}_\delta f(v)$$

which is exactly the discrete Cauchy-Riemann equation around v. □

8.3.2 Discrete primitive of f^2

Let us now show that the imaginary part of primitives of the square of s-holomorphic functions are well-defined.

Theorem 8.25. *Let $f:\Omega_\delta^\diamond\to\mathbb{C}$ be an s-holomorphic function on $\Omega_\delta^\diamond$, then there exists a unique (up to additive constant) function $H:\Omega_\delta\cup\Omega_\delta^\star\to\mathbb{C}$ such that*

$$H(b)-H(w) \;=\; \sqrt{2}\delta\left|P_{\ell(e)}[f(x)]\right|^2\;\left(=\;\sqrt{2}\delta\left|P_{\ell(e)}[f(y)]\right|^2\right)$$

for every edge $e=[xy]$ of $\Omega_\delta^\diamond$ bordered by $b\in\Omega_\delta$ and $w\in\Omega_\delta^\star$. Furthermore, for two neighboring vertices $b_1,b_2\in\Omega_\delta$, with v being the medial vertex at the center of $[b_1b_2]$,

$$H(b_1)-H(b_2) \;=\; Im\left[f(v)^2\cdot(b_1-b_2)\right], \tag{8.5}$$

the same relation holding for vertices of $\Omega_\delta^\star$.

The last relation legitimizes the fact that H is an analogue of $\mathrm{Im}\left(\int^z f^2\right)$.

Proof. Set the value of H to be $c\in\mathbb{R}$ at some vertex b_0 (or dual vertex). The uniqueness of H is straightforward since $\Omega_\delta^\diamond$ is connected, the value of H at $x\in\Omega_\delta\cup\Omega_\delta^\star$ is simply the sum of increments along an arbitrary path from b_0 to x.

To obtain the existence, construct the value at some point by summing increments along an arbitrary path from b_0 to this point. The only thing to check is that the value obtained does not depend on the path chosen to define it. Since the domain is the union of all the faces of $(\delta\mathbb{Z}^2)^\diamond$ within it, it is sufficient to check it for elementary "square" contours around each

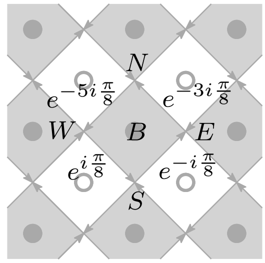

Figure 8.1: The different directions of the lines $\ell(e)$ for medial edges around a black face.

medial vertex v (these are the simplest closed contours). Therefore, we need to prove that

$$\left|P_{\ell(ne)}[f(v)]\right|^2 - \left|P_{\ell(se)}[f(v)]\right|^2 + \left|P_{\ell(sw)}[f(v)]\right|^2 - \left|P_{\ell(nw)}[f(v)]\right|^2 = 0, \quad (8.6)$$

where nw, ne, se and sw are the four medial edges with end-point v, indexed once again according to cardinal directions. Note that $\ell(ne)$ and $\ell(sw)$ (resp. $\ell(se)$ and $\ell(nw)$) are orthogonal. Hence, (8.6) follows from

$$\begin{aligned}\left|P_{\ell(ne)}[f(v)]\right|^2 + \left|P_{\ell(sw)}[f(v)]\right|^2 &= |f(v)|^2 \\ &= \left|P_{\ell(se)}[f(v)]\right|^2 + \left|P_{\ell(nw)}[f(v)]\right|^2.\end{aligned} \quad (8.7)$$

Let us now turn to (8.5). Let $b_1 \sim b_2$ be two neighboring vertices of Ω_δ and v the medial-vertex associated to $[b_1 b_2]$. Let w be one of the dual-vertices in $\Omega_\delta^\star$ adjacent to both b_1 and b_2 (there may be only one of them in $\Omega_\delta^\star$ if b_1 and b_2 are on the boundary). Let e_1 and e_2 be the two medial edges bordered by b_1 and w, and b_2 and w respectively. We find

$$\begin{aligned}H(b_1) - H(b_2) &= \sqrt{2}\delta\Big[\left|P_{\ell(e_1)}[f(v)]\right|^2 - \left|P_{\ell(e_2)}[f(v)]\right|^2\Big] \\ &= \frac{1}{2}\Big[\big(\sqrt{e_1}f(v) + \overline{\sqrt{e_1}f(v)}\big)^2 - \big(\sqrt{e_2}f(v) + \overline{\sqrt{e_2}f(v)}\big)^2\Big] \\ &= \frac{1}{2}\Big[e_1 f(v)^2 + \overline{e_1 f(v)^2} + |f(v)|^2 - e_2 f(v)^2 - \overline{e_2 f(v)^2} - |f(v)|^2\Big] \\ &= \frac{1}{2}\Big[(e_1 - e_2)f(v)^2 + \overline{(e_1 - e_2)f(v)^2}\Big] \\ &= \frac{1}{2i}\Big[(b_1 - b_2)f(v)^2 - \overline{(b_1 - b_2)f(v)^2}\Big] = \mathrm{Im}[f(v)^2 \cdot (b_1 - b_2)].\end{aligned}$$

In the second equality, we used the fact that $\frac{\delta}{\sqrt{2}} = |e_1|$ and $\frac{\delta}{\sqrt{2}} = |e_2|$. □

Even if the primitive of a discrete holomorphic map is discrete holomorphic and thus discrete harmonic, this is not the case of the primitive of the square of a discrete holomorphic map. Nonetheless, s-holomorphicity implies that H satisfies subharmonic and superharmonic properties. More precisely, denote by $H^\bullet$ and H° the restrictions of $H : \Omega_\delta \cup \Omega_\delta^\star \to \mathbb{C}$ to Ω_δ (black faces) and $\Omega_\delta^\star$ (white faces) respectively. Let $\Delta^\bullet$ and Δ° be the nearest-neighbor discrete Laplacian for functions on Ω_δ and $\Omega_\delta^\star$ respectively.

Proposition 8.26. *If $f : \Omega_\delta^\diamond \to \mathbb{C}$ is s-holomorphic, then $H^\bullet$ and H° are respectively subharmonic for $\Delta^\bullet$ on $\Omega_\delta \smallsetminus \partial\Omega_\delta$ and superharmonic for Δ° on $\Omega_\delta^\star \smallsetminus \partial\Omega_\delta^\star$.*

Proof. Let us focus on $H^\bullet$ (the proof for H° follows the same lines). Let B be a vertex of $\Omega_\delta \smallsetminus \partial\Omega_\delta$. Let N, E, S and W be the four medial-vertices adjacent to B (once again the letters refer to cardinal directions). Also set

$$\begin{aligned}
a &= e^{i\frac{\pi}{8}} P_{\ell([ES])}[f(E)] = e^{i\frac{\pi}{8}} P_{\ell([ES])}[f(S)],\\
b &= e^{-i\frac{\pi}{8}} P_{\ell([SW])}[f(S)] = e^{-i\frac{\pi}{8}} P_{\ell([SW])}[f(W)],\\
c &= e^{5i\frac{\pi}{8}} P_{\ell([WN])}[f(W)] = e^{5i\frac{\pi}{8}} P_{\ell([WN])}[f(N)],\\
d &= e^{3i\frac{\pi}{8}} P_{\ell([NE])}[f(N)] = e^{3i\frac{\pi}{8}} P_{\ell([NE])}[E)].
\end{aligned}$$

Note that a, b, c and d are real. With these definitions, we may rewrite $f(N)$, $f(E)$, $f(S)$ and $f(W)$ as follows:

$$\begin{aligned}
f(E) &= \sqrt{2}i(e^{-3i\pi/8}d + e^{-i\pi/8}a),\\
f(S) &= \sqrt{2}i(e^{3i\pi/8}a - e^{5i\pi/8}b),\\
f(W) &= \sqrt{2}i(e^{i\pi/8}b - e^{3i\pi/8}c),\\
f(N) &= \sqrt{2}i(e^{-i\pi/8}c - e^{i\pi/8}d).
\end{aligned}$$

By definition of $\Delta^\bullet$ and (8.5), we find

$$\begin{aligned}
\Delta^\bullet H^\bullet(B) &= \frac{\delta}{4}\mathrm{Im}\big[f(E)^2 - if(S)^2 - f(W)^2 + if(N)^2\big]\\
&= -\frac{\delta}{2}\mathrm{Im}\big[(e^{-3i\pi/8}d + e^{-i\pi/8}a)^2 + i(e^{3i\pi/8}a - e^{5i\pi/8}b)^2\\
&\qquad - (e^{i\pi/8}b - e^{3i\pi/8}c)^2 - i(e^{-i\pi/8}c - e^{i\pi/8}d)^2\big]\\
&= \delta\big[a^2 + b^2 + c^2 + d^2 - \sqrt{2}(ab + bc + cd - ad)\big].
\end{aligned}$$

On the other hand, let us compute

$$\begin{aligned}|f(E)-f(S)|^2+|f(W)-f(N)|^2 &= 2|e^{-3i\pi/8}d+e^{-i\pi/8}a-e^{3i\pi/8}a+e^{5i\pi/8}b|^2\\ &\quad+2|e^{i\pi/8}b-e^{3i\pi/8}c-e^{-i\pi/8}c+e^{i\pi/8}d|^2\\ &=2(d+\sqrt{2}a-b)^2+2(b-\sqrt{2}c+d)^2\\ &=4(a^2+b^2+c^2+d^2)-4\sqrt{2}(ab+bc+cd-ad).\end{aligned}$$

In conclusion,

$$4\Delta^\bullet H^\bullet(B)=\delta|f(E)-f(S)|^2+\delta|f(W)-f(N)|^2\geq 0 \tag{8.8}$$

and the claim follows. □

Similarly, we could have chosen the term $|f(S)-f(W)|^2+|f(N)-f(E)|^2$ to find

$$4\Delta^\bullet H^\bullet(B)=\delta|f(S)-f(W)|^2+\delta|f(N)-f(E)|^2\geq 0. \tag{8.9}$$

8.3.3 Precompactness for s-holomorphic maps

We plan to study BVPs. In order to do so, and for the same reason as for harmonic functions, we will need a pre compactness result. This technical theorem will be very important in the next sections. *One may skip the proof during a first reading.* Below, Q denotes a square, and $9Q$ denotes the square of same center, but 9 times bigger.

Theorem 8.27 (Precompactness for s-holomorphic maps)**.** *Let $Q\subset\Omega$ such that $9Q\subset\Omega$. Let $(f_\delta)_{\delta>0}$ be a family of s-holomorphic maps on $\Omega_\delta^\diamond$ and $(H_\delta)_{\delta>0}$ be the corresponding functions defined in the previous section. If $(H_\delta)_{\delta>0}$ is uniformly bounded on $9Q$, then $(f_\delta)_{\delta>0}$ is a precompact family of functions*[7] *on Q.*

Proof. Color the vertices of $(\delta\mathbb{Z}^2)^\diamond$ in black and white in a chessboard way (medial vertices corresponding to vertical primal edges are all colored the same, and the same for horizontal edges). The sets of black and white vertices are denoted by $(\delta\mathbb{Z}^2)^\diamond_\bullet$ and $(\delta\mathbb{Z}^2)^\diamond_\circ$ respectively.

Since f_δ is s-holomorphic, it is also holomorphic and therefore harmonic for the modified Laplacian on $\Omega_\delta^\diamond$, which corresponds to the standard Laplacian on $(\delta\mathbb{Z}^2)^\diamond_\bullet$. Imagine for a moment that the family of functions (f_δ) satisfies the second property of Proposition 8.7. In such case, Proposition 8.7 implies that the restrictions $f_\delta^\bullet$ of functions f_δ to $\Omega_\delta^\diamond\cap(\delta\mathbb{Z}^2)^\diamond_\bullet$ form a precompact family of functions.

[7] Recall that the functions f_δ are extended to the faces of Ω_δ.

Let us now use the s-holomorphicity to deduce that $(f_\delta)_{\delta>0}$ itself is precompact. Let $x \in \Omega_\delta^\diamond \cap (\delta\mathbb{Z}^2)_\circ^\diamond$. Denote the north-east and south-west neighboring vertices of x in $(\delta\mathbb{Z}^2)_\bullet^\diamond$ by y and z. The s-holomorphicity shows that

$$\begin{aligned} f_\delta(x) &= P_{\ell(xy)}(f_\delta(x)) + P_{\ell(xz)}(f_\delta(x)) \\ &= P_{\ell(xy)}(f_\delta(y)) + P_{\ell(xz)}(f_\delta(z)) \\ &= f_\delta(y) + O\big(|f_\delta(z) - f_\delta(y)|\big), \end{aligned} \tag{8.10}$$

where we used the fact $\ell(xy)$ and $\ell(xz)$ are orthogonal to each others. The previous paragraph implies that we may extract a sub-sequence $(f_{\delta_n}^\bullet)_n$ converging uniformly on every compact subset of Ω when seen as a function of $\Omega_\delta^\diamond \cap (\delta\mathbb{Z}^2)_\bullet^\diamond$. The relation (8.10) implies that (f_{δ_n}) itself converges uniformly on every compact subset of Ω.

Therefore, we would be done if we could prove the second property of Proposition 8.7.

Fix $\delta > 0$. When jumping over a medial-vertex v, the function H_δ changes by $\delta\mathrm{Re}(f_\delta^2(v))$ or $\delta\mathrm{Im}(f_\delta^2(v))$ depending on the direction (vertical or horizontal), so that

$$\delta^2 \sum_{v\in Q_\delta^\diamond} \big|f_\delta(v)\big|^2 = \delta \sum_{x\in Q_\delta} |\nabla H_\delta^\bullet(x)| + \delta \sum_{x\in Q_\delta^\star} |\nabla H_\delta^\circ(x)| \tag{8.11}$$

where $\nabla H_\delta^\bullet(x) = (H_\delta^\bullet(x+\delta) - H_\delta^\bullet(x), H_\delta^\bullet(x+i\delta) - H_\delta^\bullet(x))$, and ∇H_δ° is defined similarly for H_δ°. It follows that it is enough to prove uniform boundedness of the right-hand side in (8.11). We only treat the sum involving $H_\delta^\bullet$, the other sum can be handled similarly.

Write $H_\delta^\bullet = S_\delta + R_\delta$ where S_δ is a harmonic function with same boundary conditions on $\partial 9Q_\delta$ as $H_\delta^\bullet$. In order to prove that the sum of $|\nabla H_\delta^\bullet|$ on Q_δ is bounded by C/δ, we deal separately with $|\nabla S_\delta|$ and $|\nabla R_\delta|$. First,

$$\begin{aligned} \sum_{x\in Q_\delta} \big|\nabla S_\delta(x)\big| &\le \frac{C_1}{\delta^2}\cdot C_2\delta\left(\sup_{x\in 9Q_\delta}|S_\delta(x)|\right) = \frac{C_1}{\delta^2}\cdot C_2\delta\left(\sup_{x\in\partial 9Q_\delta}|S_\delta(x)|\right) \\ &= \frac{C_3}{\delta}\left(\sup_{x\in\partial 9Q_\delta}|H_\delta^\bullet(x)|\right) \le \frac{C_4}{\delta}, \end{aligned}$$

where in the first inequality we used derivative estimates (Proposition 8.6), in the first equality the maximum principle for S_δ (to show that the supremum is reached on the boundary), and in the second the fact that S_δ and $H_\delta^\bullet$ share the same boundary conditions on $9Q_\delta$. The last inequality comes from the fact that $H_\delta^\bullet$ remains bounded uniformly in δ.

Second, let us treat $|\nabla R_\delta|$. This function is subharmonic since S_δ is harmonic and $H_\delta^\bullet$ is subharmonic (Proposition 8.26). Recall that $G_{9Q_\delta}(\cdot, y)$

is the Green function in $9Q_\delta$ with singularity at y. Since R_δ equals 0 on the boundary, Proposition 8.11 implies

$$R_\delta(x) = \sum_{y\in 9Q_\delta} \Delta R_\delta(y) G_{9Q_\delta}(x,y), \tag{8.12}$$

thus giving

$$\nabla R_\delta(x) = \sum_{y\in 9Q_\delta} \Delta R_\delta(y) \nabla_x G_{9Q_\delta}(x,y).$$

Therefore,

$$\begin{aligned}
\sum_{x\in Q_\delta} \left|\nabla R_\delta(x)\right| &= \sum_{x\in Q_\delta} \Big| \sum_{y\in 9Q_\delta} \Delta R_\delta(y) \nabla_x G_{9Q_\delta}(x,y)\Big| \\
&\le \sum_{y\in 9Q_\delta} \Delta R_\delta(y) \sum_{x\in Q_\delta} |\nabla_x G_{9Q_\delta}(x,y)| \\
&\le \sum_{y\in 9Q_\delta} \Delta R_\delta(y)\ C_5\delta \sum_{x\in Q_\delta} G_{9Q_\delta}(x,y) \\
&= C_5\delta \sum_{x\in Q_\delta}\sum_{y\in 9Q_\delta} \Delta R_\delta(y) G_{9Q_\delta}(x,y) \\
&= C_5\delta \sum_{x\in Q_\delta} R_\delta(x) = C_6/\delta.
\end{aligned}$$

The second line uses the fact that $\Delta R_\delta \ge 0$, the third Proposition 8.12, the fifth Proposition 8.11 again, and the last equality the fact that Q_δ contains of order $1/\delta^2$ sites and the fact that R_δ is bounded uniformly in δ (since H_δ and S_δ are).

Thus, $\delta \sum_{x\in Q_\delta} |\nabla H_\delta^\bullet|$ is uniformly bounded. Since the same result holds for H_δ°, we obtain the second condition of Proposition 8.7 and we are done. □

8.3.4 Discrete version of $\mathbf{BVP}_1$

Let us now study the discretization of the problem $\mathbf{BVP}_1$. We work with Dobrushin domains $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$. For a medial vertex $x \in \partial\Omega_\delta^\diamond$ (or rather a prime end x), we define the tangent vector $\nu(x)$ as $e + e'$, where e and e' are the two medial edges of $\partial_{ab}^\diamond \cup \partial_{ba}^\diamond$ incident to x.

We say that f_δ satisfies the discrete Riemann-Hilbert $\mathbf{BVP}_1$ if
- f_δ is s-holomorphic in $\Omega_\delta^\diamond$,
- for any $x \in \partial\Omega_\delta^\diamond \smallsetminus \{a_\delta^\diamond, b_\delta^\diamond\}$, $\mathrm{Im}[f(x)\nu(x)^{1/2}] = 0$,
- $P_{\ell(e_b)}[f_\delta(b_\delta)] = \frac{1}{\sqrt{2e_b}}$, where e_b is defined as in Chapter 3.

Remark 8.28. Note that the renormalization is not exactly the same as in the continuum formulation, but we will prove that the two normalizations are equivalent as δ tends to 0. Also observe $\nu(x)$ is defined for vertices and that the definition is slightly simpler than in Section 6.1.5: the reason is that we are interested in the complex argument modulo π of $(e+e')^{1/2}$. This quantity does not depend on the choice of the argument of $e+e'$ modulo 2π, while it does for $(e+e')^\sigma$ in general. Therefore, in this context we may use $e+e'$ to define $\nu(x)$ but not for other random-cluster models.

The goal of this section is to prove the following result.

Theorem 8.29 (Smirnov [Smi10]). *Let Ω be a simply connected domain with two points a and b on its boundary. Assume that $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$ is a family of Dobrushin domains converging to (Ω, a, b) in the Carathéodory sense. Consider f_δ to be the solution of the discrete* $\mathbf{BVP}_1$ *on $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$. Then $(f_\delta)_{\delta>0}$ converges uniformly on (Ω, a, b) to $\sqrt{\phi'}$, where ϕ is any conformal map from Ω to $\mathbb{R}\times(0,1)$ mapping a to $-\infty$ and b to ∞.*

Remark 8.30. Since $\sqrt{\phi'}$ is the solution of the continuum version of $\mathbf{BVP}_1$, we simply wish to prove that the solution of the discrete BVP converges to the solution of its continuum counterpart. In order to prove this result, we mimic the continuum story and prove first the convergence of the discrete version of $\mathrm{Im}(\int^z f^2)$, namely H_δ.

In this section, let f_δ be a solution of $\mathbf{BVP}_1$ and H_δ be given by Theorem 8.25 with $H_\delta(b_\delta) = 1$.

Lemma 8.31. *The function H_δ equals 0 on $\partial_{ab}^\star$ and 1 on ∂_{ba}.*

Proof. We first prove that $H_\delta^\bullet$ is constant on ∂_{ba}. Let B and B' be two adjacent consecutive sites of ∂_{ba}, and x the medial-edge in the middle of the edge $[BB']$. Note that x is on the boundary. Since $f_\delta(x)$ is parallel to $\nu(x)^{-1/2}$, (8.5) implies that $H_\delta^\bullet(B) = H_\delta^\bullet(B')$. Hence, $H_\delta^\bullet$ is constant along the arc. Since $H_\delta^\bullet(b_\delta) = 1$, the result follows readily.

Similarly, H_δ° is constant on the arc $\partial_{ab}^\star$. Moreover, the dual white face $b_\delta^\star \in \partial_{ab}^\star$ bordering b_δ (see Fig. 3.6) satisfies

$$H_\delta^\circ(b_\delta^\star) \;=\; H_\delta^\bullet(b_\delta) - \sqrt{2}\delta|P_{\ell(e_b)}[f(b_\delta)]|^2 \;=\; 1-1 \;=\; 0. \tag{8.13}$$

In the second equality, we used the normalization hypothesis (recall that $|e_b| = \delta/\sqrt{2}$). Therefore $H_\delta^\circ = 0$ on $\partial_{ab}^\star$. □

Our goal is now to prove the following result. Let H be the solution of the Dirichlet BVP with $g = 0$ on ∂_{ab} and 1 on ∂_{ba}. We would like to prove that the function H_δ converges to H uniformly away from a and b. To get an intuition that this should be the case, observe that a subharmonic function in a domain is smaller than the harmonic function with the same boundary conditions. Therefore, $H^\bullet$ is smaller than the harmonic function $h^\bullet$ solving the same BVP. Similarly H° is bigger (since it is superharmonic) than the harmonic function h° solving the same BVP. Moreover, $H^\bullet(b)$ is larger than $H^\circ(w)$ for two neighboring faces. Hence, if $H^\bullet$ and H° are close to each other on the boundary, then they are sandwiched between two harmonic functions $h^\bullet$ and h° which are close to each other. This motivates us to understand the BVPs for $H^\bullet$ and H°. The previous lemma provides us with part of the answer (namely boundary values for $H^\bullet_\delta$ on ∂_{ba} and for H°_δ on $\partial^\star_{ba}$), but it is not clear how to obtain the relevant boundary values for $H^\bullet_\delta$ on ∂_{ab} and H°_δ for $\partial^\star_{ba}$. For this reason, we use the so-called boundary trick introduced in [CS12] and extend the functions outside $\Omega_\delta \cup \Omega^\star_\delta$.

Construct the following extension of the graph Ω_δ. First, a vertex $x \in \partial_{ab}$ can be seen as a prime-end of the domain Ω_δ very much like medial vertices on $\partial\Omega^\diamond$ may be seen as prime-ends of the domain $\Omega^\diamond_\delta$. In particular, the degree of boundary vertices seen as prime-ends is smaller than 4. Add all the edges incident to these vertices which are not already in E_{Ω_δ} together with their endpoints. We will consider all endpoints as forming different vertices of a new graph[8]. The set composed of these vertices is denoted $\widehat{\partial}_{ab}$. Consider $\widehat{\Omega}_\delta = \Omega_\delta \cup \widehat{\partial}_{ab}$. Similarly, construct $\widehat{\partial}^\star_{ba}$ and $\widehat{\Omega}^\star_\delta$.

Definition 8.32. Let $\widehat{H}^\bullet_\delta$ be the function on $\widehat{\Omega}_\delta$ equal to $H^\bullet_\delta$ on Ω_δ and 0 on $\widehat{\partial}_{ab}$. Let $\widehat{H}^\circ_\delta$ be the function on $\widehat{\Omega}^\star_\delta$ equal to H°_δ on $\Omega^\star_\delta$ and 1 on $\widehat{\partial}_{ba}$.

Define $(X^\bullet_t)_{t\geq 0}$ to be the continuous-time random walk on $\widehat{\Omega}_\delta$ that jumps with rate 1 on adjacent vertices, *except* for the vertices on $\widehat{\partial}_{ab}$ onto which it jumps with rate $\rho := 2/(\sqrt{2}+1)$. Let $(X^\circ_t)_{t\geq 0}$ denote the continuous-time random walk on $\widehat{\Omega}^\star_\delta$ that jumps with rate 1 on neighbor vertices, *except* for the dual-vertices on $\widehat{\partial}^\star_{ba}$ onto which it jumps with rate $\rho = 2/(\sqrt{2}+1)$. Let $\Delta^\bullet$ and Δ° be the generators of $X^\bullet_t$ and X°_t.

One may prefer working with the discrete time random walks $(X^\bullet_n)_{n\geq 0}$ and $(X^\circ_n)_{n\geq 0}$ induced by the continuous-time random walks above. In such case, $(X^\bullet_n)_{n\geq 0}$ jumps equally likely on each neighbor in Ω_δ, but with probability $\frac{1+\sqrt{2}}{2}$ time smaller on neighbors in $\widehat{\partial}_{ab}$. Then for any $x \in \widehat{\Omega}_\delta \setminus \partial\widehat{\Omega}_\delta$,

$$\Delta^\bullet f(x) = \sum_{y\sim x} \mathbb{P}_x(X^\bullet_1 = y)[f(y) - f(x)].$$

We are now in a position to tackle the boundary problem on the arc $\widehat{\partial}_{ab}$ and $\widehat{\partial}^\star_{ba}$. Before diving into the proof, let us make a small comment by taking the example of $\widehat{H}^\bullet_\delta$. The difficulty of the BVP does not lie in

[8] A vertex of $\mathbb{Z}^2$ could correspond to the end-points of different such edges, and for this reason we keep a clear distinction between the different end-points.

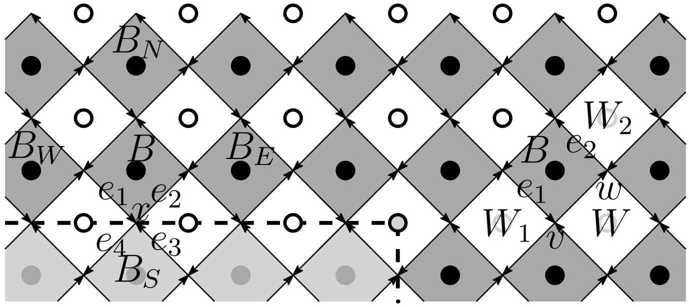

Figure 8.2: Extend $\Omega_\delta^\diamond$ by adding one extra layers of medial faces, and extend the functions $H_\delta^\bullet$ on these new medial faces. On the right, the indexation of faces and edges for the proof of Lemma 8.35.

determining boundary conditions since $\widehat{H}_\delta^\bullet$ equal 1 on ∂_{ba} and 0 on $\widehat{\partial}_{ab}$, but rather in explaining the connection between these boundary conditions (more precisely those on $\widehat{\partial}_{ab}$) and the values of $\widehat{H}_\delta^\bullet$ on $\widehat{\Omega}_\delta \smallsetminus \partial\widehat{\Omega}_\delta$. The most obvious connection would be that $\widehat{H}_\delta^\bullet$ would be subharmonic on $\widehat{\Omega}_\delta \smallsetminus \partial\widehat{\Omega}_\delta$. This is not quite the case, but we can in fact show that $\widehat{H}_\delta^\bullet$ is subharmonic for a modified Laplacian. A similar observation holds true for $\widehat{H}_\delta^\circ$.

Proposition 8.33. *If $f_\delta : \Omega_\delta^\diamond \to \mathbb{C}$ is s-holomorphic, then $\widehat{H}_\delta^\bullet$ and $\widehat{H}_\delta^\circ$ are respectively subharmonic for $\Delta^\bullet$ on $\widehat{\Omega}_\delta \smallsetminus \partial\widehat{\Omega}_\delta$ and superharmonic for Δ° on $\widehat{\Omega}_\delta^\star \smallsetminus \partial\widehat{\Omega}_\delta^\star$.*

Proof. Let us treat the case of vertices in Ω_δ. Dual vertices are treated similarly. Since we already treated vertices in $\Omega_\delta \smallsetminus \partial_{ab}$, we only need to look at a vertex $B \in \partial_{ab}$. For ease of exposition, we will assume that B is incident to three edges in Ω_δ, the vertex of $\widehat{\partial}_{ab}$ being below B (other cases can be handled similarly). Denote by B_W, B_N, B_E and B_S the black faces adjacent to B, see Figure 8.2. We claim that

$$\begin{aligned}\Delta^\bullet H_\delta^\bullet(B) &= \frac{2+\sqrt{2}}{6+5\sqrt{2}}[H_\delta^\bullet(B_W)+H_\delta^\bullet(B_N)+H_\delta^\bullet(B_E)] + \frac{2\sqrt{2}}{6+5\sqrt{2}}H_\delta^\bullet(B_S) - H_\delta^\bullet(B)\\ &\geq 0.\end{aligned}$$

Recall that $H_\delta^\bullet(B_S) = 0$ by construction, and we therefore need to prove that

$$\Delta^\bullet H_\delta^\bullet(B) = \frac{2+\sqrt{2}}{6+5\sqrt{2}}[H_\delta^\bullet(B_W)+H_\delta^\bullet(B_N)+H_\delta^\bullet(B_E)] - H_\delta^\bullet(B) \geq 0. \tag{8.14}$$

In order to do that, let us denote by e_1, e_2, e_3, e_4 the four medial edges around the medial-vertex x between B and B_S, indexed in clockwise order, with e_1 and e_2 along B, and e_3 and e_4 along B_S (see Figure 8.2) – note that e_3 and e_4 are not edges of $\Omega_\delta^\diamond$, but of $(\delta\mathbb{Z}^2)^\diamond \smallsetminus \Omega_\delta^\diamond$.

Now, since $f_\delta(x)$ is proportional to $\nu(x)^{-1/2}$, a small computation gives

$$\begin{aligned}|P_{\ell(e_3)}[f_\delta(x)]|^2 &= \left|\left(\tan\frac{\pi}{8}\right)\mathrm{e}^{\mathrm{i}\pi/4}P_{\ell(e_2)}[f_\delta(x)]\right|^2\\ &= \frac{2-\sqrt{2}}{2+\sqrt{2}}|P_{\ell(e_2)}[f_\delta(x)]|^2\\ &= \frac{2-\sqrt{2}}{2+\sqrt{2}}\frac{1}{\sqrt{2}\delta}H_\delta^\bullet(B).\end{aligned}$$

We could have chosen the medial edge e_4 instead of e_3 and we would have obtained the same result. If $\tilde{H}_\delta^\bullet$ denotes the function defined by $\tilde{H}_\delta^\bullet = H_\delta^\bullet$ on B, B_W, B_N and B_E, and by

$$\tilde{H}_\delta^\bullet(B_S) = \sqrt{2}\delta|P_{\ell(e_3)}[f_\delta(x)]|^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}}H_\delta^\bullet(B). \tag{8.15}$$

Then, $\tilde{H}_\delta^\bullet$ satisfies the same relation of Theorem 8.25 for e_3 and e_4, as inside the domain. Since f_δ verifies the same local equations, the computation performed in Proposition 8.26 applies at B (with $\tilde{H}_\delta$ instead of H_δ), and we deduce

$$\Delta\tilde{H}_\delta^\bullet(B) = \frac{1}{4}[\tilde{H}_\delta^\bullet(B_W)+\tilde{H}_\delta^\bullet(B_N)+\tilde{H}_\delta^\bullet(B_E)+\tilde{H}_\delta^\bullet(B_S)]-\tilde{H}_\delta^\bullet(B) \geq 0. \tag{8.16}$$

Using (8.15), this inequality can be rewritten as

$$\frac{1}{4}[H_\delta^\bullet(B_W)+H_\delta^\bullet(B_N)+H_\delta^\bullet(B_E)] - \frac{6+5\sqrt{2}}{4(2+\sqrt{2})}H_\delta^\bullet(B) \geq 0. \tag{8.17}$$

and the claim (8.14) follows. □

Proof of Theorem 8.29. Since $\widehat{H}_\delta^\bullet$ is sub-harmonic for $\Delta^\bullet$ and has boundary conditions 0 on $\widehat{\partial}_{ab}$, and 1 on ∂_{ba}, it is thus smaller than the $\Delta^\bullet$-harmonic function $h_\delta^\bullet$ with the same boundary conditions. Since $h_\delta^\bullet$ converges to the solution H of the continuum Dirichlet BVP with boundary condition 0 on ∂_{ab} and 1 on ∂_{ba} (one may use Theorem 8.9, or more precisely a trivial modification of it involving the modified Laplacian on the boundary), we therefore deduce that

$$\limsup_{\delta\to 0} H_\delta^\bullet \leq H.$$

Now, $\widehat{H}_\delta^\circ$ is super-harmonic for Δ° and has boundary conditions 0 on $\partial_{ab}^\star$, and 1 on $\widehat{\partial}_{ba}^\star$. It is thus larger than the Δ°-harmonic function h_δ° with the same boundary conditions. In particular, h_δ° converges to H, and therefore

$$\liminf_{\delta\to 0} H_\delta^\circ \geq H.$$

But by construction, $H^\circ(W)$ is smaller than $H^\bullet(B)$ for any neighbor B of W. Therefore,

$$H \leq \liminf_{\delta\to 0} H_\delta^\circ \leq \liminf_{\delta\to 0} H_\delta^\bullet \leq H$$

and the same holds true for the lim sup. Therefore, both $H_\delta^\bullet$ and H_δ° converge to H.

Now, let $Q \subset \Omega$ such that $9Q \subset \Omega$ (recall the definition of $9Q$ from Theorem 8.27). Since H_δ converges uniformly to a continuous function H, the family H_δ is bounded uniformly in $\delta > 0$. Theorem 8.27 thus implies that $(f_\delta)_{\delta>0}$ is a precompact family of s-holomorphic maps on Q.

Let $(f_{\delta_n})_{n\in\mathbb{N}}$ be a convergent subsequence and denote its limit by f. Note that f is holomorphic as limit of discrete holomorphic functions (Proposition 8.15). Furthermore, for two points x and y in Ω, we have:

$$H_{\delta_n}(y_{\delta_n}) - H_{\delta_n}(x_{\delta_n}) = \mathrm{Im}\left(\int_{x_{\delta_n}}^{y_{\delta_n}} f_{\delta_n}^2(z)dz\right)$$

where x_{δ_n} and y_{δ_n} denote the closest points to x and y in Ω_{δ_n}. On the one hand, the convergence of $(f_{\delta_n})_{n\in\mathbb{N}}$ being uniform on any compact subset of Ω, the right hand side converges to $\mathrm{Im}\left(\int_x^y f(z)^2dz\right)$. On the other hand, the left hand side converges to $\mathrm{Im}(\phi(y)-\phi(x))$, where ϕ is a conformal map with $\mathrm{Im}(\phi) = H$. Since both $H(y) - H(x)$ and $\mathrm{Im}\left(\int_x^y f(z)^2dz\right)$ are harmonic functions of y, there exists $C \in \mathbb{R}$ such that $\phi(y) - \phi(x) = C + \int_x^y f(z)^2dz$ for every $x, y \in \Omega$. We deduce that f equals $\sqrt{\phi'}$. Since this is true for any convergent subsequence, we find that f_δ tends to $\sqrt{\phi'}$.

It only remains to notice that a conformal map ϕ such that $\mathrm{Im}(\phi) = H$ is exactly a conformal map from Ω to $\mathbb{R}\times(0,1)$ mapping a to $-\infty$ and b to ∞. This can be done as follows. Fix a conformal map Φ from Ω to $\mathbb{R}\times(0,1)$ mapping a to $-\infty$ and b to ∞. The function $H\circ\Phi^{-1}$ is solution of the Dirichlet problem on $\mathbb{R}\times(0,1)$ with boundary condition 1 on the top and 0 on the bottom. Therefore, $(H\circ\Phi^{-1})(z) = \mathrm{Im}(z)$ which leads to $H(z') = \mathrm{Im}(\Phi(z'))$ for any $z' \in \Omega$. In particular, $\phi - \Phi$ is holomorphic and $\mathrm{Im}(\phi - \Phi) = 0$. The combination of these two facts implies that $\phi - \Phi$ is a constant function equal to a real number, which is the claim. □

8.3.5 Discrete version of $\mathbf{BVP_2}$

Let $\Omega_\delta^\diamond$ be a discrete domain and $u_\delta^\diamond, v_\delta^\diamond$ be two medial vertices on the boundary of $\Omega_\delta^\diamond$. We define the tangent vector as in the previous section (it goes counterclockwise around the boundary). The function $f_\delta : \Omega_\delta^\diamond \to \mathbb{C}$ satisfies the discrete $\mathbf{BVP}_2$ if

- f_δ is s-holomorphic on $\Omega_\delta^\diamond \setminus \{u_\delta^\diamond\}$,
- $\mathrm{Im}(f_\delta(z)\nu(z)^{1/2}) = 0$ for any $z \in \partial\Omega_\delta^\diamond \setminus \{u_\delta^\diamond\}$,
- $f_\delta(v_\delta^\diamond) = 1$.

The two following definitions correspond to the fact that the domain (respectively discrete domain) looks like the upper half-plane in a neighborhood of v (respectively $v_\delta^\diamond$). A domain Ω is *flat near* v if there exists $\varepsilon > 0$ such that

$$[-\varepsilon, \varepsilon] \times (0, \varepsilon] = (-v + \Omega) \cap [-\varepsilon, \varepsilon]^2.$$

In the discrete level, Ω_δ is *flat near* $v_\delta^\diamond$ if there exists $\varepsilon > 0$ such that

$$(\delta\mathbb{Z}^2)^\diamond \cap [-\varepsilon, \varepsilon] \times [0, \varepsilon] = (-v_\delta^\diamond - (\tfrac{1}{2}, 0) + \Omega_\delta^\diamond) \cap [-\varepsilon, \varepsilon]^2.$$

Theorem 8.34 (Chelkak, Smirnov [CS12])**.** *Let Ω be a simply connected domain with two marked points u and v on its boundary, the boundary being flat in a neighborhood of v. Let $\Omega_\delta^\diamond$ be a family of discrete simply connected domains with $u_\delta^\diamond$ and $v_\delta^\diamond$ two medial-vertices on its boundary. We assume that $(\Omega_\delta^\diamond, u_\delta^\diamond, v_\delta^\diamond)$ converges to (Ω, u, v) in the Carathéodory sense, and that the boundary of $\Omega_\delta^\diamond$ is flat near $v_\delta^\diamond$. Let f_δ be the s-holomorphic solution of* $\mathbf{BVP}_2$ *in $\Omega_\delta^\diamond$ with $u_\delta^\diamond$ and $v_\delta^\diamond$, then*

$$f_\delta(\cdot) \to \sqrt{\frac{\psi'(\cdot)}{\psi'(v)}} \quad \textit{when } \delta \to 0$$

uniformly on every compact subset of Ω, where ψ is any conformal map from Ω to the upper half-plane $\mathbb{H}$, mapping u to ∞ and v to 0.

We will also consider the discrete primitive H_δ of f_δ^2 provided by Theorem 8.25, with the condition that it is equal to 0 at some $w \in \partial\Omega_\delta^\star$.

One can check exactly as in Lemma 8.31 that $H_\delta^\circ = 0$ on $\partial\Omega_\delta^\star$. Observe that superharmonicity implies $H_\delta^\circ \geq 0$ everywhere. As before, we extend Ω_δ by adding one layer $\widehat{\partial}$ except that we do not add the other end-point of the edge passing through $u_\delta^\diamond$ (since anyway the function is not s-holomorphic at $u_\delta^\diamond$). We set $H_\delta^\bullet = 0$ on $\widehat{\partial}$. The same computation as in Proposition 8.33 implies that $H_\delta^\bullet$ is $\Delta^\bullet$-subharmonic on $\Omega_\delta \setminus \{u_\delta\}$, where u_δ is the vertex of Ω_δ bordered by $u_\delta^\diamond$.

We therefore understand the boundary conditions of $H_\delta^\bullet$ and H_δ°. Nevertheless, one may check that $H^\bullet$ does not remain bounded in a neighborhood of u (see below for a proof), and we cannot use the precompactness criterion provided by Theorem 8.27 directly. We therefore start by proving precompactness of f_δ and H_δ with our bare hands.

Lemma 8.35. *The family of functions (H_δ) and (f_δ) are precompact on any square Q such that $9Q \subset \Omega$.*

Let us first describe the proof heuristically. We wish to prove that H_δ is uniformly bounded away from u. If the value of $H_\delta^\bullet$ is equal to M at some point B_0, then subharmonicity shows that it must be larger than M on some arc γ from B_0 to u_δ (see the definition of u_δ few lines above). Furthermore, $H_\delta^\bullet \geq 0$ on the boundary. Therefore, if $H_\delta^\bullet$ was harmonic, we would deduce that $H_\delta^\bullet$ would be larger than M times the harmonic measure of the arc γ in Ω. Applying this observation to the vertex B next to $v_\delta^\diamond$, we would deduce that $\sqrt{2}\delta|F_\delta(v_\delta^\diamond)|^2 = H_\delta^\bullet(B)$ is larger than M times the harmonic measure of γ seen from B. The facts that the domain near v contains a rectangular box and that the arc γ is of macroscopic length imply that the harmonic measure of the arc seen from B would be larger than $c\delta$. This would be contradictory with the value of $F_\delta(v_\delta^\diamond)$.

Unfortunately, $H_\delta^\bullet$ is not exactly harmonic and one must use H_δ°. This renders the proof slightly more cumbersome. In particular, one has to compare the value of $H_\delta^\bullet$ and H_δ° at neighboring vertices.

Proof. We present the argument succinctly. Let $\varepsilon > 0$. We wish to prove that (H_δ) is uniformly bounded outside a ball of radius $\varepsilon > 0$ around u. If this fact can be verified for every $\varepsilon > 0$, this will imply the result directly, since by Theorem 8.27, (f_δ) would be precompact and thus (H_δ) also.

In order to prove this boundedness, consider $B_0 \in \Omega_\delta$ at distance larger than ε from u and set $H_\delta^\bullet(B_0) =: M$. By subharmonicity and the fact that the boundary conditions of $H_\delta^\bullet$ are 0 on $\widehat{\partial}$, we find that there exists a sequence γ of vertices $u_\delta = B_n \sim B_{n-1} \sim \cdots \sim B_1 \sim B_0$ such that

$$H_\delta^\bullet(u_\delta) \geq H_\delta^\bullet(B_{n-1}) \geq \cdots \geq H_\delta^\bullet(B_1) \geq H_\delta^\bullet(B_0) \geq M.$$

First, let us assume the claim below:

Claim: There exists $c_1 > 0$ such that $H_\delta^\circ(W) \geq c_1 H_\delta^\bullet(B)$ for any adjacent medial-faces B and $W \notin \partial\Omega_\delta^\star$.

Let Γ be the set of dual vertices bordering the path γ. We deduce from the claim that $H_\delta^\circ(W) \geq c_1 M$ for all $W \in \Gamma \setminus \partial\Omega_\delta^\star$.

Since H_δ° is equal to zero on $\partial\Omega_\delta^\star$ and is larger than cM on Γ, superharmonicity implies that

$$H_\delta^\circ(W_0) \geq \sum_{W \in \Gamma} H_\delta^\circ(W) \mathrm{H}_{\Omega_\delta^\star}(W_0, W) \geq c_1 M \mathrm{H}_{\Omega_\delta^\star}(W_0, \Gamma),$$

where the first inequality is due to superharmonicity and Proposition 8.5, and in the second $\mathrm{H}_{\Omega_\delta^\star}(W_0,\Gamma)$ denotes the harmonic measure of the set Γ seen from W_0 (it is the probability that a random walk on $\Omega_\delta^\star$ starting from W_0 hits Γ before reaching $\partial\Omega_\delta^\star$).

Now, the set Γ is of macroscopic length ε, therefore there exists $c_2 = c_2(\varepsilon) > 0$ such that

$$\mathrm{H}_{\Omega_\delta^\star}(V,\Gamma) \geq c_2\delta, \tag{8.18}$$

where $V \in \Omega_\delta^\star \setminus \partial\Omega_\delta^\star$ is a dual-vertex nearest to $v_\delta^\diamond$. In order to see that (8.18) is satisfied, observe that Γ has "length larger than ε", which shows the existence of $c_3 = c_3(\varepsilon) > 0$ such that $\mathrm{H}_{\Omega_\delta^\star}(W_0,\Gamma) \geq c_3$ for any W_0 at Euclidean distance ε from the boundary. Furthermore, the harmonic measure seen from V of the set of dual-vertices at Euclidean distance ε from the boundary is at least $c_4\delta$ (this follows from the gambler's ruin and the fact that the neighborhood of $v_\delta^\diamond$ in $\Omega_\delta^\diamond$ contains a rectangle of size $O(1/\delta)$). Combining these two facts together yields the existence of $c_2 > 0$ in (8.18).

Now, (8.18) implies

$$\sqrt{2}\delta = \sqrt{2}\delta|f_\delta(v_\delta^\diamond)|^2 = H_\delta^\bullet(B) \geq H_\delta^\circ(V) \geq c_1 M \mathrm{H}_{\Omega_\delta^\star}(V,\Gamma) \geq c_1 M \cdot c_2\delta,$$

where B is the vertex of Ω_δ adjacent to $v_\delta^\diamond$. This implies that $M \leq \sqrt{2}/(c_1c_2)$ and $H^\bullet$ (and therefore H°) is indeed bounded. In conclusion, we only need to prove the claim.

Proof of the Claim. In this proof, we use $\gg$ for "much larger than" and $\approx$ for "approximatively equal to" (i.e. the difference of the left and right terms is much smaller than each one of them).

We recommend that the reader takes a look at Fig. 8.2. Consider a dual-vertex $W \notin \partial\Omega_\delta^\star$ and a vertex $B \in \Omega_\delta$ which are adjacent and such that $H_\delta^\bullet(B) \gg H_\delta^\circ(W)$. Let v and w be the end-points of the medial edge between B and W and set $e = [vw]$. Also set W_1 and W_2 to be the dual-vertices such that v and w are the centers of $[WW_1]$ and $[WW_2]$, and e_1 and e_2 for the two medial-edges between B and W_1 and W_2.

First, observe that superharmonicity and $H_\delta^\circ \geq 0$ imply that

$$H_\delta^\circ(W) \geq \tfrac{1}{4}\sum_{W'\sim W} H_\delta^\circ(W') \geq \tfrac{1}{4}H_\delta^\circ(W')$$

for any $W' \sim W$. Therefore, we also have that $H_\delta^\circ(W_1) \ll H_\delta^\bullet(B)$ and $H_\delta^\circ(W_2) \ll H_\delta^\bullet(B)$. In such case,

$$\sqrt{2}\delta|P_{\ell(e)}(f_\delta(v))|^2 \approx H_\delta^\bullet(B) \approx \sqrt{2}\delta|P_{\ell(e_1)}(f_\delta(v))|^2. \tag{8.19}$$

In particular, the absolute value of the projections of $f_\delta(v)$ on $\ell(e_1)$ and $\ell(e)$ are very close to each other and therefore $f_\delta(v)$ approximately belongs to one of the bisectors of $\ell(e_1)$ and $\ell(e)$. Similarly, $f_\delta(w)$ approximately belongs to one of the bisectors of $\ell(e)$ and $\ell(e_2)$. In particular, $f_\delta(v)$ and $f_\delta(w)$ have very different complex arguments.

Thus, we find that $\delta|f_\delta(v) - f_\delta(w)|^2$ is of the order of $\delta|f_\delta(v)|^2$, which itself is of the order of $H_\delta^\bullet(B)$ by (8.19). Since $H_\delta^\circ(W') \geq 0$ for $W' \sim W$, (8.8) applied to H° gives us that

$$H_\delta^\circ(W) \geq -\Delta^\circ H_\delta^\circ(W) \geq \delta|f_\delta(v) - f_\delta(w)|^2 \geq c_4 H_\delta^\bullet(B)$$

for a universal constant $c_4 > 0$. In particular, $H_\delta^\circ(W)$ cannot be much smaller than $H_\delta^\bullet(B)$ to begin with and we get the claim. Note that we did not work with constants and used $\gg$ and $\approx$ instead, but in a language with constants, the argument gives the existence of $c_1 > 0$ as claimed. $\diamond$

$\square$

We now wish to prove that the solution to the discrete version of $\mathbf{BVP_2}$ converges to the solution to the continuum version of $\mathbf{BVP_2}$, i.e. $\sqrt{\psi'/\psi'(v)}$. In order to do that, we prove that any sub-sequential limit of (f_δ) is equal to $\sqrt{\mu\psi'}$ for some constant $\mu > 0$. We then prove that $\mu = 1/\psi'(v)$.

Proof of Theorem 8.34. Once again, the proof is presented completely but succinctly. Since $(f_\delta)_{\delta>0}$ and $(H_\delta)_{\delta>0}$ form two precompact families (the precompactness of $(H_\delta)_{\delta>0}$ follows from the one of $(f_\delta)_{\delta>0}$), consider a subsequence $(f_{\delta_n}, H_{\delta_n})$ converging to (f, H). The function H is harmonic as limit of subharmonic and superharmonic functions. Since H_δ° equals 0 on the boundary and is superharmonic, it implies that H_δ° is larger or equal to 0 everywhere and therefore $H \geq 0$ in Ω.

Let us now show that H is continuous on $\overline{\Omega} \setminus \{u\}$ and is equal to 0 on $\partial\Omega \setminus \{u\}$. The function $\widehat{H}_\delta^\bullet$ is equal to 0 on $\widehat{\partial}$ and is subharmonic. Let $z \in \Omega$ be such that $d(z, \partial\Omega) \ll 2d := d(z, u)$ and let Z_δ be the vertex of Ω_δ closest to z. Also, consider Λ to be the connected component of $z + [-d, d]^2$ in Ω containing z. The weak Beurling estimate (Lemma 8.10) together with the fact that $\widehat{H}_\delta^\bullet$ is smaller than the harmonic function with the same boundary conditions on $\partial\Lambda_\delta$ imply that

$$\widehat{H}_\delta^\bullet(Z_\delta) \leq \max\{H_\delta^\bullet(B) : B \in \partial\Lambda_\delta\} \cdot \left(\frac{d(z, \partial\Omega)}{d}\right)^\alpha.$$

Since $H^\bullet$ is uniformly bounded (as $\delta \to 0$) away from u, this implies that $H(z) \leq C\left(\frac{d(z,\partial\Omega)}{d}\right)^\alpha$ for some constant $C > 0$ depending on the distance d only. Letting z tend to the boundary, we obtain that $H \leq 0$ on $\partial\Omega \setminus \{u\}$. Since we already have $H \geq 0$, we find that $H = 0$ on $\partial\Omega \setminus \{u\}$. Furthermore, the previous displayed equation implies that H tends to 0 when approaching z.

Overall, H is a positive harmonic function on Ω which is continuous on $\overline{\Omega} \setminus \{u\}$ and equal to 0 on the boundary. Let ψ be a conformal map from Ω to $\mathbb{H}$ mapping u to ∞ and v to the origin. We now claim that the properties listed above imply that $H = \mu \mathrm{Im}(\psi)$ (this implies that $f = \sqrt{\mu\psi'}$ by the same argument as in $\mathbf{BVP}_1$) for some $\mu > 0$.

Claim: If H satisfies the properties listed above, then there exists $\mu > 0$ such that $H = \mu Im(\psi)$, where ψ is a conformal map from Ω to $\mathbb{H}$ mapping u to ∞ and v to 0.

Proof of the Claim. This fact is classical in complex analysis: it follows from the fact that positive harmonic functions in the disk can be represented as integrals of the Poisson kernel against a positive measure on the boundary[9] which can be understood as a version of Proposition 8.5 in the continuum. Let us provide a few more details. When working in Ω, this theorem asserts that a positive harmonic function h on Ω can be represented as

$$h(y) = \int_{\partial\Omega} P_\Omega(y, x) d\nu(x),$$

where ν is a measure on the boundary and $P_\Omega(\cdot,\cdot)$ is the Poisson kernel in Ω, i.e. that $P_\Omega(\cdot, x)$ is the imaginary part of a certain conformal map from Ω to $\mathbb{H}$ mapping x to infinity (as it stands, the function $P_\Omega(\cdot, x)$ is defined up to multiplication by a positive constant but this is irrelevant to the argument).

The fact that H is equal to 0 on the boundary except at u implies that the only possibility for the measure ν is that it is proportional to a Dirac mass at u, and the value $\mu > 0$ depends on this constant of proportionality. ◇

[9] One may also avoid the use of the Poisson kernel by doing the following proof. Note that by conformal invariance, we may assume that $\Omega = \mathbb{H}$, $u = \infty$ and $v = 0$. In the proof, $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$.

If H is bounded in a neighborhood of ∞, then H is bounded on the whole domain $\mathbb{H}$. Since it is equal to 0 on $\partial\mathbb{H} \setminus \{\infty\}$, this implies that $H = 0$ and we set $\mu = 0$.

If H is unbounded, let $\Psi : \mathbb{H} \to \mathbb{C}$ be an holomorphic map with $\mathrm{Im}(\Psi) = H$. The set $\Psi(\mathbb{H})$ is included in $\mathbb{H}$ since $H \geq 0$. Since $H = 0$ on the boundary except at infinity, we also get that $\Psi(\partial\mathbb{R}) \subset \mathbb{R}$.

The Cauchy-Riemann equation implies that the x-derivative of $\mathrm{Re}(\Psi)$ is equal to the y-derivative of $\mathrm{Im}(\Psi) = H$ and is therefore larger or equal to 0 since $H \geq 0$ (the fact that the derivative is well defined on the boundary follows easily from the Schwarz reflection principle for instance). We deduce that $\mathrm{Im}(\Psi)$ is increasing when going counter-clockwise along $\partial\mathbb{H}$.

Now, $\Psi(\mathbb{H})$ is unbounded since H is unbounded. Thus $\Psi(\partial\mathbb{H}) = \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$.

A slight extension of the principle of corresponding boundaries (here we do not have that the function is strictly increasing on the boundary but simply increasing), see e.g. [Lan99, Theorem 4.3], implies that Ψ is a conformal map from $\mathbb{H}$ to $\mathbb{H}$ sending ∞ to ∞. By adding a constant in $\mathbb{R}$ to H, we may fix the image of 0 to be 0 and therefore Ψ is an homothety.

Fix ψ such that $f = \sqrt{\mu\psi'}$. In order to conclude the proof, we now need to show that $\mu = 1/\psi'(v)$ by studying the behavior near v. Let $R(\alpha,\varepsilon) := (-\varepsilon,\varepsilon)\times(\alpha,\varepsilon)$ with $0 \le \alpha \ll \varepsilon \ll 1$. The assumption on Ω enables us to choose $\varepsilon > 0$ so small that $R(0,\varepsilon) \subset \Omega$. We divide the boundary of $\partial R(0,\varepsilon)_\delta$ into three pieces: the bottom side ∂_1, the part ∂_2 intersecting $\partial R(\alpha,\varepsilon)$, and the rest of the boundary ∂_3 (made of two vertical segments).

Subtract the constant $\sqrt{\mu\psi'(v)}$ from the function f_δ and consider the function $\tilde{H}_\delta$ constructed from the s-holomorphic function $f_\delta - \sqrt{\mu\psi'(v)}$ via Theorem 8.25. We wish to prove that $\tilde{H}_\delta$ is small.

Since $\tilde{H}_\delta^\bullet$ is subharmonic, we find that

$$\tilde{H}_\delta^\bullet(B_0) \le \sum_{B\in\partial R(0,\varepsilon)_\delta} \tilde{H}_\delta(B)\,\mathrm{H}_{\Omega_\delta}(B_0,B), \tag{8.20}$$

where we remind the reader that $\mathrm{H}_{\Omega_\delta}(\cdot,\cdot)$ denotes the harmonic measure in Ω_δ. The gambler's ruin together with traditional estimates on exit probabilities gives us that $\mathrm{H}_{\Omega_\delta}(B_0,\partial_2) \le c_1\delta/\varepsilon$, and $\mathrm{H}_{\Omega_\delta}(B_0,\partial_3) \le c_2\alpha\delta/\varepsilon$.

The boundary conditions of $\tilde{H}_\delta$ are not difficult to estimate. First, $\tilde{H}_\delta$ equals 0 on ∂_1. Second, at fixed α we deduce from the uniform convergence of f_δ that

$$\tilde{H}_\delta(z) \longrightarrow \mathrm{Im}\left[\int_0^z \left(\sqrt{\mu\psi'(x)} - \sqrt{\mu\psi'(v)}\right)^2 dx\right] = o(\mathrm{Im}(z)) = o(\varepsilon)$$

on $R(\alpha,\varepsilon)$, where $o(t)$ means that the term tends to 0 as t tends to 0. Finally, note that $\tilde{H}_\delta$ is also uniformly bounded by a constant $c_3 > 0$ on $\partial R(\alpha,\varepsilon)$ and therefore the boundary conditions are uniformly bounded on ∂_3.

Altogether, by distinguishing in (8.20) between the three parts ∂_1, ∂_2 and ∂_3 of the boundary, we find that

$$\tilde{H}_\delta^\bullet(B_0) \le 0 + o(\varepsilon)\cdot c_1\delta/\varepsilon + c_3\cdot c_2\delta\alpha/\varepsilon = \delta(c_1 o(1) + c_2c_3\tfrac{\alpha}{\varepsilon}),$$

where the term $o(1)$ tends to zero as ε tends to 0. This implies that

$$\left|F_\delta(v_\delta^\diamond) - \sqrt{\mu\psi'(v)}\right|^2 := \tilde{H}_\delta^\bullet(B_0) \le c_1 o(1) + c_2c_3\tfrac{\alpha}{\varepsilon}.$$

When δ, and then α and ε tend to 0, we obtain that $1 = \lim_{\delta\to 0} F_\delta(v_\delta^\diamond) = \sqrt{\mu\psi'(v)}$. This implies that $\mu = 1/\psi'(v)$. □

8.3.6 Discrete version of $\mathbf{BVP}_3$

We will sometimes be facing s-holomorphic functions with singularities, i.e. medial-vertices inside $\Omega_\delta^\diamond$ where the function is not s-holomorphic. We briefly explain how the problem of discrete singularities can be addressed. We refer to [HS11] and [Hon10a] for a complete study of this case and we focus on $\mathbf{BVP}_3$ as an example. Let $\Omega_\delta^\diamond$ be a discrete domain.

Definition 8.36. Let $x_\delta^\diamond \in \Omega_\delta^\diamond$ and $f_\delta : \Omega_\delta^\diamond \setminus \{x_\delta^\diamond\} \to \mathbb{C}$. We say that f_δ has a simple pole at $x_\delta^\diamond$ with discrete residue μ if

- f_δ is s-holomorphic on $\Omega_\delta^\diamond \setminus \{x_\delta^\diamond\}$.
- There exists $\lambda \in \mathbb{C}$ such that the function f_δ satisfies:

$$\begin{aligned} P_{\ell(e_{NE})}[\lambda] &= P_{\ell(e_{NE})}[f_\delta(NE)], \\ P_{\ell(e_{SE})}[\lambda] &= P_{\ell(e_{SE})}[f_\delta(SE)], \\ P_{\ell(e_{NW})}[\lambda + 2\pi\mu] &= P_{\ell(e_{NW})}[f_\delta(NW)], \\ P_{\ell(e_{SW})}[\lambda + 2\pi\mu] &= P_{\ell(e_{SW})}[f_\delta(SW)], \end{aligned}$$

where NE, NW, SW and SE are the medial vertices adjacent to $x_\delta^\diamond$, and e_{NE}, e_{NW}, e_{NE} and e_{SE} are the medial edges between $x_\delta^\diamond$ and NE, NW, SW and SE respectively (once again the indexation refers to cardinal directions).

The function f_δ can be thought of as a s-holomorphic function on a graph where the medial-vertex $x_\delta^\diamond$ is split into two end-points x^+ and x^- of degree 2, with $f_\delta(x^+) = \lambda$ and $f_\delta(x^-) = \lambda + 2\pi\mu$. With this interpretation, one may easily check that the integrals along discrete contours (see (8.4) for the definition of the integral of a discrete contour) surrounding the singularity $x_\delta^\diamond$ are equal to $2\pi i\mu$. Indeed, one may check this fact by looking at the contour composed of the vertical medial-edge passing by x^+ and coming back by x^-. Then, the integral equals $i(\lambda + 2\pi\mu) - i\lambda = 2\pi i\mu$. Any other contour integral can be obtained by adding integrals of contours not surrounding the singularity which are therefore equal to 0.

The definition above thus corresponds to a discrete version of residues. In particular, if $\mu = 0$, then one may extend f_δ at $x_\delta^\diamond$ by setting $f_\delta(x_\delta^\diamond) = \lambda$.

In the continuum, singularities are usually removed by subtracting Green functions. In the discrete context, we will do the same and it is therefore necessary to introduce a discrete s-holomorphic Green function. Discrete holomorphic Green functions were already constructed in [Ken00, Proposition 10] using dimers. Namely, consider the function of $(t, s) \in \mathbb{Z}^2$ defined by

$$C[0,(s,t)] := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{i(s\theta - t\phi)}}{2i\sin(\theta) + 2\sin(\phi)} d\theta d\phi$$

and set $C[z_1, z_2] = C[0, z_2 - z_1]$. Kenyon studied the asymptotic of this function and proved that it is discrete holomorphic. The functions $C(\cdot, z)$ are not s-holomorphic but relevant linear combinations of them are, as noticed by Hongler and Smirnov in [HS11].

Definition 8.37 (**s**-holomorphic Green function)**.** For any $x_\delta^\diamond \neq z_\delta^\diamond$ on

$(\delta\mathbb{Z}^2)^\diamond$, we set $G_\delta(x_\delta^\diamond, z_\delta^\diamond) = G(\frac{x_\delta^\diamond}{\delta}, \frac{z_\delta^\diamond}{\delta})$, where

$$\begin{aligned} G(x,z) &= 4\pi\cos\left(\tfrac{\pi}{8}\right)e^{i\frac{\pi}{8}}\left(C_0(2x+1,2z)+C_0(2x-i,2z)\right) \\ &\quad + 4\pi\sin\left(\tfrac{\pi}{8}\right)e^{-i\frac{3\pi}{8}}\left(C_0(2x-1,2z)+C_0(2x+i,2z)\right). \end{aligned}$$

The fact that the formula is explicit allows us to derive the convergence of this Green function from the convergence result of Kenyon.

Proposition 8.38. *Let $x \neq z$, then*

$$G_\delta(x_\delta^\diamond, z_\delta^\diamond) \longrightarrow \frac{1}{z-x} \qquad \textit{when } \delta \to 0$$

uniformly on any compact subset of $\mathbb{C} \setminus \{x\}$.

We are now in a position to state and solve $\mathbf{BVP}_3$. Let $x_\delta^\diamond \in \Omega_\delta^\diamond$. The function $f_\delta : \Omega_\delta^\diamond \to \mathbb{C}$ satisfies the discrete $\mathbf{BVP}_3$ if

- f_δ is s-holomorphic on $\Omega_\delta^\diamond \setminus \{x_\delta^\diamond\}$,
- $\mathrm{Im}(f_\delta(z)\nu(z)^{1/2}) = 0$ for any $z \in \partial\Omega_\delta^\diamond$,
- f_δ has a singularity at $x_\delta^\diamond$ with residue 1.

In the following statement, we extend the notion of Carathéodory convergence to a simply-connected domain Ω with a marked point x inside. A sequence of domains $(\Omega_\delta^\diamond, x_\delta^\diamond)$ converges to (Ω, x) if the conformal map $g_\delta : \mathbb{D} \mapsto \Omega_\delta^\diamond$ (here $\Omega_\delta^\diamond$ is seen as a subdomain of the plane as explained in Chapter 3) with $g_\delta(0) = x_\delta^\diamond$ and $g_\delta'(0) > 0$ converges on every compact of $\mathbb{D}$ to the conformal map $g : \mathbb{D} \mapsto \Omega$ with $g(0) = x$ and $g'(0) > 0$.

Theorem 8.39 (Hongler, Smirnov [HS11])**.** *Let Ω be a simply connected domain with a marked point x inside Ω. Consider a family of discrete simply connected domains $\Omega_\delta^\diamond$ with $x_\delta^\diamond$ a medial-vertex in $\Omega_\delta^\diamond$. We assume that $(\Omega_\delta^\diamond, x_\delta^\diamond)$ converges to (Ω, x) in the Carathéodory sense. Let f_δ be the s-holomorphic solution of $\mathbf{BVP}_3$, then $f_\delta - g_\delta$ can be extended at $x_\delta^\diamond$ and the extension satisfies that*

$$f_\delta - G_\delta \to \begin{cases} \sqrt{\phi'(x)}\sqrt{\phi'(z)}\left(\dfrac{1}{\phi(z)} - \mathrm{i}\right) - \dfrac{1}{z-x} & \textit{if } z \neq x, \\ -\mathrm{i}\phi'(x) & \textit{otherwise} \end{cases} \qquad \textit{when } \delta \to 0$$

uniformly on every compact subset of Ω, where ϕ is the unique *conformal map from Ω to the unit disk $\mathbb{D}$, with $\phi(x) = 0$ and $\phi'(x) > 0$.*

Since G_δ itself converges to $1/(z-x)$, the previous theorem implies that

$$f_\delta \longrightarrow \sqrt{\phi'(x)}\sqrt{\phi'(z)}\left(\frac{1}{\phi(z)} - \mathrm{i}\right)$$

uniformly on every compact subset of $\Omega \setminus \{x\}$.

Proof. **(sketch)** By linearity the discrete residue of $h_\delta := f_\delta - G_\delta$ is 0. Therefore, this function can be extended to an s-holomorphic on $\Omega_\delta^\diamond$. As a consequence, h_δ is solving a Riemann-Hilbert BVP which is a discretization of the BVP

$$h \text{ holomorphic on } \Omega \text{ and } \mathrm{Im}\Big[\big(h+g\big)\nu^{1/2}\Big] = 0 \text{ on } \partial\Omega$$

with $g = 1/(z-x)$. Since G_δ converges to $\frac{1}{z-x}$ when δ tends to zero, a trivial extension of Theorem 8.27 implies that h_δ is precompact on every compact subset of Ω. One may then show that any sub-sequential limit of h_δ tends to a solution of the Riemann-Hilbert BVP listed above. This part of the proof, which is omitted, is the most technical one of course.

We conclude by checking that

$$h_{\Omega,x}(z) := \sqrt{\phi'(x)}\sqrt{\phi'(z)}\Big(\frac{1}{\phi(z)} - \mathrm{i}\Big) - \frac{1}{(z-x)}$$

is the unique holomorphic solution of the Riemann-Hilbert BVP above with $g = 1/(z-x)$. Solutions of continuous Riemann-Hilbert BVP of this kind are classically unique: simply subtract two solutions to obtain a solution with $g = 0$, and then use the fact that the imaginary part of the primitive of the square is harmonic and constant on the boundary (it is thus constant everywhere and the square-root of the derivative is equal to 0). We therefore focus on the fact that $h_{\Omega,x}$ is indeed a solution. Obviously, it is holomorphic inside Ω. Since solutions to this Riemann-Hilbert BVP are conformally covariant (with covariance exponent 1/2), it is in fact sufficient to check that $h_{\Omega,x}$ is a solution for $x = 0$ and $\Omega = \mathbb{D}$. We then have $\phi(z) = z$ and therefore at $z = \mathrm{e}^{\mathrm{i}\theta}$, we have that $\nu(z) = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}$ and

$$\mathrm{Im}\Big(\sqrt{\phi'(x)}\sqrt{\phi'(z)}\Big(\frac{1}{\phi(z)} - \mathrm{i}\Big)\nu(z)^{1/2}\Big) = \mathrm{Im}\Big(\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\theta/2} + \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}\theta/2}\Big) = 0.$$

Therefore, $h_{\mathbb{D},0}$ is indeed a solution of the BVP when $\Omega = \mathbb{D}$ and $x = 0$ and the result follows. □

Chapter 9

Conformal invariance of the FK-Ising and Ising models

There are many different definitions of conformal invariance for a model. For instance, one may speak of conformal invariance of interfaces. Alternatively, conformal invariance can also refer to the fact that relevant observables of the model are conformally covariant in the scaling limit. In this chapter, we explore these different aspects of the conformal invariance of the Ising and FK-Ising models. *We only deal with critical models and we therefore fix $p = p_c(2)$ and $\beta = \beta_c$ in the whole chapter.*

The chapter is organized as follows. The two first sections are devoted to the proof of conformal invariance of interfaces. This proof follows a program whose scope exceeds the case of the Ising and FK Ising model. The program proceeds in two main steps.

1. First, one proves that a certain observable of the model is conformally invariant in the scaling limit. In order to do so, we show that the observable is solution of a certain discrete BVP, and we harness the theory of discrete holomorphic functions to prove that the solution of this discrete BVP problem converges as the mesh size tends to 0 to the solution of its continuum counterpart.
2. Second, we show that the conformal invariance of this observable is sufficient to prove conformal invariance of interfaces or other measurable quantities of the scaling limit.

The last section presents a brief summary of other Ising properties which have been proved to be conformally invariant in the last few years.

Before starting, let us recall the definition of a conformally covariant family of functions.

Definition 9.1. A family of functions $F_{\Omega,a_1,\dots,a_n} : \Omega \to \mathbb{C}$ indexed by simply connected domains with marked points $a_1,\dots,a_n \in \overline{\Omega}$ is *conformally covariant* if there exist $\alpha,\alpha',\beta_1,\beta_1',\dots,\beta_n',\beta_n > 0$ such that for any domain Ω and any conformal map $\psi : \Omega \to \mathbb{C}$ (i.e. holomorphic and one-to-one), for every $z \in \Omega$,

$$\begin{aligned}&F_{\psi(\Omega),\psi(a_1),\dots,\psi(a_n)}\big(\psi(z)\big)\\ &= \psi'(z)^{\alpha}\overline{\psi'(z)^{\alpha'}}\cdot\psi'(a_1)^{\beta_1}\overline{\psi'(a_1)^{\beta_1'}}\cdots\psi'(a_n)^{\beta_n}\overline{\psi'(a_n)^{\beta_n'}}\cdot F_{\Omega,a_1,\dots,a_n}\big(z\big).\end{aligned}$$

If $\alpha = \beta_1 = \beta_1' = \dots = \beta_n = \beta_n' = 0$, the family is said to be *conformally invariant.*

Example. An archetype of a conformally covariant family of functions is the solution to boundary problems such as Dirichlet or Riemann-Hilbert BVPs.

9.1 Conformal invariance of the Ising and FK-Ising fermionic observables

A family of observables for random-cluster models with general cluster-weights q were introduced in Chapter 6. It was argued that the scaling limits of these observables should be holomorphic when $q \in [0,4]$. The boundary conditions can be determined and correspond to discrete Riemann-Hilbert BVPs. It provides a good hint that the scaling-limit of the observable is conformally covariant. Unfortunately, the observables are not entirely determined by their boundary conditions and the local relations that they satisfy and it is therefore not possible at the moment to prove the convergence to a conformally covariant family of functions.

When $q = 2$ (the case of FK-Ising), the fermionic observable satisfies specific additional integrability properties that allow us to prove its s-holomorphicity. The Ising model is also conformally invariant in this sense: the conformally covariant observable is the fermionic observable introduced in Chapter 7. We now discuss these two cases.

9.1.1 Convergence of the FK fermionic observable

In this section, we consider a simply connected domain Ω with two marked points a and b on its boundary. We will work with discretizations of this Dobrushin domain. For $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$, define the *vertex fermionic observable* by

$$F_\delta(v) = F(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond, p_c, 2, v) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{u\sim v} F([uv]) & \text{if } v \in \Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond, \\ \dfrac{2}{2+\sqrt{2}}\displaystyle\sum_{u\sim v} F([uv]) & \text{if } v \in \partial\Omega_\delta^\diamond, \end{cases}$$

where $F_\delta([uv])$ is the edge fermionic observable at the edge $[uv]$ defined in Chapter 6.

Theorem 9.2 (Smirnov [Smi10]). *Let (Ω, a, b) be a simply connected domain with two marked points on its boundary. Let $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$ be a family of Dobrushin domains converging to (Ω, a, b) in the Carathéodory sense. Let $F_\delta(v)$ be the* vertex fermionic observable *in $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$. We have*

$$\frac{1}{\sqrt{2e_b}} F_\delta(\cdot) \;\to\; \sqrt{\phi'(\cdot)} \quad \text{when } \delta \to 0 \tag{9.1}$$

uniformly on any compact subset of Ω, where ϕ is any conformal map from Ω to the strip $\mathbb{R}\times(0,1)$ mapping a to $-\infty$ and b to ∞.

Recall the definition of e_b from Chapter 3 and note that $|2e_b| = \sqrt{2}\delta$. Therefore, when rotating the lattice as in [Smi10], we find the original formulation back.

We aim at proving that $\frac{1}{\sqrt{2e_b}} F_\delta$ is a s-holomorphic solution of $\mathbf{BVP}_1$. The key ingredient is the fact that the spin σ takes the special value $1/2$. This enables us to determine the complex argument (modulo π) of the observable.

Beware of the fact that F_δ is the vertex fermionic observable. We will use the same notation for the edge fermionic observable $F_\delta(e)$ defined on the edges of $\Omega_\delta^\diamond$.

Lemma 9.3. *For an edge $e \in \Omega_\delta^\diamond$, $\frac{1}{\sqrt{2e_b}} F_\delta(e)$ belongs to $\ell(e)$.*

Proof. The winding $W_{\gamma(\omega)}(e, e_b)$ at an edge e can only take its value in the set $W + 2\pi\mathbb{Z}$ where W is the winding at e of an arbitrary oriented path going from e to e_b. Therefore, the winding weight $\mathrm{e}^{\mathrm{i}W_{\gamma(\omega)}(e,e_b)/2}$ involved in the definition of $F_\delta(e)$ is always equal to $\mathrm{e}^{\mathrm{i}W/2}$ or $-\mathrm{e}^{\mathrm{i}W/2}$, *ergo* $F_\delta(e)$ is proportional to $e^{iW/2}$. Since $\frac{1}{\sqrt{2e_b}} e^{iW/2}$ belongs to $\ell(e)$ for any e, so does $\frac{1}{\sqrt{2e_b}} F_\delta(e)$. □

We are now in a position to prove s-holomorphicity.

Proposition 9.4. *The vertex fermionic observable $\frac{1}{\sqrt{2e_b}} F_\delta$ is s-holomorphic.*

Proof. Consider a medial vertex $v \in \Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond$ first. Four medial vertices are adjacent to v. We index them by NW, NE, SE and SW (the notation refers to cardinal directions once more). Write $\sigma = 1/2 = 1 - \sigma$. When rewriting (6.2) of Lemma 6.10 by setting $1/2 = 1 - \sigma$, we find

$$\overline{F_\delta(NW)} + \overline{F_\delta(SE)} = \overline{F_\delta(NE)} + \overline{F_\delta(SW)}$$

and therefore

$$F_\delta(NW) + F_\delta(SE) = F_\delta(NE) + F_\delta(SW).$$

The previous equation and the definition of the vertex fermionic observable imply

$$F_\delta(v) := \frac{1}{2}\sum_{u\sim v} F_\delta([uv]) = F_\delta(NW) + F_\delta(SE) = F_\delta(NE) + F_\delta(SW).$$

Using Lemma 9.3, $\frac{1}{\sqrt{2e_b}}F_\delta(NW)$ and $\frac{1}{\sqrt{2e_b}}F_\delta(SE)$ belong to $\ell(NW)$ and $\ell(SE)$ (they are in particular orthogonal to each other), so that $\frac{1}{\sqrt{2e_b}}F_\delta(NW)$ is the projection of $\frac{1}{\sqrt{2e_b}}F_\delta(v)$ on $\ell(NW)$ (and similarly for other edges). Therefore, for a medial edge $e = [xy]$, $\frac{1}{\sqrt{2e_b}}F_\delta(e)$ is the projection of $\frac{1}{\sqrt{2e_b}}F_\delta(x)$ and $\frac{1}{\sqrt{2e_b}}F_\delta(y)$ with respect to $\ell(e)$. A direct consequence is that the two projections are equal, a fact which implies that the vertex fermionic observable is s-holomorphic.

Let us now treat the case of $v \in \partial\Omega_\delta^\diamond$. We assume without loss of generality that $v \in \partial_{ab}^\diamond$ and we set x to be the primal-vertex bordered by v. Let e and e' be the two medial edges of $\Omega_\delta^\diamond$ incident to v. Lemma 6.11 implies that

$$\frac{1}{\sqrt{2e_b}}F_\delta(e') = \frac{1}{\sqrt{2e_b}}\exp\left[\tfrac{\mathrm{i}}{2}W_{\partial_{ab}^\diamond}(e,e_b)\right]\cdot \phi_{p_c,2,\Omega}^{a,b},$$
$$\frac{1}{\sqrt{2e_b}}F_\delta(e') = \frac{1}{\sqrt{2e_b}}\exp\left[\tfrac{\mathrm{i}}{2}W_{\partial_{ab}^\diamond}(e',e_b)\right]\cdot \phi_{p_c,2,\Omega}^{a,b}.$$

We deduce that

$$\begin{aligned} P_{\ell(e)}(F_\delta(v)) &= \frac{2}{2+\sqrt{2}}\Big[P_{\ell(e)}(F_\delta(e)) + P_{\ell(e)}(F_\delta(e'))\Big] \\ &= \frac{2}{2+\sqrt{2}}\Big[F_\delta(e) + \cos(\tfrac{\pi}{4})F_\delta(e)\Big] = F_\delta(e). \end{aligned}$$

(The normalization $\frac{2}{2+\sqrt{2}}$ was introduced in order to have this property.) If $e = [vw]$, we deduce that

$$P_{\ell(e)}(F_\delta(v)) = F_\delta(e) = P_{\ell(e)}(F_\delta(w)),$$

where in the second equality we have used the fact that w belongs to $\Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond$ (and we can therefore apply what we proved previously). A similar statement can be proved for e', and we deduce that F_δ is s-holomorphic at v, thus concluding the proof. $\square$

Proof of Theorem 9.2. The previous proposition shows that $\frac{1}{\sqrt{2e_b}}F_\delta$ is s-holomorphic. By construction, the exploration path must go through e_b so that $F_\delta(e_b) = 1$. Furthermore, we know from the previous proof that $\frac{1}{\sqrt{2e_b}}F_\delta(e_b)$ is the projection of $\frac{1}{\sqrt{2e_b}}F_\delta(b_\delta^\diamond)$ on $\ell(e_b)$, so that

$$\mathrm{P}_{\ell(e_b)}\left[\frac{1}{\sqrt{2e_b}}F_\delta(b_\delta^\diamond)\right] = \frac{1}{\sqrt{2e_b}}F_\delta(e_b) = \frac{1}{\sqrt{2e_b}}.$$

Finally, consider a medial vertex $v \in \partial\Omega_\delta^\diamond \setminus \{a_\delta^\diamond, b_\delta^\diamond\}$ incident to two medial edges e and e' of $\Omega_\delta^\diamond$. Assume that $v \in \partial_{ab}^\diamond$ (the case of $v \in \partial_{ba}^\diamond$ can be treated similarly). Lemma 6.11 once more shows that

$$\begin{aligned}\frac{1}{\sqrt{2e_b}}F_\delta(v) &= \frac{1}{\sqrt{2e_b}}\frac{2}{2+\sqrt{2}}\left[F_\delta(e) + F_\delta(e')\right]\\ &= \frac{1}{\sqrt{2e_b}}\frac{2}{2+\sqrt{2}}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}W_{\partial_{ab}^\diamond}(e,e_b)} + \mathrm{e}^{\frac{\mathrm{i}}{2}W_{\partial_{ab}^\diamond}(e',e_b)}\right)\phi_{\Omega_\delta,p_c,2}^{a_\delta,b_\delta}[e\in\gamma]\\ &= \frac{1}{\sqrt{2e_b}}\cdot\frac{4\cos(\pi/8)}{2+\sqrt{2}}\cdot\mathrm{e}^{\frac{\mathrm{i}}{4}\left(W_{\partial_{ab}^\diamond}(e,e_b)+W_{\partial_{ab}^\diamond}(e,e_b)\right)}\cdot\phi_{\Omega_\delta,p_c,2}^{a_\delta,b_\delta}[e\in\gamma].\end{aligned}$$

In particular, $\frac{1}{\sqrt{2e_b}}F_\delta(v)$ is collinear with $\nu_v^{-1/2}$ (recall the definition from Section 8.3.4).

Overall, $\frac{1}{\sqrt{2e_b}}F_\delta$ satisfies $\mathbf{BVP}_1$. Theorem 8.29 guarantees the convergence of $\frac{1}{\sqrt{2e_b}}F_\delta$ towards $\sqrt{\phi'}$, where ϕ is the conformal map from Ω to $\mathbb{R}\times(0,1)$ mapping a to $-\infty$ and b to ∞. This gives us the result. □

9.1.2 Convergence of the spin fermionic observable

Let us turn our attention to the spin fermionic observable. Recall that Ω is flat near v if there exists $\varepsilon > 0$ such that

$$v + [-\varepsilon,\varepsilon]\times(0,\varepsilon] = \Omega\cap(v+[-\varepsilon,\varepsilon]^2).$$

Theorem 9.5 (Chelkak, Smirnov [CS12]). *Let Ω be a simply connected domain with two marked points u and v on its boundary; we assume that the boundary is flat in a neighborhood of v. Let $\Omega_\delta^\diamond$ be a family of discrete domains with two medial-vertices $u_\delta^\diamond$ and $v_\delta^\diamond$ on their boundary. We assume that $(\Omega_\delta^\diamond, u_\delta^\diamond, v_\delta^\diamond)$ converges to (Ω, u, v) in the Carathéodory sense and that the boundary of $\Omega_\delta^\diamond$ is flat near $v_\delta^\diamond$. Let $F_\delta = F_{\Omega_\delta^\diamond,u_\delta^\diamond,v_\delta^\diamond}$ be the fermionic spin observable defined in Chapter 7, then*

$$F_\delta(\cdot) \to \sqrt{\frac{\psi'(\cdot)}{\psi'(v)}} \quad \textit{when } \delta\to 0 \tag{9.2}$$

uniformly on every compact subset of Ω, *where* ψ *is any conformal map from* Ω *to the upper half-plane* $\mathbb{H}$, *mapping* u *to* ∞ *and* v *to* 0.

Proof. We wish to prove that F_δ is the solution of $\mathbf{BVP}_2$. Theorem 8.34 will then imply the result immediately. Recall the definition of $\nu(z)$ from Section 8.3.4.

Let us prove that for $\delta > 0$,

- F_δ is s-holomorphic on $\Omega_\delta^\diamond$,
- $\mathrm{Re}(F_\delta(z)\nu(z)^{1/2}) = 0$ for any $z \in \partial\Omega_\delta^\diamond \smallsetminus \{u_\delta^\diamond\}$,
- $F_\delta(v_\delta^\diamond) = 1$.

The third condition is guaranteed by the normalization. The second condition follows from considerations close to the case of the FK-Ising model. Indeed, any interface ending at $x_\delta^\diamond \in \partial\Omega_\delta^\diamond$ has the same winding denoted by $W(x_\delta^\diamond)$ (once again, no curve can wind around the boundary before arriving at a boundary vertex). Since this winding does not depend on the loop-configuration in $\widehat{\mathcal{E}}_\Omega(u_\delta^\diamond, x_\delta^\diamond)$, the complex argument of the observable is equal to $-\frac{1}{2}[W(x_\delta^\diamond) - W(v_\delta^\diamond)]$ modulo π. Now, the flatness condition implies that $v_\delta^\diamond$ is the medical-vertex south of a boundary vertex. The second condition then follows from the fact that $\nu(x_\delta^\diamond)^{-1/2} = \mathrm{e}^{-\mathrm{i}\frac{1}{2}[W(x_\delta^\diamond)-W(v_\delta^\diamond)]}$ in this case.

Let us now prove the first condition. Let x and y be two adjacent medial-vertices connected by the medial-edge $e = [xy]$. Let B be the vertex of Ω_δ bordering the medial-edge e. Let us further assume that x and y are the vertices respectively south and east of B (other cases may be handled similarly). Set x_ω and y_ω for the contribution of ω to $F_\delta(x)$ and $F_\delta(y)$. We wish to prove that

$$\sum_\omega P_{\ell(e)}(x_\omega) = \sum_\omega P_{\ell(e)}(y_\omega). \tag{9.3}$$

The curve $\gamma(\omega)$ finishes at x_ω or at y_ω so that ω cannot contribute to both $F_\delta(x)$ and $F_\delta(y)$ at the same time. Thus, it is sufficient to partition the set of configurations into pairs of configurations (ω, ω'), one contributing to x, the other one to y, such that $P_{\ell(e)}(x_\omega) = P_{\ell(e)}(y_{\omega'})$.

There are six types of pairs that one can create depending on what $\gamma(\omega)$ does, where $\gamma(\omega)$ denotes *any* non-self-crossing path from $u_\delta^\diamond$ to x. In each case of the following list, there is a trivial way of associating configurations in the two sets; see Fig. 9.1 for more details. Cases C1a and C1b (resp. C3a and C3b) can be obtained from each other by exchanging the roles of x and y.

C1a
- $\gamma(\omega)$ reaches x before B and stops, and there is a loop going through B but not through y.
- $\gamma(\omega)$ reaches x and then y in one step (more precisely there is a choice of $\gamma(\omega)$ doing so).

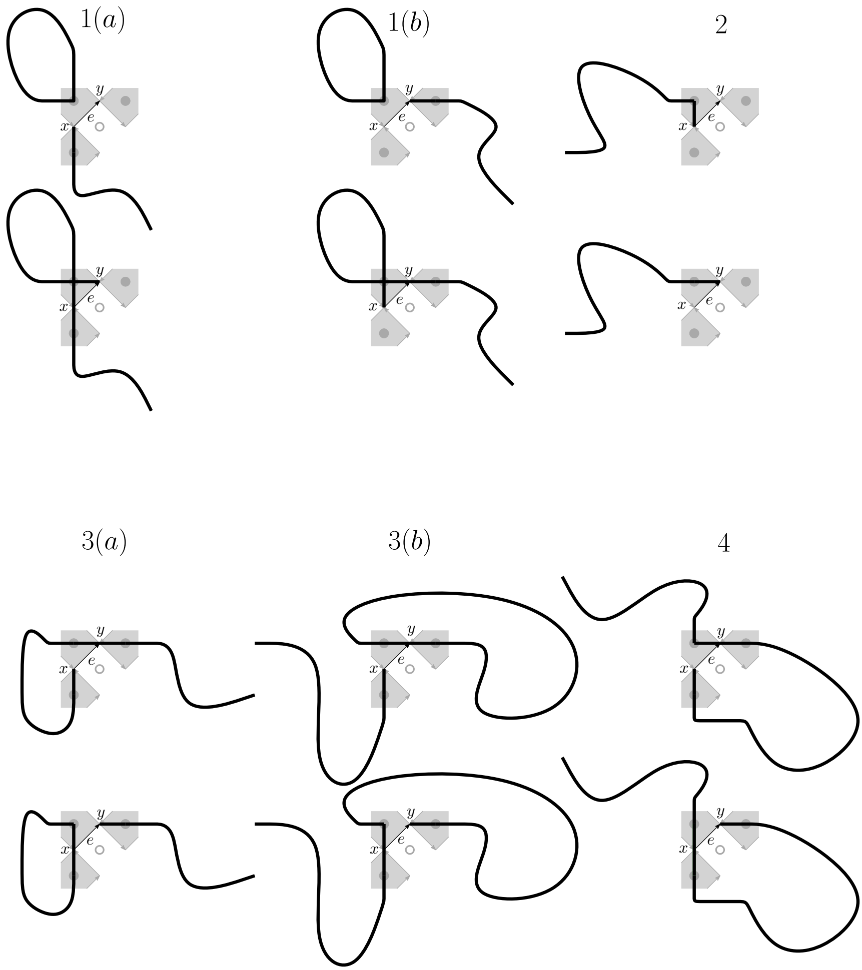

Figure 9.1: The different possible cases in the proof of Theorem 9.5: ω is depicted at the top, and ω' at the bottom of each case.

C1b – $\gamma(\omega)$ reaches y before B and stops, and there is a loop going through B but not through x.
– $\gamma(\omega)$ reaches y and then x in one step (more precisely there is a choice of $\gamma(\omega)$ doing so).

C2 – $\gamma(\omega)$ reaches B first and then makes a half-step to finish at x.
– $\gamma(\omega)$ reaches B first and then makes a half-step to finish at y.

C3a – $\gamma(\omega)$ reaches y before B and stops, and there is a loop in ω passing by B and x.

– $\gamma(\omega)$ reaches y, B, then leaves before coming back to x.

C3b – $\gamma(\omega)$ reaches x before B and stops, and there is a loop in ω passing by B and y.
– $\gamma(\omega)$ reaches x, B, then leaves before coming back to y.

C4 – $\gamma(\omega)$ reaches B before reaching x and then goes around before coming back to y.
– $\gamma(\omega)$ reaches B before reaching y and then goes around before coming back to x.

Now, the formulæ for x_ω and $y_{\omega'}$ enable us to express $y_{\omega'}$ in terms of x_ω in each case (very much in the same way as for random-cluster models). We obtain the following table for $y_{\omega'}$ in terms of x_ω. Moreover, the argument modulo π of contributions x_ω is known in each case since the orientation of e is known and $v_\delta^\diamond$ is a boundary medial-vertex south of a vertex in the domain. When projecting on $e^{-i\pi/8}\mathbb{R}$, the result follows (we use that $\tan(\pi/8) = \sqrt{2} - 1$ in Cases 1(a) and 1(b)).

config.	Case 1(a)	Case 1(b)	Case 2	Case 3(a)	Case 3(b)	Case 4
$y_{\omega'}$	$\frac{\sqrt{2}-1}{e^{-i\pi/4}}x_\omega$	$\frac{e^{i\pi/4}}{\sqrt{2}-1}x_\omega$	$e^{-i\pi/4}x_\omega$	$e^{3i\pi/4}x_\omega$	$e^{3i\pi/4}x_\omega$	$e^{-5i\pi/4}x_\omega$
$\arg x_\omega$ mod π	$\pi/2$	0	0	$\pi/2$	$\pi/2$	$\pi/2$

□

9.2 Conformal invariance of interfaces

Let us now discuss conformal invariance of interfaces. Before starting proving the convergence of interfaces in the FK-Ising and the Ising models, we would like to identify a family of curves which would be a natural candidate for the scaling limit of such interfaces.

Recall that a *domain* is a simply connected open set not equal to $\mathbb{C}$. We will consider families of random curves $(\Gamma_{(\Omega,a,b)})$ indexed by domains with two marked points a and b on their boundaries. The curves $\Gamma_{(\Omega,a,b)}$ are random non-self-crossing continuous curves in $\overline{\Omega}$ parametrized in such a way that they start at a and end at b. Recall the following two notions.

Definition 9.6 (Conformal invariance)**.** A family of random non-self-crossing continuous curves $\gamma_{(\Omega,a,b)}$, going from a to b and contained in Ω, indexed by simply connected domains with two marked points on the boundary (Ω, a, b) is *conformally invariant* if for any (Ω, a, b) and any conformal map $\psi : \Omega \to \mathbb{C}$,

$$\psi(\gamma_{(\Omega,a,b)}) \text{ has the same law as } \gamma_{(\psi(\Omega),\psi(a),\psi(b))}.$$

The domain Markov property (for random-cluster models) and the DLR condition (for Ising or other spin models) imply that interfaces in lattice models naturally satisfy the following property.

Definition 9.7 (Domain Markov property)**.** A family of random continuous curves $\Gamma_{(\Omega,a,b)}$, going from a to b and contained in Ω, indexed by simply connected domains with two marked points on the boundary (Ω, a, b), satisfies the domain Markov property if for every (Ω, a, b) and every $t > 0$, the law of the curve $\Gamma_{(\Omega,a,b)}[t, \infty)$ conditionally on $\Gamma_{(\Omega,a,b)}[0, t]$ is the same as the law of $\Gamma_{(\Omega_t,\Gamma_t,b)}$, where Ω_t is the connected component of $\Omega \setminus \Gamma[0, t]$ having b on its boundary.

As discussed in the introduction, Schramm proposed a natural candidate for the possible conformally invariant families of non-self-crossing continuous curves satisfying the domain Markov property, called the Schramm-Loewner Evolution (SLE). These random curves are indexed by a parameter $\kappa \geq 0$ and we usually write SLE(κ) for the Schramm-Loewner Evolution with parameter κ.

Remark 9.8. There exist different kinds of SLE. The curve can go from boundary point to boundary point (in this case it is called *chordal*) or from a boundary point to a point inside the domain (in this case it is called *radial*). The curve can also have driving points (we then speak of SLE($\kappa, \overline{\rho}$)). In this book, we will only deal with chordal SLEs.

One of the first and most fundamental models for which convergence to SLE is known is site percolation on the triangular lattice [Smi01, Smi05, CN07] (in such case it converges to SLE(6)). In [LSW11], loop-erased random walks were shown to converge to SLE(2). In [SS05], an *ad-hoc* model, called the harmonic explorer, was shown to converge to SLE(4).

In the next sections, we will show that the interfaces of the FK-Ising and Ising model converge to SLE(16/3) and SLE(3) respectively but let us start first by briefly describing SLEs.

9.2.1 A crash-course on Schramm-Loewner Evolution

In this section, several non-trivial concepts about Loewner chains are used and we refer to [Law05] for details. We do not aim for completeness (see [Law05, Wer04, Wer05] for deeper expositions). We simply introduce notions needed in the next sections. We first explain how a curve between two points on the boundary of a domain can be encoded via a real function, called the driving process. We then explain how the procedure can be reversed. Finally, we define the Schramm–Loewner Evolution.

From curves in domains to the driving process. Set $\mathbb{H}$ to be the upper half-plane $\mathbb{R}\times(0,\infty)$. Fix a compact set $K\subset\overline{\mathbb{H}}$ such that $H=\mathbb{H}\smallsetminus K$ is simply connected. Riemann's mapping theorem guarantees the existence of a conformal map from H onto $\mathbb{H}$. Moreover, there are *a priori* three real degrees of freedom in the choice of the conformal map, so that it is possible to fix its asymptotic behavior as z goes to ∞. Let g_K be the *unique* conformal map from H onto $\mathbb{H}$ such that

$$g_K(z) \ := \ z+\frac{C}{z}+O\Big(\frac{1}{z^2}\Big).$$

The proof of the existence of this map is not completely obvious and requires Schwarz's reflection principle. The constant C is called the *h-capacity* of K (it acts like a capacity: it is increasing in K and the h-capacity of λK is λ^2 times the h-capacity of K).

There is a natural way to parametrize certain continuous non-self-crossing curves $\Gamma:\mathbb{R}_+\to\overline{\mathbb{H}}$ with $\Gamma(0)=0$ and with $\Gamma(s)$ going to ∞ when $s\to\infty$. For every s, let H_s be the connected component of $\mathbb{H}\smallsetminus\Gamma[0,s]$ containing ∞. We denote by K_s the *hull created by* $\Gamma[0,s]$, *i.e.* the compact set $\overline{\mathbb{H}}\smallsetminus H_s$. By construction, K_s has a certain h-capacity C_s. The continuity of the curve guarantees that C_s grows continuously, so that it is possible to parametrize the curve via a time-change $s(t)$ in such a way that $C_{s(t)}=2t$. This parametrization is called the *h-capacity parametrization.*

From now on, we will assume that the parametrization is the h-capacity, and reflect this by using the letter t for the time parameter from now on.

Remark 9.9. In general, the previous time change is not a proper parametrization. For instance, the h-capacity is not necessarily increasing since any part of the curve "hidden from ∞" will not make the h-capacity grow. It might also be the case that t does not go to infinity along the curve (e.g. if Γ "crawls" along the boundary of the domain).

With this notation, the curve can be encoded via the family of conformal maps $g_t:=g_{K_t}$ from H_t to $\mathbb{H}$, in such a way that

$$g_t(z) \ := \ z+\frac{2t}{z}+O\Big(\frac{1}{z^2}\Big).$$

Under mild conditions, the infinitesimal evolution of the family (g_t) implies the existence of a continuous real valued function W_t such that for every t and $z\in H_t$,

$$\partial_t g_t(z) \ = \ \frac{2}{g_t(z)-W_t}. \tag{9.4}$$

The function W_t is called the *driving function* of Γ. The typical required hypothesis for W to be well-defined is the following *Local Growth Condition*:

For any $t \geq 0$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $0 \leq s \leq t$, the diameter of $g_s(K_{s+\delta} \setminus K_s)$ is smaller than ε.

This condition is always satisfied in the case of continuous curves (in general, Loewner chains can be defined for families of growing hulls, see [Law05] for additional details).

From a driving function to curves. It is important to notice that the procedure to obtain W from γ is reversible under mild assumptions on the driving function: if a continuous function $(W_t)_{t>0}$ is given, it is possible to reconstruct H_t as the set of points z for which the differential equation (9.4) with initial condition z admits a solution defined on $[0,t]$. We then set $K_t = \overline{\mathbb{H}} \setminus H_t$. The family of hulls $(K_t)_{t>0}$ is said to be the *Loewner Evolution with driving function* $(W_t)_{t>0}$.

So far, we did not refer to any curve in this reverse construction. If there exists a parametrized curve $(\Gamma_t)_{t>0}$ such that for any $t > 0$, H_t is the connected component of $\mathbb{H} \setminus \Gamma[0,t]$ containing ∞, the Loewner chain $(K_t)_{t>0}$ is said to be *generated by a curve.* In such case, $(\Gamma(t))_{t>0}$ is called the *trace* of $(K_t)_{t>0}$.

A general necessary and sufficient condition for a parametrized family of growing hulls in (Ω, a, b) to be the time-change of the trace of a Loewner chain is:
(C1) Its h-capacity is continuous;
(C2) Its h-capacity is strictly increasing;
(C3) The hull satisfies the Local Growth Condition.

The Schramm–Loewner Evolution. We are now in a position to define Schramm–Loewner Evolutions:

Definition 9.10 (SLE in the upper half-plane)**.** The Schramm–Loewner Evolution in $\mathbb{H}$ with parameter $\kappa > 0$ is the (random) Loewner chain with driving process $W_t := \sqrt{\kappa} B_t$, where B_t is a standard Brownian motion.

Loewner chains in other domains are defined using conformal maps.

Definition 9.11 (SLE in a general domain)**.** Fix a domain Ω with two points a and b on the boundary and assume it has a nice boundary (for instance a Jordan curve). The Schramm–Loewner evolution with parameter $\kappa > 0$ in (Ω, a, b) is the image of the Schramm–Loewner evolution in the upper half-plane by a conformal map[1] from $\mathbb{H}$ onto Ω mapping 0 to a and ∞ to b.

[1]The scaling properties of Brownian motion ensure that the definition does not depend on the choice of the conformal map involved; equivalently, the definition is consistent in the case $\Omega = \mathbb{H}$.

Defined as such, SLE is only a random family of growing hulls, but the Loewner chain can in fact be shown to be generated by a curve (see [RS05] for $\kappa \neq 8$ and [LSW11] for $\kappa = 8$).

Let us conclude this section by mentioning the following result from Schramm [Sch00] which justifies why SLE traces arise in planar statistical physics. Interfaces in lattice models usually satisfy the domain Markov property, due for instance to the DLR condition or the domain Markov property of the lattice model itself. Therefore, the following result justifies SLE processes as the only natural candidates[2] for such scaling limits.

Theorem 9.12 (Schramm [Sch00]). *Every family of random curves $\Gamma_{(\Omega,a,b)}$ which*

- *is conformally invariant,*
- *satisfies the domain Markov property,*
- *satisfies that $\Gamma_{(\mathbb{H},0,\infty)}$ is scale invariant,*

is the trace of a Schramm–Loewner evolution with a certain parameter $\kappa \in [0,\infty)$.

Remark 9.13. It is formally not necessary to assume scale invariance of the curve in the case of the upper-half plane, because it can be seen as a particular case of conformal invariance; we keep it nevertheless in the previous statement because it is potentially easier, while still informative, to prove.

9.2.2 Statements of conformal invariance for FK-Ising interfaces

We are now in a position to state the theorem of conformal invariance for interfaces of the FK-Ising and Ising models. Let us start by the former. Let X be the set of continuous parametrized curves and d be the distance on X defined for $\gamma_1 : I \to \mathbb{C}$ and $\gamma_2 : J \to \mathbb{C}$ by

$$d(\gamma_1,\gamma_2) = \min_{\substack{\varphi_1:[0,1]\to I\\ \varphi_2:[0,1]\to J}} \sup_{t\in[0,1]} |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|,$$

where the minimization is over *increasing bijective functions* φ_1 and φ_2. Note that I and J can be equal to $\mathbb{R}_+ \cup \{\infty\}$. The topology on (X,d) gives rise to a notion of weak convergence for random curves on X.

Recall that we are at criticality, i.e. $p = p_c(2)$.

[2]In fact, this is a bit of an exaggeration: some variants of SLE can also rise as scaling limits of interfaces.

Theorem 9.14 (Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov [CDCH⁺13])**.** *Let Ω be a simply connected domain with two marked points a and b on its boundary. Let $(\Omega_\delta, a_\delta, b_\delta)$ be a family of Dobrushin domains converging to (Ω, a, b) in the Carathéodory sense. The exploration path γ_δ of the* critical *FK-Ising model with Dobrushin boundary conditions in $(\Omega_\delta, a_\delta, b_\delta)$ converges weakly to SLE(16/3) as $\delta \to 0$.*

The strategy to prove that a family of parametrized curves converges to SLE(κ) follows two steps:

- First, we prove that the family (γ_δ) is tight for the weak convergence.
- Second, we identify the possible sub-sequential limits.

This second step is based on the fermionic observable. More precisely, imagine for a moment that a sub-sequential limit γ can be parametrized by a Loewner chain, and that its driving process is given by W. We will show that the fermionic observable may be seen as a martingale for the exploration process, a fact which implies that its limit is a martingale for γ. This martingale property, together with Itô's formula, will allow us to prove that W_t and $W_t^2 - \kappa t$ are martingales (where κ equals 16/3 for the FK-Ising model). Lévy's theorem thus implies that $W_t = \sqrt{\kappa} B_t$. This identifies SLE(κ) as being the only possible sub-sequential limit, which proves that (γ_δ) converges to SLE(κ).

In order to apply the second step it is crucial to have a strong notion of tightness in the first step to ensure that any sub-sequential limit can be parametrized by its h-capacity in such a way that a driving process is well-defined and continuous. This first step can be performed for any *critical* random-cluster model with $1 \le q \le 4$ and is based on Property **P5** of Corollary 6.16. We present it here.

9.2.3 Sub-sequential limits of critical random-cluster interfaces are Loewner chains for $1 \le q \le 4$

Since the result is valid for any random-cluster model with $q \in [1, 4]$, we state it in this degree of generality. *One may skip this section altogether and simply assume that the following informal statement is known.* We will consider convergence for the metric space (X, d) and therefore tightness with respect to this topology.

Pseudo-theorem. The family of exploration paths (γ_δ) is tight and any sub-sequential limit can be properly parametrized as a Loewner chain (generated by a curve) with continuous driving process.

The precise version of this pseudo-theorem is the following much more technical result. We include it for completeness but this may be skipped in a first reading. The proof itself is based on the strong form of the Russo-Seymour-Welsh theorem proved in the previous chapters and on a highly non-trivial result of Kemppainen and Smirnov [KS12].

Theorem 9.15 (Duminil-Copin, Sidoravicius, Tassion [DCST13]). *Fix $1 \le q \le 4$, $p = p_c(q)$ and a simply connected domain Ω with two marked points on its boundary a and b. Let $(\Omega_\delta, a_\delta, b_\delta)$ be a sequence of Dobrushin domains converging in the Carathéodory sense towards (Ω, a, b). Define γ_δ to be the exploration path in $(\Omega_\delta, a_\delta, b_\delta)$ with Dobrushin boundary conditions. Then, the family (γ_δ) is tight and any sub-sequential limit γ satisfies the following properties:*

R1 *γ is almost surely a continuous non-intersecting curve from a to b staying in $\overline{\Omega}$.*

R2 *For any parametrization $\gamma : [0,1] \to \overline{\mathbb{R}}_+$, b is a simple point, in the sense that $\gamma(t) = b$ implies $t = 1$. Furthermore, almost surely $\gamma(t)$ is on the boundary of $\Omega \smallsetminus \gamma[0,t]$ for any $t \in [0,1]$.*

R3 *Let Φ be a conformal map from Ω to the upper half-plane $\mathbb{H}$ sending a to 0 and b to ∞. For any parametrization $\gamma : [0,1] \to \overline{\mathbb{R}}_+$, the h-capacity of the hull $\widehat{K}_s$ of $\Phi(\gamma[0,s])$ tends to ∞ when s approaches 1. Furthermore, if $(\widehat{K}_t)_{t\ge 0}$ denotes $(\widehat{K}_s)_{s\in[0,1]}$ parametrized by h-capacity, then $(\widehat{K}_t)_{t\ge 0}$ is a Loewner chain with a driving process $(W_t)_{t\ge 0}$ which is α-Hölder for any $\alpha < 1/2$ almost surely. Furthermore, there exists $\varepsilon > 0$ such that for any $t > 0$, $\mathbb{E}[\exp(\varepsilon W_t/\sqrt{t})] < \infty$.*

Condition R3 guarantees that any sub-sequential limit can be parametrized by its h-capacity and is obtained by the Loewner Evolution with a certain continuous driving process. It also shows that the convergence of driving processes is equivalent to the weak convergence in (X, d). Finally, R2 guarantees that the Loewner chain is generated by a curve.

Proof. In order to prove Theorem 9.15, [KS12] shows that we only need to check the condition **G2** defined now. Consider a fixed domain (Ω, a, b) and a parametrized continuous curve Γ from a to b in Ω. A connected set C is said to disconnect $\Gamma(t)$ from b if it disconnects a neighborhood of $\Gamma(t)$ from a neighborhood of b in $\Omega \smallsetminus \Gamma[0,t]$.

Fig. 9.2 will help the reader here. For any annulus $A = A(z, r, R) := z + (\Lambda_R \smallsetminus \Lambda_r)$, let A_t be the subset of Ω satisfying $A_t := \varnothing$ if $\partial(z + \Lambda_r) \cap \partial(\Omega \smallsetminus \Gamma[0,t]) = \varnothing$, and otherwise

$$A_t := \left\{ \begin{array}{l} z \in A \smallsetminus \Gamma[0,t] \text{ such that the connected component of } z \\ \text{in } A \smallsetminus \Gamma[0,t] \text{ does not disconnect } \Gamma(t) \text{ from } b \text{ in } \Omega \smallsetminus \Gamma[0,t] \end{array} \right\}.$$

Consider the exploration path γ_δ as a continuous curve from $a_\delta^\diamond$ to $b_\delta^\diamond$ parametrized in such a way that it goes along one medial vertex in time

1 (in particular, after time n the path explored n medial-vertices). For simplicity, once the path reaches $b_\delta^\diamond$, it remains at $b_\delta^\diamond$ for any subsequent time (and therefore γ_δ is a curve parametrized by positive real numbers).

Condition G2. *There exists $C<1$ such that for any (γ_δ) in $(\Omega_\delta, a_\delta, b_\delta)$, for any stopping time τ and any annulus $A = A(z,r,R)$ with $0 < Cr < R$,*

$$\phi_{p_c(q),q,\Omega_\delta}^{a_\delta,b_\delta}\Big(\gamma_\delta[\tau,\infty] \text{ makes a crossing of } A \text{ contained in } A_\tau \Big| \gamma_\delta[0,\tau]\Big) < \tfrac{1}{2}$$

where a crossing is a portion of the path connecting the inner and outer parts of A.

We now prove this condition. Let $A(z,r,R)$ and A_τ as defined above. We can fix a realization of $\gamma_\delta[0,\tau]$, and work in the slit Dobrushin domain $(\Omega_\delta \setminus \gamma_\delta[0,\tau], c_\delta, b_\delta)$ (see the next paragraph for a precise definition of a slit domain[3]).

See A_τ as the union of connected components of the Dobrushin domain $\Omega_\delta^\diamond$ seen as an open domain of $\mathbb{R}^2$ minus the path $\gamma_\delta[0,\tau]$. We denote generically a connected component by $\mathcal{C}$ (we see it as a subset of $\mathbb{R}^2$).

The connected components can be divided into three classes:

- $\partial\mathcal{C}$ intersects both $\partial^\diamond_{c_\delta^\diamond b_\delta^\diamond}$ and $\partial^\diamond_{b_\delta^\diamond c_\delta^\diamond}$;
- $\partial\mathcal{C}$ intersects $\partial^\diamond_{b_\delta^\diamond c_\delta^\diamond}$ but not $\partial^\diamond_{c_\delta^\diamond b_\delta^\diamond}$;
- $\partial\mathcal{C}$ intersects $\partial^\diamond_{c_\delta^\diamond b_\delta^\diamond}$ but not $\partial^\diamond_{b_\delta^\diamond c_\delta^\diamond}$.

In fact, there cannot be any connected component of the first type. Indeed, let us assume that such a connected component $\mathcal{C}$ does exist. Let γ be a self-avoiding path in $\mathcal{C}$ going from $\partial^\diamond_{b_\delta^\diamond c_\delta^\diamond}$ to $\partial^\diamond_{c_\delta^\diamond b_\delta^\diamond}$. Topologically, $c_\delta^\diamond$ and $b_\delta^\diamond$ must be on two different sides of γ in $\Omega_\tau^\diamond \setminus \gamma$. But this means that $\mathcal{C}$ disconnects $c_\delta^\diamond(=\gamma_\delta(\tau))$ from b_δ, and therefore that $\mathcal{C}$ is not part of A_τ, which is contradictory.

We can therefore safely assume that the connected components are either of the second or third types. We now come back to the interpretation in terms of graphs.

Let S be the subgraph of Ω_τ given by the union of the connected components (seen as primal graphs this time) of the second type (see Fig. 9.2). This set is a subset of A_τ. Furthermore, the boundary conditions induced by the conditioning on $\gamma_\delta[0,\tau]$ are wired on $\partial S \setminus \partial A_\tau$. Therefore, conditioned on $\gamma_\delta[0,\tau]$ and the configuration outside $A(z,r,R)$, the configuration ω in S stochastically dominates $\omega'_{|S}$, where ω' follows the law of a random-cluster model in $A(z,r,R)$ with free boundary conditions (we faced similar argument before in this book and we therefore omit the

[3] We apologize for sending the reader to a forthcoming paragraph, but some readers will have skipped this part of the book and we preferred to define this object in a more central part of the book.

details). In particular, if there exists an open circuit in ω' surrounding $z + \Lambda_r$ in $A(z,r,R)$, then the restriction of this path to S is also open in ω and it disconnects $z + \partial\Lambda_r$ from $z + \partial\Lambda_R$ in S. In particular, the exploration path $\gamma_\delta[\tau,\infty]$ cannot cross A_τ inside S since this would require the existence of a dual-open path from the outer to the inner part of A_τ.

Property **P5'** of Corollary 6.16 implies that this open circuit exists in ω' with probability larger than a constant $c > 0$ not depending on δ, and that therefore $\gamma_\delta[\tau,\infty]$ cannot cross A_τ inside S with probability larger than c uniformly on the configuration outside A_τ.

Let now $S^\star$ be the subgraph of $\Omega_\tau^\star$ given by the union of the connected components (seen as dual graphs) of the third type. The same reasoning for the dual model implies that with probability $c > 0$, the exploration path $\gamma_\delta[\tau,\infty]$ cannot cross A_τ inside $S^\star$.

Altogether, $\gamma_\delta[\tau,\infty]$ cannot cross A_τ with probability c^2. Now, Proposition 5.33 shows that c can be taken to be equal to $1-(1-c_2)^{\lfloor \log_2(R/r)\rfloor}$. Since $R/r \geq C$, we can guarantee that $c^2 \geq 1/2$ by choosing C large enough. □

9.2.4 Convergence of FK-Ising interfaces to SLE(16/3)

We are now ready to implement the last step of our program and prove Theorem 9.14.

Let us first start by proving that the observable may be seen as a martingale. In order to do so, let us introduce the notion of *slit domain.* Fig. 9.2 may give a good idea of what it is.

Fix a Dobrushin domain $(\Omega_\delta, a_\delta, b_\delta)$ and consider the exploration path γ_δ in the loop representation on Ω_δ. The path γ_δ can be seen as a random parametrized curve (the parametrization being simply given by the number of steps along the curve between a medial-vertex in γ_δ and $a_\delta^\diamond$).

Definition 9.16. The *slit domain* $\Omega_\delta \smallsetminus \gamma_\delta[0,n]$ is defined as the subdomain of Ω_δ constructed by removing all the primal edges crossed by $\gamma_\delta[0,n]$ and by keeping only the connected component of the remaining graph containing b_δ. It is seen as a Dobrushin domain by fixing the points c_δ and b_δ, where c_δ is the vertex of $\delta\mathbb{Z}^2$ bordered by the last medial edge of $\gamma_\delta[0,n]$.

One may associate a dual Dobrushin domain to $\Omega_\delta \smallsetminus \gamma_\delta[0,n]$. The marked point is then $c_\delta^\star$, where $c_\delta^\star$ is the dual-vertex of $(\delta\mathbb{Z}^2)^\star$ bordered by the last medial edge of $\gamma_\delta[0,n]$. It is worth mentioning that the construction is symmetric for the dual Dobrushin domain: the dual of the slit domain $\Omega_\delta \smallsetminus \gamma[0,n]$ is simply the subgraph of $\Omega_\delta^\star$ obtained by removing the dual-edges crossed by the curve.

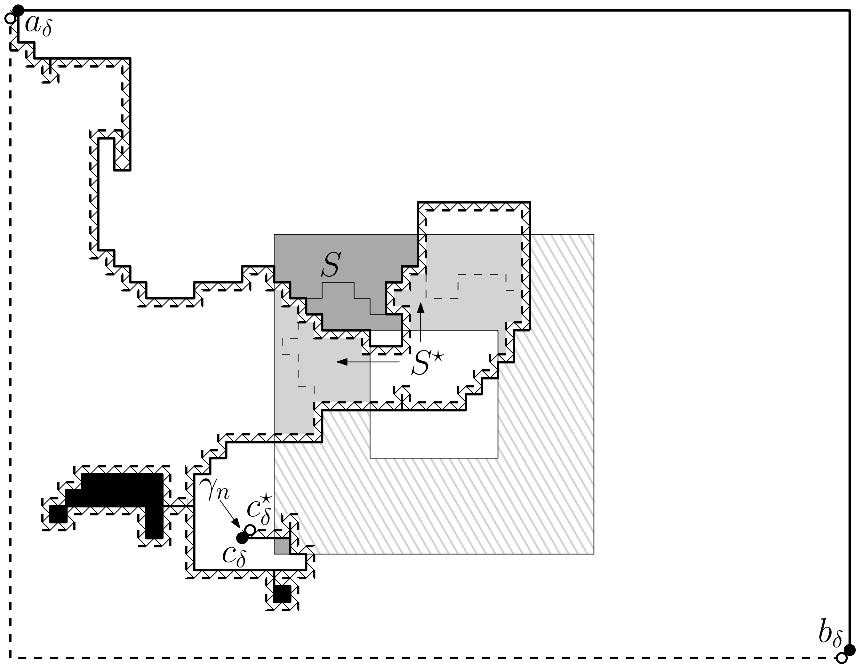

Figure 9.2: In this picture, t or τ are equal to n. The dashed area is a connected component of $A \setminus \gamma_\delta[0,n]$ which is disconnecting $\gamma_\delta(n)$ from $b_\delta^\diamond$, or equivalently c_δ from b_δ, and which is therefore not in A_τ. The black parts are not included in the slit domain since they correspond to connected components that are not containing $b_\delta^\diamond$. Conditioning on $\gamma_\delta[0,n]$ induces Dobrushin boundary conditions in the new domain. The dark grey area is S and the light-gray $S^\star$. We depicted a blocking open path in S and a dual-open path in each connected component of $S^\star$.

Remark 9.17. The notation $\Omega_\delta \setminus \gamma_\delta[0,n]$ could be somewhat misleading since Ω_δ is a subset of $\delta\mathbb{Z}^2$ and $\gamma_\delta[0,n]$ is a path of medial edges. Nevertheless, we allow ourselves some latitude here since we find this notation both concise and intuitive.

If one starts with Dobrushin boundary conditions on $(\Omega_\delta, a_\delta, b_\delta)$, then conditionally on $\gamma_\delta[0,n]$ the law of the configuration inside $\Omega_\delta \setminus \gamma_\delta[0,n]$ is a FK-Ising model with wired boundary conditions on $\partial_{b_\delta c_\delta}$ and free elsewhere. This comes from the fact that the exploration path $\gamma_\delta[0,n]$ "slides between open edges and dual-open dual-edges" and therefore the edges on its left must be open and the dual-edges on its right dual-open. This implies that the arc $\partial_{a_\delta c_\delta}$ must be wired (and therefore $\partial_{b_\delta c_\delta}$ is since $\partial_{b_\delta a_\delta}$ was already wired to start with) and the dual arc $\partial_{c^\star_\delta a^\star_\delta}$ is dual-wired.

Remark 9.18. Dobrushin domains of the type $(\Omega_\delta \setminus \gamma_\delta[0,n], a_\delta, b_\delta)$ have specific properties: $\partial^\diamond_{\gamma_\delta(n) b_\delta^\diamond}$ is self-touching on the right but not on the left and $\partial^\diamond_{b_\delta^\diamond \gamma_\delta(n)}$ is self-touching on the left but not on the right. In other words, all doubly-visited vertices on the medial boundary are pinched points and there is no doubly-visited vertex corresponding to two prime-ends.

We are now in a position to state the martingale-property of the observable.

Lemma 9.19. *Let $\delta > 0$. The random variable*

$$M_n^\delta(z) = \frac{1}{\sqrt{2e_b}} F_{\Omega_\delta \setminus \gamma_\delta[0,n], \gamma_\delta(n), b_\delta}(z)$$

is a martingale with respect to $(\mathcal{F}_n)$ where $\mathcal{F}_n$ is the σ-algebra generated by $\gamma_\delta[0,n]$.

Proof. The fact that conditionally on $\gamma_\delta[0,n]$, the law in the slit domain $\Omega_\delta \setminus \gamma_\delta[0,n]$ is a FK-Ising model with Dobrushin boundary conditions implies that $M_n^\delta(z)$ is the random variable $\frac{1}{\sqrt{2e_b}} \mathbf{1}_{z \in \gamma_\delta} e^{\frac{1}{2} i W_{\gamma_\delta}(z, b_\delta^\diamond)}$ conditionally on $\mathcal{F}_n$, therefore it is automatically a closed martingale. $\square$

Proposition 9.20. *Any sub-sequential limit of $(\gamma_\delta)_{\delta>0}$ which is a Loewner chain with has a continuous driving process (more precisely satisfying Property R3 of Theorem 9.15) is the Schramm-Loewner Evolution with parameter $\kappa = 16/3$.*

Proof. Consider a sub-sequential limit γ in the domain (Ω, a, b) which is a Loewner chain. Let ϕ be a map from (Ω, a, b) to $(\mathbb{H}, 0, \infty)$. Our goal is to prove that $\tilde{\gamma} = \phi(\gamma)$ is a SLE(16/3) in the upper half-plane.

Since γ is assumed to be a Loewner chain, $\tilde{\gamma}$ is a growing hull from 0 to ∞ parametrized by its h-capacity. Let W_t be its continuous driving process. Also define g_t to be the conformal map from $\mathbb{H} \setminus \tilde{\gamma}[0,t]$ to $\mathbb{H}$ such that $g_t(z) = z + 2t/z + O(1/z^2)$ when z goes to ∞.

Fix $z' \in \Omega$. For $\delta > 0$, recall that $M_n^\delta(z')$ is a martingale for γ_δ. Since the martingale is bounded, the stopping time theorem implies that $M_{\tau_t}^\delta(z')$ is a martingale with respect to $\mathcal{F}_{\tau_t}$, where τ_t is the first time at which $\phi(\gamma_\delta)$ has an h-capacity larger than t. Now, we use that $M_{\tau_t}^\delta(z')$ converges in the scaling limit thanks to Theorem 9.14. Since the convergence is uniform[4], $M_t(z') := \lim_{\delta\to 0} M_{\tau_t}^\delta(z')$ is a martingale with respect to $\mathcal{G}_t$, where $\mathcal{G}_t$ is

[4]One would have every right to be troubled by this claim. Hopefully, the doubts should vanish when reading the paragraph ten lines below.

the σ-algebra generated by the curve $\tilde{\gamma}$ up to the first time its h-capacity exceeds t. By definition, this time is t, and $\mathcal{G}_t$ is the σ-algebra generated by $\tilde{\gamma}[0,t]$.

Recall that $M_t(z')$ is related to $\phi(z')$ via the conformal map from $\mathbb{H} \setminus \tilde{\gamma}[0,t]$ to $\mathbb{R} \times (0,1)$, normalized to send $\tilde{\gamma}_t$ to $-\infty$ and ∞ to ∞. This last map is exactly $\frac{1}{\pi}\ln(g_t - W_t)$. Setting $z = \phi(z')$, we obtain that

$$\sqrt{\pi} M_t^z \;:=\; \sqrt{\pi} M_t(z') \;=\; \sqrt{[\ln(g_t(z) - W_t)]'} \;=\; \sqrt{\frac{g_t'(z)}{g_t(z) - W_t}} \tag{9.5}$$

is a martingale.

Formally, the previous reasoning is not quite rigorous. Indeed, in order to apply Theorem 9.14, one needs z' and the hull of $\gamma[0,\tau_t]$, or equivalently z and the hull of $\tilde{\gamma}[0,t]$ to be well apart. For this reason, we only obtain that $M_{t\wedge\sigma}^z$ is a martingale for $\mathcal{G}_{t\wedge\sigma}$, where σ is the hitting time of the boundary of the ball of size $R < |z|$ by the curve $\tilde{\gamma}$.

Recall that, when z goes to infinity,

$$g_t(z) \;=\; z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \quad \text{and} \quad g_t'(z) \;=\; 1 - \frac{2t}{z^2} + O\left(\frac{1}{z^3}\right). \tag{9.6}$$

Thus, for $s \le t$,

$$\begin{aligned}
&\sqrt{\pi}\cdot\mathbb{E}[M_{t\wedge\sigma}^z|\mathcal{G}_{s\wedge\sigma}]\\
&= \mathbb{E}\left[\sqrt{\frac{1 - 2(t\wedge\sigma)/z^2 + O(1/z^3)}{z - W_{t\wedge\sigma} + 2(t\wedge\sigma)/z + O(1/z^2)}} \;\Big|\; \mathcal{G}_{s\wedge\sigma}\right]\\
&= \frac{1}{\sqrt{z}}\;\mathbb{E}\left[1 + \frac{1}{2}W_{t\wedge\sigma}/z + \frac{1}{8}\left(3W_{t\wedge\sigma}^2 - 16(t\wedge\sigma)\right)/z^2 + O\left(1/z^3\right) \;\Big|\; \mathcal{G}_{s\wedge\sigma}\right]\\
&= \frac{1}{\sqrt{z}}\left(1 + \frac{1}{2}\mathbb{E}[W_{t\wedge\sigma}|\mathcal{G}_{s\wedge\sigma}]/z + \frac{1}{8}\mathbb{E}[3W_{t\wedge\sigma}^2 - 16(t\wedge\sigma)|\mathcal{G}_{s\wedge\sigma}]/z^2 + O\left(1/z^3\right)\right).
\end{aligned}$$

Taking $s = t$ yields

$$\sqrt{\pi}\cdot M_{s\wedge\sigma}^z \;=\; \frac{1}{\sqrt{z}}\left(1 + \frac{1}{2}W_{s\wedge\sigma}/z + \frac{1}{8}\Big(3W_{s\wedge\sigma}^2 - 16(s\wedge\sigma)\Big)/z^2 + O(1/z^3)\right).$$

Since $M_{t\wedge\sigma}^z$ is a martingale, $\mathbb{E}[M_{t\wedge\sigma}^z|\mathcal{G}_{s\wedge\sigma}] = M_{s\wedge\sigma}^z$. Therefore, terms in the previous asymptotic development can be matched together by letting z tend to infinity so that

$$\mathbb{E}[W_{t\wedge\sigma}|\mathcal{G}_{s\wedge\sigma}] = W_{s\wedge\sigma} \quad \text{and} \quad \mathbb{E}[W_{t\wedge\sigma}^2 - \tfrac{16}{3}(t\wedge\sigma)|\mathcal{G}_{s\wedge\sigma}] = W_{s\wedge\sigma}^2 - \frac{16}{3}(s\wedge\sigma).$$

Since the identification only uses the fact that z tends to infinity, one can now let R go to infinity (and thus σ go to infinity as well) to obtain

$$\mathbb{E}[W_t|\mathcal{G}_s] = W_s \quad \text{and} \quad \mathbb{E}[W_t^2 - \tfrac{16}{3}t|\mathcal{G}_s] = W_s^2 - \tfrac{16}{3}s.$$

(Note that the integrability condition $\mathbb{E}[\exp(\varepsilon W_t/\sqrt{t})] < \infty$ was necessary to justify passing to the limit here.) Since W_t is continuous, Lévy's theorem implies that $W_t = \sqrt{\frac{16}{3}} B_t$ where B_t is a standard Brownian motion.

In conclusion, γ is the image by ϕ^{-1} of the Schramm-Loewner Evolution with parameter $\kappa = 16/3$ in the upper half-plane. This is exactly the definition of the Schramm-Loewner Evolution with parameter $\kappa = 16/3$ in the domain (Ω, a, b). □

Proof of Theorem 9.14. Theorem 9.15 applied to $q = 2$ implies that the family (γ_δ) is tight for the weak convergence in (X, d) and that any sub-sequential limit can be parametrized by a Loewner chain with continuous driving process. Proposition 9.20 implies that any sub-sequential limit is SLE(16/3), and therefore the sequence (γ_δ) converges weakly to SLE(16/3). □

9.2.5 Convergence to SLE(3) for Ising interfaces

Let $\Omega_\delta^\diamond$ be a medial discrete domain with two boundary medial-vertices $u_\delta^\diamond$ and $v_\delta^\diamond$. We consider the critical Ising model on $\Omega_\delta^\star$ with Dobrushin boundary conditions defined as follows. As seen in Chapter 7 (one may look at Remark 7.29 and Fig. 7.2), the boundary $\partial\Omega_\delta^\star$ is naturally divided by $u_\delta^\diamond$ and $v_\delta^\diamond$ into two arcs ∂_- and ∂_+ when going along $\partial\Omega_\delta^\star$ counterclockwise[5]. Fix the spins of the vertices to be +1 on ∂_+ and −1 on ∂_-.

Now that we have the Ising measure, we define the *interface*. The path γ_δ is constructed as follows. It starts from $u_\delta^\diamond$, lies on the primal lattice and turns at every vertex of Ω_δ in such a way that it has always dual vertices with spin −1 on its left and +1 on its right. If there is an indetermination when arriving at a vertex (when going counterclockwise around this vertex, spins could be +1, −1, +1 and −1), turn left. The process thus obtained is the interface between spins +1 $\star$-connected[6] to ∂_+ and spins −1 connected to ∂_-.

Theorem 9.21 (Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov [CDCH+13]). *Let Ω be a simply connected domain with two marked points u and v on its boundary. We assume that the boundary of Ω is flat near v. Let $\Omega_\delta^\diamond$ be a family of discrete domains with $u_\delta^\diamond$ and $v_\delta^\diamond$ on its boundary. We assume that $(\Omega_\delta^\diamond, u_\delta^\diamond, v_\delta^\diamond)$ converges to (Ω, u, v) in the Carathéodory sense and that the boundary of $\Omega_\delta^\diamond$ is flat near $v_\delta^\diamond$. Then, the interface γ_δ of the critical Ising model on $\Omega_\delta^\star$ with Dobrushin boundary conditions converges weakly to SLE(3) in Ω from u to v.*

[5] We refer the reader to the construction (presented in Chapter 7) of Dobrushin boundary conditions on $\partial_e\Omega$ when the Ising model is considered on the primal lattice. This is the relevant adaptation to the dual graph.

[6] Once again, two vertices are $\star$-neighbors if they are at $\|\cdot\|_\infty$-distance 1.

Remark 9.22. The choice of turning left when there is an indetermination was arbitrary. One may equivalently turn right when there is an indetermination and obtain a different interface $\widehat{\gamma}_\delta$. This would correspond to the interface between spins 1 connected to ∂_+ and spins -1 $\star$-connected to ∂_-. In the scaling limit, $\widehat{\gamma}_\delta$ also converges weakly to SLE(3). Now, the SLE(3) is almost surely a simple curve (see e.g. [Law05]) and therefore in the scaling limit, γ_δ and $\widehat{\gamma}_\delta$ converge to the same parametrized curve. This yields the fact that in the scaling limit, the standard notion of connectivity and the $\star$-connectivity are equivalent for the Ising model.

An equivalent of Theorem 9.21 can also be proved using the same argument as in the proof of Theorem 9.14, except that the condition **G2** follows from the following result.

Proposition 9.23. *Let $\mathcal{E}_n$ be the event that there exists a circuit of vertices with spins +1 in the annulus $\Lambda_{8n} \smallsetminus \Lambda_n$ surrounding the origin. Then there exists $c > 0$ such that for any $n \geq 1$,*

$$\mu^-_{\beta_c, \Lambda_{8n} \smallsetminus \Lambda_n}\left[\mathcal{E}_n\right] \geq c.$$

Proof. We use the Edwards-Sokal coupling $\mathbf{P}$ between the Ising measure with -1 boundary conditions and the FK-Ising measure with wired boundary conditions obtained by assigning random cluster-spins to clusters except for the clusters touching $\partial\Lambda_n$ or $\partial\Lambda_{8n}$ which automatically receive cluster-spin -1.

Divide the annulus $\Lambda_{8n} \smallsetminus \Lambda_n$ into three annuli $A_1 = \Lambda_{2n} \smallsetminus \Lambda_n$, $A_2 = \Lambda_{4n} \smallsetminus \Lambda_{2n}$ and $A_3 = \Lambda_{8n} \smallsetminus \Lambda_{4n}$. Let $\mathcal{F}_n$ be the event that A_1 and A_3 contains dual-open dual-circuits surrounding the origin, and A_2 contains an open circuit surrounding the origin. Property **P5'** for crossing probabilities in the critical FK-Ising model shows the existence of $c_1 > 0$ such that

$$\phi^1_{p_c, 2, \Lambda_{8n} \smallsetminus \Lambda_n}[\mathcal{F}_n] \geq c_1$$

for every $n \geq 1$. Now, the Edwards-Sokal coupling guarantees that conditionally on a configuration in $\mathcal{F}_n$, the cluster of the primal circuit in A_2 receives spin +1 with probability 1/2 (since it is not connected to the boundary). This implies that

$$\mu^-_{\beta_c, \Lambda_{8n} \smallsetminus \Lambda_n}[\mathcal{E}_n] \geq c_1/2.$$

$\square$

We now turn our attention to the spin fermionic observable. We prove that it is a martingale for the curve. Before stating the lemma, let us write a few things in more detail. Assume we work in $(\Omega^\diamond_\delta, u^\diamond_\delta, v^\diamond_\delta)$. The curve γ_δ lives on Ω_δ, except the half-edges at its beginning and its end (therefore the starting and ending points are medial-vertices). Let us parametrize the curve in such a way that at time n, the curve contains n vertices on it (in such case it contains $n+1$ medial-vertices).

As before, we define *slit domains*. We work directly with medial discrete domains $\Omega_\delta^\diamond$ with two marked points $u_\delta^\diamond$ and $v_\delta^\diamond$ and a curve $\gamma_\delta[0,n]$. For such a domain, the slit domain $\Omega_\delta^\diamond \smallsetminus \gamma_\delta[0,n]$ is obtained in three steps:

1. Remove all the vertices of Ω_δ visited by $\gamma_\delta[0,n]$ and all the edges emanating from these vertices;
2. Take the medial graph of each connected component of the new graph;
3. Keep the connected component containing v_δ.

One may see by trying to construct slit domains on Fig. 7.2 that the definition corresponds to the intuition of removing the curve $\gamma_\delta[0,n]$.

Lemma 9.24. *Consider the critical Ising model on $(\Omega_\delta^\diamond, u_\delta^\diamond, v_\delta^\diamond)$ with Dobrushin boundary conditions. Let γ_δ be the interface defined above. For any $z \in \Omega_\delta^\diamond$, the spin fermionic observable $M_n^\delta(z) = F_{\Omega_\delta^\diamond \smallsetminus \gamma_\delta[0,n], \gamma_\delta(n), v_\delta^\diamond}(z)$ is a martingale with respect to $(\mathcal{F}_n)$, where $\mathcal{F}_n$ be the σ-algebra generated by $\gamma_\delta[0,n]$.*

The proof is slightly more intricate than for the FK-Ising model due to the lack of a direct interpretation of the observable in terms of the expectation of a random variable. For this reason, we need to work slightly more but the philosophy is still the same.

Proof. Let $\mu_{\beta_c}^{u,v}$ be the critical Ising measure with Dobrushin boundary conditions on $\Omega_\delta^\star$. In this proof, we work mostly on the medial lattice and we therefore drop the subscript δ and the superscript $\diamond$.

It is sufficient to check that $M_n(z)$ has the martingale property when $\gamma = \gamma(\omega)$ makes one step γ_1. In this case $\mathcal{F}_0$ is the trivial σ-algebra, so that we wish to prove that

$$\mu_{\beta_c}^{u,v}\left[F_{\Omega \smallsetminus [u\gamma_1], \gamma_1, v}(z)\right] = F_{\Omega, u, v}(z). \tag{9.7}$$

Set $\gamma_1 =: x$ and Z and Z_x for the partition functions of the Ising model with Dobrushin boundary conditions on $\Omega_\delta^\diamond$ with marked points u and v, and in $\Omega_\delta \smallsetminus [ux]$ with marked points x and v. The high-temperature expansion described in Chapter 7 implies that

$$\begin{aligned}Z\mu^{u,v}_{\beta_c}(\gamma_1=x) &= (\sqrt{2}-1)Z_x\\ &= (\sqrt{2}-1)e^{i\frac{1}{2}W_\gamma(x,v)}\frac{\sum_{\omega\in\widehat{\mathcal{E}}_{\Omega\smallsetminus[ux]}(x,z)} e^{-i\frac{1}{2}W_\gamma(x,z)}(\sqrt{2}-1)^{|\omega|}}{F_{\Omega\smallsetminus[ux],x,v}(z)}\\ &= e^{i\frac{1}{2}W_\gamma(u,v)}\frac{\sum_{\omega\in\widehat{\mathcal{E}}_{\Omega}(u,z)} e^{-i\frac{1}{2}W_\gamma(u,z)}(\sqrt{2}-1)^{|\omega|}1_{\{\gamma_1=x\}}}{F_{\Omega\smallsetminus[ux],x,v}(z)}.\end{aligned}$$

In the third equality, we used the fact that $\widehat{\mathcal{E}}_{\Omega\smallsetminus[ux]}(x,z)$ is in bijection with configurations of $\widehat{\mathcal{E}}_\Omega(u,z)$ such that $\gamma_1=x$. There is still a difference of weight of $\sqrt{2}-1$ between two associated configurations that explains the disappearance of $\sqrt{2}-1$ between the second and the third line. Thus,

$$\mu^{u,v}_{\beta_c}(\gamma_1=x)\,F_{\Omega\smallsetminus[ux],x,v}(z)=\frac{\sum_{\omega\in\mathcal{E}(u,z)} e^{-i\frac{1}{2}W_\gamma(u,z)}(\sqrt{2}-1)^{|\omega|}1_{\{\gamma_1=x\}}}{e^{-i\frac{1}{2}W_\gamma(u,v)}Z}.$$

The same holds for all possible first steps. Summing over all possibilities, we obtain (9.7) (on the right, we indeed obtain $F_{\Omega,u,v}(z)$, and on the left, the expectation of $F_{\Omega\smallsetminus[u\gamma_1],\gamma_1,v}(z)$). □

We then prove the equivalent of Proposition 9.20 by expanding $\sqrt{\psi'/\psi'(v)}$ instead of $\sqrt{\phi'}$. If a sub-sequential limit is a Loewner chain with continuous driving process W_t, the development implies that W_t and W_t^2-3t are martingales for the curve. This implies that $W_t=\sqrt{3}B_t$, and therefore this sub-sequential limit is SLE(3). The theorem follows.

Remark 9.25. Slit discrete domains $\Omega^\diamond_\delta\smallsetminus\gamma_\delta[0,n]$ have a specific structure: the boundary ∂ is self-touching on the left but not on the right. In other words, doubly-visited medial vertices are all corresponding to two prime ends and there is no pinched vertices.

9.3 The energy and spin fields

From a physics point of view, the scaling limit of the Ising model corresponds to a minimal model of Conformal Field Theory with central charge $c=1/2$. Let us discuss two so-called primary fields: the *energy field* and the *spin field*. The n-point correlations of these two fields were computed in the full plane in recent years (see [BdT10, BdT11] for the energy field, and [Dub11] for the spin field). We wish to discuss these fields in simply connected domains (in particular their conformal covariance structure). We focus on + boundary conditions, but free boundary conditions and more general boundary conditions can be treated (at the cost of additional technical difficulties in the latter).

9.3.1 Energy field

Before stating the theorem, let us recall what the Pfaffian of a matrix is. For an anti-symmetric matrix $\mathbf{A} \in M_{2n}(\mathbb{C})$, set

$$\mathrm{Pfaff}(\mathbf{A}) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} \mathbf{A}_{\sigma(2j-1),\sigma(2j)},$$

where S_{2n} is the set of permutations of the set $\{1,\dots,2n\}$, and $\mathrm{sgn}(\sigma)$ denotes the signature of the permutation σ. This definition makes sense for any matrix. In the case of complex valued entries, we can simply check that

$$\mathrm{Pfaff}(\mathbf{A})^2 = \det(\mathbf{A}).$$

For $2n$ distinct points $x_1,\dots,x_{2n}$ in $\mathbb{C}$, define the matrix

$$\mathbf{K}(x_1,x_2,\dots,x_{2n})_{ij} = \begin{cases} \frac{1}{x_j - x_i} & \text{if } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

The *energy density* at an edge $e = [xy]$ is given by the formula[7]

$$\varepsilon_e = \frac{\sqrt{2}}{2} - \sigma_x \sigma_y.$$

The value $\frac{\sqrt{2}}{2}$ is determined by the fact that in infinite volume, $\mu_{\beta_c}(\sigma_x\sigma_y) = \frac{\sqrt{2}}{2}$.

For $a \in \Omega$, let $e(a^\delta)$ be an edge having a^δ for an endpoint (there are a priori four edges like that, but the choice of the edge is irrelevant).

Theorem 9.26 (Hongler [Hon10a]). *Let Ω be a simply connected domain and $a_1,\dots,a_n \in \Omega$. Consider a sequence of simply connected domains Ω_δ with marked points $a_1^\delta,\dots,a_n^\delta$ converging to $(\Omega,a_1,\dots,a_n)$ in the Carathéodory sense*[8]*. Then,*

$$\lim_{\delta\to 0} \frac{1}{(\sqrt{2}\delta)^n} \mu^+_{\beta_c,\Omega_\delta}\left[\varepsilon_{e(a_1^\delta)} \cdots \varepsilon_{e(a_n^\delta)}\right] = \langle \varepsilon_{a_1} \cdots \varepsilon_{a_n} \rangle_\Omega,$$

where $\langle \varepsilon_{a_1} \cdots \varepsilon_{a_n} \rangle_\Omega$ satisfies

$$\langle \varepsilon_{a_1} \cdots \varepsilon_{a_n} \rangle_\Omega = |\phi'(a_1)| \cdots |\phi'(a_n)| \langle \varepsilon_{\phi(a_1)} \cdots \varepsilon_{\phi(a_n)} \rangle_{\phi(\Omega)}$$

[7] One may wonder why we do not defined ε_e as $\frac{\sqrt{2}}{2} + \sigma_x\sigma_y$ instead of $\frac{\sqrt{2}}{2} - \sigma_x\sigma_y$ to avoid some unpleasant signs: the reason is that we wish to keep the physical intuition of an energy.

[8] In this context, the Carathéodory convergence corresponds to the fact that $(\Omega_\delta, a_1^\delta)$ converges to (Ω, a_1), and $a_i^\delta \longrightarrow a_i$ for any $2 \le i \le n$.

for any simply connected domain Ω, any $a_1, \dots, a_n \in \Omega$, and any conformal map ϕ on Ω.

Furthermore, for the upper half-plane, we possess the following explicit formula:

$$\langle \varepsilon_{a_1} \cdots \varepsilon_{a_n} \rangle_{\mathbb{H}} := \tfrac{1}{(\pi i)^n} \cdot \mathrm{Pfaff}\big[\mathbf{K}(a_1, \dots, a_n, \overline{a_n}, \dots, \overline{a_1})\big],$$

where $\mathbf{K}$ is the matrix defined above.

Observe that the answer does not depend on the choice of the orientation of $e(a^\delta)$. The result can also be obtained for more general boundary conditions, and for points on the boundary of smooth domains. Let us mention that for $n = 1$, we find the following formula.

Corollary 9.27 (Hongler, Smirnov [HS11]). *Let Ω be a simply connected domain and $a \in \Omega$. Then*

$$\langle \varepsilon_a \rangle_\Omega = -\frac{\phi'(a)}{\pi},$$

where ϕ is the conformal map from Ω to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ sending a to 0 and such that $\phi'(a) > 0$.

We will not prove the theorem, but we will sketch the proof of Corollary 9.27. We refer to the original articles for more details. Our goal is to highlight the fact that the different steps of the proof are similar to the proof of conformal invariance of the spin fermionic observable.

Proof. For simplicity, the Ising model itself will lie on the dual graph and we are interested in the energy-density for the dual-edge $[xy]$ passing through the medial-vertex $a_\delta^\diamond$. We assume that the dual-edge is vertical.

So far, we considered observables depending on a point $u_\delta^\diamond$ on the boundary of a domain, but we could allow more flexibility and move $u_\delta^\diamond$ inside the domain: we define the fermionic observable $F_{\Omega_\delta^\diamond, a_\delta^\diamond}(z_\delta^\diamond)$ by

$$F_{\Omega_\delta^\diamond, a_\delta^\diamond}(z_\delta^\diamond) := \begin{cases} \dfrac{\sum\limits_{\omega \in \widehat{\mathcal{E}}_{\Omega_\delta}(a_\delta^\diamond, z_\delta^\diamond)} \mathrm{e}^{-\frac{1}{2} i W_{\gamma(\omega)}(a_\delta^\diamond, z_\delta^\diamond)} (\sqrt{2}-1)^{|\omega|}}{\sum\limits_{\omega \in \mathcal{E}_{\Omega_\delta}} (\sqrt{2}-1)^{|\omega|}} & \text{if } z_\delta^\diamond \neq a_\delta^\diamond, \\[2em] \dfrac{\sum\limits_{\omega \in \mathcal{E}_{\Omega_\delta} : e \notin \omega} (\sqrt{2}-1)^{|\omega|}}{\sum\limits_{\omega \in \mathcal{E}_{\Omega_\delta}} (\sqrt{2}-1)^{|\omega|}} & \text{if } z_\delta^\diamond = a_\delta^\diamond. \end{cases} \tag{9.8}$$

The edge e is the primal edge passing through $a_\delta^\diamond$. The choice of the value of the observable at $z_\delta^\diamond = a_\delta^\diamond$ will be justified by the next paragraphs.

The function $F_{\Omega_\delta^\diamond, a_\delta^\diamond}$ can be shown to be s-holomorphic on $\Omega_\delta \setminus \{a_\delta^\diamond\}$ (the argument is similar to the proof of Theorem 9.5). Now, by computing the local contributions of the function near $a_\delta^\diamond$, the function can be shown to have a singularity at $a_\delta^\diamond$ with discrete residue $\frac{1}{-2\pi \mathrm{i}}$. Furthermore, $-2\pi\mathrm{i}F_{\Omega_\delta^\diamond, a_\delta^\diamond}$ satisfies some Riemann-Hilbert boundary conditions and therefore $-2\pi\mathrm{i}F_{\Omega_\delta^\diamond, a_\delta^\diamond}$ satisfies $\mathbf{BVP}_3$.

Thus, Theorem 8.39 implies that $\frac{1}{-2\pi\mathrm{i}}F_{\Omega_\delta^\diamond, a_\delta^\diamond} - G_{\Omega_\delta^\diamond}(\cdot, a_\delta^\diamond)$ can be extended at $a_\delta^\diamond$ into a s-holomorphic function. Some tedious computations of the projections around the singularity show that we must set $G(x_\delta^\diamond, x_\delta^\diamond) = -\mathrm{i}(2+\sqrt{2})\pi/2$ in order to extend the function correctly.

The uniform convergence guaranteed by Theorem 8.39 applied to $z_\delta^\diamond = a_\delta^\diamond$ implies that

$$\frac{1}{\delta}\Big[F_{\Omega_\delta^\diamond, a_\delta^\diamond}(a_\delta^\diamond) - \frac{2+\sqrt{2}}{4}\Big] = \frac{1}{\delta}\Big[F_{\Omega_\delta^\diamond, a_\delta^\diamond}(a_\delta^\diamond) - \tfrac{1}{-2\pi\mathrm{i}}G_\delta(a_\delta^\diamond, a_\delta^\diamond)\Big] \longrightarrow \frac{1}{2\pi}\phi'(a).$$

We now translate this convergence into a result for the energy-density. The expectation of the energy-density[9] $\varepsilon(a_\delta^\diamond) := \frac{\sqrt{2}}{2} - \sigma_x\sigma_y$ can be expressed in terms of the existence or not of the edge e via the low-temperature expansion of the Ising model on the dual graph. Indeed, if $\sigma_x = \sigma_y$, the edge e separating the two vertices is not present in the low-temperature expansion and thus

$$\mu^+_{\beta_c,\Omega_\delta^\star}(\sigma_x = \sigma_y) = \frac{\displaystyle\sum_{\omega\in\mathcal{E}_{\Omega_\delta}: e\notin\omega} (\sqrt{2}-1)^{|\omega|}}{\displaystyle\sum_{\omega\in\mathcal{E}_{\Omega_\delta}} (\sqrt{2}-1)^{|\omega|}} = F_{\Omega_\delta^\diamond, a_\delta^\diamond}(a_\delta^\diamond)$$

which in turn implies that

$$\begin{aligned}\mu^+_{\beta_c,\Omega_\delta^\star}[\varepsilon(a_\delta^\diamond)] &= \frac{\sqrt{2}}{2} - \mu^+_{\beta_c,\Omega_\delta^\star}[\sigma_x\sigma_y] = \frac{\sqrt{2}}{2} - \big(2\mu_{\beta_c,\Omega_\delta^\star}(\sigma_x = \sigma_y) - 1\big)\\ &= -2\Big(F_{\Omega_\delta^\diamond, a_\delta^\diamond}(a_\delta^\diamond) - \frac{2+\sqrt{2}}{4}\Big).\end{aligned}$$

Hence, the previous convergence result implies the result. □

The general results follow after introducing a similar fermionic observable F_δ depending on $2n$ medial vertices $(a_1^\diamond)_\delta, (b_1^\diamond)_\delta, \dots, (a_n^\diamond)_\delta, (b_n^\diamond)_\delta$ which can be expressed in terms of

$$\mu^+_{\beta_c,\Omega_\delta^\star}[\varepsilon(a^\diamond_{1\,\delta})\cdots\varepsilon(a^\diamond_{n\,\delta})]$$

[9] Recall that in this proof the Ising model is defined on the dual graph so that the formula $\varepsilon(a_\delta^\diamond) = \frac{\sqrt{2}}{2} - \sigma_x\sigma_y$ corresponds exactly to the definition of the energy-density of the dual-edge passing through $a_\delta^\diamond$.

when taking $(a_{j\,\delta})^\diamond$ and $(b_{j\,\delta})^\diamond$ two medial vertices which are neighbors of a_j. The fermionic observable can then be proved to satisfy recursive relations (in terms of $a_1,\dots,a_n$) which are also satisfied by Pfaffians[10]. This implies that the n-point energy correlation also satisfy these relations. These recursive formulas allow a drastic simplification: it is in fact sufficient to treat the case $n = 2$. For this case, technicalities arise but the general philosophy is the same: one proves that the observable is solution of a discrete Riemann-Hilbert BVP. Its convergence in the scaling limit then follows from the convergence of solutions of a discrete Riemann-Hilbert BVP to its continuum counterpart.

9.3.2 Spin-spin correlations

In the past paragraph, we explored the case of the energy density of the Ising model. We now focus on other important quantities, namely the spin-spin correlations. Let $C = 2^{1/6}\exp[-\frac{3}{2}\zeta'(-1)]$ be a (lattice-dependent) constant.

Theorem 9.28 (Chelkak, Hongler, Izyurov [CHI12]). *Let Ω be a simply connected domain and $a_1,\dots,a_n \in \Omega$. Consider a sequence of simply connected domains Ω_δ with marked points $a_1^\delta,\dots,a_n^\delta$ converging to $(\Omega, a_1,\dots,a_n)$. Then,*

$$\lim_{\delta\to 0}\frac{1}{(\sqrt{2}\,\delta)^{n/8}}\mu^+_{\beta_c,\Omega_\delta}\big[\sigma_{a_1^\delta}\cdots\sigma_{a_n^\delta}\big] = C^k\langle\sigma_{a_1}\cdots\sigma_{a_n}\rangle_\Omega,$$

where $\langle\sigma_{a_1}\cdots\sigma_{a_n}\rangle_\Omega$ satisfies

$$\langle\sigma_{a_1}\cdots\sigma_{a_n}\rangle_\Omega = |\phi'(a_1)|^{1/8}\cdots|\phi'(a_n)|^{1/8}\langle\sigma_{\phi(a_1)}\cdots\sigma_{\phi(a_n)}\rangle_{\phi(\Omega)}$$

for any simply connected domain Ω, and $a_1,\dots,a_n \in \Omega$, and any conformal map ϕ on Ω.

The general form of $\langle - - \rangle_\Omega$ was predicted by means of Conformal Field Theory in [BG87]. The method of [CHI12] gives another formula (which is slightly less explicit). At the moment, there is no direct proof that the two formulas coincide though it can be checked in several situations. For instance, when $n = 1$ or 2, its value can be computed.

[10] Let us quote these formulas for completeness: For a matrix $\mathbf{A}$ in $M_{2n}(\mathbb{C})$ and $j,k \in \{1,\dots,2n\}$, let $\mathbf{A}^{jk}$ be the matrix of $M_{2n-2}(\mathbb{C})$ obtained by removing the j-th and k-th lines and columns. We have

$$\mathrm{Pfaff}(\mathbf{A}) = \sum_{j\geq 1}(-1)^j\mathbf{A}_{j1}\mathrm{Pfaff}(\mathbf{A}^{j1}).$$

Theorem 9.29 (Chelkak, Hongler, Izyurov [CHI12]). *Let Ω be a simply connected domain and $a, b \in \Omega$. Then,*

$$\langle \sigma_a \rangle_\Omega = \phi'(a)^{1/8} \quad \textit{and}$$
$$\langle \sigma_a \sigma_b \rangle_\Omega = \langle \sigma_a \rangle_\Omega \langle \sigma_b \rangle_\Omega \Big[1 - e^{-2d_\Omega(a,b)} \Big]^{-1/4},$$

where ϕ is the unique conformal map from Ω to the unit disk $\mathbb{D}$ with $\phi(a) = 0$ and $\phi'(a) > 0$, and where d_Ω is the hyperbolic metric on Ω.

Let us highlight one important aspect of the proof of this theorem (we also refer to [Hon10b] for another summary of the strategy). For a function $f : \delta\mathbb{Z}^2 \longrightarrow \mathbb{C}$, let us introduce the following modified discrete gradient

$$\widetilde{\nabla} f(a) = \big(f(a + (\delta, \delta)) - f(a)\, ,\, f(a + (\delta, -\delta)) - f(a) \big).$$

For a function $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, we define the equivalent notion in the continuum $\widetilde{\nabla} f(a) = \big((\partial_x + \partial_y) f(a), (\partial_x - \partial_y) f(a) \big)$, where ∂_x and ∂_y are the directional derivatives in the first and second coordinates.

Proposition 9.30. *With the same notation and assumptions as in Theorem 9.28, we find*

$$\lim_{\delta \to 0} \frac{1}{\sqrt{2}\delta} \left(\frac{\widetilde{\nabla}_{a_1^\delta}\ \mu^+_{\beta_c,\Omega_\delta}(\sigma_{a_1^\delta}\sigma_{a_2^\delta} \dots \sigma_{a_n^\delta})}{\mu^+_{\beta_c,\Omega_\delta}(\sigma_{a_1^\delta}\sigma_{a_2^\delta} \dots \sigma_{a_n^\delta})} \right) = \widetilde{\nabla}_{a_1} \log \langle \sigma_{a_1}\sigma_{a_2} \dots \sigma_{a_n} \rangle_\Omega.$$

Then, one may integrate the relation to obtain the ratio for correlations of n points at macroscopic distances from each other. Interestingly, Proposition 9.30 does not give the result directly since it provides information on the ratio only. It is therefore necessary to identify the multiplicative normalization. More formally the result implies the existence of a sequence $(\rho_n(\Omega, \delta))_{n \geq 1}$ such that

$$\rho_n(\Omega, \delta) \cdot \mu^+_{\beta_c,\Omega_\delta}(\sigma_{a_1^\delta}\sigma_{a_2^\delta} \dots \sigma_{a_n^\delta}) \longrightarrow \langle \sigma_{a_1} \dots \sigma_{a_n} \rangle_\Omega,$$

and we need to identify $\rho_n(\Omega, \delta)$. In order to do so, one uses a result by McCoy and Wu to conclude. Let us focus on the $n = 2$ case first. The RSW theorem for the FK-Ising model easily implies (exercise) that when a_1 and a_2 are merged together

$$\rho_2(\Omega, \delta) \langle \sigma_{a_1}\sigma_{a_2} \rangle_\Omega \sim \mu^+_{\beta_c,\Omega_\delta}(\sigma_{a_1}\sigma_{a_2}) \sim \mu_{\beta_c,\delta\mathbb{Z}^2}(\sigma_{a_1}\sigma_{a_2}) \sim C^2 \left(\frac{\sqrt{2}\,\delta}{|a_1 - a_2|} \right)^{1/4},$$

where the sign $\sim$ means that the ratio tends as $\delta \to 0$ to a term which itself tends to 1 as a_1 tends to a_2. The last $\sim$ comes from the classical computation of the critical two-point Ising function from McCoy and Wu [MW73]. This gives $\rho_2(\Omega,\delta) = C^2(\sqrt{2}\,\delta)^{1/4}$. Note that the constant C defined above appears because of this asymptotics. In order to deduce the general $n \geq 2$ cases, observe that the RSW theorem implies that when sending a_n to the boundary of Ω,

$$\mu^+_{\beta_c,\Omega_\delta}(\sigma_{a_1}\cdots\sigma_{a_{n-1}}\sigma_{a_n}) \sim \mu^+_{\beta_c,\Omega_\delta}(\sigma_{a_1}\cdots\sigma_{a_{n-1}})\mu^+_{\beta_c,\Omega_\delta}(\sigma_{a_n}).$$

In fact, one may also prove directly that when a_n tends to the boundary,

$$\langle\sigma_{a_1}\cdots\sigma_{a_n}\rangle_\Omega \sim \langle\sigma_{a_1}\cdots\sigma_{a_{n-1}}\rangle_\Omega\langle\sigma_{a_n}\rangle_\Omega.$$

This implies that $\rho_n(\Omega,\delta) \sim \rho_1(\Omega,\delta)\rho_{n-1}(\Omega,\delta) \sim \rho_1(\Omega,\delta)^n$ and the result follows from the computation of $\rho_2(\Omega,\delta)$.

In order to prove Proposition 9.30, Chelkak, Hongler and Izyurov use fermionic observables once again. Let us focus on the $n = 1$ case. We will not go through the whole discussion yet again. Let us simply mention that the observable considered here is a modification of the standard observable. Let us define it for completeness.

Let $a \in \Omega^\star$. Consider the double cover $\overline{\Omega}$ of the graph Ω with a ramification at a constructed as follows. Let $\mathbb{U}_2$ be the graph $\mathbb{U}$ introduced in Chapter 6 quotiented by the equivalence relation $x = (x_1,x_2,x_3) \sim y = (y_1,y_2,y_3)$ if $x_1 = y_1$, $x_2 = y_2$ and $x_3 - y_3$ is even. Then,

$$\overline{\Omega} = \{x = (x_1,x_2,x_3) \in (\tilde{a} + \mathbb{U}_2) : (x_1,x_2) \in \Omega\},$$

where $\tilde{a}$ is chosen in such a way that the branching point is at a instead of $(-\frac{1}{2},-\frac{1}{2})$. Each vertex of $\overline{\Omega}$ has a natural projection onto Ω, and every path on Ω has a natural lift on $\overline{\Omega}$.

One may consider the high-temperature expansion on Ω. Recall that $\widehat{\mathcal{E}}_\Omega(u^\diamond,z^\diamond)$ denotes the set of loop configurations with one path $\gamma(\omega)$ from $u^\diamond$ to $z^\diamond$. Recall that the decomposition in loops and one path is not unique.

For $\omega \in \widehat{\mathcal{E}}_\Omega(u^\diamond,z^\diamond)$ and $\overline{z} \in \overline{\Omega}$ one of the two medial-vertices with projection $z^\diamond$, define:

- $\ell(\overline{\omega})$ to be the number of loops of ω that have a non-trivial lift on $\overline{\Omega}$, meaning that when drawn on $\overline{\Omega}$, they start and end at two different points (note that these two points necessarily have the same projection onto Ω);
- $s(\omega,z)$ to be 1 if $\gamma(\omega)$ ends on $\overline{z}$, and -1 if it ends on the other medial-vertex of $\overline{\Omega}$ with projection equal to $z^\diamond$.

Definition 9.31. Let Ω be a discrete domain with $a \in \Omega^\star$. Let $u^\diamond$ and $z^\diamond$ two medial vertices of $\Omega^\diamond$. The *spinor observable* at $\overline{z}$ (with projection $z^\diamond$) is defined by

$$F_{\Omega_\delta,a,u^\diamond}(\overline{z}) = \sum_{\omega \in \widehat{\mathcal{E}}_\Omega(u^\diamond,z^\diamond)} \mathrm{e}^{-\frac{1}{2}\mathrm{i}W_{\gamma(\omega)}(u^\diamond,z^\diamond)}(-1)^{\ell(\omega)}s(\omega,\overline{z})(\sqrt{2}-1)^{|\omega|},$$

where $\gamma(\omega)$ is a non-self-crossing self-avoiding path in ω going from $u^\diamond$ to $z^\diamond$.

Note that the observable changes sign when changing sheet (in this respect it behaves like $\sqrt{z-a}$). Also observe that while we proved that $\mathrm{e}^{-\frac{1}{2}\mathrm{i}W_{\gamma(\omega)}(u^\diamond,z^\diamond)}$ does not depend on the choice of $\gamma(\omega)$, it is unclear whether $(-1)^{\ell(\omega)}s(\omega,\overline{z})$ also does. We refer to [CI12] for a proof of this fact.

This observable can also be proved to be s-holomorphic on the lattice. The convergence in the scaling limit follows from considerations of Riemann-Hilbert BVPs once again.

9.3.3 Magnetization field

The physics approach to scaling limits deals with so-called fields. Let us describe some of the recent mathematical results in this direction. We discuss the specific case of the primary field called the *magnetization field* (sometimes, it is also called the *spin field*).

At the discrete level (i.e. on Ω_δ), the following field[11] encodes the quantitative information given by the spins

$$\Phi_{\Omega_\delta} := \delta^{15/8} \sum_{x \in \Omega_\delta} \sigma_x \delta_x,$$

where δ_x is the Dirac mass at x. For any $\delta > 0$, the field Φ_{Ω_δ} can be seen as a random function. Nevertheless, when $\delta \to 0$, this random function becomes rougher and rougher. Therefore, we prefer to think of Φ_{Ω_δ} as a random distribution. The space of distributions will be the Sobolev space $\mathcal{H}^{-3}(\Omega)$ with the norm $\|\cdot\|_{\mathcal{H}^{-3}}$. For those who were not born close to Sobolev spaces, we recall the definition of this space and this norm for $\Omega = [0,1]^2$. For any $k, \ell \in \mathbb{N}$, define

$$e_{k,\ell}(x,y) = 2\sin(k\pi x)\sin(\ell\pi y).$$

This family of functions forms an orthogonal basis of $\mathcal{C}^\infty([0,1]^2,\mathbb{R})$ endowed with the L^2-norm. Any function $f \in \mathcal{C}^\infty([0,1]^2,\mathbb{R})$ admits a

[11] The normalization by $\delta^{15/8}$ is connected to the fact that there are $O(1/\delta^2)$ vertices in Ω_δ and that the magnetization of each vertex is of order $\delta^{1/8}$ (one should be careful for vertices close to the boundary, but this technical issue can be easily handled).

unique decomposition in this basis:

$$f(x,y)=\sum_{k,\ell}a_{k,\ell}\cdot e_{k,\ell}(x,y).$$

We define $\mathcal{H}^3([0,1]^2)$ as the closure of $\mathcal{C}^\infty([0,1]^2,\mathbb{R})$ under the norm

$$\|f\|_{\mathcal{H}^3}=\sum_{k,\ell}(k+\ell)^3a_{k,\ell}^2.$$

The space $\mathcal{H}^{-3}([0,1]^2)$ of distributions on $[0,1]^3$ is defined as the dual of $\mathcal{H}^3$. The norm on this space is defined by the operator norm

$$\|\Phi\|_{\mathcal{H}^{-3}}=\sup\Big\{|\langle\Phi,f\rangle|\ :\ f\in\mathcal{C}^\infty([0,1]^2,\mathbb{R})\text{ such that }\|f\|_{\mathcal{H}^3}\le 1\Big\}.$$

For general simply connected domains, one may partition the domain into squares.

The following theorem has been proved in [CGN12, CGN13]. The convergence in law is in the space of distributions $\mathcal{H}^{-3}(\Omega)$ with the topology induced by the norm $\|\cdot\|_{\mathcal{H}^{-3}}$.

Theorem 9.32 (Camia, Garban, Newman [CGN12, CGN13])**.** *Let Ω be a simply connected domain. Consider a sequence of simply connected graphs Ω_δ converging to Ω. The sequence (Φ_{Ω_δ}) converges (as $\delta\to 0$) in law to a conformally covariant random distribution Φ_Ω.*

Furthermore, Φ_Ω is conformally covariant in the following sense: for any conformal map $\psi:\Omega\to\mathbb{C}$ and for any $f\in\mathcal{C}^\infty(\psi(\Omega),\mathbb{R})$,

$$\langle\Phi_\Omega\circ\psi^{-1},f\rangle=\langle\Phi_{\psi(\Omega)},|\psi'|^{15/8}f\rangle.$$

Remark 9.33. The n-point correlations of the spin field are given by the n-point correlations of Theorem 9.28. The conformal covariance shows that the model is not Gaussian and is therefore different from the infamous Gaussian Free Field (see [She07] for instance).

Chapter 10

Crossing probabilities for the critical FK-Ising model

This chapter and the next one are more specialized and non-experts may be willing to skip them. Nevertheless, they are self-contained and can be read after having followed the previous chapters of this book.

The present chapter is devoted to bounds on crossing probabilities in topological rectangles in the critical FK-Ising model. We already encountered slightly weaker bounds for standard rectangles (i.e. of the form $[0,n] \times [0,m]$) in Chapter 5. In this chapter, we will extend these bounds to possibly fractal domains and to boundary conditions which are directly on the boundary of the topological rectangle (in opposition to the results in Chapter 5 which were restricted to boundary conditions at "macroscopic distance"). We start by illustrating an approach for bounding crossing probabilities from above and below which is based on discrete holomorphicity by gtreating the case of standard rectangles. In a second time, we will use this approach to derive the improved result for general topological rectangles. In order to illustrate how useful this last result is, we will study arm-events in the last section of this chapter.

In this chapter, we fix $q = 2$ and $p = p_c(2)$ and we drop the dependency on p and $q = 2$ in the measure. For instance, $\phi^{\xi}_{p_c(2),2,\Omega}$ will be denoted by ϕ^{ξ}_{Ω}.

10.1 RSW theory via discrete holomorphicity

10.1.1 Statement of the theorem

Recall that a *rectangle* R is a subgraph of $\mathbb{Z}^2$ of the form $[0,n]\times[0,m]$ for $n,m>0$, or a translation of one of these graphs. Also recall that the event that there exists a *vertical crossing* in R, *i.e.* an open path from the bottom side $[0,n]\times\{0\}$ to the top side $[0,n]\times\{m\}$, is denoted by $\mathcal{C}_v(R)$.

Theorem 10.1 (Duminil-Copin, Hongler, Nolin [DCHN11])**.** *Let $\beta\in(0,\infty)$. There exists $c_1=c_1(\beta)>0$ such that for any rectangle R with side lengths n and βn and any boundary condition ξ on ∂R,*

$$c_1 \;\le\; \phi_R^{\xi}(\mathcal{C}_v(R)) \;\le\; 1-c_1.$$

This theorem is an improvement (for $q=2$) of Property **P5** of Corollary 6.16 since boundary conditions are now allowed to be taken directly on the boundary of the domain. The proof is based on the fermionic observable which is used to express macroscopic quantities such as connection probabilities in terms of discrete harmonic measures.

Let (Ω,a,b) be a Dobrushin domain. Recall the construction of $(X_t^{\bullet})_{t\ge0}$ and $(X_t^{\circ})_{t\ge0}$ on $\widehat{\Omega}$ from Chapter 8. For $B\in\Omega$, let $\mathbf{HM}_{\bullet}(B)$ denote the probability that the random walk $X_t^{\bullet}$ starting from B hits ∂_{ba} before hitting $\widehat{\partial}_{ab}$. Similarly, for $W\in\Omega^{\star}$, let $\mathbf{HM}_{\circ}(W)$ denote the probability that the random walk X_t° starting from W hits $\partial_{ab}^{\star}$ before hitting $\widehat{\partial}_{ba}^{\star}$. Note that there is no extra difficulty in defining these quantities for infinite discrete domains as well.

For simplicity, we will often refer to ∂_{ab} and ∂_{ba} as being the free and wired arcs respectively.

Proposition 10.2 (uniform comparability)**.** *Let (Ω,a,b) be a discrete Dobrushin domain. Let $B\in\partial_{ab}$ and $W\in\Omega^{\star}\smallsetminus\partial_{ab}^{\star}$ adjacent to B. Then we have*

$$\sqrt{\mathbf{HM}_{\circ}(W)}\le\phi_{\Omega}^{a,b}(B\longleftrightarrow\text{wired arc})\le\sqrt{\mathbf{HM}_{\bullet}(B)}. \tag{10.1}$$

Proof. Let $e\in\partial_{ab}^{\diamond}$ bordering B. Now consider the fermionic observable F in the domain (Ω,a,b) and the function H associated to it. The study of its boundary conditions (Lemma 6.11) implies that

$$|F(e)|=\phi_{\Omega}^{a,b}(B\longleftrightarrow\text{wired arc}).$$

By definition of H, we have $|F(e)|^2=H^{\bullet}(B)$ and $H^{\circ}(W)=|F(e)|^2-|F(e')|^2\le|F(e)|^2$, where e' is the medial edge between B and

W: it is therefore sufficient to recall that $H^\bullet(B) \le \mathbf{HM}_\bullet(B)$ and $H^\circ(W) \ge \mathbf{HM}_\circ(W)$. □

The inequalities (10.1) will allow us to use a second-moment method on the number of pairs of connected vertices. Before implementing this second-moment method, we provide bounds on the harmonic measures $\mathbf{HM}_\bullet$ and $\mathbf{HM}_\circ$ in specific domains that will be used in the proof of Theorem 10.1.

10.1.2 Some estimates on the harmonic measures $\mathbf{HM}_\bullet$ and $\mathbf{HM}_\circ$

Consider *only* Dobrushin domains (Ω, a, b) that *contain the origin* on the free arc, and are *subsets* of the medial lattice $\mathbb{H}^\diamond$, where $\mathbb{H} = \{(x_1, x_2) \in \mathbb{Z}^2, x_2 \ge 0\}$ denotes the upper half plane. In this case, Ω is said to be a *Dobrushin* $\mathbb{H}$*-domain*. For the following estimates on harmonic measures, the Dobrushin domains that are considered can also be infinite. We are interested in the harmonic measure of the wired arc seen from the origin. Let W_0 be a dual vertex of $\Omega^\star \setminus \partial^\star_{ab}$ adjacent to the origin. We first prove a lower bound on the harmonic measure. For that, introduce, for $k \in \mathbb{Z}$ and $n \ge 0$, the segments

$$l_n(k) = \{k\} \times [0, n] \quad (= \{(k, j) : 0 \le j \le n\}).$$

Lemma 10.3. *There exists a constant $c_2 > 0$ such that for any Dobrushin $\mathbb{H}$-domain (Ω, a, b), we have*

$$\mathbf{HM}_\circ(W_0) \ge \frac{c_2}{k}, \tag{10.2}$$

provided that, in Ω, the segment $l_k(-k)$ disconnects the intersection of the free arc with the upper half-plane from the origin (see Figure 10.1).

Proof. The arc $l_k(-k)$ disconnects the origin from the part of the free arc that lies in the upper half-plane. Let us thus consider the connected component of $\Omega \setminus l_k(-k)$ that contains the origin. Boundary conditions along $l_k(-k)$ are free. In this new Dobrushin domain Ω_0, the harmonic measure of the wired arc is smaller than the harmonic measure of the wired arc in the original domain Ω. On the other hand, the harmonic measure of the wired arc in Ω_0 is larger than the harmonic measure of the wired arc in the slit domain $(\mathbb{H} \setminus l_k(-k), (-k, k), \infty)$, which has respectively wired and free boundary conditions to the left and to the right of $(-k, k)$ (see Figure 10.1). Estimating this harmonic measure is relatively straightforward and we leave this task as an exercise. □

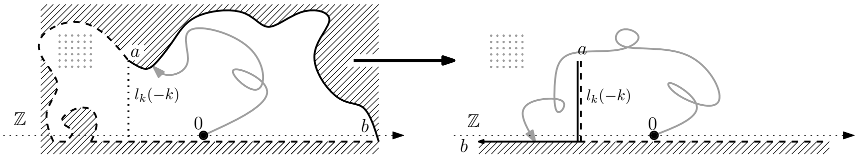

Figure 10.1: The two domains involved in the proof of Lemma 10.3.

Upper bounds on the harmonic measures are now derived. Estimates of two different types will be needed. The first one takes into account the distance between the origin and the wired arc, while the second one requires the existence of a segment $l_n(k)$ disconnecting the wired arc from the origin (still inside the domain).

Lemma 10.4. *There exist constants $c_3, c_4 > 0$ such that for any Dobrushin $\mathbb{H}$-domain (Ω, a, b),*

- *if $d_1(0)$ denotes the graph distance between the origin and the wired arc,*

$$\mathbf{HM}_\bullet(0) \le \frac{c_3}{d_1(0)} \ ; \tag{10.3}$$

- *and if the segment $l_n(k)$ disconnects the wired arc from the origin inside Ω,*

$$\mathbf{HM}_\bullet(0) \le c_4 \frac{n}{|k|^2}. \tag{10.4}$$

Proof. Let us first consider (10.3). For $d = d_1(0)$, define the Dobrushin domain $(\mathcal{B}_d, (-d, 0), (d, 0))$, where $\mathcal{B}_d$ is the set of vertices in $\mathbb{H}$ at a graph distance at most d from the origin (it has a $\diamond$ shape; see Figure 10.2). The harmonic measure of the wired arc in (Ω, a, b) is smaller than the harmonic measure of the wired arc in this new domain $\mathcal{B}_d$, and, as before, this harmonic measure is easy to estimate.

Let us now turn to (10.4). Since $l_n(k)$ disconnects the wired arc from the origin, the harmonic measure of the wired arc is smaller than the harmonic measure of $l_n(k)$ inside Ω, and this harmonic measure is smaller than it is in the domain $\mathbb{H} \setminus l_n(k)$ with wired boundary conditions on the left side of $l_n(k)$ – right side if $k < 0$ (see Figure 10.2). Once again, the estimates are easy to perform in this domain. □

10.1.3 Proof of Theorem 10.1

We now prove Theorem 10.1. The main step is to prove the uniform lower bound for rectangles of bounded aspect ratio with free boundary

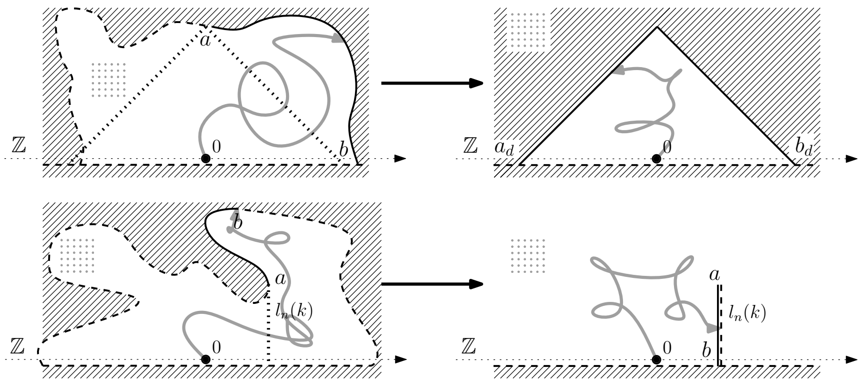

Figure 10.2: The two different upper bounds (10.3) and (10.4) of Lemma 10.4.

conditions. We then use monotonicity to compare boundary conditions and obtain the desired result. In the case of free boundary conditions, the proof relies on a second moment estimate on the number N of pairs of vertices (x, u), on the top and bottom sides of the rectangle respectively, that are connected by an open path.

The organization of this section follows the second-moment estimate strategy. In Proposition 10.5, we first prove a lower bound on the probability of a connection from a given vertex on the bottom side of a rectangle to a given vertex on the top side. This estimate gives a lower bound on the expectation of N. Then, Proposition 10.6 provides an upper bound on the probability that two vertices on the bottom side of a rectangle are connected to the top side. This proposition is the core of the proof: it provides the right bound for the second moment of N. We conclude this section and the proof of Theorem 10.1 by using the second moment estimate method.

We start by a lower bound on crossing probabilities. Let us introduce the definition of R_n^β:

$$R_n^\beta = [-\beta n, \beta n] \times [0, 2n]. \tag{10.5}$$

Let $\partial_+ R_n^\beta$ (resp. $\partial_- R_n^\beta$) be the *top side* $[-\beta n, \beta n] \times \{2n\}$ (resp. *bottom side* $[-\beta n, \beta n] \times \{0\}$) of the rectangle R_n^β. We begin with a lower bound on connection probabilities.

Proposition 10.5 (connection probability for one point on the bottom side)**.** *Let $\beta > 0$, there exists a constant $c = c(\beta) > 0$ such that for any $n \geq 1$,*

$$\phi^0_{R_n^\beta}(x \leftrightarrow u) \geq \frac{c}{n} \tag{10.6}$$

for all $x = (x_1, 0) \in \partial_- R_n^\beta$, $u = (u_1, 2n) \in \partial_+ R_n^\beta$, *satisfying* $|x_1|, |u_1| \le \beta n/2$.

Proof. The probability that x and u are connected in the rectangle with free boundary conditions can be written as the probability that x is connected to the wired arc in (R_n^β, u, u) (where the wired arc consists of a single vertex). Proposition 10.2 gives that $\phi^0_{R_n^\beta}(x \leftrightarrow u) \ge \sqrt{\mathbf{HM}_\circ(W_x)}$. Since x and u are at distance $\beta n/2$ from the left and right sides of R_n^β, the lower bound $\mathbf{HM}_\circ(W_x) \ge \frac{c_1}{n^2}$ on the harmonic measure imply the result. This last estimate follows from standard results on simple random walks (gambler's ruin type estimates). We leave this proof as an exercise. □

We now study the probability that two boundary points on the bottom side of R_n^β are connected to the top side, with boundary conditions wired on the top side and free on the other sides.

Proposition 10.6 (connection probability for two points on the bottom side)**.** *There exists a constant* $c > 0$ *such that for any rectangle* R_n^β *and any two points* x, y *on the bottom side* $\partial_- R_n^\beta$,

$$\phi^{a_n,b_n}_{R_n^\beta}(x, y \longleftrightarrow \textit{wired arc}) \le \frac{c}{\sqrt{|x-y|n}}, \tag{10.7}$$

where a_n *and* b_n *denote respectively the top-left and top-right corners of the rectangle* R_n^β.

The proof is based on the following lemma, which is a strong form of the so-called half-plane one-arm probability estimate. For x on the bottom side of R_n^β and $k \ge 1$, denote by $\mathcal{B}_k(x)$ the box centered at x of radius k for the *graph distance.*

Lemma 10.7. *There exists a constant* $c_5 > 0$ *such that for any box* R_n^β, *any* x *on the bottom side* $\partial_- R_n^\beta$ *and any* $k \ge 0$,

$$\phi^{a_n,b_n}_{R_n^\beta}(\mathcal{B}_k(x) \leftrightarrow \text{wired arc}) \le c_5 \sqrt{\frac{k}{n}}. \tag{10.8}$$

Proof. Consider $n, k, \beta > 0$, and the box R_n^β with one point $x \in \partial_- R_n^\beta$. The inequality (10.8) becomes trivial if $k \ge n$, so we can assume that $k \le n$. For any choice of $\beta' \ge \beta$, the monotonicity between boundary conditions implies that the probability that $\mathcal{B}_k(x)$ is connected to the wired arc $\partial_+ R_n^\beta$ in (R_n^β, a_n, b_n) is smaller than the probability that $\mathcal{B}_k(x)$ is connected to the wired arc in the Dobrushin domain $(R_n^{\beta'}, c_n, d_n)$, where c_n and d_n are the bottom-left and bottom-right corners of $R_n^{\beta'}$. From now on, replace β by $\beta + 2$, and consider the new domain (R_n^β, c_n, d_n).

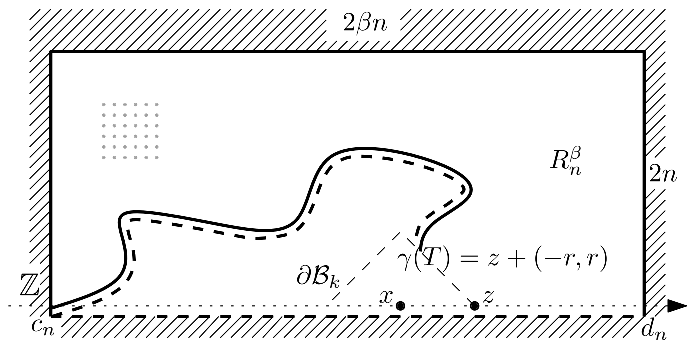

Figure 10.3: The Dobrushin domain (R_n^β, c_n, d_n), together with the exploration path up to time T.

Since we are working in a Dobrushin domain, we may consider the exploration path, denoted γ, which goes from $c_n^\diamond$ to $d_n^\diamond$. Let T denote the hitting time – for γ naturally parametrized by the number of steps from $c_n^\diamond$ – of the set of medial edges bordering the vertices of $\mathcal{B}_k(x)$; set $T = \infty$ if the exploration path never reaches this set, so that $\mathcal{B}_k(x)$ is connected to the wired arc if and only if $T < \infty$. Our goal is to bound $\phi_{R_n^\beta}^{c_n,d_n}(T < \infty)$ from above.

The right-most vertex of $\mathcal{B}_k(x)$ will be denoted by z until the end of this proof (note that it is in R_n^β thanks to the new choice of β). Consider now the event $\{z \leftrightarrow \text{wired arc}\}$. By conditioning on the curve up to time T (and on the event $\{\mathcal{B}_k(x) \leftrightarrow \text{wired arc}\}$), we obtain

$$\begin{aligned}\phi_{R_n^\beta}^{c_n,d_n}(z \leftrightarrow \text{wired arc}) &= \phi_{R_n^\beta}^{c_n,d_n}\big[\mathbf{1}_{T<\infty} \cdot \phi_{R_n^\beta}^{c_n,d_n}(z \leftrightarrow \text{wired arc} \,|\, \gamma[0,T])\big] \\ &= \phi_{R_n^\beta}^{c_n,d_n}\big[\mathbf{1}_{T<\infty} \cdot \phi_{R_n^\beta \smallsetminus \gamma[0,T]}^{\gamma(T),d_n}(z \leftrightarrow \text{wired arc})\big],\end{aligned}$$

where the second inequality used the domain Markov property and the fact that it is sufficient for z to be connected to the wired arc in the new domain (since it is then automatically connected to the wired arc of the original domain). Note that we used the notation for slit domains introduced in Definition 9.16 when writing $R_n^\beta \smallsetminus \gamma[0,T]$.

On the one hand, since z is at a distance at least n from the wired arc in R_n^β (thanks to the new choice of β again), Proposition 10.2 can be combined with Item (10.3) of Lemma 10.4 to obtain

$$\phi_{R_n^\beta}^{c_n,d_n}(z \leftrightarrow \text{wired arc}) \le \frac{c_3}{\sqrt{n}}.$$

On the other hand, if $\gamma(T)$ can be written as $\gamma(T) = z + (-r, r)$, with $0 \le r \le k$, then the arc $z + l_r(-r)$ disconnects the free arc from z in the

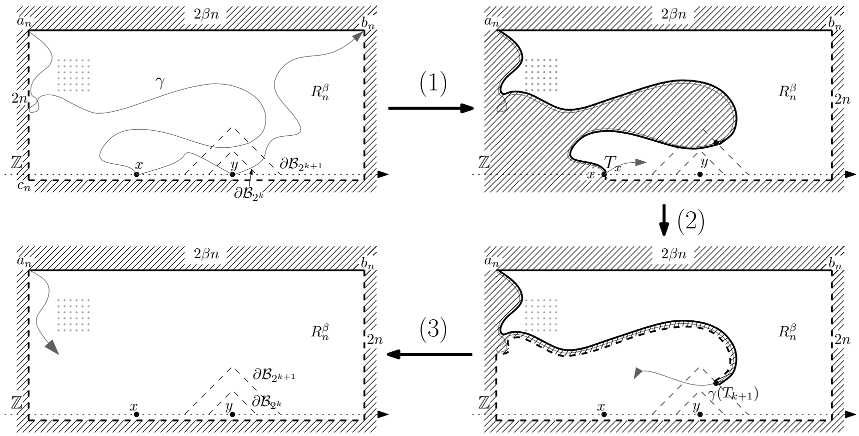

Figure 10.4: This picture presents the different steps in the proof of Proposition 10.6: we first (1) condition on $\gamma[0, T_x]$ and use the uniform estimate (10.3) of Lemma 10.4, then (2) condition on $\gamma[0, T_{k+1}]$ and use the estimate (10.4) of Lemma 10.4, in order to (3) conclude with Lemma 10.7.

domain $R_n^\beta \setminus \gamma[0,T]$, while if $\gamma(T) = z + (-r, 2k - r)$, with $k + 1 \le r \le 2k$, then the arc $z + l_r(-r)$ still disconnects the free arc from z. Using once again Proposition 10.2, this time with Lemma 10.3, we obtain that a.s.

$$\phi^{\gamma(T),d_n}_{R_n^\beta \setminus \gamma[0,T]}(z \leftrightarrow \text{wired arc}) \ge \frac{c_4}{\sqrt{r}} \ge \frac{c_4}{\sqrt{2k}}.$$

This estimate being uniform in the realization of $\gamma[0,T]$, we obtain

$$\frac{c_4}{\sqrt{2k}} \phi^{c_n,d_n}_{R_n^\beta}(T < \infty) \le \phi^{c_n,d_n}_{R_n^\beta}(z \leftrightarrow \text{wired arc}) \le \frac{c_3}{\sqrt{n}},$$

which implies the desired claim (10.8). □

Proof of Proposition 10.6. Let us take two vertices x and y on $\partial_- R_n^\beta$. As in the previous proof, the larger the β, the larger the corresponding probability. Hence, β can be chosen in such a way that there are no "boundary effects". In order to prove the estimate, we express the event considered in terms of the exploration path γ in the Dobrushin domain R_n^β with a being the bottom-left corner and b the bottom-right one. If x and y are connected to the wired arc, γ must go through two boundary edges which are adjacent to x and y, which we denote by e_x and e_y. Notice that e_x has to be discovered by γ before e_y is.

Now, define T_x to be the hitting time of e_x, and T_k to be the hitting time of $\mathcal{B}_{2^k}(y)^\diamond$, for $k \le K = \lfloor \log_2 |x-y| \rfloor$ – where $\lfloor \cdot \rfloor$ is the integer part of a real number. If the exploration path does not cross this ball before hitting e_x, set $T_k = \infty$. With these definitions, the probability that e_x and e_y are both on γ can be expressed as

$$\begin{aligned}
&\phi_{R_n^\beta}^{a_n,b_n}(x, y \leftrightarrow \text{wired arc}) = \phi_{R_n^\beta}^{a_n,b_n}(e_x, e_y \in \gamma)\\
&= \sum_{k=0}^{K} \phi_{R_n^\beta}^{a_n,b_n}(e_y \in \gamma, T_x < \infty, T_{k+1} < T_k = \infty)\\
&= \sum_{k=0}^{K} \phi_{R_n^\beta}^{a_n,b_n}\big[\mathbf{1}_{T_{k+1}<T_k=\infty} \cdot \mathbf{1}_{T_x<\infty} \cdot \phi_{R_n^\beta}^{a_n,b_n}(e_y \in \gamma \,|\, \gamma[0, T_x])\big], \qquad (10.9)
\end{aligned}$$

where the third equality is obtained by conditioning on the exploration path up to time T_x. Recall that e_y belongs to γ if and only if y is connected to the wired arc. Moreover, if $\{T_k = \infty\}$, y is at a distance at least 2^k from the wired arc in the slit domain $R_n^\beta \setminus \gamma[0, T_x]$. Hence, the domain Markov property together with (10.3) from Lemma 10.4 and Proposition 10.2 give that, on $\{T_k = \infty\}$,

$$\phi_{R_n^\beta}^{a_n,b_n}(e_y \in \gamma \,|\, \gamma[0, T_x]) = \phi_{R_n^\beta \setminus \gamma[0,T_x]}^{x,b_n}(y \leftrightarrow \text{wired arc}) \le \frac{c_3}{\sqrt{2^k}} \quad a.s.$$

By plugging this uniform estimate into (10.9), and removing the condition on $T_k = \infty$ in the first indicator, we obtain

$$\phi_{R_n^\beta}^{a_n,b_n}(e_x, e_y \in \gamma) \le \sum_{k=0}^{K} \frac{c_3}{\sqrt{2^k}}\, \phi_{R_n^\beta}^{a_n,b_n}\big[\mathbf{1}_{T_{k+1}<\infty} \cdot \phi_{R_n^\beta}^{a_n,b_n}(T_x < \infty \,|\, \gamma[0, T_{k+1}])\big],$$

where we conditioned on the path up to time T_{k+1}. Now, e_x belongs to γ if and only if x is connected to the wired arc. Assuming $\{T_{k+1} < \infty\}$, the vertical segment connecting $\gamma(T_{k+1})$ to $\mathbb{Z}$ – of length at most 2^{k+1} – disconnects the wired arc from x in the domain $R_n^\beta \setminus \gamma[0, T_{k+1}]$. For $k+1 < K$, this vertical segment is at distance at least $\frac{1}{2}|x-y|$ from x. Applying the domain Markov property and item (10.4) of Lemma 10.4, we deduce that, for $k+1 < K$, on $\{T_{k+1} < \infty\}$,

$$\phi_{R_n^\beta}^{a_n,b_n}(T_x < \infty \,|\, \gamma[0, T_{k+1}]) = \phi_{R_n^\beta \setminus \gamma[0,T_{k+1}]}^{\gamma(T_{k+1}),b_n}(x \leftrightarrow \text{wired arc}) \le 2c_4 \frac{\sqrt{2^{k+1}}}{|x-y|} \ a.s.$$

Making use of this uniform bound, we obtain

$$\begin{aligned}
&\phi_{R_n^\beta}^{a_n,b_n}(x,y \leftrightarrow \text{wired arc}) \\
&\le\ 2c_3c_4\sum_{k=0}^{K-2}\frac{\sqrt{2^{k+1}}}{\sqrt{2^k}|x-y|}\phi_{R_n^\beta}^{a_n,b_n}(T_{k+1}<\infty)\ +\ 2c_3\frac{\phi_{R_n^\beta}^{a_n,b_n}(T_x<\infty)}{\sqrt{2^{K-1}}} \\
&\le\ \frac{\sqrt{2}c_3c_4c_5}{|x-y|\sqrt{n}}\sum_{k=0}^{K-2}\sqrt{2^k}\ +\ \frac{2c_3c_5}{\sqrt{n2^{K-1}}} \\
&\le\ \frac{c}{\sqrt{n|x-y|}},
\end{aligned}$$

using also Lemma 10.7 (twice) for the second inequality. □

We are now in a position to prove our result.

Proof of Theorem 10.1. Fix $\beta > 0$ and $n > 0$.

Step 1: lower bound for free boundary conditions. Let N_n be the number of connected pairs (x,u), with $x \in \partial_- R_n^\beta$, and $u \in \partial_+ R_n^\beta$. The expected value of this quantity is equal to

$$\phi_{R_n^\beta}^0[N_n] = \sum_{\substack{u\in\partial_+R_n^\beta\\ x\in\partial_-R_n^\beta}} \phi_{R_n^\beta}^0(x \leftrightarrow u).$$

Proposition 10.5 directly provides the following lower bound on the expectation by summing over the $(\beta n)^2$ pairs of points (x,u) far enough from the corners, *i.e.* satisfying the condition of the proposition:

$$\phi_{R_n^\beta}^0[N_n] \ge c_6(\beta)n$$

for some $c_6(\beta) > 0$.

On the other hand, if x and u (resp. y and v) are pair-wise connected, then they are also connected to the horizontal line $\mathbb{Z}\times\{n\}$ which is (vertically) at the middle of R_n^β. Moreover, the comparison between boundary conditions implies that the probability – in R_n^β with free boundary conditions – that x and y are connected to this line is smaller than the probability of this event in the rectangle of half height with wired boundary conditions on the top side. In the following, assume without loss of generality that n is even[1] and set $m = n/2$, so that the previous rectangle is $R_m^{2\beta}$, and define a_m and b_m as before. Using the comparison between boundary conditions, and also the symmetry of the lattice, we get

$$\phi_{R_n^\beta}^0(x\leftrightarrow u, y\leftrightarrow v) \le \phi_{R_m^{2\beta}}^{a_m,b_m}(x,y\leftrightarrow \text{wired arc})\,\phi_{R_m^{2\beta}}^{a_m,b_m}(\bar u,\bar v\leftrightarrow \text{wired arc}),$$

[1] The argument may be adapted to the case of odd integers.

where $\bar{u}$ and $\bar{v}$ are the projections on the real axis of u and v. Summing the bound provided by Proposition 10.6 on all vertices $x, y \in \partial_- R_n^\beta$ and $u, v \in \partial_+ R_n^\beta$, we obtain

$$\phi^0_{R_n^\beta}[N_n^2] \leq c_7 m^2 \leq c_7 n^2$$

for some constant $c_7 > 0$. Now, by the Cauchy-Schwarz inequality,

$$\phi^0_{R_n^\beta}(\mathcal{C}_v(R_n^\beta)) = \phi^0_{R_n^\beta}(N_n > 0) = \phi^0_{R_n^\beta}[(\mathbf{1}_{N_n>0})^2] \geq \frac{\phi^0_{R_n^\beta}[N_n]^2}{\phi^0_{R_n^\beta}[N_n^2]} \geq c_6(\beta)^2/c_7$$

(since $\phi^0_{R_n^\beta}[N_n] = \phi^0_{R_n^\beta}[N_n \mathbf{1}_{N_n>0}]$). We have thus reached the claim.

Step 2: lower and upper bounds for general boundary conditions. As already shown in Chapter 5, the lower bound that was previously proved for free boundary conditions actually implies the lower bound for any boundary conditions ξ by using the ordering between boundary conditions.

For the upper bound, use duality to find that

$$\phi^\xi_R(\mathcal{C}_v(R)) \leq \phi^1_R(\mathcal{C}_v(R)) = 1 - \phi^0_{R^\star}(\mathcal{C}_h(R^\star)) \leq 1 - c_1, \tag{10.10}$$

where the notation $\mathcal{C}_h^\star(R^\star)$ is used for the existence of a horizontal dual crossing in the dual rectangle $R^\star$ is as usual the dual graph of R (note that the invariance by $\pi/2$-rotations was implicitly used). This fact implies Theorem 10.1 readily.

□

Remark 10.8. As a by-product of our proofs, one can also obtain the value of the critical exponent for the boundary magnetization in the Ising model, near a free horizontal boundary arc, and the corresponding one-arm half-plane exponent for the FK-Ising model.

Indeed, consider the rectangle $R_n = [-n, n] \times [0, n]$. There exist strictly positive constants c_1 and c_2 such that for the boundary conditions *dobr* free on the bottom and wired everywhere else, one has

$$c_1 n^{-1/2} \leq \phi^{\mathrm{dobr}}_{R_n}(0 \leftrightarrow \text{wired arc}) \leq c_2 n^{-1/2},$$

uniformly over all n. This translates via the Edwards-Sokal coupling into the following inequality for the Ising model: for every $n \geq 1$,

$$c_1 n^{-1/2} \leq \mu^f_{R_n}[\sigma_0 | \sigma_x = 1, \forall x \in \partial^+ R_n] \leq c_2 n^{-1/2}.$$

10.2 RSW in general topological rectangle

We now extend crossing estimates to a more general class of domains. Such crossing probabilities bounds, uniform with respect to the boundary conditions, have been obtained in standard rectangles in Theorem 10.1. In this section, the crossing bounds are proved to hold in *general topological rectangles with general boundary conditions*, and are independent of the local geometry of the boundary. This generalization is needed when dealing with domains generated by random interfaces (which usually have fractal scaling limits).

Definition 10.9. A *topological rectangle* is given by (Ω, a, b, c, d), where Ω is a discrete domain and a, b, c and d are four boundary vertices on $\partial\Omega$ found in counterclockwise order. For two points $x, y \in \partial\Omega$, we denote by $(xy) \subset \partial\Omega$ the counterclockwise arc of $\partial\Omega$ from x to y (including x and y). We will frequently identify $x \in \partial\Omega$ with the arc (xx).

Before going further, we need an equivalent of the aspect ratio between the width and the height of a standard rectangle that we could apply for topological rectangles to measure "how flat they are". We use the *extremal length* defined as follows. Recall that the medial graph $\Omega^\diamond$ of a discrete domain Ω may be seen as a subdomain of $\mathbb{C}$ by adding small patches around pinched points. Furthermore, a, b, c and d are naturally associated to four medial vertices $a^\diamond$, $b^\diamond$, $c^\diamond$ and $d^\diamond$ of $\partial\Omega^\diamond$ (for instance, every vertex is bordered by two medial-vertices in $\partial\Omega^\diamond$; pick the second one when going in counterclockwise order. If $a = b$ say, then pick $a^\diamond$ to be the first of the two medial-vertices bordering a, and $b^\diamond$ to be the second one). Let us denote by $\ell_\Omega\left[(ab),(cd)\right]$ the standard extremal length between $(a^\diamond b^\diamond)$ and $(c^\diamond d^\diamond)$ in $\Omega^\diamond$, i.e.

$$\begin{aligned}&\ell_\Omega\left[(ab),(cd)\right]\\ &:= \sup_\rho \frac{\left(\inf\left\{\int_\gamma \rho|dz| : \gamma \text{ rectifiable path from } (a^\diamond b^\diamond) \text{ to } (c^\diamond d^\diamond) \text{ in } \Omega^\diamond\right\}\right)^2}{\int_{\Omega^\diamond} \rho dxdy},\end{aligned}$$

where the supremum is taken over measurable functions $\rho : \Omega^\diamond \mapsto \mathbb{R}_+$, and $|dz|$ denotes the Euclidean element of length. Informally speaking, $\ell_\Omega\left[(ab),(cd)\right]$ measures the distance between (ab) and (cd) from a conformal invariance point of view. The notion was introduced by Ahlfors and Beurling in a more general context and we refer to [Ahl73] for its basic properties. Also, some reader will have encountered the inverse of the extremal length which is sometimes called the modulus of the rectangle.

Given a topological rectangle (Ω, a, b, c, d), let $\{(ab) \leftrightarrow (cd)\}$ be the event that there is an open path in Ω between (ab) and (cd). We are now in a position to state the theorem.

Theorem 10.10 (Chelkak, Duminil-Copin, Hongler [CDCH12])**.** *Let $M > 0$. There exists $c = c(M) > 0$ such that for any boundary conditions ξ and for any topological rectangle (Ω, a, b, c, d) with $\ell_\Omega[(ab),(cd)] \in \left[\frac{1}{M}, M\right]$,*

$$c \le \phi_\Omega^\xi[(ab) \leftrightarrow (cd)] \le 1 - c.$$

The condition above is a conformally invariant version of the condition that the aspect ratio of rectangles remains bounded away from 0 and 1.

In the rest of this chapter, for two functions f and g, $f \asymp g$ means that $cg \le f \le Cg$, where $c > 0$ and $C > 0$ are universal constants (we will precise each time on what they depend).

10.2.1 More involved discrete complex analysis

In the previous section, we use the comparison between discrete harmonic measures and the probability that a vertex on the free arc is connected to the wired arc. This relation is not restricted to standard rectangles of the form $[0, n] \times [0, m]$, and we now wish to exploit this comparison for more general domains. In order to do so, we need a few technical results on harmonic measures that we list below (this section should be understood as a toolbox). These results were obtained in [Che11] from discrete complex analysis considerations.

Before starting, let us change slightly the notation for harmonic measures to be more coherent with [Che11]. For two vertices x and y of a discrete domain Ω, let $Z_\Omega^\bullet[x, y]$ be the partition function of the random walks $(X_n^\bullet)$ (introduced in Chapter 8) on $\widehat{\Omega}$ from x to y, killed when reaching $\partial\widehat{\Omega} \cup \{y\}$. In other words,

$$Z_\Omega^\bullet[x, y] := \mathbb{P}\Big[(X_n^\bullet) \text{ starting from } x \text{ hits } y \text{ before } \partial\widehat{\Omega}\Big] = \mathbf{HM}_\bullet(x),$$

where $\mathbf{HM}_\bullet(x)$ is taken in the Dobrushin domain (Ω, y, y). For two boundary arcs (ab) and $(cd) \subset \partial\Omega$ and $x \in \partial\Omega$, define $Z_\Omega^\bullet[x,(cd)] := \sum_{y\in(cd)} Z_\Omega^\bullet[x, y]$ and $Z_\Omega^\bullet[(ab),(cd)] := \sum_{x\in(ab)} Z_\Omega^\bullet[x,(cd)]$.

It is time to list several important properties of $Z_\Omega^\bullet$. The first one yields that whenever the extremal length is of order 1, then so are the partition functions of $(X_n^\bullet)$ under consideration.

Theorem 10.11 ([Che11])**.** *Let $M > 1$. For any topological rectangle (Ω, a, b, c, d), the following properties are equivalent:*

1. $\ell_\Omega[(ab),(cd)] \asymp 1$,
2. $\ell_\Omega[(bc),(da)] \asymp 1$,
3. $Z_\Omega^\bullet[(ab),(cd)] \asymp 1$,
4. $Z_\Omega^\bullet[(bc),(da)] \asymp 1$,

where the constants in $\asymp$ never depend on (Ω, a, b, c, d) *but simply on the constants in other* $\asymp$.

We now describe factorization properties of these partition functions (or equivalently of discrete harmonic measures). While in the continuum the results below are rather easy to derive (for instance using conformal invariance and explicit expressions in the upper half-plane), obtaining them at the discrete level requires a much more delicate analysis.

Theorem 10.12 ([Che11])**.** *For any topological rectangle* (Ω, a, b, c, d), *we have*

$$\begin{aligned} Z^{\bullet}_{\Omega}[a,(bc)] &\asymp \sqrt{\frac{Z^{\bullet}_{\Omega}[a,b]\,Z^{\bullet}_{\Omega}[a,c]}{Z^{\bullet}_{\Omega}[b,c]}}, \\ Z^{\bullet}_{\Omega}[(ab),(cd)] &\asymp \sqrt{\frac{Z^{\bullet}_{\Omega}[a,d]\,Z^{\bullet}_{\Omega}[b,c]}{Z^{\bullet}_{\Omega}[a,b]\,Z^{\bullet}_{\Omega}[c,d]}}, \qquad \textit{if} \quad \ell_{\Omega}[(ab),(cd)] \le M, \end{aligned} \tag{10.11}$$

where the constants in $\asymp$ *are universal in the first equality, and depend on* M *only in the second.*

We now focus on the notion of separator. If (Ω, a, b, c, d) is a topological rectangle, a *separating curve* between (ab) and (cd) is a self-avoiding discrete path Γ in Ω separating (ab) from (cd). Let $\Omega[\Gamma,(ab)]$ be the union of Γ with the connected component of $\Omega \smallsetminus \Gamma$ containing (ab).

Theorem 10.13 ([Che11])**.** *Let* $M > 1$. *There exists* $\varepsilon = \varepsilon(M) \in (0,1)$ *such that for any topological rectangle* (Ω, a, b, c, d) *with* $Z^{\bullet}_{\Omega}[(ab),(cd)] \le M$ *and any* $\kappa \in \left[\frac{Z^{\bullet}_{\Omega}[(ab),(cd)]}{\varepsilon}, \varepsilon\right]$, *there exists a separating curve* $\Gamma \subset \Omega$ *between* (ab) *and* (cd) *with*

$$Z^{\bullet}_{\Omega[\Gamma,(ab)]}[(ab),\Gamma] \cdot Z^{\bullet}_{\Omega[\Gamma,(cd)]}[\Gamma,(cd)] \asymp Z^{\bullet}_{\Omega}[(ab),(cd)] \tag{10.12}$$

and

$$Z^{\bullet}_{\Omega[\Gamma,(cd)]}[\Gamma,(cd)] \in [\varepsilon\kappa, \kappa], \tag{10.13}$$

where the constants in $\asymp$ *depend on* M *only.*

A separating curve satisfying the first part of (10.12) will be called a *separator*. Informally speaking, separators are discrete curves that separate domains in two pieces in a "good" manner from the harmonic measure point of view: the product of partition functions of random walks in the two pieces is of the same order as the partition function of random walks in the original domain.

We will also need the following corollary, which yields that a topological rectangle can be split in "fair" shares.

Corollary 10.14. *Let $M > 1$. For any topological rectangle (Ω, a, b, c, d) with $M^{-1} \leq \ell_\Omega[(ab),(cd)] \leq M$, there exists a separating curve $\Gamma \subset \Omega$ between (ab) and (cd) such that*

$$\ell_{\Omega[\Gamma,(ab)]}[(ab),\Gamma] \asymp \ell_{\Omega[\Gamma,(cd)]}[(cd),\Gamma] \asymp \ell_\Omega[(ab),(cd)],$$

where the constants in $\asymp$ depend on M only.

Proof. By 1. $\Rightarrow$ 3. of Theorem 10.11, we have that $Z^\bullet_\Omega[(ab),(cd)] \asymp 1$ (where the constant depends on M only). Applying Theorem 10.13 with $\kappa = \varepsilon = \varepsilon(M)$, we obtain a simple curve Γ separating (ab) from (cd) with

$$Z^\bullet_{\Omega[\Gamma,(ab)]}[(ab),\Gamma] \asymp Z_{\Omega[\Gamma,(cd)]}[\Gamma,(cd)] \asymp Z^\bullet_\Omega[(ab),(cd)],$$

where the constants in $\asymp$ depend on M only. We then get the result by applying 3. $\Rightarrow$ 1. of Theorem 10.11 in $\Omega[\Gamma,(cd)]$ and $\Omega[\Gamma,(ab)]$. □

10.2.2 Proof of Theorem 10.10

Before presenting the proof of Theorem 10.10, we need two more lemmata. Recall from Proposition 10.2 that for a Dobrushin domain (Ω, a, b) and $c \in \partial_{ab}$,

$$\phi^{a,b}_\Omega(c \leftrightarrow \partial_{ba}) \leq \sqrt{\mathbf{HM}_\bullet(c)} = \sqrt{Z^\bullet_\Omega[c,(ba)]} \tag{10.14}$$

(the second equality is just the translation between the notations from Chapters 6 and 8 and the notations of this section). This fact can be extended to the following context. Let $\phi^{(ab),(cd)}_\Omega$ be the FK-Ising measure with wired boundary conditions on (ab) and (cd), and free elsewhere. To avoid confusion, we now use the notation $\phi^{(ab)}_\Omega$ for the Dobrushin boundary conditions in the domain (Ω, a, b) instead of $\phi^{a,b}_\Omega$ (this could indeed be confused with $\phi^{(aa),(bb)}_\Omega$).

Lemma 10.15. *For any $M > 0$, there exists $C_1 = C_1(M) > 0$ such that for any topological rectangle (Ω, a, b, c, d) with $Z^\bullet_\Omega[(ab),(cd)] \leq M$,*

$$\phi^{(ab),(cd)}_\Omega[(ab) \leftrightarrow (cd)] \leq C_1\sqrt{Z^\bullet_\Omega[(ab),(cd)]}.$$

Proof. (sketch) The proof follows the ideas of the proof of [CS12, Theorem 6.1], where the above crossing probability is computed in the scaling limit. We only sketch it here and refer to [CDCH12] for precisions.

Fix a topological rectangle (Ω, a, b, c, d) and consider the critical FK-Ising model with wired boundary conditions on (ab) and (cd) and free elsewhere. In [CS12, Proof of Theorem 6.1], two discrete holomorphic observables F_1 and F_2 are introduced in this context. Furthermore, Chelkak and Smirnov showed that there exists a unique linear combination F of F_1 and F_2, and a unique $\kappa \in \mathbb{R}$ such that a discrete version H of $\mathrm{Im}(\int^z F^2)$ satisfies the following boundary conditions:

$$H = 0 \text{ on } (da), \quad H = 1 \text{ on } (cd) \quad \text{and} \quad H = \kappa \text{ on } (ab)_{\mathrm{ext}} \cup (bc)_{\mathrm{ext}},$$

where $(ab)_{\mathrm{ext}}$ and $(bc)_{\mathrm{ext}}$ denote the set of vertices of $\partial\widehat{\Omega}$ adjacent to (ab) and (bc) respectively.

This discrete function H is $\Delta^\bullet$-subharmonic on $\Omega \setminus ((cd) \cup (da))$. Furthermore, the constant κ is shown to be in one-to-one correspondence with the quantity $\phi_\Omega^{(ab),(cd)}[(ab) \leftrightarrow (cd)]$; from [CS12, Formula 6.6], we get in particular that

$$\sqrt{\kappa} \asymp \phi_\Omega^{(ab),(cd)}[(ab) \leftrightarrow (cd)], \tag{10.15}$$

where the constants in $\asymp$ are universal.

By construction of H (see [CS12, Proof of Theorem 6.1]), we have that $H(w) = \kappa$ for some dual vertex on the dual arc $(ac)^\star$ adjacent to the vertex b. Since $H(b) - H(w) = |F(e)|^2 \geq 0$, where e is the medial edge bordering b and w, we find

$$0 = H(w) - \kappa \leq H(b) - \kappa \leq (1-\kappa)\, Z_\Omega^\bullet[b,(cd)] - \kappa Z_\Omega^\bullet[b,(da)].$$

In the second equality, we used the fact that $H - \kappa$ is $\Delta^\bullet$-subharmonic on Ω and the boundary conditions of $H - \kappa$. This leads to

$$\kappa \leq \frac{Z_\Omega^\bullet[b,(cd)]}{Z_\Omega^\bullet[b,(da)]}.$$

Using the factorization result for the harmonic measure (Proposition 10.12), we obtain that

$$\kappa \leq \frac{Z_\Omega^\bullet[b,(cd)]}{Z_\Omega^\bullet[b,(da)]} \asymp \sqrt{\frac{Z_\Omega^\bullet[b,c]\, Z_\Omega^\bullet[b,d]\, Z_\Omega^\bullet[d,a]}{Z_\Omega^\bullet[c,d]\, Z_\Omega^\bullet[b,d]\, Z_\Omega^\bullet[b,a]}} \asymp \sqrt{\frac{Z_\Omega^\bullet[a,d]\, Z_\Omega^\bullet[b,c]}{Z_\Omega^\bullet[a,b]\, Z_\Omega^\bullet[c,d]}}.$$

In the second $\asymp$, we use that $Z_\Omega^\bullet[a,b] \asymp Z_\Omega^\bullet[b,a]$ for two vertices a and b on the boundary[2]. Using the assumption $Z_\Omega^\bullet[(ab),(cd)] \leq M$, we get by

[2]Observe that the ratio $Z_\Omega^\bullet[a,b]/Z_\Omega^\bullet[b,a]$ is not equal to 1 but to $G_{\widehat{\Omega}\setminus\{b\}}(a,a)/G_{\widehat{\Omega}\setminus\{a\}}(b,b)$, where G is the Green function defined in Chapter 8. The Green function $G_{\widehat{\Omega}\setminus\{b\}}(a,a)$ is equal to the number of visits of a before reaching b for a random walk starting from a in $\widehat{\Omega}$. If a is on the boundary, one may easily see that this expected number of visits is bounded uniformly in the shape of the domain (every excursion away from a has positive probability of hitting the boundary before coming back).

Theorem 10.12 that

$$\kappa \leq \sqrt{\frac{Z_\Omega^\bullet[a,d]\, Z_\Omega^\bullet[b,c]}{Z_\Omega^\bullet[a,b]\, Z_\Omega^\bullet[c,d]}} \asymp Z_\Omega^\bullet[(ab),(cd)].$$

Hence, (10.15) implies the existence of $C_1 = C_1(M) < \infty$ such that

$$\phi_\Omega^{(ab),(cd)}[(ab) \leftrightarrow (cd)] \asymp \sqrt{\kappa} \leq C_1 \sqrt{Z_\Omega^\bullet[(ab),(cd)]}.$$

□

This leads to the following lemma.

Lemma 10.16. *Let $M > 1$. There exists $C_2 = C_2(M) > 0$ such that for any (Ω, a, b, c, d) with $Z_\Omega^\bullet[(ab),(cd)] \leq M$,*

$$\phi_\Omega^{(cd)}\big(a \leftrightarrow (cd), b \leftrightarrow (cd)\big) \leq C_2 \sqrt{\frac{Z_\Omega^\bullet[a,(cd)] Z_\Omega^\bullet[b,(cd)]}{Z_\Omega^\bullet[(ab),(cd)]}}.$$

Proof. Constants in $\asymp$ depend on M only. Note that $Z_\Omega^\bullet[a,(cd)] \leq Z_\Omega^\bullet[(ab),(cd)] \leq M$. Fix $\varepsilon = \varepsilon(M) \in (0,1/3)$ as given by Theorem 10.13. Then we have two cases:

Case 1: $Z_\Omega^\bullet[a,(cd)] > \frac{\varepsilon^2}{M} Z_\Omega^\bullet[(ab),(cd)]$ or $Z_\Omega^\bullet[b,(cd)] > \frac{\varepsilon^2}{M} Z_\Omega^\bullet[(ab),(cd)]$.

Suppose we are in the first case (the other case is handled similarly). Theorem 10.12 and (10.14) imply that

$$\begin{aligned}\phi_\Omega^{(cd)}\big(a, b \leftrightarrow (cd)\big) \leq \phi_\Omega^{(cd)}\big(b \leftrightarrow (cd)\big) &\leq \sqrt{Z_\Omega^\bullet[b,(cd)]}\\ &\leq \sqrt{\frac{M}{\varepsilon^2} \cdot \frac{Z_\Omega^\bullet[a,(cd)] Z_\Omega^\bullet[b,(cd)]}{Z_\Omega^\bullet[(ab),(cd)]}}.\end{aligned}$$

Case 2: $Z_\Omega^\bullet[a,(cd)] \leq \frac{\varepsilon^2}{M} Z_\Omega^\bullet[(ab),(cd)]$ and $Z_\Omega^\bullet[b,(cd)] \leq \frac{\varepsilon^2}{M} Z_\Omega^\bullet[(ab),(cd)]$.

Set $\kappa := \frac{\varepsilon}{M} Z_\Omega^\bullet[(ab),(cd)]$. Note that κ is smaller than ε and larger than $Z_\Omega^\bullet[a,(cd)]/\varepsilon$ and $Z_\Omega^\bullet[b,(cd)]/\varepsilon$. By Theorem 10.13 applied to κ in (Ω, a, a, c, d), there exists a separator Γ_a between a and (cd) such that

$$\frac{\varepsilon^2}{M} Z_\Omega^\bullet[(ab),(cd)] \;\leq\; Z_\Omega^\bullet[\Gamma_a,(cd)] \;\leq\; \frac{\varepsilon}{M} Z_\Omega^\bullet[(ab),(cd)]. \qquad (10.16)$$

Doing the same in (Ω, b, b, c, d), there exists a separator Γ_b of b and (cd) such that

$$\frac{\varepsilon^2}{M} Z_\Omega^\bullet[(ab),(cd)] \;\leq\; Z_\Omega^\bullet[\Gamma_b,(cd)] \;\leq\; \frac{\varepsilon}{M} Z_\Omega^\bullet[(ab),(cd)]. \qquad (10.17)$$

Set $\Omega_a = \Omega[a, \Gamma_a]$ and $\Omega_b = \Omega[b, \Gamma_b]$. Note that the two separators do not intersect otherwise their union would separate the whole arc (ab) from (cd), which is contradictory since $2\varepsilon/M < 1$ and

$$Z_\Omega^\bullet[\Gamma_a \cup \Gamma_b, (cd)] \;\le\; Z_{\Omega_a}^\bullet[\Gamma_a, (cd)] + Z_{\Omega_b}^\bullet[\Gamma_b, (cd)] \;\le\; \frac{2\varepsilon}{M} \cdot Z_\Omega^\bullet[(ab), (cd)].$$

We are thus facing the following topological picture: the two arcs Γ_a and Γ_b are not intersecting and are separating a, b and (cd). Let Γ be the arc composed of the arcs Γ_a, Γ_b and (ab). Wiring Γ, we find:

$$\phi_\Omega^{(cd)}[a, b \leftrightarrow (cd)] \;\le\; \phi_{\Omega_a}^{\Gamma_a}[a \leftrightarrow \Gamma_a]\phi_{\Omega_b}^{\Gamma_b}[b \leftrightarrow \Gamma_b]\phi_{\Omega[\Gamma,(cd)]}^{\Gamma,(cd)}[\Gamma \leftrightarrow (cd)].$$

Let us deal with the first term on the right-side. Using (10.14) and the fact that Γ_a is a separator between a and (cd), we obtain

$$\phi_{\Omega_a}^{\Gamma_a}[a \leftrightarrow \Gamma_a] \;\le\; \sqrt{Z_\Omega^\bullet[a, \Gamma_a]} \;\asymp\; \sqrt{\frac{Z_\Omega^\bullet[a, (cd)]}{Z_\Omega^\bullet[\Gamma_a, (cd)]}} \;\le\; C\sqrt{\frac{Z_\Omega^\bullet[a, (cd)]}{Z_\Omega^\bullet[(ab), (cd)]}},$$

where in the last inequality we used (10.16). Similarly:

$$\phi_{\Omega_b}^{\Gamma_b}[b \leftrightarrow \Gamma_b] \;\le\; C\sqrt{\frac{Z_\Omega^\bullet[b, (cd)]}{Z_\Omega^\bullet[(ab), (cd)]}}.$$

For the last term, Proposition 10.15 gives that

$$\begin{aligned}\phi_{\Omega[\Gamma,(cd)]}^{\Gamma,(cd)}[\Gamma \leftrightarrow (cd)] \;&\le\; \sqrt{Z_{\Omega[\Gamma,(cd)]}^\bullet[\Gamma, (cd)]}\\ &\le\; \sqrt{(1 + \tfrac{2}{3M})Z_\Omega^\bullet[(ab), (cd)]},\end{aligned}$$

where in the second inequality we used the fact that Γ is included in the union of (ab), Γ_a and Γ_b, and the two inequalities (10.16) and (10.17). Putting everything together we find

$$\phi_\Omega^{(cd)}[a, b \leftrightarrow (cd)] \;\le\; C_2\sqrt{\frac{Z_\Omega^\bullet[a, (cd)]Z_\Omega^\bullet[b, (cd)]}{Z_\Omega^\bullet[(ab), (cd)]}}.$$

□

Proof of Theorem 10.10. Let $M > 1$. Once again, constants in $\asymp$ depend only on $M > 0$. Fix a domain (Ω, a, b, c, d) with $\ell_\Omega[(ab), (cd)] \in [M^{-1}, M]$. The monotonicity allows us to treat free boundary conditions in order to get a uniform lower bound. As usual, obtaining an upper bound for the probability of a crossing $(ab) \longleftrightarrow (cd)$ on Ω is equivalent to obtaining a lower bound for the probability of a crossing $(bc)^\star \overset{\star}{\longleftrightarrow} (da)^\star$ for the critical FK-Ising model on $\Omega^\star$ by duality. Indeed, it is enough to bound from below the probability $\phi^0_{\Omega^\star}[(bc)^\star \overset{\star}{\longleftrightarrow} (da)^\star]$ of a crossing from $(bc)^\star$ to $(da)^\star$ in $\Omega^\star$ (by a constant depending on M only). Theorem 10.11 implies that whenever $\ell_\Omega[(ab),(cd)]$ is of order 1, $\ell_\Omega[(bc),(da)]$ is of order one, and it is easy to check that $\ell_{\Omega^\star}[(ab)^\star,(cd)^\star]$ is then also of order 1. In conclusion, the lower bound for the dual model follows from the lower bound for the primal one, and it implies the upper bound of Theorem 10.10.

We now focus on the proof of the lower bound. In order to do so, we use a second-moment estimate on the random variable

$$N := \sum_{u\in(ab)} \sum_{v\in(cd)} \phi^0_\Omega[u \leftrightarrow v]\, \mathbf{1}_{u\leftrightarrow v}. \tag{10.18}$$

Step 1: First moment of N. Proposition 10.2 implies

$$\phi^0_\Omega[N] = \sum_{u\in(ab)} \sum_{v\in(cd)} \phi^0_\Omega[u \leftrightarrow v]^2 \geq \sum_{w\in(ab)^\star, t\in(cd)^\star} \mathbf{HM}_\circ(w,t),$$

where $\mathbf{HM}_\circ(w,t)$ is the Δ°-harmonic measure of w seen from t in $\Omega^\star$ as defined in Chapter 8.

We now wish to prove that the right-hand side is of order 1. We only sketch the proof since the details are slightly tedious: one has to switch between several domains with extremal lengths of order 1.

The quantity $\mathbf{HM}_\circ(\cdot,\cdot)$ may be seen as $\mathbf{HM}_\bullet(\cdot,\cdot)$ in the dual graph $\tilde\Omega^\star$ obtained by putting dual-vertices in the middle of every face of Ω (then the extension of $\tilde\Omega^\star$ is exactly $\Omega^\star$). We have already mentioned that when $\ell_\Omega[(ab),(cd)]$ is of order 1, so is $\ell_{\Omega^\star}[(ab)^\star,(cd)^\star]$. It is then easy to check that the extremal length remains of order 1 in the slightly different dual graph $\Omega^\star$ mentioned just before. Theorem 10.11 thus implies that

$$\sum_{w\in(ab)^\star, t\in(cd)^\star} \mathbf{HM}_\circ(w,t) \asymp 1.$$

In conclusion, we find the existence of $c_1 = c_1(M) > 0$ such that

$$\phi^0_\Omega[N] \geq c_1.$$

Step 2: Second moment of N. Corollary 10.14 applied in (Ω,a,b,c,d) gives a separator $\Gamma \subset \Omega$ between (ab) and (cd) splitting Ω in two parts of comparable sizes (in terms of harmonic measure):

$$Z^\bullet_\Omega[(ab),\Gamma] \asymp Z^\bullet_\Omega[\Gamma,(cd)] \asymp Z^\bullet_\Omega[(ab),(cd)] \asymp 1. \tag{10.19}$$

We find:

$$\begin{aligned}\phi^0_\Omega[N^2] &= \sum_{u,v\in(ab)}\sum_{u',v'\in(cd)}\phi^0_\Omega[u\leftrightarrow v]\phi^0_\Omega[u'\leftrightarrow v']\phi^0_\Omega[u\leftrightarrow v,u'\leftrightarrow v']\\ &\le \sum_{u,v\in(ab)}\sum_{u',v'\in(cd)}\phi^0_\Omega[u\leftrightarrow \Gamma]\phi^0_\Omega[u'\leftrightarrow \Gamma]\\ &\qquad\phi^0_\Omega[v\leftrightarrow \Gamma]\phi^0_\Omega[v'\leftrightarrow \Gamma]\phi^0_\Omega[u,u'\leftrightarrow \Gamma,v,v'\leftrightarrow\Gamma].\end{aligned}$$

Let $\Omega_1=\Omega[\Gamma,(ab)]$ and $\Omega_2=\Omega[\Gamma,(cd)]$. Wiring the arc Γ, the right-hand side factorizes into the product of two terms

$$\begin{aligned}S_{\Omega_1} &= \sum_{u,v\in(ab)}\phi^\Gamma_{\Omega_1}[u\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[v\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[u,v\leftrightarrow\Gamma],\\ S_{\Omega_2} &= \sum_{u',v'\in(cd)}\phi^\Gamma_{\Omega_2}[u'\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_2}[v'\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[u',v'\leftrightarrow\Gamma].\end{aligned}$$

Assume for a moment that we possess the bounds

$$S_{\Omega_1}\le CZ^\bullet_\Omega[(ab),\Gamma]^{3/2}\qquad\text{and}\qquad S_{\Omega_2}\le CZ^\bullet_\Omega[\Gamma,(cd)]^{3/2}.\tag{10.20}$$

They imply, thanks to the definition of separators,

$$\phi^0_\Omega[N^2]\le\big(Z^\bullet_\Omega[(ab),\Gamma]\cdot Z^\bullet_\Omega[\Gamma,(cd)]\big)^{3/2}\le C'Z^\bullet_\Omega[(ab),(cd)]^{3/2}\le C'M^{3/2}.\tag{10.21}$$

The last inequality follows from Theorem 10.11. The end of the proof is split into two steps: in the first one we prove (10.20), and in the last step we implement the second-moment argument.

Step 3: Let us prove the two estimates in (10.20). We only show the first one, since the second one is handled similarly. Using (10.14) and then Lemma 10.16, we find

$$\begin{aligned}S_{\Omega_1} &= \sum_{u,v\in(ab)}\phi^\Gamma_{\Omega_1}[u\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[v\leftrightarrow\Gamma]\phi^\Gamma_{\Omega_1}[u,v\leftrightarrow\Gamma]\\ &\le C\sum_{u,v\in(ab)}\frac{Z^\bullet_\Omega(u,\Gamma)Z^\bullet_\Omega(v,\Gamma)}{\sqrt{Z^\bullet_\Omega[(uv),\Gamma]}}.\end{aligned}$$

Now, for any sequence of positive real numbers $(u_n)_{n\ge0}$, and $\alpha>0$, a comparison between series and integral implies

$$\sum_{k=1}^n u_k\left(\sum_{j=1}^k u_j\right)^{\alpha-1}\le\frac{1}{\alpha}\left(\sum_{k=1}^n u_k\right)^\alpha.\tag{10.22}$$

Say that $u\prec v$ if u and v are found in this order when going along the arc (ab) in the counterclockwise order. Recall that $Z^\bullet_\Omega[(uv),\Gamma]=$

$\sum_{x\in(uv)} Z_\Omega^\bullet[x,\Gamma]$. Therefore, (10.22) implies in this case that

$$\begin{aligned}
\sum_{u,v\in(ab)} \frac{Z_\Omega^\bullet[u,\Gamma]Z_\Omega^\bullet[v,\Gamma]}{\sqrt{Z_\Omega^\bullet[(uv),\Gamma]}} &\le 2\sum_{u<v\in(ab)} \frac{Z_\Omega^\bullet[u,\Gamma]Z_\Omega^\bullet[v,\Gamma]}{\sqrt{Z_\Omega^\bullet[(uv),\Gamma]}}\\
&= 2\sum_{v\in(ab)} Z_\Omega^\bullet[v,\Gamma]\sum_{u\in(av)} \frac{Z_\Omega^\bullet[u,\Gamma]}{\sqrt{Z_\Omega^\bullet[(uv),\Gamma]}}\\
&\le \sum_{v\in(ab)} Z_\Omega^\bullet[v,\Gamma]\sqrt{Z_\Omega^\bullet[(av),\Gamma]}\\
&\le \sum_{v\in(ab)} Z_\Omega^\bullet[v,\Gamma]\sqrt{Z_\Omega^\bullet[(ab),\Gamma]}\\
&\le Z_\Omega^\bullet[(ab),\Gamma]^{\frac{3}{2}},
\end{aligned}$$

thus giving (10.20).

Step 4: We finish the proof by implementing the second-moment estimate to obtain a lower bound on crossing probabilities with free boundary conditions. By the Cauchy-Schwarz inequality,

$$\phi_\Omega^0\big((ab)\leftrightarrow(cd)\big) = \phi_\Omega^0(N>0) = \phi_\Omega^0[(\mathbf{1}_{N>0})^2] \ge \frac{\phi_\Omega^0[N]^2}{\phi_\Omega^0[N^2]} \ge c,$$

where we used the two first steps to show the existence of $c = c(M) > 0$. This concludes the proof. □

10.3 Applications to arm exponents

To quantify connectivity properties at p_c, we introduce the notion of *arm-event*. In this section, we describe one application to arm-events of the previous theorem on crossing probabilities.

Below, ϕ denotes the unique infinite-volume measure at $q = 2$ and $p = p_c(2)$. Recall that $\Lambda_n = [-n,n]^2$ and define $\Lambda_n(x) := x + \Lambda_n$. In this section, $\Lambda_n^\star$ is seen as the dual graph of discrete domain Λ_n (dual-vertices correspond to the centers of faces of $\mathbb{Z}^2$ touching Λ_n, and dual-edges connect nearest neighbors).

Definition 10.17. An *arm crossing the annulus* $\Lambda_N \setminus \Lambda_n$ is either a self-avoiding path in $\Lambda_N \setminus \Lambda_{n-1}$ from $\partial\Lambda_n$ to $\partial\Lambda_N$ or a self-avoiding dual-open path in $\Lambda_N^\star \setminus \Lambda_{n-1}^\star$ from $\partial\Lambda_n^\star$ to $\partial\Lambda_N^\star$.

Fix a configuration ω. An arm is said to be of *type* 1 if it is composed of open edges in ω only. An arm is said to be of *type* 0 if it is composed of open dual-edges in $\omega^\star$ only. Note that an arm of type 1 is defined on $\mathbb{Z}^2$ while an arm of type 0 is defined on $(\mathbb{Z}^2)^\star$.

Definition 10.18. Fix a sequence $\sigma \in \{0,1\}^j$ and $n < N$. Let $A_\sigma(n,N)$ be the event that j *mutually edge-avoiding* arms crossing the annulus $\Lambda_N \setminus \Lambda_n$ of respective types $\sigma_1, \dots, \sigma_j$ can be found in the counterclockwise order.

For instance, $A_1(n,N)$ is the one-arm event corresponding to the existence of an open path from the inner to the outer boundaries of $\Lambda_N \setminus \Lambda_n$. For an example of $A_{1010}(n,N)$, see Fig. 10.5.

Proposition 10.19. *For any σ, there exist $\beta_\sigma > 0$ and $\gamma_\sigma > 0$ such that*

$$(n/N)^{\beta_\sigma} \;\le\; \phi[A_\sigma(n,N)] \;\le\; (n/N)^{\gamma_\sigma}.$$

Let us remark that the proof of this proposition applies *mutatis mutandis* to any random-cluster model with $1 \le q \le 4$.

Proof. **(sketch)** The upper bound has been treated in Lemma 5.35 since $A_\sigma(n,N)$ is either included in $A_1(n,N)$ or $A_0(n,N)$, so that one can take $\xi(\sigma,q) = \xi(1,q) > 0$. For the complementary bound, fix σ of length j and $n \le N$. Consider the cones

$$C_i = \Big\{(r\cos(\theta), r\sin(\theta)) : r > 0 \text{ and } \theta \in [\tfrac{2\pi i}{2j}, \tfrac{2\pi(i+1)}{2j}]\Big\},$$

where $0 \le i \le 2j$. Let $\mathcal{E}_i$ be the event that there exists an arm of type σ_i crossing $\Lambda_N \setminus \Lambda_n$ which is included in C_i. Observe that

$$\phi[\mathcal{E}_i | \mathcal{E}_{i'} : i' < i] \ge \left(\frac{n}{N}\right)^{c_1},$$

where $c_1 = c_1(j) > 0$. The existence of c_1 is guaranteed by successive applications of Theorem 10.1 in the following topological rectangles for k between $\lfloor \log_2(n) \rfloor$ and $\lfloor \log_2(N) \rfloor$:

- $T_k := C_i \cap (\Lambda_{2^{k+2}} \setminus \Lambda_{2^k})$ are crossed from "inner to outer boundary" (i.e. from $\partial\Lambda_{2^k}$ to $\partial\Lambda_{2^{k+2}}$);
- $S_k := C_i \cap (\Lambda_{2^{k+1}} \setminus \Lambda_{2^k})$ are crossed between the two sides of S_k (the sides shared with ∂C_i).

At the end, we find

$$\phi[A_\sigma(n,N)] \ge \phi\Big[\bigcap_{i \le j} \mathcal{E}_i\Big] \ge \left(\frac{n}{N}\right)^{jc_1}.$$

□

Remark 10.20. Note that the conditioning on $\mathcal{E}_{i'}$ for $i' < i$ enforces boundary conditions on C_ℓ for $\ell < 2i-1$. In particular, these boundary conditions remain bounded away from C_i, thus allowing us to apply **P5**

of Theorem 5.24 instead of Theorem 10.1 for $1 \le q \le 4$. For $q = 2$, this requirement is not necessary, since the boundary conditions are allowed to be right on the boundary of $\mathcal{E}_{i'}$. As a consequence, one may simply consider j cones instead of $2j$. We chose to expose the proof like this to illustrate that the result is not specific to $q = 2$.

It is natural to predict that there exists $\xi_\sigma \in (0, \infty)$ such that

$$\phi[A_\sigma(n, N)] = (n/N)^{\xi_\sigma + o(1)},$$

where $o(1)$ is a quantity converging to 0 as n/N goes to 0. The quantity ξ_σ is called a *critical arm-exponent*. Predictions from physics provide us with exact conjectures for these exponents, but except in special cases, even the existence of this exponent is unknown mathematically[3].

Remark 10.21. Before going further, let us mention that the probability of arm-events does not really depend on the boundary conditions. Indeed, Corollary 6.16 together with Theorem 5.45 imply that there exist $c, C > 0$ such that for any $n < N$ and any boundary conditions on $\partial\Lambda_{2N}$,

$$c\phi[A_\sigma(n, N)] \le \phi^\xi_{\Lambda_{2N}}\big[A_\sigma(n, N)\big] \le C\phi[A_\sigma(n, N)]. \tag{10.23}$$

From now on, we will state all results in infinite-volume directly.

10.3.1 Quasi-multiplicativity and localization

The following statements yield two important properties of arm-events. These properties are crucial when working with probabilities of arm-events. Let us start by the so-called "quasi-multiplicativity property".

Theorem 10.22 (Quasi-multiplicativity)**.** *Fix a sequence σ. There exist $c = c(\sigma) > 0$ and $C = C(\sigma) < \infty$ such that*

$$c\phi\big[A_\sigma(n_1, n_2)\big]\phi\big[A_\sigma(n_2, n_3)\big] \le \phi\big[A_\sigma(n_1, n_3)\big] \le C\phi\big[A_\sigma(n_1, n_2)\big]\phi\big[A_\sigma(n_2, n_3)\big]$$

for every $n_1 < n_2 < n_3$.

The second property is the following "localization of arms". Let $\delta > 0$; for a sequence σ of length j, consider $2j$ points $x_1, x_2, \dots, x_{2j}$ found in clockwise order on the boundary of Λ_n, with the additional condition that $|x_{k+1} - x_k| \ge \delta n$ for any $k < 2j$ and $|x_{2j} - x_1| \ge \delta n$. Similarly, consider $2j$ points $y_1, \dots, y_{2j}$ found in clockwise order on the boundary of Λ_N, with the additional condition that $|y_{k+1} - y_k| \ge \delta N$ for any $k < 2j$ and $|y_{2j} - y_1| \ge \delta N$. The sequence of intervals $(I_k = [x_{2k-1}, x_{2k}])_{k \le j}$ and $(J_k = [y_{2k-1}, y_{2k}])_{k \le j}$ are called *δ-well separated landing sequences.*

[3]For $q = 2$, ξ_1 as well as $\xi_{101\dots1}$ and $\xi_{101\dots10}$ (with respectively $2n - 1$ and $2n$ arms) can be computed using SLE(16/3).

Let $A_\sigma^{I,J}(n,N)$ be the event that $A_\sigma(n,N)$ occurs but that in addition to this, the arms can be chosen in such a way that the k-th arm goes from I_k to J_k.

Theorem 10.23. *Let σ be a sequence. For any $\delta > 0$ there exists $C = C(\sigma) < \infty$ such that, for any $2n \le N$ and any choice of δ-well-separated landing sequences I, J at radii n and N,*

$$\phi\big[A_\sigma^{I,J}(n,N)\big] \le \phi\big[A_\sigma(n,N)\big] \le C\phi\big[A_\sigma^{I,J}(n,N)\big].$$

The theorem asserts that forcing arms to start and finish in some prescribed areas of the boundary does not cost more than a bounded multiplicative constant to the probability.

10.3.2 Proofs of Theorems 10.22 and 10.23

In this section we sketch the proof of Theorem 10.22. At the heart of the proof is the notion of well-separated arms. In words, well-separated arms are arms whose end-points are at macroscopic distance (we also add the technical condition that they extend slightly outside the boxes, see Fig. 10.5). In what is next, let x_k and y_k be the end-points[4] of the arm γ_k on the inner and outer boundary respectively. The paths $\gamma_1, \dots, \gamma_j$ are said to be *well-separated* if

- points y_k are at distance larger than $2\delta N$ from each others,
- points x_k are at distance larger than $2\delta n$ from each others,
- for every k, y_k is σ_k-connected to distance δN of $\partial\Lambda_N$ in $\Lambda_{\delta N}(y_k)$,
- for every k, x_k is σ_k-connected to distance δn of $\partial\Lambda_n$ in $\Lambda_{\delta n}(x_k)$.

Let $A_\sigma^{\rm sep}(n,N)$ be the event that $A_\sigma(n,N)$ holds true and there exist arms realizing $A_\sigma(n,N)$ which are δ-well-separated. Note that while the notation does not suggest it, the event depends on δ.

Most of this section will be devoted to the following result yielding that forcing the arm to be well-separated changes the probability by a bounded multiplicative constant only.

Theorem 10.24. *Fix σ and $\delta > 0$ small enough. There exists $C = C(\sigma) > 0$ such that for every $n < N$,*

$$\phi\big[A_\sigma^{\rm sep}(n,N)\big] \le \phi\big[A_\sigma(n,N)\big] \le C\phi\big[A_\sigma^{\rm sep}(n,N)\big].$$

This theorem is classical in the theory of percolation. It has been proved several times. Theorem 10.10 will exactly be the tool required to adapt the

[4] Since an arm is self-avoiding, x_k and y_k are uniquely defined. Furthermore, x_k and y_k are on the primal graph if the path is of type 1, and on the dual graph it is of type 0.

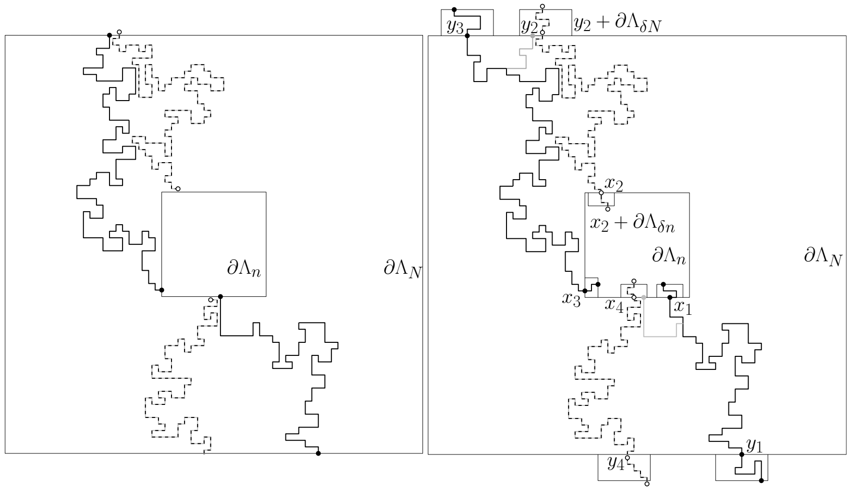

Figure 10.5: On the left, the four-arm event $A_{1010}(n,N)$. On the right, the event $A^{\text{sep}}_{1010}(n,N)$ with well-separated arms. Note that these arms are not at macroscopic distance of each other inside the domain, but only at their end-points.

proofs valid for percolation to the context of the FK-Ising model. *Since the proof for percolation can be found in the literature and since this proof is a modification of it, we will only sketch it here.* In what follows, we keep the notation $\asymp$ introduced in the previous section.

Before doing so, let us mention that this theorem is extremely useful. For instance, Theorems 10.22 and 10.23 follow classically from it. We illustrate this fact by proving Theorem 10.22 (the derivation of Theorem 10.23 is left to the reader).

Proof of Theorem 10.22 (sketch). There exists $c > 0$ such that for $n_1 \leq n_2 \leq n_3$:

$$\begin{aligned}
\phi\big[A_\sigma(n_1,n_3)\big] &= \phi\big[A_\sigma(n_1,n_3)\big|A_\sigma(2n_2,n_3)\big]\cdot\phi\big[A_\sigma(2n_2,n_3)\big]\\
&\leq \phi\big[A_\sigma(n_1,n_2)\big|A_\sigma(2n_2,n_3)\big]\cdot\phi\big[A_\sigma(2n_2,n_3)\big]\\
&\asymp \phi\big[A_\sigma(n_1,n_2)\big]\cdot\phi\big[A_\sigma(2n_2,n_3)\big]\\
&\asymp \phi\big[A_\sigma(n_1,n_2)\big]\cdot\phi\big[A^{\text{sep}}_\sigma(2n_2,n_3)\big]\\
&\leq c\phi\big[A_\sigma(n_1,n_2)\big]\cdot\phi\big[A_\sigma(n_2,n_3)\big],
\end{aligned}$$

where in the third line we used the domain Markov property and (10.23), in the fourth, Theorem 10.24, and in the fifth, the following claim:

Claim: There exists $c>0$ such that for any $n \le N$,

$$\phi\big[A_\sigma(n,N)\big] \ge c\,\phi\big[A_\sigma^{\mathrm{sep}}(2n,N)\big].$$

Proof of the Claim. We only sketch the proof. Condition on the well-separated arms reaching $\partial\Lambda_{2n}$. Let x_i be the end-points of the paths and consider j thin rectangles of length n and width $4n\delta$, with one of the short edges centered on x_i, for $1 \le i \le j$.

Theorem 10.10 implies that there exists a path of type σ_i in each of these rectangles from $\partial\Lambda_{2n}$ to $\partial\Lambda_n$ with positive probability. Furthermore, the end-points x_i are σ_i-connected to distance $2\delta n$ in Λ_{2n} thanks to the conditioning on $A_\sigma^{\mathrm{sep}}(2n,N)$. The two paths are therefore connected to the path of type σ_i from x_i to distance $2\delta n$ with positive probability, thus giving the claim. ◇

Now, there exists $c>0$ such that

$$\begin{aligned}
\phi\big[A_\sigma(n_1,n_3)\big] &\ge \phi\big[A_\sigma^{\mathrm{sep}}(n_1,n_2)\cap A_\sigma^{\mathrm{sep}}(2n_2,n_3)\cap A_\sigma(n_1,n_3)\big]\\
&\ge c_1\phi\big[A_\sigma^{\mathrm{sep}}(n_1,n_2)\cap A_\sigma^{\mathrm{sep}}(2n_2,n_3)\big]\\
&\asymp\ \phi\big[A_\sigma^{\mathrm{sep}}(n_1,n_2)\big]\cdot\phi\big[A_\sigma^{\mathrm{sep}}(2n_2,n_3)\big]\\
&\asymp\ \phi\big[A_\sigma(n_1,n_2)\big]\cdot\phi\big[A_\sigma(2n_2,n_3)\big]\\
&\ge\ \phi\big[A_\sigma(n_1,n_2)\big]\cdot\phi\big[A_\sigma(n_2,n_3)\big].
\end{aligned}$$

Once again, in the second line, we used the fact that well-separated arms can be glued together in the annulus $\Lambda_{2n_2}\smallsetminus\Lambda_{n_2}$. The proof is similar to the proof of the claim, where the thin rectangles are replaced by thin disjoint tubes (of "width" δn_2) going around x_i to y_i for every $1 \le i \le j$. □

We now sketch the proof of Theorem 10.24. Let us start with the following two lemmata.

Lemma 10.25. *For any $\varepsilon>0$, there exists $T>0$ such that for any $n>0$ and any boundary conditions ξ,*

$$\phi^\xi_{\Lambda_{2n}\smallsetminus\Lambda_n}\big[\text{there exist } T \text{ disjoint arms crossing } \Lambda_{2n}\smallsetminus\Lambda_n\big] \le \varepsilon.$$

Proof. If T arms are crossing from the inner to the outer sides, then $T/4$ arms are actually crossing one of the following rectangles: $[-2n,2n]\times[n,2n]$, $[-2n,2n]\times[-2n,-n]$, $[n,2n]\times[-2n,2n]$ and $[-2n,-n]\times[-2n,2n]$. It is thus sufficient to show that for $\varepsilon>0$, there exists $T>0$ such that the probability of T disjoint vertical crossings of $[0,4n]\times[0,n]$ is bounded by ε uniformly in n and the boundary conditions. In fact, we only need to prove that conditionally on the existence of k

crossings, the existence of another crossing is bounded from above by some constant $c < 1$, since the probability of T crossings is then bounded by c^T.

In order to prove this statement, condition on the k-th left-most crossing γ_k. Assume without loss of generality that γ_k is a dual crossing. Construct a subdomain of $[0,4n]\times[0,n]$ by considering the connected component of $[0,4n]\times[0,n]\setminus\gamma_k$ containing $\{4n\}\times[0,n]$. The configuration in Ω is a random-cluster configuration with boundary conditions ξ on the outside and free elsewhere (*i.e.* on the arc bordering the dual arc γ_k). Now, Theorem 10.10 implies that Ω is crossed from left to right by a primal *and* a dual crossing with probability bounded from below by a universal constant. Indeed, cut the domain Ω into two domains $\Omega_1 = \Omega\cap[0,4n]\times[0,n/2]$ and $\Omega_2 = \Omega\cap[0,4n]\times[n/2,n]$ and assume Ω_1 is horizontally crossed and Ω_2 is horizontally dual crossed. This prevents the existence of an additional vertical crossing or dual crossing, therefore implying the claim. □

Remark 10.26. The previous proof harnesses Theorem 10.10 in a crucial way, the left boundary of Ω being possibly very rough. Crossing estimates for standard rectangles (even with uniform boundary conditions) would not have been strong enough.

Let $\delta > 0$ and $n \geq 1$. Define B_n to be the event that for some $j \geq 1$, the annulus $\Lambda_{2n}\setminus\Lambda_n$ is crossed by disjoint arms $\gamma_1,\dots,\gamma_j$ of type $\sigma_1,\dots,\sigma_j$ but there is no δ-well-separated arms $\tilde\gamma_1,\dots,\tilde\gamma_j$ of type $\sigma_1,\dots,\sigma_j$ such that $\tilde\gamma_i$ is in the σ_i-cluster of γ_i for every $i \leq j$ (σ_i-cluster means primal cluster if $\sigma_i = 1$ and dual cluster otherwise).

Lemma 10.27. *Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\phi(B_n) \leq \varepsilon$ for any $n \geq 1$.*

Proof. Consider T large enough so that there exist more than T disjoint arms of $\Lambda_{2n}\setminus\Lambda_n$ with probability less than ε. From now on, we assume that there are at most T disjoint arms crossing the annulus.

Fix $\delta > 0$ such that uniformly in *any subdomain* $D \subset \Lambda_n\setminus\Lambda_{\delta n}$ and any boundary conditions on ∂D, there is no crossing from $\partial\Lambda_{\delta n}$ to $\partial\Lambda_n$ in D with probability $1-\varepsilon$ [5]. The existence of $\delta > 0$ can be proved easily using Theorem 10.10.

We may therefore assume that no arm ends at distance less than δn of a corner of $\Lambda_{2n}\setminus\Lambda_n$ with probability $1-8\varepsilon$. This enables us to restrict our attention to vertical crossings in the rectangle $[-2n,2n]\times[n,2n]$ for instance.

[5]Note that this claim is slightly stronger than simply the fact that the annulus is not crossed. Indeed, even if the crossing is forced to remain in D, the boundary conditions on ∂D could help the existence of a crossing.

Figure 10.6: The construction of open and closed paths extending the crossing and preventing other crossings from finishing close to the path.

Now, condition on the left-most crossing γ of $R_n = [-2n,2n]\times[n,2n]$ and set y to be the ending point of γ on the top. Without loss of generality, let us assume that the crossing is of type 1. As before, construct the domain Ω to be the connected component of the right side of R_n in $R_n \setminus \gamma$.

For $k \geq 1$, let $A_k = \Lambda_{\delta^k n}(y) \setminus \Lambda_{\delta^{k+1} n}(y)$. We can assume with probability $1-\varepsilon/T$ that no vertical crossing lands at distance $\delta^3 n$ of y by making the following construction:

- $\Omega \cap A_1$ contains an open path disconnecting y from the right-side of R_n;
- $\Omega \cap A_2$ contains a dual-open path disconnecting y from the right-side of R_n.

By choosing $\delta > 0$ small enough, Theorem 10.10 shows that the paths in this construction exist with probability $1 - \varepsilon/T > 0$ independent of the shape of Ω.

We may also show that γ can be extended to the top of A_j by constructing a path in $A_j \setminus (R_n \setminus \Omega)$ from γ to the top of A_j (this occurs once again with probability $c > 0$ independently of Ω and the configuration outside A_j). Therefore, the probability that there exists $4 \leq j \leq k$ such that this happens is larger than $1 - (1-c)^{k-3}$. We find that with probability $1-\varepsilon/T-(1-c)^{k-3}$ the path γ can be modified into a self-avoiding crossing which is well-separated (on the outside) from any crossing on the right of it by a distance at least $(\delta^3 - \delta^4)n$ and that this crossing is extended to distance at least δ^k above its end-point. We may choose k large enough that the previous probability is larger than $1-2\varepsilon/T$. One may also do the same for the inner boundary. Iterating the construction T times, we find that $\phi(B_n) \leq 12\varepsilon$ with δ^k as a distance of separation. □

Proof of Theorem 10.24 (sketch). The lower bound $\phi[A_\sigma^{\mathrm{sep}}(n,N)] \leq \phi[A_\sigma(n,N)]$ is straightforward. Let L and K be such that $4^{L-1} < n \leq 4^L$

and $4^{K+1} \leq N < 4^{K+2}$.

Recall the definition of B_k and choose δ small enough that $\phi(B_k) \leq \varepsilon$ (the existence of such δ is guaranteed by Lemma 10.27). Set $\tilde{B}_k = B_{2k}$. We may decompose the event $A_\sigma(n,N)$ with respect to the smallest and largest scales at which the event $\tilde{B}_k^c$ occurs. We find

$$\begin{aligned}&\phi\big[A_\sigma(n,N)\big]\\ &\leq \sum_{L\leq \ell\leq k\leq K} \phi\big[\tilde{B}_L \cap\cdots\cap \tilde{B}_{\ell-1}\cap \tilde{B}_\ell^c \cap A_\sigma(2^{2\ell},2^{2k+1})\cap \tilde{B}_k^c\cap \tilde{B}_{k+1}\cap\cdots\cap \tilde{B}_K\big].\end{aligned}$$

Since the annuli $\Lambda_{2^{2k}} \setminus \Lambda_{2^{2k-1}}$ are separated by macroscopic areas, we can use (10.23) repeatedly to find the existence of $C > 0$ such that

$$\begin{aligned}&\phi(A_\sigma(n,N))\\ &\leq \sum_{L\leq \ell\leq k\leq K} C^{K-L-(k-\ell)}\phi[\tilde{B}_L]\cdots\phi[\tilde{B}_{\ell-1}]\phi[\tilde{B}_\ell^c\cap A_\sigma(2^{2\ell},2^{2k+1})\cap \tilde{B}_k^c]\phi[\tilde{B}_{k+1}]\cdots\phi[\tilde{B}_K]\\ &\leq \sum_{L\leq \ell\leq k\leq K} (C\varepsilon)^{N-L-(k-\ell)}\phi[\tilde{B}_\ell^c\cap A_\sigma(2^{2\ell},2^{2k+1})\cap \tilde{B}_k^c].\end{aligned}$$

Now, we see that

$$\tilde{B}_\ell^c\cap A_\sigma(2^{2\ell},2^{2k+1})\cap \tilde{B}_k^c \;\subset\; A_\sigma^{\mathrm{sep}}(2^{2\ell},2^{2k+1}).$$

Furthermore, the construction used in the proof of Proposition 10.19 can easily be adapted to show that

$$\phi\big[A_\sigma^{\mathrm{sep}}(2^{2\ell},2^{2k+1})\big] \leq C_1\alpha^{K-L-(k-\ell)}\phi\big[A_\sigma^{\mathrm{sep}}(n,N)\big]$$

for some universal constant $\alpha > 1$ and $C_1 = C_1(\delta) > 0$. Altogether, we find that

$$\phi[A_\sigma(n,N)] \leq \phi[A_\sigma^{\mathrm{sep}}(n,N)]C_1 \sum_{L\leq \ell\leq k\leq K}(C\varepsilon\alpha)^{N-L-(k-\ell)} \leq C_2\phi[A_\sigma^{\mathrm{sep}}(n,N)]$$

provided that $\varepsilon > 0$ is small enough, which can be guaranteed by taking δ small enough. The constant C_2 depends on δ only. □

10.3.3 Universal exponents

Theorem 10.28. *For every $1 \leq n \leq N$,*

$$\phi\big[A_{10110}(n,N)\big] \asymp \Big(\frac{n}{N}\Big)^2, \quad \phi\big[A_{10}^{\mathbb{H}}(n,N)\big] \asymp \frac{n}{N}, \quad \phi\big[A_{101}^{\mathbb{H}}(n,N)\big] \asymp \Big(\frac{n}{N}\Big)^2,$$

where $A_\sigma^{\mathbb{H}}(n,N)$ is the existence of j arms crossing $\Lambda_N \setminus \Lambda_n$ and contained in the upper-half plane $\mathbb{H} = \mathbb{Z}\times\mathbb{N}$.

Proof. (sketch) We only give a sketch of the proof of the first statement; the others are derived from similar arguments (actually the arguments are slightly simpler). By quasi-multiplicativity (Theorem 10.22), we only need to show that $\phi\big[A_{10110}(0,N)\big] \asymp N^{-2}$.

Lower bound. Fix $N > 0$. Consider the following construction: assume that there exists a dual-open dual-path crossing $[-2N,2N] \times [-N,0]$ horizontally and an open path crossing $[-2N,2N] \times [0,N]$ horizontally. This happens with probability bounded from below by $c_1 > 0$ not depending on N. By conditioning on the lowest open self-avoiding path Γ crossing horizontally, the configuration in the domain Ω above Γ is a random-cluster configuration with wired boundary conditions on Γ and undetermined boundary conditions on the other three sides (i.e. $\partial\Omega \cap \partial\Lambda_{2N}$).

Assume that $[-N,0] \times [-2N,2N]$ is crossed vertically by an open path staying in Ω, and that $[\frac{1}{2}, N-\frac{1}{2}] \times [-2N+\frac{1}{2}, 2N-\frac{1}{2}]$ is crossed vertically by a dual-open path staying in $\Omega^\star$. The probability of this event is once again bounded from below uniformly in N and Ω thanks to Theorem 10.10. Here again, uniform crossing estimates for standard rectangles would not have been sufficiently strong to imply this result.

Summarizing, all these events occur with probability larger than $c_2 > 0$. Moreover, the existence of all these crossings implies the existence of a vertex in Λ_N with five arms emanating from it, since one may observe that $[-N,N] \times [-2N,2N]$ is crossed by both a primal and a dual vertical crossing, and that there exists x on Γ at the interface between two such crossings. Such an x has five arms emanating from it and going to distance at least N[6]. The union bound implies

$$N^2 \phi[A_{10110}(0,N)] \geq c_2.$$

Upper bound. Recall that it suffices to show the upper bound for chosen landing sequences thanks to Theorem 10.23. Consider the event A_x, see Fig. 10.7, that five mutually edge-avoiding arms $\gamma_1, \ldots, \gamma_5$ of respective types 10110 are in such a way that

- γ_1 starts at x and finishes on $\{N\} \times [\frac{N}{4}, \frac{N}{2}]$;
- γ_2 starts at $x + (\frac{1}{2}, \frac{1}{2})$ and finishes on $[-\frac{N}{2} - \frac{1}{2}, \frac{N}{2} + \frac{1}{2}] \times \{N + \frac{1}{2}\}$;
- γ_3 starts at x and finishes on $\{-N\} \times [-\frac{N}{2}, \frac{N}{2}]$;
- γ_4 starts at x and finishes on $[-\frac{N}{2}, \frac{N}{2}] \times \{-N\}$;
- γ_5 starts at $x + (\frac{1}{2}, \frac{1}{2})$ and finishes on $\{N + \frac{1}{2}\} \times [-\frac{N}{2} + \frac{1}{2}, -\frac{N}{4} + \frac{1}{2}]$.

[6]The path Γ provides us with two primal paths going from x to the boundary. Since Γ is the lowest crossing of $[-N,0] \times [-2N,2N]$, there is an additional dual crossing below Γ. Finally, since x is at the interface between a primal and a dual crossing above Γ, we obtain the two additional paths. Since x is in Λ_N and that arms connect x and $\partial\Lambda_{2N}$, we deduce that these arms extend to distance at least N.

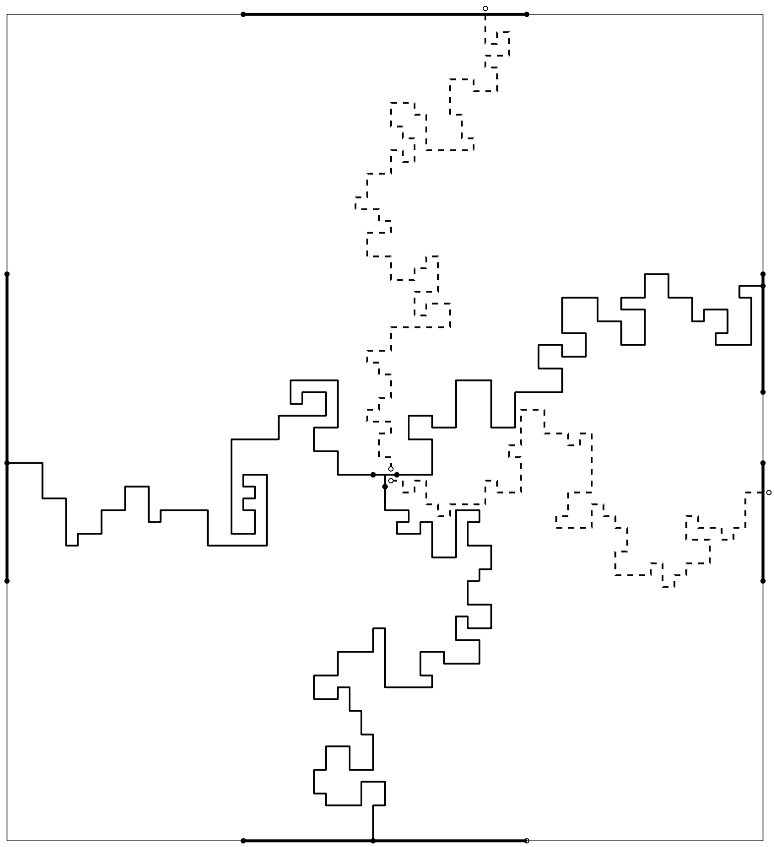

Figure 10.7: Only one vertex per box can satisfy the following topological picture.

One may easily show using the same techniques as in the previous section that $\phi[A_x] \asymp \phi[A_{10110}(0,N)]$ for every $x \in \Lambda_{N/2}$. In particular,

$$N^2\phi[A_{10110}(0,N)] \asymp \sum_{x\in\Lambda_{N/2}} \phi[A_x] \leq 1.$$

The last inequality is due to the fact that the events A_x are disjoint (topologically no two vertices in Λ_N can satisfy the events in question). $\square$

This result has an interesting corollary.

Corollary 10.29. *There exist $\alpha > 0$ and $c, C > 0$ such that for every $0 < n < N$,*

$$\phi\big[A_{101010}(n,N)\big] \le C\left(\frac{n}{N}\right)^{2+\alpha},$$
$$\phi\big[A_{1010}(n,N)\big] \ge c\left(\frac{n}{N}\right)^{2-\alpha}.$$

The "six-arm" event $A_{101010}(n,N)$ is related to the property R2 in Theorem 9.15. The "four-arm event" $A_{1010}(n,N)$ is important for the existence of so-called pivotal vertices (see Chapter 11).

Proof. **(sketch)** Fix $n < N$, we have

$$\phi\big[A_{101010}(n,N)\big] \asymp \phi\big[A_{101010}(n,N) \cap \{\Lambda_n \not\leftrightarrow [-N,N] \times \{-N\}\}\big].$$

The fact that no arm finishes on $[-N,N] \times \{-N\}$ enables us to condition on five arms as follows. Start by conditioning on the two lowest primal open arms. This discovers two primal arms and one dual arm below them. Above these primal arms, condition on the two "lowest" dual arms starting from the origin (meaning the first one discovered when going respectively clockwise and counter-clockwise). It can be shown that the probability of the sixth arm in the undiscovered area is smaller than the probability of $\phi^0_{\Lambda_N}\big[A_0(n,N)\big]$ uniformly in the five arms on which we conditioned. Therefore,

$$\begin{aligned}\phi\big[A_{101010}(n,N) \cap \{\Lambda_n \not\leftrightarrow [-N,N] \times \{-N\}\}\big]&\\ \le \phi^0_{\Lambda_N}\big[A_0(n,N)\big]\phi\big[A_{10101}(n,N)\big].&\end{aligned}$$

The result follows from Theorem 10.28. $\square$

Chapter 11

The FK-Ising model away from criticality

Similarly to the previous chapter, this one is devoted to a deeper study of the FK-Ising model and is aimed at specialists.

It is now time to leave the critical regime of the FK-Ising model to explore the off-critical regime. More precisely, we discuss two *a priori* different notions of "correlation length". We then study their behavior when p approaches criticality. After a small detour where we present a monotone coupling for random-cluster configurations, we discuss the violation of a classical scaling relation valid for Bernoulli percolation.

In this section, we fix $q = 2$ and we drop it from the notations.

11.1 Correlation length of the Ising model

Theorem 5.18 implies that correlations decay exponentially when $p < p_c$, but at which speed? In this section, we answer this question by computing the *correlation length*

$$\xi_p(x) = -\Big[\lim_{n\to\infty} \tfrac{1}{n}\log\phi_p(0 \longleftrightarrow nx)\Big]^{-1}.$$

Recall that Lemma 5.31 implies that this quantity is well defined.

11.1.1 Short proof of exponential decay of correlations

We start by providing an alternative proof of Theorem 5.18 (in the special case of $q = 2$) based on the fermionic observable. Let $p < p_c$.

Consider the FK-Ising model on the strip $\mathcal{S}_\ell$ of height $\ell > 0$ with wired boundary conditions on the bottom and free boundary conditions on the top. In what follows, F denotes the edge fermionic observable in this Dobrushin domain. As before, one should not be alerted by the fact that the domain is infinite. Indeed, one may easily define the fermionic observable by taking the limit as $n \to \infty$ of the observable in domains $[-n, n] \times [0, \ell]$ with $a = (n, 0)$ and $b(-n, 0)$.

Define e_k to be medial-edge bordering $(0, k)$ on the north-west side. Label some of the medial-edges around $(0, k)$ and $(0, k+1)$ as e_k, e_{k+1}, e, e', e'', f and f' as shown in Figure 11.1.

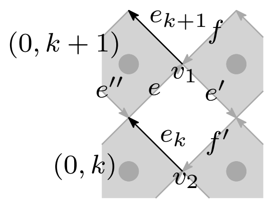

Figure 11.1: The labeling of medial-edges around e_k and e_{k+1} used in Step 1.

Proposition 6.10 and Lemma 9.3 have a very important consequence: around a medial-vertex, the value of the (edge) fermionic observable on one medial-edge can be expressed in terms of its values on *any two other medial-edges*. For instance, (6.2) can be projected around v_1 orthogonally to $F(f)$, so that a relation is obtained between projections of $F(e)$, $F(e')$ and $F(e_{k+1})$. Moreover, the complex argument (modulo π) of F is known (Lemma 9.3) for each edge so that the relation between projections can be written as a relation between $F(e)$, $F(e')$ and $F(e_{k+1})$ themselves. Doing the same with v_2, we find two relations

$$e^{-i\pi/4}F(e) = \cos(\pi/4 - \alpha)F(e_{k+1}) - \cos(\pi/4 + \alpha)e^{-i\pi/2}F(e'), \qquad (11.1)$$

$$e^{-i\pi/4}F(e) = \cos(\pi/4 + \alpha)F(e_k) - \cos(\pi/4 - \alpha)e^{-i\pi/2}F(e''). \qquad (11.2)$$

The invariance (in the strip) under horizontal translations gives

$$F(e') = F(e''). \qquad (11.3)$$

Moreover, the symmetry under the orthogonal symmetry with respect to the imaginary axis implies that

$$F(e) = e^{i\pi/4}\overline{F(e')} = e^{-i\pi/4}F(e'). \qquad (11.4)$$

(Indeed, if for a configuration ω, e belongs to γ and the winding is equal to W, in the reflected configuration ω', e' belongs to $\gamma(\omega')$ and the winding is equal to $\pi/2 - W$.) Plugging (11.3) and (11.4) into (11.1) and (11.2) leads to

$$\begin{aligned} F(e_{k+1}) &= \mathrm{e}^{-\mathrm{i}\pi/4}\frac{1+\cos(\pi/4+\alpha)}{\cos(\pi/4-\alpha)}F(e) \\ &= \frac{[1+\cos(\pi/4+\alpha)]\cos(\pi/4+\alpha)}{[1+\cos(\pi/4-\alpha)]\cos(\pi/4-\alpha)}F(e_k) = c_1 F(e_k), \end{aligned}$$

where $\alpha = \alpha(p)$ was defined in (6.2) and $c_1 = c_1(p) > 0$ is a universal constant. Remember that $\alpha(p) > 0$ since $p < p_c$, so that the multiplicative constant is less than 1. Using Lemma 6.11 and the previous equality inductively, we find that for every $\ell > 0$,

$$\phi^{\mathrm{dobr}}_{p,\mathcal{S}_\ell}[(0,\ell) \leftrightarrow \mathbb{Z}] = |F(e_\ell)| = c_1^\ell |F(e_1)| \le c_1^\ell, \tag{11.5}$$

where $\phi^{\mathrm{dobr}}_{p,\mathcal{S}_\ell}$ is the FK-Ising measure in $\mathcal{S}_\ell$ with Dobrushin boundary conditions. The last inequality is due to the fact that the observable has complex modulus less than 1.

The comparison between boundary conditions implies that for any $\ell \ge 0$ and $v \in \partial\Lambda_\ell$,

$$\phi^0_{p,\Lambda_\ell}(0 \longleftrightarrow v) \le \phi^{\mathrm{dobr}}_{p,\mathcal{S}_\ell}[(0,\ell) \leftrightarrow \mathbb{Z}] \le \exp[-c_1\ell].$$

We may once again use Lemma 4.23 (more precisely the steps in (6.14)) to conclude the proof.

11.1.2 Correlation length of the FK-Ising model

The computation of the correlation length of the FK-Ising model is known for a long time thanks to the Ising model. In [MW73], McCoy and Wu derived a closed formula for the correlation length of the Ising model which leads to an explicit formula for the correlation length of the FK-Ising model using the Edwards-Sokal coupling. Nevertheless, in a recent paper [Mes06], Messikh raised a surprising new connection between the Ising model and random walks by noticing that this formula is connected to large deviations estimates for the simple random walk. In the following, we exhibit what we believe to be the first derivation of the correlation length based on this connection with the simple random walk.

In order to state the connection between the correlation length of the FK-Ising model and random walks, we introduce the *massive Green function* G_m, defined in the bulk as

$$G_m(x,y) := \mathbb{E}^x\Big[\sum_{n\ge 0} m^n \mathbf{1}_{X_n=y}\Big],$$

where $m < 1$ and $\mathbb{E}^x$ is the law of a simple random walk starting at x. This quantity can be interpreted as the expected number of visits to y for a random walk starting from x and killed at each step with probability $1-m$. It is very convenient to encode the large deviations behavior of the simple random walk.

We are now in a position to state the main result of this section.

Theorem 11.1 (Beffara, Duminil-Copin [BDC12b]). *For $p < p_c$ and any $x \in \mathbb{Z}$,*

$$\xi_{p,2}(x) = \Big[\lim_{n\to\infty} -\tfrac{1}{n}\log G_{\cos[2\alpha(p)]}(0,nx)\Big]^{-1} \tag{11.6}$$

where $\alpha(p)$ is defined in Lemma 6.10.

The behavior of the simple random walk is very well understood. In particular, one can compute the right-hand side of (11.6) explicitly (we refer to standard text books on large deviations). Furthermore, the Edwards-Sokal coupling together with the previous theorem leads to the following consequence for the Ising model (we omit the details).

Corollary 11.2. *Let $\beta < \beta_c$ and let μ_β denote the (unique) infinite-volume Ising measure at inverse temperature β; fix $a = (a_1, a_2) \in \mathbb{Z}^2$. Then,*

$$\tau_\beta(a) = \lim_{n\to\infty} -\tfrac{1}{n}\ln\left(\mu_\beta[\sigma(0)\sigma(na)]\right) = a_1\text{arcsinh } sa_1 + a_2\text{arcsinh } sa_2,$$

where s solves the equation

$$\sqrt{1+(sa_1)^2} + \sqrt{1+(sa_2)^2} = \sinh 2\beta + \frac{1}{\sinh 2\beta}.$$

This result is exactly the formula found by McCoy and Wu. The quantity $\tau_\beta(a)$ is called the *inverse correlation length* of the Ising model in direction a and is equal to 1 divided by the so-called correlation length of the Ising model (which is itself equal to $\xi_{p(\beta)}(a)$ by the Edwards-Sokal coupling).

The interpretation in terms of random walks has a neat corollary. The convergence of random walks to Brownian motion and the isotropy of the latter imply that the *Wulff shape* (see [DKS92] for details on this beautiful object)

$$W_\beta := \{x \in \mathbb{C} : \langle x|u\rangle \le \tau_\beta(u), u \in \mathbb{U}\},$$

of the 2D Ising model becomes (when properly rescaled) a Euclidean ball when approaching criticality. Indeed, the mass m tends to zero and the massive Green function thus converges to the Green function itself. This isotropy (as $\beta \nearrow \beta_c$) of the Wulff shape can be thought of as a glimpse of conformal invariance. More precisely, for every $a \in \mathbb{C}$, one may expand the equation defining s in $\beta - \beta_c$ to find that $s \sim 4|\beta-\beta_c|/|a|$ as β tends to β_c.

Inserting this asymptotic in the formula for the inverse correlation length leads to the following result.

Corollary 11.3. *For $z \in \mathbb{C}$, if $\tau_\beta(z)$ denotes the quantity defined in the previous corollary,*

$$\lim_{\beta \nearrow \beta_c} \frac{\tau_\beta(z)}{(\beta_c - \beta)} = 4|z|. \tag{11.7}$$

Before diving into the actual proof, here is a short outline of the strategy. Exponential decay in the strip was already shown in the previous section: it is an essentially one-dimensional computation. We now aim to refine it into a two-dimensional version for correlations between two points 0 and a in the bulk, and once again the observable is used in a crucial way. The basic step, namely obtaining local linear relations between the values of the observable, is the same, although it is complicated by the lack of translation invariance. The point is that the relations thus obtained will be interpreted as follows: the observable is also massive harmonic when $p \neq p_c$ (see Lemma 11.4 below). Since $G_m(x,y)$ is massive harmonic in the variable x for every $x \neq y$, it is possible to compare both quantities.

While the strategy is fairly clear, some technical issues appear very quickly. The main problem is that we are interested in correlations in the bulk. The observable can be defined directly in the bulk (see below) but it only provides us with a lower bound on the correlations. In order to obtain an upper bound, we have to introduce an "artificial" domain (that will be $T(a)$ below), which needs to have two features: the observable in it can be well estimated, and at the same time correlations inside it have comparable probabilities to correlations in the bulk.

There is a natural recipe to construct such a domain: the second condition is in fact equivalent to impose that the Wulff shape centered at 0 and having a on its boundary is contained in the domain in the neighborhood of a. From convexity, it is then natural to construct $T(a)$ as the whole plane minus two wedges, one with vertex at 0 and the other with vertex at a.

The proof is rather tedious since one needs to deal with the behavior of the observable on the boundary of the domain $T(a)$. This was also an issue in the proof of conformal invariance (in which case the domains are even more general). At criticality, the difficulty was overcome by working with the discrete primitive H of F^2. Unfortunately, there is no nice equivalent of H to work with away from criticality. The solution is to use a representation of F in terms of a massive random walk. This representation extends to the boundary and allows us to control the behavior of F everywhere.

The proof may be skipped during a first reading.

Proof. Let $p < p_c$. Without loss of generality, consider $a = (a_1, a_2) \in \mathbb{Z}^2$ satisfying $a_2 \geq a_1 \geq 0$. In this proof, a vertex $u \in \mathbb{Z}$ will sometimes be identified with the unique medial-vertex e_u pointing north-west and bordering u. In other words $F(u)$ and $\{u \in \gamma\}$ should be understood as $F(e_u)$ and $\{e_u \in \gamma\}$. We will also index the neighbors of U by N, W, S and E (the indices refer to the cardinal directions).

The lower bound. Consider the observable F in the bulk defined as follows: for every edge $e \neq e_0$,

$$F(e) := \phi_p^0 \left(\mathrm{e}^{\frac{\mathrm{i}}{2} \mathrm{W}_\gamma(e, e_0)} \mathbf{1}_{e \in \gamma} \right), \tag{11.8}$$

where γ is the unique loop passing through e_0. The fact that F is defined even though we are working on an infinite graph is justified by the fact that $p < p_c$. Note that F is not well defined at e_0. Indeed, e_0 can be thought of as the start of the loop γ or its end. In other words, F is multi-valued at e_0, with value 1 or -1. Proposition 6.10 can be extended to this context following a very similar proof, but taking into account that F is multi-valued at e_0. More precisely, let $e_0 = [xy]$. Around any vertex $v \notin \{x, y\}$ the relation in Proposition 6.10 still holds; besides,

$$\begin{cases} F(se) + 1 = \mathrm{e}^{\mathrm{i}\alpha(p)} \left[F(sw) + F(ne) \right] & \text{if } v = y, \\ F(sw) + F(ne) = \mathrm{e}^{\mathrm{i}\alpha(p)} \left[-1 + F(se) \right] & \text{if } v = x, \end{cases}$$

where the ne (resp. se, sw) is the medial-edge at v pointing to the north-east (resp. south-east, south-west). In other words, the statement of Proposition 6.10 still formally holds if the convention becomes $F(e_0) = 1$ when considering the relation around x, and $F(e_0) = -1$ when considering the relation around y.

Let us now show that F is massive harmonic.

Lemma 11.4. *Let $p < p_{sd}$ and consider the observable F in the bulk. For any vertex X not equal to 0, we have*

$$\Delta_\alpha F(X) \ := \ \frac{\cos 2\alpha}{4} \Big[F(W) + F(S) + F(E) + F(N) \Big] - F(X) \ = \ 0,$$

where W, S, E and N are the four neighbors of X.

Proof. Consider a vertex X inside the domain and recall that X is identified with the corresponding medial-edge of the medial lattice pointing north-west. Index the edges around X in the same way as in Case 1 of Figure 11.2. By considering the six equations corresponding to vertices that end one of the edges $e_1, \ldots, e_6$, the following linear system can be obtained:

$$\begin{cases} F(X)+F(e_7) &=& e^{i\alpha}\ [F(e_1)+F(e_6)], \\ F(e_8)+F(e_1) &=& e^{i\alpha}\ [F(e_2)+F(W)], \\ F(S)+F(e_2) &=& e^{i\alpha}\ [F(e_9)+F(e_3)], \\ F(e_3)+F(e_4) &=& e^{i\alpha}\ [F(e_{10})+F(X)], \\ F(E)+F(e_5) &=& e^{i\alpha}\ [F(e_4)+F(e_{11})], \\ F(e_6)+F(e_{12}) &=& e^{i\alpha}\ [F(e_5)+F(N)]. \end{cases}$$

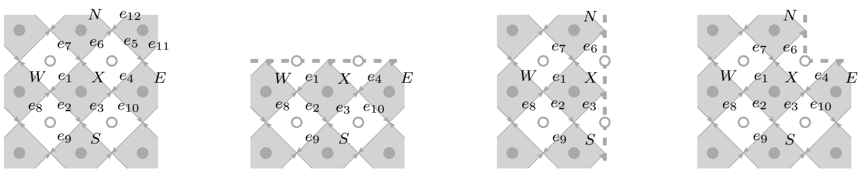

Figure 11.2: Indexation of the edges around vertices in the different cases.

Lemma 9.3 identifies the complex arguments modulo π of F for different edges ($F(e)$ belongs to $e^{-i\pi/8}\mathbb{R}$, $e^{i\pi/8}\mathbb{R}$, $e^{3i\pi/8}\mathbb{R}$ or $e^{5i\pi/8}\mathbb{R}$ depending on the direction). For an edge e, set $f(e)=\sqrt{e}F(e)$. By projecting orthogonally to the $F(y_i)$, $i=1\dots6$, the system becomes:

$$\begin{cases} f(X) &=& \cos(\pi/4+\alpha)f(e_1) &+& \cos(\pi/4-\alpha)f(e_6) & (1) \\ f(e_1) &=& \cos(\pi/4+\alpha)f(e_2) &+& \cos(\pi/4-\alpha)f(W) & (2) \\ f(e_3) &=& \cos(\pi/4-\alpha)f(S) &+& -\cos(\pi/4+\alpha)f(e_2) & (3) \\ f(X) &=& \cos(\pi/4+\alpha)f(e_3) &+& \cos(\pi/4-\alpha)f(e_4) & (4) \\ f(e_4) &=& \cos(\pi/4+\alpha)f(E) &+& \cos(\pi/4-\alpha)f(e_5) & (5) \\ f(e_6) &=& -\cos(\pi/4-\alpha)f(e_5) &+& \cos(\pi/4+\alpha)f(N) & (6) \end{cases}$$

By adding (2) to (3), (5) to (6) and (1) to (4), we find

$$\begin{cases} f(e_3)+f(e_1) &=& \cos(\pi/4-\alpha)[f(W)+f(S)] & (7) \\ f(e_6)+f(e_4) &=& \cos(\pi/4+\alpha)[f(E)+f(N)] & (8) \\ 2f(X) &=& \cos(\pi/4+\alpha)[f(e_3)+f(e_1)] & \\ && +\cos(\pi/4-\alpha)[f(e_6)+f(e_4)] & (9) \end{cases}$$

Plugging (7) and (8) into (9) leads to

$$2f(X)=\cos(\pi/4+\alpha)\cos(\pi/4-\alpha)[f(W)+f(S)+f(E)+f(N)].$$

The edges $X, \dots, N$ are pointing in the same direction so the previous equality becomes an equality with F in place of f (use Lemma 9.3 one more time). A simple trigonometric identity then leads to the claim. □

Define the Markov process with generator Δ_α, which one can see either as a branching process or as the random walk of a massive particle. We choose the latter interpretation and write this process (X_n, m_n) where X_n is a random walk with jump probabilities defined in terms of Δ_α — the proportionality between jump probabilities is the same as the proportionality between coefficients — and m_n is the mass associated to this random walk. The law of the random walk starting at x is denoted $\mathbb{P}^x$. Note that the mass of the walk decays by a factor $\cos 2\alpha$ at each step.

Denote by τ the hitting time of 0. The last lemma translates into the following formula for any a and any t,

$$F(a) = \mathbb{E}^a[F(X_{t\wedge\tau})m_{t\wedge\tau}]. \tag{11.9}$$

The sequence $(F(X_t)m_t)_{t\le\tau}$ is obviously uniformly integrable (it is bounded deterministically by 1), so that (11.9) can be improved to

$$F(a) = \mathbb{E}^a[F(X_\tau)m_\tau]. \tag{11.10}$$

Equation (11.10) together with Lemma 11.5 below gives

$$\phi_p^0(0 \leftrightarrow a) \ge \phi_p^0(e_a \in \gamma) \ge |F(a)| \ge \frac{c}{|a|}G_{\cos 2\alpha}(0,a),$$

which implies the lower bound.

Lemma 11.5. *There exists $c > 0$ such that, for every a in the upper-right quadrant,*

$$\left|\mathbb{E}^a[F(X_\tau)m_\tau]\right| \ge \frac{c}{|a|}G_{\cos 2\alpha}(0,a).$$

Proof. Recall that $F(X_\tau)$ is equal to 1 or -1 depending on the last step taken by the walk before reaching 0. Let us rewrite $\mathbb{E}^a[F(X_\tau)m_\tau]$ as

$$\mathbb{E}^a[m^\tau, X_{\tau-1} \in \{W,S\}] - \mathbb{E}^a[m^\tau, X_{\tau-1} \in \{N,E\}].$$

Now, let d be the line $y = -x$ and let T be the time of the *last* visit[1] of d by the walk before time τ (set $T = \infty$ if it does not exist). On the event that $X_{\tau-1} = W$ or S, this time is finite, and reflecting the part of the path between T and τ across Δ_α produces a path from a to 0 with $X_{\tau-1} = E$ or N. This transformation is one-to-one, so summing over all paths, we obtain

$$\mathbb{E}^a[m^\tau \mathbf{1}_{X_{\tau-1}\in\{W,S\}}] - \mathbb{E}^a[m^\tau \mathbf{1}_{X_{\tau-1}\in\{N,E\}}] = -\mathbb{E}^a[m^\tau \mathbf{1}_{X_{\tau-1}\in\{N,E\}}\mathbf{1}_{\{T=\infty\}}]$$

which in turn is equal to $-\mathbb{E}^a[m^\tau \mathbf{1}_{\{T=\infty\}}]$. General arguments of large deviation theory imply that $\mathbb{E}^a[m^\tau \mathbf{1}_{\{T=\infty\}}] \ge \frac{c}{|a|}G_{\cos 2\alpha}(0,a)$ for some universal constant c. □

[1] It is not a stopping time.

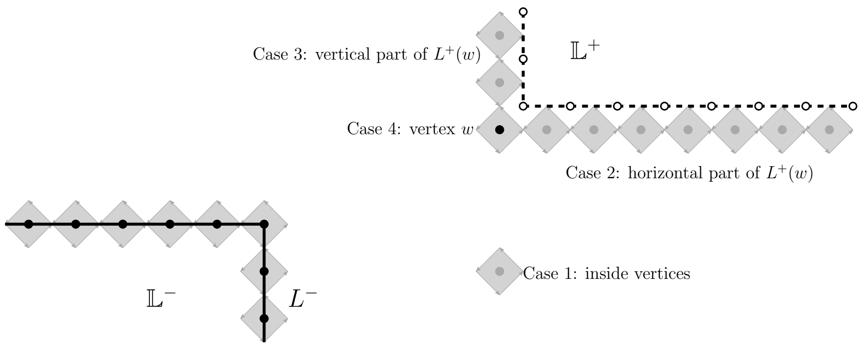

Figure 11.3: The set $T(w)$. The different cases listed in the definition of the Laplacian are also depicted.

The upper bound. Assume that 0 is connected to a in the bulk. We first show how to reduce the problem to estimations of correlations for points on the boundary of domains.

For every $u = (u_1, u_2)$ and $v = (v_1, v_2)$ two vertices of $\mathbb{Z}^2$, write $u \prec v$ if $u_1 < v_1$ and $u_2 < v_2$. This relation is a partial ordering of $\mathbb{Z}^2$. Consider the following sets

$$\mathbb{L}^+(u) = \{x \in \mathbb{Z}^2 : u \prec x\} \quad \text{and} \quad \mathbb{L}^- = \{x \in \mathbb{Z}^2 : x \prec 0\};$$

and

$$T(u) = \mathbb{Z}^2 \setminus (\mathbb{L}^+(u) \cup \mathbb{L}^-).$$

In the following, $L^+(u)$ and L^- will denote the boundaries of $T(u)$ near $\mathbb{L}^+(u)$ and $\mathbb{L}^-$ respectively, see Figure 11.3. The measure with wired boundary conditions on $\mathbb{L}^-$ and free boundary conditions on $\mathbb{L}^+(u)$ is denoted $\phi_{T(u)}$.

Assume that a is connected to 0 in the bulk. By conditioning on the site w that maximizes the partial $\prec$-ordering in the cluster of 0 [2], we obtain the following:

$$\phi_p^0(a \leftrightarrow 0) \le \sum_{w \succ a} \phi_{T(w)}(w \leftrightarrow \mathbb{L}^-) \le C_3 |a| \max_{w \succ a, |w| \le c_3 |a|} \phi_{T(w)}(w \leftrightarrow \mathbb{L}^-) \tag{11.11}$$

for c_3 and C_3 two large universal constants. The existence of c_3 is given by the fact that the two-point function decays exponentially fast: *a priori* estimates on the correlation length show that the maximum above cannot be reached at any w which is much further away from the origin than a,

[2]It is the same reasoning as in Lemma 4.23. Note that w may be non unique, in such case, we choose one such site, for instance by taking the minimal one for the lexicographical ordering.

and even that the sum of the corresponding probabilities is actually of a smaller order than the remaining terms. Summarizing, it is sufficient to estimate the probability of the right-hand side of (11.11).

Observe that w is on the free arc of $T(w)$, so that, using Lemma 6.11, we find

$$\phi_{T(w)}(w \leftrightarrow L^-) = |F(w)|, \tag{11.12}$$

where F is the observable in the infinite Dobrushin domain $T(w)$ (the winding is fixed in such a way that it equals 0 at e_w). Now, similarly to Lemma 11.4, F satisfies local relations in the domain $T(w)$:

Lemma 11.6. *The observable F satisfies $\Delta_\alpha F = 0$ for every vertex* not on the wired arc, *where the massive Laplacian Δ_α on $T(w)$ is defined by the following relations: for all $g : T(w) \mapsto \mathbb{R}$, $(g + \Delta_\alpha g)(X)$ is equal to:*

$$\begin{cases} \dfrac{\cos 2\alpha}{4}[g(W)+g(S)+g(E)+g(N)] & \textit{inside the domain;}\\ \dfrac{\frac{\cos 2\alpha}{2}[g(W)+g(S)]+\cos(\frac{\pi}{4}+\alpha)g(E)}{1+\cos(\frac{\pi}{4}-\alpha)} & \textit{on hor. part of } L^+(w);\\ \dfrac{\frac{\cos 2\alpha}{2}[g(W)+g(S)]+\cos(\frac{\pi}{4}+\alpha)g(N)}{1+\cos(\frac{\pi}{4}-\alpha)} & \textit{on vert. part of } L^+(w);\\ \dfrac{\frac{\cos 2\alpha}{2}[g(W)+g(S)]+\cos(\frac{\pi}{4}-\alpha)[g(E)+g(N)]}{2} & \textit{at } w; \end{cases}$$

with N, E, S and W being the four neighbors of X.

Proof. When the vertex is inside the domain, the proof is the same as in Lemma 11.4. For boundary vertices, a similar computation can be done. For instance, consider Case 2 in Fig. 11.2. Equations (3) and (7) in the proof of Lemma 11.4 are preserved. Furthermore, Lemma 6.11 implies that

$$f(X) = f(e_1) = \phi_{T(w)}(X \leftrightarrow L^-)$$

and similarly $f(e_4) = f(E)$ (where f is still as defined in the proof of Lemma 11.4). Plugging all these equations together, we obtain the second equality. The other cases are handled similarly. □

Now, we aim to use a representation with massive random walks similar to the proof of the lower bound for free boundary conditions. One technical point is the fact that, if we do it naively, the mass at w is larger than 1. This could *a priori* prevent $(F(X_t)m_t)_t$ from being uniformly integrable. Therefore, the behavior at w needs to be treated separately. Denote by τ_1 the hitting time (for $t > 0$) of w, and by τ the hitting time of L^-. Since the masses are smaller than 1, except at w, $(F(X_t)m_t)_{t\le\tau\wedge\tau_1}$ is uniformly integrable and we can apply the stopping theorem to obtain:

$$F(w) = \mathbb{E}^w[F(X_{\tau\wedge\tau_1})m_{\tau\wedge\tau_1}] = \mathbb{E}^w[F(X_{\tau_1})m_{\tau_1}\mathbf{1}_{\tau_1<\tau}] + \mathbb{E}^w[F(X_\tau)m_\tau\mathbf{1}_{\tau<\tau_1}].$$

Since $X_{\tau_1} = w$, the previous formula can be rewritten as

$$F(w) = \frac{\mathbb{E}^w[F(X_\tau)m_\tau \mathbf{1}_{\tau<\tau_1}]}{1 - \mathbb{E}^w(m_{\tau_1}\mathbf{1}_{\tau_1<\tau})}. \tag{11.13}$$

When w goes to infinity in a prescribed direction, $[1 - \mathbb{E}^w(m_{\tau_1}\mathbf{1}_{\tau_1<\tau})]$ converges to the analytic function $h : [0,1] \to \mathbb{R}$, $p \mapsto 1 - \mathbb{E}^w(m_{\tau_1})$ (since the function is translation-invariant). The function h is not equal to 0 when $p = 0$, implying that it is equal to 0 for a discrete set $\mathcal{P}$ of points. In particular, for $p \notin \mathcal{P}$, the first term in the right hand side stays bounded by a constant $C_4 = C_4(p) < \infty$ when w goes to infinity. Recalling that $|F| \le 1$ and that the mass is smaller than 1 except at w, (11.13) becomes

$$\begin{aligned} |F(w)| &\le C_4 \left|\mathbb{E}^w[F(X_\tau)m_\tau \mathbf{1}_{\tau<\tau_1}]\right| \le \mathbb{E}^w[m_\tau \mathbf{1}_{\tau<\tau_1}] \\ &\le C_4 \sum_{w<x} \mathbb{E}^x[(\cos 2\alpha)^\tau \mathbf{1}_{\tau<\tau_1}\mathbf{1}_{\{(X_t) \text{ avoids } L^+(w)\}}] \le C_4 \sum_{w<x} G_{\cos 2\alpha}(0,x) \end{aligned} \tag{11.14}$$

where the last inequality is due to the release of the conditioning on avoiding $L^+(w)$.

Finally, it only remains to bound the right hand side. From (11.14), we deduce

$$|F(w)| \le C_5|w|G_{\cos 2\alpha}(0,w), \tag{11.15}$$

where the existence of C_5 is due to the exponential decay of $G_{\cos 2\alpha}(\cdot,\cdot)$ and the fact that $G_{\cos 2\alpha}(0,x) \le G_{\cos 2\alpha}(0,w)$ whenever $w \prec x$. We deduce from (11.11), (11.12) and (11.15) that

$$\phi_p(0 \leftrightarrow a) \le C_3C_5|a|^2 \max_{w \prec a, |w|_\infty \le c_5|a|_\infty} G_m(0,w) \le C_6|a|^2 G_m(0,a).$$

Taking the logarithm and passing to the limit, the claim is obtained for all $p < p_c$ not in the discrete set $\mathcal{P}$. The result follows for every p using the fact that the correlation length is increasing in p. □

11.2 Characteristic length of the FK-Ising model

Beyond the understanding of the critical and non-critical phases (which was the subject of previous chapters and the previous section respectively), the principal goal of statistical physics is to study the phase transition itself, and in particular the behavior of macroscopic properties such as the density of the infinite-cluster for $p > p_c$ near the critical point.

It is usually possible to relate the critical regime to these thermodynamical properties via the study of the so-called *near-critical* *regime*. This regime was investigated in [Kes87] in the case of percolation. Many works followed afterward, culminating in a good understanding of *near-critical phenomena* in Bernoulli percolation [GPS10, NW09, GPS13]. The goal of this section is to discuss the near-critical regime in the random-cluster case, and more precisely in the FK-Ising case.

11.2.1 Definition of the characteristic length

Informally, the near-critical regime is the study of the FK-Ising model of edge-parameter p in the box of size L when (p, L) goes to (p_c, ∞). Consider $L = L(p)$ and let $p \nearrow p_c$. Note that, on the one hand, if $L(p)$ goes to ∞ too slowly, the configuration in the box of size L will look critical. On the other hand, if $L(p)$ goes to ∞ (from above) too quickly, the random-cluster model will look supercritical. The typical scale $L(p)$ separating these two regimes is called the *characteristic length*, also called *finite-size scaling correlation length* or simply *correlation length* (which is unfortunate since a priori it is not defined as the correlation length introduced in the previous section).

Let us define the characteristic length more formally. As illustrated before, the critical regime is often characterized by the fact that crossing probabilities remain bounded away from 0 and 1 (at least for continuous phase transitions). It is therefore natural to introduce the following definition.

Definition 11.7 (Characteristic length)**.** Fix $\rho > 0$ and $\varepsilon > 0$. Define

$$L_{\rho,\varepsilon}(p) \;:=\; \begin{cases} \inf\left\{n > 0 \,:\, \phi^0_{p,R_n}\big(\mathcal{C}_v(R_n)\big) \le \varepsilon\right\} & \text{if } p < p_c, \\ \inf\left\{n > 0 \,:\, \phi^1_{p,R_n}\big(\mathcal{C}_v(R_n)\big) \ge 1-\varepsilon\right\} & \text{if } p > p_c, \end{cases}$$

where $R_n = [0, n] \times [0, \rho n]$.

Theorem 10.1 shows that for $\varepsilon < c_1(\rho)$ (where c_1 is given by the theorem), $L_{\rho,\varepsilon}(p)$ tends to infinity as p tends to p_c. Furthermore, we have

$$\varepsilon \le \phi^\xi_{p,R_n}\big(\mathcal{C}_v(R_n) \le 1-\varepsilon$$

for any $n \le L_{\rho,\varepsilon}(p)$ and any boundary conditions ξ on ∂R_n. In some way, this justifies the fact that the configuration resembles the critical one below $L_{\rho,\varepsilon}(p)$.

Remark 11.8. The notion of characteristic length can be useful for any random-cluster model with $1 \le q \le 4$. For Bernoulli percolation ($q = 1$), Smirnov and Werner [SW01] showed that $L_{\rho,\varepsilon}(p)$ was behaving like $|p - \frac{1}{2}|^{-4/3+o(1)}$.

Remark 11.9. The characteristic length is expected to be roughly the same for different ε and ρ, in the sense that there should exist universal constants $0 < c, C < \infty$ such that

$$cL_{\rho,\varepsilon}(p) \le L_{\rho',\varepsilon'}(p) \le CL_{\rho,\varepsilon}(p)$$

uniformly as p tends to p_c. This result is known for percolation.

Remark 11.10. One may ask why we chose free boundary conditions for $p < p_c$, and wired boundary conditions for $p > p_c$. It can in fact be shown that boundary conditions are not really relevant by harnessing Theorem 10.1. We omit the proof of this fact here.

11.2.2 Behavior of the FK-Ising characteristic length

Theorem 11.11 (Duminil-Copin, Garban, Pete [DCGP11])**.** *Fix $q = 2$. For every $\varepsilon, \rho > 0$, there is a constant $c = c(\varepsilon, \rho)$ such that for all $p \ne p_c$,*

$$c\frac{1}{|p-p_c|} \le L_{\rho,\varepsilon}(p) \le c^{-1}\frac{1}{|p-p_c|}\log\left(\frac{1}{|p-p_c|}\right).$$

Remark 11.12. The characteristic length behaves like $|p - p_c|^{-1+o(1)}$ exactly as the correlation length studied in the previous section. This correspondence between the two notions of correlation length is expected to be true for most models of planar statistical physics, even though it is known in a few cases only.

11.2.3 Heuristic of the proof of Theorem 11.11

The proof is very technical and we do not present it here. For more details, we refer to the original article [DCGP11]. We focus on $p < p_c$. The goal of the proof is to show that crossing probabilities remain bounded away from 0 for $n \le \frac{c}{|p-p_c|}$ and that they tend to 0 for $n \ge \frac{C}{|p-p_c|}\log\frac{1}{|p-p_c|}$.

Recall from (11.5) that

$$\phi^{\text{dobr}}_{p,\mathcal{S}_n}[(0,n) \longleftrightarrow \text{wired arc}] = \left(\frac{[1+\cos(\pi/4+\alpha)]\cos(\pi/4+\alpha)}{[1+\cos(\pi/4-\alpha)]\cos(\pi/4-\alpha)}\right)^n |F(e_1)|.$$

Also note that as p tends to p_c, we have that $\alpha(p) \approx p_c - p$ and therefore

$$\phi^{\text{dobr}}_{p,\mathcal{S}_n}[(0,n) \leftrightarrow \text{wired arc}] \approx \left(1 - O\left(\tfrac{p-p_c}{p_c}\right)\right)^n |F(e_1)| \approx \exp\left(-O\left(\tfrac{p-p_c}{p_c}\right)n\right)|F(e_1)|.$$

Since $|F(e_1)| \le 1$, we immediately obtain that there exist $c, C > 0$ such that for $n \ge \frac{C}{|p-p_c|}\log\frac{1}{|p-p_c|}$,

$$\phi^{\text{dobr}}_{p,\mathcal{S}_n}\left[(0,n) \longleftrightarrow \text{wired arc}\right] \le n^{-c}.$$

The invariance under translation implies that

$$\phi^{\text{dobr}}_{p,\mathcal{S}_n}\big[[0,\kappa n]\times\{n\}\longleftrightarrow\text{ wired arc}\big]\le\kappa n^{1-c}.$$

Using the comparison between boundary conditions, we get

$$\phi^{\text{dobr}}_{p,R_n}[\mathcal{C}_v(R_n)]\le\kappa n^{1-c}.$$

In particular, crossing probabilities tend to zero when the boundary conditions are free.

Remark 11.13. In particular, in this regime the largest cluster in R_n is no longer of diameter of order n but rather smaller than $\sqrt{n}$. Therefore, if we are presented with such a box of size n, the configuration inside the domain does not look critical anymore. This justifies the fact that *above the critical length, the configuration looks subcritical.*

Let us turn ourselves to the lower bound. Recall that Dobrushin boundary conditions on the strip are wired on the bottom and free on the top. For $n\le 1/|p-p_c|$, we find that

$$\phi^{\text{dobr}}_{p,\mathcal{S}_n}\big[(0,n)\longleftrightarrow\text{ wired arc}\big]\approx|F(e_1)|.$$

Now, $|F(e_1)|$ is simply the probability that a dual-vertex on the dual-free arc is connected to the dual-wired arc. Duality and the fact that this probability decays as $C/\sqrt{n}$ at criticality (see Remark 10.8 of last chapter) imply that there exists a constant $c>0$ not depending on p (and on $n\le 1/|p-p_c|$) such that

$$\phi^{\text{dobr}}_{p,\mathcal{S}_n}\big[(0,n)\longleftrightarrow\text{ wired arc}\big]\ge c/\sqrt{n}.$$

Let us now use a second-moment method on the number N of vertices of $[0,n]\times\{n\}$ connected to the wired arc. The previous displayed equation implies that the expectation of N is larger than $c\sqrt{n}$. For the expectation of N^2, we can simply remark that it is smaller than the expectation at criticality, and therefore use the computation that we did in the last chapter to find that it is smaller than Cn. As a consequence, we obtain that there exists a constant $c'>0$ not depending on p (and on $n\le 1/|p-p_c|$) such that

$$\phi^{\text{dobr}}_{p,\mathcal{S}_n}\big[[0,n]\times\{n\}\longleftrightarrow\text{ wired arc}\big]\ge c'.$$

We therefore get that the probability of some crossing probabilities remains bounded away from 0. Getting rid of the specific boundary conditions of the strip is the tricky point. Modulo this difficulty, we have a proof of the lower bound.

Let us finish this section by mentioning that getting rid of the specific boundary conditions is extremely tedious. One needs to use the representation of the observable in terms of massive random walks in specific domains, a little bit like in the previous section. We skipped the details but we hope that the previous paragraphs illustrated the fact that the characteristic length can be computed using the observable.

11.3 Monotone coupling for the random cluster model with $q \geq 1$

11.3.1 Definition

It is natural to study the random-cluster model through its phase transition by constructing a monotone coupling of random-cluster models with fixed cluster-weight $q \geq 1$ and different edge-weights. The Holley criterion provides us with a monotone coupling of two random-cluster model measures with the same cluster-weight and different edge-weights, but it does not construct a coupling of random-cluster measures for all edge-weights simultaneously. In the case of Bernoulli percolation, such a monotone coupling simply consists of i.i.d. Uniform$[0,1]$ labels on the edges, and a percolation configuration ω_p of density p is the set of bonds with labels at most p. For $q > 1$, the construction is more difficult and we describe it now (see [Gri95, Gri06, HJL02] for a detailed exposition).

Fix $q \geq 1$ and a finite subgraph G of $\mathbb{Z}^2$. The goal is to find a measure μ_G on $[0,1]^{E_G}$ in a such a way that all the projections $\omega_p(Z) \in \{0,1\}^{E_G}$ with $Z \sim \mu$, defined by

$$\omega_p(Z)(e) := 1_{Z(e) \leq p}\,, \qquad p \in [0,1],\ e \in E\,,$$

follow the random-cluster probability measures of parameters (p,q) on $\{0,1\}^{E_G}$ with some given boundary conditions. For simplicity, we will focus on the free boundary conditions.

The measure μ_G will be constructed as the invariant measure of a natural Markov process Z_t on the space $[0,1]^{E_G}$ constructed as follows. The labels of edges are updated at exponential rate of mean 1. Once the clock of an edge $e = [xy]$ rings (let us say at time t), resample the label according to the law on $[0,1]$ with partition function

$$\mathbb{P}(\mathsf{U}_e \leq p) := \begin{cases} p & \text{if } p \geq P(Z_{t-}), \\ \dfrac{p}{p + (1-p)q} & \text{if } p < P(Z_{t-}), \end{cases} \tag{11.16}$$

where Z_{t-} is the limit of Z_s for $s \nearrow t$ and P is the random variable defined by

$$P = P(Z_{t-}) := \inf\Big\{p \in [0,1] \text{ s.t. } \omega_p(Z_{t-}) \in \{x \longleftrightarrow y \text{ in } E_G \setminus \{e\}\}\Big\}.$$

The condition $q \geq 1$ implies that this is a valid distribution function, hence we can simply define U_e to be a sample from this distribution.

Proposition 11.14. *Let G be a graph and $q \geq 1$. The random-variable $(\omega_p(Z))_{0 \leq p \leq 1}$ provides us with an increasing coupling of random-cluster models with cluster-weight q.*

Proof. The coupling thus obtained is increasing by construction. For a fixed p, the process $(\omega_p(Z_t))_{t\geq 0}$ is a Markov process on $\{0,1\}^{E_G}$ whose invariant law is $\omega_p(Z)$. Now, $(\omega_p(Z_t))_{t\geq 0}$ is obtained by resampling (the state of) each edges at exponential rate 1 according to the law

$$\mathbb{P}(\mathsf{V}_e = 1) := \begin{cases} p & \text{if } x \longleftrightarrow y \text{ in } E_G \setminus \{e\}, \\ \dfrac{p}{p+(1-p)q} & \text{otherwise}\,. \end{cases}$$

One may easily check that $\phi^0_{p,q,G}$ is invariant under this Markov process. It is therefore equal to the law of $\omega_p(Z)$. □

Constructing an infinite-volume version of the previous dynamics is not straightforward. Nevertheless, one has the following asymptotic statement from [Gri95, Gri06].

Proposition 11.15 (Infinite Volume Limit [Gri95])**.** *Let ξ be some initial configuration in $[0,1]^{E_{\mathbb{Z}^2}}$. For $q \geq 1$, consider the above dynamics $Z_t^{\Lambda_n}$ on Λ_n with free boundary conditions, which starts from the initial state $Z_0^{\Lambda_n} \equiv \xi_{|\Lambda_n}$. Then:*

- *As $n \to \infty$, the process $(Z_t^{\Lambda_n})$ weakly converges to a Markov process $(Z_t^{\mathrm{free}})_{t\geq 0}$ which starts from the initial configuration $Z_0^{\mathrm{free}} = \xi$.*
- *As $t \to \infty$, Z_t^{free} weakly converges to an invariant measure μ on $[0,1]^{E_{\mathbb{Z}^2}}$.*
- *If, in the limiting procedure, one uses wired boundary conditions instead, one obtains at the limit a Markov process $(Z_t^{\mathrm{wired}})_{t\geq 0}$. The processes Z_t^{wired} and Z_t^{free} might possibly have different transition kernels but they both have the same μ as the unique invariant measure. For $Z \sim \mu$, the projections $\omega_p(Z)$ given by (11.3.1) have the law of a random-cluster model with parameters p and q and boundary conditions ξ.*

11.3.2 Existence of emerging clouds for $q > 1$

While no two edges appeared simultaneously in the standard monotone coupling for Bernoulli percolation, several edges may appear at the same p in monotone couplings for random-cluster models with $q > 1$. We will see later that this phenomenon is crucial in order to explain the behavior of the near-critical regime.

Given a sample $Z \in [0,1]^{E_G}$ from Grimmett's monotone coupling μ_G, for an edge $e \in E_G$, let $\mathrm{cloud}(e)$ be the set of edges which appear simultaneously with e:

$$\mathrm{cloud}(e) := \{f \in E_G \text{ such that } Z(f) = Z(e)\}\,.$$

Proposition 11.16. *Fix $q > 1$. For any $N \geq 1$, let $(\omega_p(Z)_{p\in[0,1]}$ be a monotone coupling in the box Λ_n. The probability that clouds of at least N edges appear simultaneously in $\omega_p(Z)$ at some $p \in (0,1)$ converges to 1 when n tends to $+\infty$.*

We will exploit the fact that U_e has an absolutely continuous part plus a Dirac point mass for $q > 1$ on P (the smallest p such that $x \overset{\omega_p}{\longleftrightarrow} y$ in E_G), namely $[P - \frac{P}{P+(1-P)q}]\delta_P$.

Proof. Fix $N > 0$. Let us consider the sets

$$\begin{aligned} E_{\text{hor}} &= \{\text{horizontal edges of } [0,1] \times [0,N]\}, \\ E_{\text{vert}} &= \{\text{vertical edges of } [0,1] \times [0,N]\}, \\ E_{\text{ext}} &= \{\text{edges with exactly one end-point in } [0,1] \times [0,N]\}. \end{aligned}$$

We also set e_0 to be the edge between the origin and the vertex $(1,0)$.

Let us sample $Z_{t=0}$ according to the invariant measure μ_G, and let us run the dynamics given by the Markov process for a unit time. With positive probability, all edges in $E := E_{\text{hor}} \cup E_{\text{vert}} \cup E_{\text{ext}}$ are updated and their labels at time 1 satisfy the following:

- All labels in E_{vert} are smaller than $1/4$;
- The label of the edge e_0 lies in $(1/4, 1/2)$;
- All other labels in $E_{\text{hor}} \cup E_{\text{ext}}$ are larger than $3/4$.

Under such circumstances, all edges $e \in E_{\text{hor}} \setminus \{e_0\}$ are such that $P_e(Z_{t=1}) = Z_{t=1}(e_0)$. It could be that this situation evolves later on, but we have that, with positive probability, none of the edges in $E_{\text{vert}} \cup E_{\text{ext}} \cup \{e_0\}$ are updated from time 1 to time 2. Knowing this, again with positive probability, all edges in E_{vert} are updated from time 1 to time 2 and all of them take exactly the value $u := Z_{t=1}(e_0)$ (this is due to the Dirac mass in the law $\mathcal{U}_e$). Since we started at equilibrium, $Z_{t=2}$ has the equilibrium law, and edges in E_{vert} are all open or all closed in the projections of $Z_{t=2}$. This shows that with positive probability, at least N edges appear simultaneously as one raises p.

Now, the previous procedure showed that with probability bounded away from 0 uniformly in the state of edges outside E, the edges in E_{vert} are all open or all closed. When $G = \Lambda_n$ is getting very large, we can divide the box into translates of E. Since in each of these boxes will have positive chance to see edges appearing simultaneously, the result follows. By changing slightly the argument, one can show that there are such clouds for any open interval of $p \in [0,1]$. □

We end this subsection by a hand-waving argument why these clouds appear and may play an important role in the understanding of the near-critical regime. Due to the factor $q^{k(\omega)}$ in the partition function, random-cluster configurations tend to have as many clusters as possible. When

$q > 1$ and p is increased, there is a fight between the entropy under the product measure $p^{o(\omega)}(1-p)^{c(\omega)}$ pushing to open edges and the energy corresponding here to $-k(\omega)\log(q)$ trying to maximize the number of clusters.

A good strategy for adding many edges without a significant increase in energy is the following storing mechanism. Say we have two touching clusters with a certain number of closed edges going from one to the other. Once one of these edges becomes open, then there is no energy cost to open other edges. At this point, it is unclear whether this storing mechanism can happen or not.

Now, Grimmett's coupling $p \longmapsto \omega_p(Z)$ is in fact a continuous-time (here p is the time-parameter) monotone Markov process (we refer to [DCGP11] for details) and therefore the only way for this storing mechanism to actually happen is to have some values of p where the system can simultaneously open several edges. This indeed can happen, due to the atomic part of the update distribution, as shown in Lemma 11.16, and the construction there was indeed a simple example of edges arriving simultaneously between two neighboring large clusters (the two components of E_{vert}).

It is worth noticing that this heuristic explanation hints that the storing mechanism should be much stronger near the critical point. Indeed, near $p_c(q)$, there are many neighboring large clusters which makes the storing mechanism more efficient.

11.4 Violation of Kesten's scaling relation for $4 \geq q > 1$

We go back to the random-cluster for a moment. Let $q \in [1,4]$. In this section, we postulate the existence of the four following exponents:

$$\begin{aligned}
\phi_{p_c,q}[A_1(0,n)] &= n^{-\xi_1+o(1)} && \text{as } n \to \infty,\\
\phi_{p_c,q}[A_{1010}(0,n)] &= n^{-\xi_{1010}+o(1)} && \text{as } n \to \infty,\\
\phi_{p,q}(0 \longleftrightarrow \infty) &= (p-p_c)^{\beta+o(1)} && \text{as } p \searrow p_c,\\
L(p,q) &= |p-p_c|^{-\nu+o(1)} && \text{as } p \searrow p_c
\end{aligned}$$

(this last definition is similar to the FK-Ising case). In the case of percolation, Kesten proved that $(2-\xi_{1010})\nu = 1$ and $\beta = \nu\xi_1$ in [Kes87] (also see [Nol08, NW09, GPS13, GP11]). These relations are very useful since they allow one to compute the near-critical exponents ν and β describing the behavior of thermodynamical quantities in terms of the critical exponents ξ_1 and ξ_{1010} describing fractal properties of the critical regime (which may be obtained by SLE computations for instance).

It would be interesting to see if these relations are still valid for other values of $q \in [1,4]$. Unfortunately, this is not the case as we can see for the FK-Ising model. Indeed, in this case $\beta = \frac{1}{8}$, $\xi_1 = \frac{1}{8}$ (these exponents were computed by Onsager), $\nu = 1$ (this exponent was computed in the previous section) and $\xi_{1010} = \frac{24}{13}$ [Gar]. When putting these exponents in $(2 - \xi_{1010})\nu = 1$, we see that the equality is not satisfied.

The exact value of the previous exponents enabled us to show that Kesten's relation is violated. However, it does not tell us why. We therefore propose to give a heuristic explanation for this violation now by first explaining this relation in the case of Bernoulli percolation, and then explain what is different for random-cluster models with $q > 1$.

Recall that for $p > p_c$, $L(p,1)$ is roughly the scale at which the configuration starts to look supercritical. The geometry of the large clusters in the box depends on the fact that so-called *macroscopic pivotal edges* (by macroscopic pivotal edge we informally mean an edge which is pivotal in the sense of Section 5.2.3 for crossing of some say large rectangle in the box; we omit the rigorous definition here and keep only an intuitive approach) are rather open or closed.

Let ω_p be the monotone coupling of Bernoulli percolation. Getting from ω_{p_c} to ω_p ($p > p_c$) in the box of size n, roughly $n^2(p - p_c)$ edges are switched from closed to open. Now, the probability that an edge is pivotal for the event of being crossed is of order $\xi_4(n) := \phi_{p_c,q}[A_{1010}(0,n)]$. The expected number of opened edges that were closed macroscopic pivotals in the *initial* configuration (preventing macroscopic open paths) is about $n^2\xi_4(n)(p-p_c)$. Now, if $n^2\xi_4(n)(p-p_c) \gg 1$, it is not very hard to show that many of these initial macroscopic pivotals have become open with good probability (not only their expected number is large), and this implies that the window of size n has become well-connected. That is, we have left the near-critical regime. But this is only a very rough cartoon. Indeed, maybe many pivotal edges were destroyed during the dynamics, and therefore the counting argument outlined above is not accurate.

On the other hand, the regime $n^2\xi_4(n)(p - p_c) \ll 1$ is more difficult to understand. The number of initial macroscopic pivotals that have switched is small and the macroscopic connectivity of the configuration have not changed. In particular, the crossing probability is still bounded away from 1. Once again, this is only a cartoon. Indeed, even though the number of macroscopic pivotal edges that switched from closed to open is small, maybe many new pivotal edges have appeared during the dynamics, which could have switched then, establishing macroscopic open connections.

Anyway, the following scaling relation holds:

$$L(p,1)^2 \cdot \xi_4(L(p,1)) \cdot (p - p_c) \asymp 1, \tag{11.17}$$

where $\asymp$ means that the quantity remains bounded away from 0 and ∞ uniformly in p. This relation leads to $(2 - \xi_{1010})\nu = 1$.

To our knowledge, it has been widely believed in the community that basically the same mechanism should hold in the case of random-cluster models. Namely, once we understand the geometry of the set of pivotal points at criticality, we may readily deduce information on the near-critical behavior.

The reason why the relation is violated when $q > 1$ (and in particular when $q = 2$) comes from the intrinsic difference between monotone couplings for $q = 1$ and $q > 1$. Let us explain a little bit.

First, there is a basic phenomenon in the coupling for the random-cluster models with $q \geq 2$ that is very relevant to the above discussion: the difference between the average densities of edges between p_c and p is not proportional to $p - p_c$, but larger than that, with an exponent given by the so-called *specific heat* of the model (see the exponent α in Section 13.2.3). A first guess could be that the discrepancy in (11.17) for $q \geq 2$ is a result of the fact that $p - p_c$ is not the density of the new edges arriving, and this should have been taken into account in the computation using the pivotal exponent. However, this is only partially right: the specific heat exponent itself is not large enough to account for this discrepancy (in fact, for $q = 2$ it equals 0: there is only a logarithmic blow-up).

The main reason for the discrepancy is that the storing mechanism mentioned in the previous section kicks in. In standard percolation, new edges arrive in a "Poissonian" way. If this were true in the Grimmett coupling for the random-cluster model with $q > 1$, then an argument similar to the Bernoulli percolation case could be performed and Kesten's relation would be valid. But this is not the case: new edges arrive in a correlated fashion. It becomes possible for the arriving edges to prefer "strategic" locations, creating and then opening new pivotal edges at large scales, thereby speeding up the dynamics compared to what could be guessed from the number of pivotal edges at criticality. In other words, near p_c, the arriving edges depend in a very sensitive way on the current configuration. This balance between the current configuration and the conditional law of the arriving edges is representative of a self-organized mechanism behind the phenomenology of clouds and the way the configuration passes from a subcritical state to a supercritical one. Understanding this mechanism seems to be a very interesting challenge.

Part IV

What's next?

Chapter 12

What about other graphs and other models?

This chapter is devoted to the study of the parafermionic observable in other models. We will not present any detailed proof but we wish to illustrate the fact that the theory of parafermionic observables is not restricted to specific planar lattices and to the random-cluster model, the Ising model and the self-avoiding walk. We expect that new applications of discrete observables should be found in a broader context.

12.1 A glimpse of universality: statistical physics on isoradial graphs

When speaking about random-cluster models and the Ising model in this book, we chose to restrict ourselves to the square lattice $\mathbb{Z}^2$. However, one may ask the same questions for the triangular, the hexagonal or even more general planar lattices. In this section, we discuss an important class of graphs, called *isoradial graphs*, to which most of what has been described above can be extended.

Definition 12.1. An *isoradial graph* is a planar graph admitting an embedding in the plane in such a way that every face is inscribed in a circle of radius 1, see Fig. 12.1. In such cases, we will say that the embedding is *isoradial*.

The triangular, hexagonal and square lattices are isoradial graphs. Each of these graphs possesses in fact a whole family of isoradial embeddings. For instance, any distortion of the square lattice, where each square is stretched into a rectangle, is still an isoradial embedding. We would like to point out that isoradial graphs form a rather large family of graphs. Kenyon and Schlenker [KS05] gave a simple necessary and sufficient topological condition on planar graphs for the existence of an isoradial embedding.

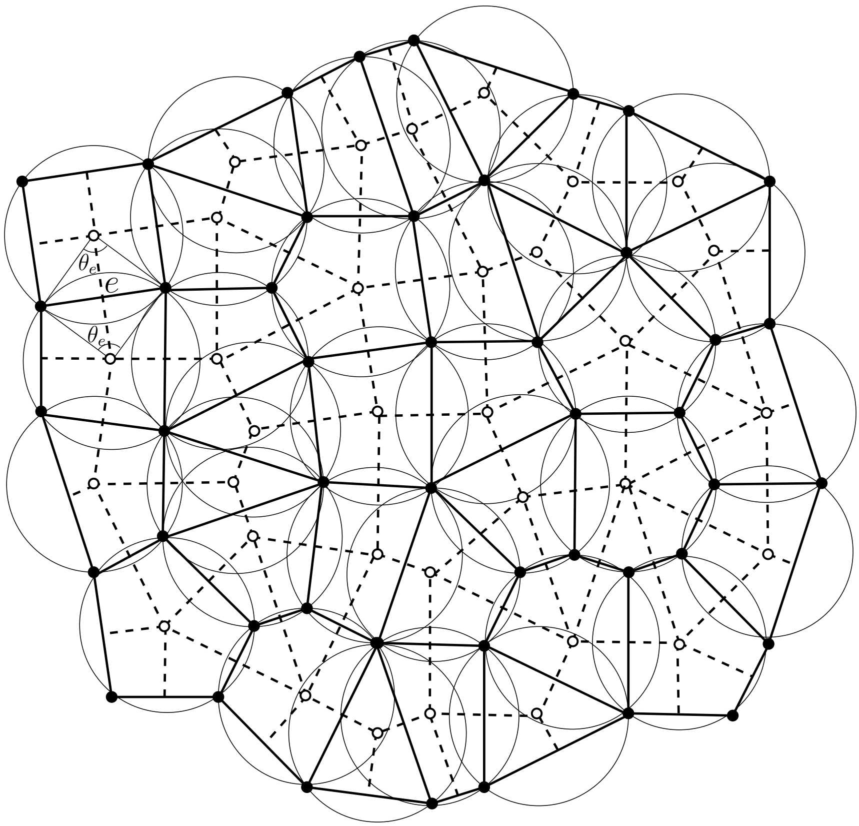

Figure 12.1: Black vertices and plain lines form the isoradial graph. White vertices and dashed lines form the dual graph. Dual vertices have been drawn in such a way that they are the centers of circles of radius 1. On the top left, an edge e with the angle θ_e on each sides.

Isoradial graphs were introduced by Duffin [Duf68] who extended the definition of discrete holomorphic functions to their embeddings. For these graphs, the Cauchy-Riemann operator admits a nice discretization. In particular, restrictions of holomorphic functions to such graphs are discrete holomorphic to higher orders (meaning that they are closer to discrete holomorphic functions than on other graphs). We refer to [Mer01, Ken02, CS11] for an analysis of discrete holomorphicity on isoradial graphs.

It seems that the first appearance of a related family of graphs in the probabilistic context was in the work of Baxter [Bax78], where the so-called eight vertex model and the Ising model were considered on Z-invariant graphs arising from planar line arrangements[1].

When working with isoradial graphs, one does not usually require invariance under translations but the following condition is classical. Let e be an edge of an isoradial embedding of a graph G. It subtends an angle $\theta_e \in (0, \pi)$ at the center of the circle corresponding to any of the two faces bordered by e; see Fig. 12.1.

Definition 12.2. Fix $\theta > 0$, and let $\mathcal{G}$ be an infinite isoradial graph. The graph is said to satisfy the *bounded-angle property* if the following condition holds:

$$(\mathrm{BAP}_\theta) \qquad \text{For any } e \in E_{\mathcal{G}}, \quad \theta \le \theta_e \le \pi - \theta.$$

12.1.1 Random-cluster model on isoradial graphs

We consider a random-cluster model with edge-weights depending on each edge. Let $\mathcal{G}$ be an infinite isoradial graph and $p = (p_e)_{e\in E_{\mathcal{G}}}$ a family of edge-weights $p_e \in [0,1]$. For a finite subgraph G of $\mathcal{G}$, the probability of a configuration is proportional to

$$\prod_{e\in E_G} p_e^{\omega(e)}(1-p_e)^{1-\omega(e)} q^{k(\omega)}.$$

In this section, the weighted graph $(\mathcal{G}, p)$ will be assumed to be *periodic*, in the sense that it will carry an action of the square lattice $\mathbb{Z}^2$ with finitely many orbits. For $q \ge 1$, this model can be extended to $\mathcal{G}$ where it exhibits a phase transition. In general, there is no conjecture for the value of the critical surface, i.e. the set of $(p_e)_{e\in E_{\mathcal{G}}}$ for which the model is critical and getting a good understanding of the general case seems very challenging. For this reason, we restrict our attention to specific random-cluster measures with cluster-weight $q \ge 4$ on G. For $\beta > 0$, define the edge-weight $p_e(\beta) \in [0,1]$ for $e \in E_G$ by the formula

$$\frac{p_e(\beta)}{[1-p_e(\beta)]\sqrt{q}} = \beta\,\frac{\sinh[\frac{\sigma(\pi-\theta_e)}{2}]}{\sinh[\frac{\sigma\theta_e}{2}]},$$

[1] These graphs are topologically the same as the isoradial ones, and though they are embedded differently into the plane in Baxter's work, by [KS05] they always admit isoradial embeddings. In [Bax78], Baxter was not considering scaling limits, and so the actual choice of embedding was immaterial for his results.

where the *spin* σ is given by the relation

$$\cosh\left(\frac{\sigma\pi}{2}\right) = \frac{\sqrt{q}}{2}.$$

Note that $p_e(\beta)$ is uniquely defined by this relation, and that the edge-weight depends both on β and θ_e (and therefore on the whole rhombic embedding). Obviously, not every periodic edge-weight $(p_e)_{e\in E_{\mathcal{G}}}$ can a priori be written in the form above since it would fix the values of θ_e and therefore the embedding which would not always be a rhombic embedding. Nevertheless, this specific choice of edge-weights $(p_e(\beta))_{e\in E_{\mathcal{G}}}$ possesses very interesting properties as illustrated below.

The infinite-volume measure on $\mathcal{G}$ with cluster-weight $q \geq 4$, edge-weights $(p_e(\beta))_{e\in E_{\mathcal{G}}}$ and free boundary conditions is denoted by $\phi^0_{\beta,q,\mathcal{G}}$.

Theorem 12.3 (Beffara, Duminil-Copin, Smirnov [BDCS12])**.** *Let $q \geq 4$, $\theta > 0$ and $\beta < 1$. There exists $c = c(\beta, q, \theta) > 0$ such that for any infinite isoradial graph $\mathcal{G}$ satisfying (BAP_θ) and for any $u, v \in \mathcal{G}$*

$$\phi^0_{\beta,q,\mathcal{G}}(u \longleftrightarrow v) \leq \exp[-c|u - v|].$$

The proof is based on a parafermionic observable defined on isoradial graphs. The integral along discrete contours can be proved to vanish when $\beta = 1$. Furthermore, for $\beta < 1$, an adaptation of the proof of Section 6.3 allows one to show that the parafermionic observables decay exponential fast in the distance to the boundary.

The dual of an isoradial graph $\mathcal{G}$ is also an isoradial graph $\mathcal{G}^\star$. Furthermore, with the choice of edge-weights $(p_e(\beta))_{e\in E_{\mathcal{G}}}$, the edge-weights of the dual measure on $\mathcal{G}^\star$ are $(p_{e^\star}(1/\beta))_{e^\star\in E_{\mathcal{G}^\star}}$ (since $\theta_{e^\star} = \pi - \theta_e$; see Fig. 12.1). Therefore, the previous theorem implies that there is exponential decay in the dual graph for $\beta > 1$, and therefore there exists an infinite cluster almost surely[2]. In conclusion, edge-weights $(p_e)_{e\in E_{\mathcal{G}}} = (p_e(1))_{e\in E_{\mathcal{G}}}$ are critical in the following sense.

Theorem 12.4. *Let $q \geq 4$, $\theta > 0$. For any* periodic *isoradial graph $\mathcal{G}$:*

1. *The infinite-volume measure is unique whenever $\beta \neq 1$. We denote it by $\phi_{\beta,q,\mathcal{G}}$.*
2. *For $\beta < 1$, there is $\phi_{\beta,q,\mathcal{G}}$-almost surely no infinite-cluster.*
3. *For $\beta > 1$, there is $\phi_{\beta,q,\mathcal{G}}$-almost surely a unique infinite-cluster.*

The equivalent of Theorems 12.3 and 12.4 was previously known for two other choices of q:

[2] We applied this reasoning in the proof of Theorem 6.18 in Section 6.3: the Borel-Cantelli lemma shows that there are finitely many dual circuits surrounding the origin almost surely.

- When $q = 2$, the model was studied in [Bax78]; Also see Theorem 12.6.
- When $q = 1$ (for percolation), Manolescu and Grimmett [GM13b, GM13a, GM12] showed the corresponding statements and much more. We refer to [BDCS12] and [DCM13b] for more details on these results.

The previous theorem has a nice byproduct. Attribute to each edges of the square, triangular and hexagonal lattices $\mathbb{Z}^2$, $\mathbb{T}$ and $\mathbb{H}$ an edge-weight depending only on the orientation of the edge (for instance on $\mathbb{Z}^2$, the edge-weight of vertical edges is p_1 and the one of horizontal edges is p_2). The inhomogeneous random-cluster models on the square, the triangular and the hexagonal lattices can be seen (after some straightforward computations) as random-cluster models on periodic isoradial graphs (the isoradial embedding depends on the proportionality between weights). Thus, Theorem 12.4 provides us with an explicit expression for the critical surfaces that we mention below.

Corollary 12.5. *The inhomogeneous random-cluster model with cluster-weight $q \geq 4$ on the square, triangular and hexagonal lattices have the following critical surfaces:*

$$\begin{aligned}
&\text{on } \mathbb{Z}^2 && \frac{p_1}{1-p_1}\frac{p_2}{1-p_2} = q,\\
&\text{on } \mathbb{T} && \frac{p_1}{1-p_1}\frac{p_2}{1-p_2}\frac{p_3}{1-p_3} + \frac{p_1}{1-p_1}\frac{p_2}{1-p_2} + \frac{p_1}{1-p_1}\frac{p_3}{1-p_3} + \frac{p_2}{1-p_2}\frac{p_3}{1-p_3} = q,\\
&\text{on } \mathbb{H} && \frac{p_1}{1-p_1}\frac{p_2}{1-p_2}\frac{p_3}{1-p_3} = q\frac{p_1}{1-p_1} + q\frac{p_2}{1-p_2} + q\frac{p_3}{1-p_3} + q^2,
\end{aligned}$$

where p_1, p_2 (resp. p_1, p_2, p_3) are the edge-weights of the different orientations of edges.

For percolation, the previous corollary was predicted in [SE63] and proved in [Kes80, Section 3.4] for the case of the square lattice and [Gri99, Section 11.9] for the case of triangular and hexagonal lattices.

Let us also mention that the critical parameter of the so-called continuum random-cluster model can be computed using the fact that it is the limit of inhomogeneous random-cluster models on the square lattice with $(p_1, p_2) \to (0, 1)$. We refer to [GOS08] for a precise definition of the models and their connection to Quantum Potts models. The parameters of the models are usually referred to as $\lambda, \delta > 0$, where λ and δ are the intensities of the Poisson Point Process of so-called births and deaths respectively. In such case, Theorem 12.4 implies that the critical point is given by $\lambda/\delta = q$ for $q \geq 1$.

12.1.2 Ising on isoradial graphs

Historically, the weights of Baxter's models suggested an isoradial embedding. This led Mercat [Mer01], Boutillier and de Tilière [BdT10, BdT11] to study the Ising model on isoradial graphs[3]. Note that since isoradial graphs are fundamental for both discrete holomorphicity and random-cluster models, it is not surprising that the Ising model on isoradial graphs also satisfies very specific integrability properties.

A natural critical model can be defined by taking the following Hamiltonian:

$$H_G^{\mathrm{f}} = -\sum_{x\sim y} J_{xy}\sigma_x\sigma_y,$$

where

$$J_{xy} = \tfrac{1}{2}\log\left(\frac{1+\cos(\theta_{[xy]}/2)}{\sin(\theta_{[xy]}/2)}\right),$$

and $\theta_{[xy]}$ is yet again the angle associated to the edge $[xy]$ in the isoradial embedding. One may observe that the random-cluster representation of this Ising model is a random-cluster model on G with critical isoradial weights and therefore the model is critical (it is a simple exercise on trigonometric functions). In fact, this model is not only critical but it is also conformally invariant in the following sense. A graph Ω_δ will denote the image by $z \mapsto \theta z$ of an isoradial embedding (in other words it is an isoradial embedding but with circles of radius δ instead of 1).

Theorem 12.6 (Chelkak, Smirnov [CS12]). *Let Ω be a simply connected domain with two marked points u and v on its boundary such that $\partial\Omega$ is flat near v. Let $(\Omega_\delta, u_\delta, b_\delta)$ be a family of discrete approximations of (Ω, u, v) converging in the Carathéodory sense to (Ω, u, v). We further assume that the boundary is "flat" near v_δ (see [CS12] for a precise definition). Then the spin fermionic observable on $(\Omega_\delta, u_\delta, v_\delta)$ converges in the scaling limit to the same limit as for the square lattice.*

The convergence is in fact uniform in the possible choices of Ω_δ (in particular they can come form different pieces of isoradial graphs). Even though details were not provided, the techniques exposed in this book extend to the Ising model on isoradial graphs and lead to the convergence of interfaces for instance.

The proof of Theorem 12.6 is based on a generalization of the spin fermionic observable to isoradial graphs. The techniques are similar to the case of the square lattice and the architecture of the proof is the same (though additional technicalities arise).

[3]The dimer model and the uniform spanning tree models on such graphs were also studied: they share nice properties, see e.g. [Ken02].

12.2 The Potts models

The Potts model is a model of random coloring of $\mathbb{Z}^2$ introduced as a generalization of the Ising model to more-than-two components spin systems. In this model, each vertex of $\mathbb{Z}^2$ receives a spin among q possible colors. The energy of a configuration is proportional to the number of frustrated edges, meaning edges whose endpoints have different spins. Since its introduction by Potts [Pot52] (after a suggestion of his adviser Domb), the model has been a laboratory for testing new ideas and developing far-reaching tools. In two dimensions, it exhibits a rich panel of possible critical behaviors depending on the number of colors, and despite the fact that the model is exactly solvable, the mathematical understanding of its phase transition remains restricted to a few cases (namely $q = 2$ and q large). We refer to [Wu82] for a review on this model.

Consider an integer $q \geq 2$ and a subgraph G of the square lattice. Let $\tau \in \{1,\dots,q\}^{\mathbb{Z}^2}$. The *$q$-state Potts model on G with boundary conditions τ* is defined as follows. The space of configurations is $\Omega = \{1,\dots,q\}^{\mathbb{Z}^2}$. For a configuration $\sigma = (\sigma_x : x \in \mathbb{Z}^2) \in \Omega$, the quantity σ_x is called the *spin* at x (it is sometimes interpreted as being a color). The *energy* of a configuration $\sigma \in \Omega$ is given by the Hamiltonian

$$H^\tau_{q,G}(\sigma) := \begin{cases} -2 \displaystyle\sum_{\substack{x\sim y \\ \{x,y\}\cap G \neq \varnothing}} \delta_{\sigma_x,\sigma_y} & \text{if } \sigma_x = \tau_x \text{ for } x \notin G, \\ \infty & \text{otherwise.} \end{cases}$$

Above, $\delta_{a,b}$ denotes the Kronecker symbol equal to 1 if $a = b$ and 0 otherwise.

For $q = 2$, we recognize the Ising model (the multiplicative constant 2 in the Hamiltonian is introduced in such a way that the Ising model and the 2-state Potts model can be identified together).

The spin-configuration is sampled proportionally to its Boltzmann weight: at an inverse-temperature β, the probability $\mu^\tau_{\beta,q,G}$ of a configuration σ satisfies

$$\mu^\tau_{\beta,q,G}[\sigma] := \frac{\mathrm{e}^{-\beta H^\tau_{q,G}(\sigma)}}{Z^\tau_{\beta,q,G}} \quad \text{where} \quad Z^\tau_{\beta,q,G} := \sum_{\sigma\in\Omega} \mathrm{e}^{-\beta H^\tau_{q,G}(\sigma)}$$

is the so-called *partition function* defined in such a way that the sum of the weights over all possible configurations equals 1. By construction, configurations that do not coincide with τ outside of G have probability 0.

Infinite-volume Gibbs measures can be defined by taking limits, as G tends to $\mathbb{Z}^2$, of finite-volume measures $\mu^\tau_{\beta,q,G}$. In particular, if $(i) := \tau$ denotes the constant configuration equal to $i \in \{1,\dots,q\}$, the sequence of measures $\mu^{(i)}_{\beta,q,G}$ can be proved to converge, as G tends to infinity, to a Gibbs measure denoted by $\mu^{(i)}_{\beta,q}$ (see one possible justification in Remark 12.10). This measure is called the *infinite-volume Gibbs measure with monochromatic boundary conditions i.*

The Potts models undergo a phase transition in infinite volume at a certain *critical inverse-temperature* $\beta_c(q) \in (0,\infty)$ in the following sense

$$\mu^{(i)}_{\beta,q}[\sigma_0 = i] = \begin{cases} \frac{1}{q} & \text{if } \beta < \beta_c(q), \\ \frac{1}{q} + m_\beta > \frac{1}{q} & \text{if } \beta > \beta_c(q). \end{cases}$$

One may ask what is the value of $\beta_c(q)$, and what is happening at criticality. In [LMR86, LMMS+91, KS82], the Potts model was proved to undergo a discontinuous phase transition at criticality when $q > 25$, i.e. that $\mu^{(i)}_{\beta_c(q),q}[\sigma_0 = i] > \frac{1}{q}$. The following result provides one of the first theorems treating small $q \neq 2$ values.

Theorem 12.7. *For any $q \geq 2$ and for any $i \in \{1,\dots,q\}$, we have (Beffara, Duminil-Copin [BDC12a])*

$$\beta_c(q) = \frac{1}{2}\log(1+\sqrt{q}).$$

Furthermore (Duminil-Copin, Sidoravicius, Tassion [DCST13]), if $q \in \{2,3,4\}$,

$$\mu^{(i)}_{\beta_c(q),q}[\sigma_0 = i] = \tfrac{1}{q}.$$

Before showing this theorem, let us mention that the proof is based on the random-cluster model (more precisely on Theorem 5.10 and Corollary 6.16) and a coupling between the random-cluster model with cluster-weight q and the q-state Potts model. The fact that Corollary 6.16 is based on the parafermionic observable implies that Theorem 12.7 is an example of application of the parafermionic observable to Potts models.

The value of the critical point was known in the $q = 2$ case and for large q. The second property yields the continuity of the phase transition (in opposition to a discontinuous phase transition for which $\mu^{(i)}_{\beta_c(q),q}[\sigma_0 = i] > \frac{1}{q}$). The proof of the continuity of the phase transition for q equal to 3 or 4 appears to be new (for $q = 2$, it is known since Onsager and Yang). Let us mention that Baxter [Bax71, Bax73, Bax78, Bax89] used a mapping between the Potts model and solid-on-solid ice-models to compute the free energy at criticality. He was able to predict that the phase transition was continuous for $q \leq 4$ and discontinuous for $q \geq 5$. While this computation gives a good insight on the behavior of the model, it relies on unproved assumptions which, forty years after their formulation, seem still very difficult to justify rigorously. Unfortunately, we are currently unable to show rigorously that the phase transition is discontinuous for every $q \geq 5$.

Remark 12.8. In dimension $d \geq 3$, the phase transition is expected to be continuous if and only if $q = 2$. The best results in this direction are the following ones. On the one hand, the fact that the phase transition is continuous for the Ising model ($q = 2$) is known for any $d \geq 3$ [ADCS13] (in fact, the critical exponents are known to be taking their mean-field value [AF86] for $d \geq 4$). On the other hand, mean-field considerations combined with Reflection-Positivity enabled [BCC06] to prove that for any $q \geq 3$, the q-state Potts model undergoes a discontinuous phase transition above some dimension $d_c(q)$. Finally, [KS82] used Reflection-Positivity to prove that for any $d \geq 2$, the phase transition is discontinuous provided q is large enough (the Pirogov-Sinai theory was used in [LMR86] to obtain the same result).

Theorem 12.7 follows directly from Theorem 5.10 and Corollary 6.16 via the following extension of the Edwards-Sokal coupling to q-colorings. Let $q \geq 2$ and let G be a finite graph. Assume a configuration ω of open and closed edges on G is given. One can deduce a q-coloring σ of the graph G by assigning independently to each cluster of ω a color among the q possible colors (here again, we mean that we give this color to each spin on the cluster), each with probability $1/q$, except for the cluster of the boundary which receives color i.

Proposition 12.9. *Fix an integer $q \geq 2$. Let $p \in (0,1)$ and G a finite graph. If the configuration ω is distributed according to a random-cluster measure with parameters (p,q) and wired boundary conditions, then the coloring σ is distributed according to a q-state Potts measure with inverse temperature $\beta = -\frac{1}{2}\ln(1-p)$ and monochromatic boundary conditions i.*

Proof. The proof of the Edwards-Sokal coupling works *mutatis mutandis* in this case. □

Remark 12.10. The coupling has many other important implications. It allows us to sample efficiently Potts configurations via algorithms on the random-cluster model such as Swendsen-Wang [SW87] (see also [Gri06, Section 8] and references therein). It also leads to additional correlation inequalities which are not available directly for Potts models: for instance, the FKG inequality for the random-cluster representation often replaces the FKG inequality for spins which fails whenever $q \geq 3$. To provide yet another example of applications, let us justify the definition of $\mu^{(i)}_{\beta,q}$. For $q \geq 2$, the existence of the limit of $\mu^{+}_{\beta,2,\Lambda_n}$ could follow from the stochastic ordering between different measures with + boundary conditions[4]. For $q \geq 3$, this stochastic ordering does not exist. Nevertheless, one may look at random-cluster models on Λ_n with wired boundary conditions, and use the Edwards-Sokal coupling to sample the measure $\mu^{(i)}_{\beta,q,\Lambda_n}$. It is then easy to show that the coupling converges in the limit towards the following coupling: consider the infinite-volume random-cluster model with wired boundary conditions, and color each finite cluster uniformly and independently, except for the infinite cluster which is colored in color i. We therefore obtained the convergence result, and an extension of the Edwards-Sokal coupling to the infinite volume.

12.3 The spin $O(n)$-models

12.3.1 Spin $O(n)$-models

After the introduction of the Ising model by Lenz [Len20], and the conjecture by Ising that no phase transition was occurring, many physicists tried to find natural generalizations of the model exhibiting a phase transition. In [HK34], Heller and Kramers described the classical version of the celebrated quantum Heisenberg model where spins are vectors of the three-dimensional sphere $\mathbb{S}^3$. Later, Stanley generalized this model by allowing spins to be on the sphere $\mathbb{S}^n$ of radius $\sqrt{n}$ in dimension n [Sta68]. This model is now known as the *spin $O(n)$-model*. Therefore, the $O(1)$-model is the Ising model, the $O(2)$-model is the so-called XY-model (it was introduced in [VL66] two years before the general model), and the $O(3)$-model is the classical Heisenberg model. We refer to [DG76] for a history of the subject.

The spin $O(n)$-model can be defined on any graph, however, we restrict ourselves to the hexagonal lattice $\mathbb{H}$. Let G be a finite subgraph of $\mathbb{H}$. The spin $O(n)$-model with *free boundary conditions* is a random assignment $\sigma \in (\mathbb{S}^n)^{V_G}$ of spins $\sigma_x \in \mathbb{S}^n$, where σ_x denotes the spin at vertex x. The *Hamiltonian* of the model is defined by

$$H^{\mathrm{f}}_{n,G}(\sigma) := -\sum_{x\sim y} \langle \sigma_x | \sigma_y \rangle ,$$

where the summation is over all pairs of neighboring vertices x, y in G, and $\langle \cdot | \cdot \rangle$ is the scalar product in dimension n. The *partition function* of the model is

$$Z^{\mathrm{f}}_{\beta,n,G} := \int_{\sigma\in(\mathbb{S}^n)^{V_G}} d\sigma \exp\left[-\beta H^{\mathrm{f}}_{n,G}(\sigma)\right], \qquad (12.1)$$

[4]We did not use this fact to prove the existence of Gibbs measures for the Ising model in this book.

where β is the inverse temperature of the model and $d\sigma$ the tensor product of $|V_G|$ measures $k_n dx$, where dx is the Lebesgue measure on $\mathbb{S}^n$ and k_n a normalization factor introduced in such a way that $\int_{\mathbb{S}^n} d\sigma = 1$. The spin $O(n)$-measure is given by the measure $\mu^{\mathrm{f}}_{\beta,n}$ with density

$$\frac{1}{Z^{\mathrm{f}}_{\beta,n,G}} \exp\left[-\beta H^{\mathrm{f}}_{n,G}(\sigma)\right].$$

12.3.2 Phase transition in planar $O(n)$-models

The planar spin $O(1)$-model being the Ising model, we already discussed its phase transition extensively[5]. The phase transition in the spin $O(n)$-model when $n \geq 2$ is very different from the phase transition in the Ising model. The case of the $O(2)$-model is already interesting: the planar XY-model is never ordered at any temperature. Nevertheless, there is a qualitative change of behavior in the model:

- At very low inverse-temperature, spin correlations $\mu^{\mathrm{f}}_{\beta,n}(\langle\sigma_a|\sigma_b\rangle)$ decay exponentially fast in the distance between the vertices [MS77].
- At very high inverse-temperature, spin correlations $\mu^{\mathrm{f}}_{\beta,n}(\langle\sigma_a|\sigma_b\rangle)$ decay as an inverse power in the distance between the vertices [FS81].

Moreover, there exists a critical inverse-temperature β_c separating the two phases: for $\beta > \beta_c$, correlations decay as inverse power laws while for $\beta < \beta_c$, they decay exponentially fast. Several papers are studying this value β_c, and we refer to [Cha98] for a more detailed discussion. A phase transition of this type is called *Berezinsky-Kosterlitz-Thouless* (it is named after Berezinsky and Kosterlitz-Thouless who introduced it non rigorously for the planar XY-model in two independent papers [Ber72] and [KT73]). The main differences with phase transitions previously described in this document is the absence of an ordered phase with global symmetry breaking. Moreover, the order of the phase transition is infinite (the free energy is infinitely differentiable but not analytic at the transition).

Other values of n are very different: Polyakov conjectured in 1975 that no phase transition occurs whenever $n \geq 3$ [Pol75]. Polyakov's conjecture is generally accepted, even so it is not completely unanimous. We mention that the existence/absence of phase transitions is still an open mathematical question of great interest even for very large n. The only known results deal with the $n \to \infty$ limit which corresponds to the spherical model of Berlin and Kac [BK52]: in [Kup80], the absence of a phase transition was established when $n \to \infty$ at fixed β/n by using an asymptotic expansion in $1/n$. Note that it does not imply the absence of a phase transition at any fixed n.

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[5] Most of the book was devoted to the Ising model on the square lattice, however the last section shows that the same results hold on the hexagonal lattice.

12.4 Loop $O(n)$-model on the hexagonal lattice

12.4.1 Definition

This model, introduced in [DMNS81] on the hexagonal lattice, is a lattice gas of non-intersecting loops. More precisely, consider configurations of non-intersecting simple loops on a finite subgraph of the hexagonal lattice and introduce two parameters: a loop-weight $n \geq 0$ (in fact after suitable modifications, one may work with $n \geq -2$) and an edge-weight $x > 0$, and ask the probability of a configuration to be proportional to $n^{\#\,\text{loops}} x^{\#\,\text{edges}}$. From time to time, an interface between two boundary points could be added: in this case configurations are composed of non-intersecting simple loops and one self-avoiding interface (avoiding all the loops) from a to b.

Two values of n are of special interest. Since no loop is allowed, the $O(0)$-model with an interface is the self-avoiding walk from a to b, as first mentioned in [DG72]. In the next paragraph, the loop $O(1)$-model will be related to the high-temperature expansion of the Ising model on the hexagonal lattice.

12.4.2 Connection between the spin and loop $O(n)$-models

In fact, the loop $O(n)$-model was introduced as an approximation of the high-temperature expansion of the spin $O(n)$-model on the hexagonal lattice. Instead of the partition function in (12.1), consider the simplified partition function

$$\tilde{Z}^{\mathrm{f}}_{x,G} := \int_{\sigma \in (\mathbb{S}^n)^{V_G}} d\sigma \prod_{[ab] \in E_G} \big(1 + x \,\langle \sigma_a | \sigma_b \rangle\big), \tag{12.2}$$

where $x > 0$. Strictly speaking, the partition functions $Z^{\mathrm{f}}_{x,G}$ (here x replaces β in (12.1)) and $\tilde{Z}^{\mathrm{f}}_{x,G}$ coincide only in the limit x approaching 0 yet the two models are expected to belong to the same universality class. In the Ising case, $Z^{\mathrm{f}}_{\beta,G}$ is the integral of

$$\prod_{[ab] \in E_G} \Big(\frac{e^{\beta} + e^{-\beta}}{2} + \frac{e^{\beta} - e^{-\beta}}{2} \,\langle \sigma_a | \sigma_b \rangle\Big).$$

Thus,

$$Z^{\mathrm{f}}_{\beta,G} := \cosh(\beta)^{|E_G|} \tilde{Z}^{\mathrm{f}}_{x,G}$$

for $x := \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}}$. In other words, when $n = 1$, the previous replacement is not an approximation.

As in the high-temperature expansion of the Ising model, $\tilde{Z}_{x,G}$ can be expanded in powers of x.

$$\begin{aligned}\tilde{Z}^{\mathrm{f}}_{x,G} &= \int_{\sigma\in(\mathbb{S}^n)^{V_G}} d\sigma \prod_{[ab]\in E_G} \big(1 + x\,\langle\sigma_a|\sigma_b\rangle\big) \\ &= \int_{\sigma\in(\mathbb{S}^n)^{V_G}} d\sigma \sum_{\omega\subset E_G}\prod_{[ab]\in\omega} x\,\langle\sigma_a|\sigma_b\rangle \\ &= \sum_{\omega\subset E_G} x^{|\omega|}\int_{\sigma\in(\mathbb{S}^n)^{G}} d\sigma \prod_{[ab]\in\omega}\langle\sigma_a|\sigma_b\rangle,\end{aligned}$$

where $|\omega|$ is the number of edges in ω. Recall that $\mathcal{E}_\Omega$ is the set of even subgraphs of Ω. In the case of the hexagonal lattice, it is simply the set of collections of non-intersecting self-avoiding loops. Now,

$$\int_{\sigma\in(\mathbb{S}^n)^{G}} d\sigma \prod_{[ab]\in\omega}\langle\sigma_a|\sigma_b\rangle = \begin{cases} n^{\#\,\text{loops}} & \text{if } \omega\in\mathcal{E}_\Omega \\ 0 & \text{otherwise.}\end{cases} \tag{12.3}$$

In order to see that, first observe that if there exists a vertex x of degree 1 in ω (let us assume that it belongs to the edge $[xy]$ in ω), then

$$\begin{aligned}\int_{\sigma\in(\mathbb{S}^n)^{G}} d\sigma \prod_{[ab]\in\omega}\langle\sigma_a|\sigma_b\rangle &= \int_{\sigma\in(\mathbb{S}^n)^{G\smallsetminus\{x\}}} d\sigma \prod_{[ab]\in\omega\smallsetminus\{[xy]\}}\langle\sigma_a|\sigma_b\rangle \int d\sigma_x\langle\sigma_x|\sigma_y\rangle \\ &= 0\end{aligned}$$

where the last equality is due to the fact that the integral on the right equals 0 by symmetry of $d\sigma_x$ with respect to $\sigma_x \longleftrightarrow -\sigma_x$. This computation implies that the only configurations for which the integral is non-zero are the elements of $\mathcal{E}_\Omega$.

Let us now compute the integral in the case of a configuration composed of one loop γ only. First, a simple computation leads[6] to

$$\int d\sigma_x\langle\sigma_z|\sigma_x\rangle\langle\sigma_x|\sigma_y\rangle = \langle\sigma_z|\sigma_y\rangle.$$

By using this formula iteratively, we deduce that

$$\int_{\sigma\in(\mathbb{S}^n)^{G}} d\sigma \prod_{[ab]\in\gamma}\langle\sigma_a|\sigma_b\rangle = \int d\sigma_c\langle\sigma_c|\sigma_c\rangle = \int d\sigma_c n = n,$$

[6] First observe that the two terms are rotationally invariant so that we may assume that $\sigma_z = (\sqrt{n}, 0, \dots, 0)$. Now if $\sigma_y = (y_1, \dots, y_n)$ and $\sigma_x = (x_1, \dots, x_n)$, we get that $\langle\sigma_z|\sigma_x\rangle\langle\sigma_x|\sigma_y\rangle = x_1\sqrt{n}(x_1y_1 + \cdots + x_ny_n)$. When developing the previous expression and then integrating with respect to σ_x, all the terms containing x_1 to the power 1 disappear by symmetry and computing the original integral reduces to the computation of the integral of $\sqrt{n}x_1^2y_1$ with respect to σ_x. Now, integrating $n = \|x\|^2 = x_1^2 + \cdots + x_n^2$ on the sphere gives n thanks to the normalization of the measure. Since the coordinates of σ_x play symmetric roles, we get that the integral of x_1^2 is 1. Overall, the original integral equals $\sqrt{n}y_1 = \langle z|y\rangle$.

where c is any vertex on γ. We used that σ_x belongs to the sphere of radius $\sqrt{n}$ and the normalization of $\int d\sigma_c = 1$. The formula for an arbitrary loop-configuration follows by integration on each loops. This gives (12.7). Altogether,

$$\tilde{Z}^{\mathrm{f}}_{x,G} = \sum_{\omega\in\mathcal{E}_G} x^{|\omega|} n^{\#\text{ loops}}$$

and we thus obtain the partition function of the loop $O(n)$-model.

We may insert $\langle\sigma_c|\sigma_d\rangle$ in the integral. Doing the same computation as above, we obtain the correlation functions $\tilde{\mu}^{\mathrm{f}}_{x,n}(\langle\sigma_a|\sigma_b\rangle)$ for this new model:

$$\tilde{\mu}^{\mathrm{f}}_{x,n}(\langle\sigma_c|\sigma_d\rangle) := \frac{\displaystyle\int_{\sigma\in(\mathbb{S}^n)^{V_G}} d\sigma\ \langle\sigma_c|\sigma_d\rangle \prod_{[ab]\in E_G} \big(1 + x\,\langle\sigma_a|\sigma_b\rangle\big)}{\displaystyle\int_{\sigma\in(\mathbb{S}^n)^{V_G}} d\sigma \prod_{[ab]\in E_G} \big(1 + x\,\langle\sigma_a|\sigma_b\rangle\big)}$$

$$= \frac{\displaystyle\sum_{\omega\in\mathcal{E}_G(c,d)} x^{|\omega|} n^{\#\text{ loops}}}{\displaystyle\sum_{\omega\in\mathcal{E}_G} x^{|\omega|} n^{\#\text{ loops}}},$$

where $\mathcal{E}_\Omega(c,d)$ is the set of configurations of loops with one interface from c to d (this justifies the introduction of configurations with one additional self-avoiding path).

12.4.3 Parafermionic observable and phase transition

The loop $O(n)$-model exhibits a greater variety of critical behavior than the spin $O(n)$-model, since n is not constrained to be an integer. Similarly to the spin $O(n)$-model, the loop $O(n)$-model is expected to have a Berezinsky-Kosterlitz-Thouless phase transition for some range of $n \le n_c$, while for $n > n_c$, no phase transition occurs.

In the former case, the definition of the phase transition corresponds to the existence of $x_c \in (0,\infty)$ such that

- For $x < x_c$, the probability that vertices a and b are on the same loop decays exponentially fast in the distance between a and b.
- For $x > x_c$, the probability that vertices a and b are on the same loop decays as an inverse power in the distance between a and b.

Bernard Nienhuis [Nie82, Nie84] suggested that $n_c = 2$ and proposed the following conjecture, supported by physics arguments.

Conjecture 12.11 (Nienhuis [Nie82, Nie84])**.** *For $n \in [-2,2]$, the critical value is given by $x_c(n) = 1/\sqrt{2+\sqrt{2-n}}$.*

The conjecture is rigorously established in two cases only:

- When $n = 1$, the critical value is related to the critical temperature of the Ising model, since the $O(1)$-model is exactly the high-temperature expansion of the spin Ising model and the computation of the critical value for this model gives us that indeed $x_c(1) = 1/\sqrt{3}$;
- When $n = 0$, it was proved in Chapter 2 that $\sqrt{2+\sqrt{2}}$ is the connective constant of the hexagonal lattice. Remembering the discussion on the self-avoiding walk model seen as a model of statistical physics in the introduction, this computation implies that $x_c(0) = 1/\sqrt{2+\sqrt{2}}$.

For other values of $n \in [0,2]$, this prediction can be understood using parafermionic observables. Exactly as in the case of the random-cluster model, one can extend the definition of the spin fermionic observable, as shown by Smirnov in [Smi06]. For a discrete domain Ω (see Chapter 3) with two *mid-edges* on the boundary u and v, the *parafermionic observable* is defined for a mid-edge z in Ω by

$$F(z) = F(z,u,v,x,n,\sigma) = \frac{\sum\limits_{\omega\in\mathcal{E}_\Omega(u,z)} e^{-\sigma \mathrm{i} W_\gamma(u,z)} x^{|\omega|} n^{\#\text{ loops}}}{\sum\limits_{\omega\in\mathcal{E}_\Omega(u,v)} e^{-\sigma \mathrm{i} W_\gamma(u,v)} x^{|\omega|} n^{\#\text{ loops}}}, \tag{12.4}$$

where $\mathcal{E}_\Omega(u,v)$ is the set of configurations of loops with one interface from the mid-edge u to the mid-edge v. It is important to notice that exactly as for the computation of the connective constant of the hexagonal lattice, the model is slightly modified: the interface starts and ends at mid-edges instead of vertices. One can easily prove using the same argument as in Chapter 2 (see Fig. 12.2) that the observable satisfies local relations similar to the self-avoiding walk case at the conjectured critical value if σ is chosen carefully.

Proposition 12.12. *Let Ω be a finite domain of $\mathbb{H}$ (see Chapter 2) and a,b two boundary mid-edges. If $x = x_c(n) = 1/\sqrt{2+\sqrt{2-n}}$ and $\sigma = \sigma(n) = 1 - \frac{3}{4\pi}\arccos(-n/2)$, let F be the parafermionic observable, then*

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) \;=\; 0 \tag{12.5}$$

where p, q and r are the three mid-edges adjacent to a vertex v.

Recall that these relations imply that discrete contour integrals vanish. They may also be understood as discretizations of the Cauchy-Riemann equations around vertices of $\mathbb{H}$ (unfortunately we have no information for discretizations of Cauchy-Riemann equations around dual-vertices of

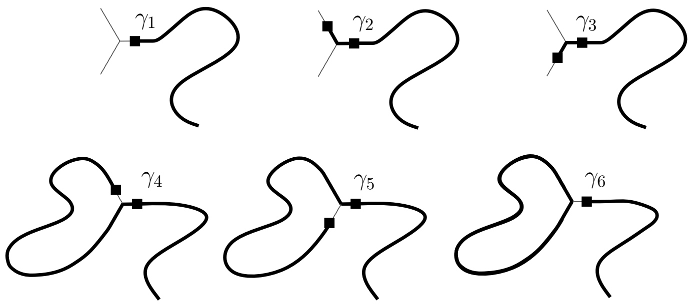

Figure 12.2: Following the same proof as for self-avoiding walks, one may associate configurations γ_1, γ_2 and γ_3 (resp. γ_4, γ_5 and γ_6) together to show that the sum of contributions vanishes in Proposition 12.12 (note the additional configuration with one self-avoiding walk visiting one mid-edge around the vertex and a loop facing it). We leave the proof as an exercise.

$(\mathbb{Z}^2)^\star$, and therefore F is not (a priori) discrete holomorphic in the classical sense). In the next chapter, we will come back to this point and discuss the scaling limit of F.

12.5 $O(n)$-model on the square lattice

It is tempting to extend the definition of the $O(n)$-model to the square lattice in order to obtain a family of models containing self-avoiding walks and the high-temperature expansion of the Ising model on $\mathbb{Z}^2$. Nevertheless, difficulties arise when dealing with the $O(n)$-model on graphs which are not trivalent. Indeed, the indeterminacy when counting intersecting loops in even subgraphs prevents us from defining the model as in the previous paragraph. One can still define a model of loops on $G^\star \subset \mathbb{Z}^2$ by distinguishing between possible intersection patterns: faces of $G \subset \mathbb{Z}^2$ are filled with one of the nine plaquets in Fig. 12.3. A weight p_v is associated to every face v depending on its plaquet. The probability of a configuration is then proportional to $n^{\#\text{ loops}} \prod_{v \in G^\star} p_v$ (we use that faces of $\mathbb{Z}^2$ are in correspondence with vertices of the dual lattice). This model was studied in [Nie90]. We refer to [IC09] for references and additional details.

Remark 12.13. Several special cases are noteworthy:

- $u_1 = u_2 = v = x$, $t = 1$ and $w_1 = w_2 = n = 0$ corresponds to vertex self-avoiding walks on the square lattice.

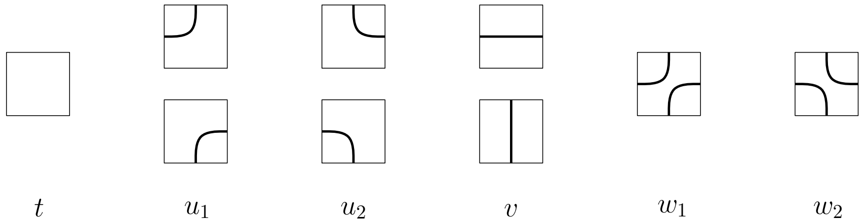

Figure 12.3: Different possible plaquets with their associated weights.

- $u_1 = u_2 = v = \sqrt{w_1} = \sqrt{w_2} = x$ and $n = t = 1$ corresponds to the high-temperature expansion of the Ising model at inverse temperature $\beta = \frac{1}{2}\log[(1+x)/(1-x)]$.
- $t = u_1 = u_2 = v = 0$, $w_1 = w_2 = 1$ and $n > 0$ corresponds to the loop representation of the random-cluster model at criticality with $q = n^2$.

A parafermionic observable can also be defined on the medial lattice:

$$F(z) = \frac{\sum\limits_{\omega\in\widetilde{\mathcal{E}}_\Omega(a,z)} e^{-i\sigma W_\gamma(a,z)}\, n^{\#\text{ loops}} \prod\limits_{v\in\Omega^\star} p_v}{\sum\limits_{\omega\in\widetilde{\mathcal{E}}_\Omega} n^{\#\text{ loops}} \prod\limits_{v\in\Omega^\star} p_v} \tag{12.6}$$

where $\widetilde{\mathcal{E}}_\Omega$ and $\widetilde{\mathcal{E}}_\Omega(a,z)$ correspond to families of plaquets satisfying that the configuration is respectively a family of non-intersecting loops and a family of non-intersecting loops plus a self-avoiding path from a to z.

Let us study the existence of local relations for this observable. We will allow ourselves an additional geometric degree of freedom: the lattice can be twisted, meaning that each rhombus is not a square anymore but a rhombus with inside angle θ (this modifies the winding term); see Fig. 12.4 for an example when $\theta = \pi/3$.

One can then look for a local relation for F around a vertex $v \in \Omega^\star$, which would be a discrete analogue of the Cauchy-Riemann equation. As in the case of random-cluster models and the Ising model, one can associate configurations by small groups, and try to check the equation for each of these groups, thus leading to a certain number of complex equations. For each choice of the weights of the model, of the spin σ and of the geometric parameter θ, one may check these equations.

We present a solution and its parametrization due to Alexander Glazman (personal communication). Fix n, there exists a solution for $\sigma = 1 + s$ satisfying $n = -2\cos[\frac{4\pi}{3}s]$. Note that there are a priori four possible choices for σ. In general, the unique solution is given by the following weights:

$$\begin{cases} t &= \dfrac{\sin[\frac{2\pi}{3}s]^3}{\sin[\frac{\pi}{3}s]} + \sin[(\theta-\frac{\pi}{3})s]\ \sin[(\frac{2\pi}{3}-\theta)s], \\ u_1 &= \sin[(\pi-\theta)s]\ \sin[\frac{2\pi}{3}s], \\ u_2 &= \sin[\theta s]\ \sin[\frac{2\pi}{3}s], \\ v &= \sin[\theta s]\ \sin[(\pi-\theta)s], \\ w_1 &= \sin[(\frac{2\pi}{3}-\theta)s]\ \sin[(\pi-\theta)s], \\ w_2 &= \sin[(\theta-\frac{\pi}{3})s]\ \sin[\theta s]. \end{cases} \tag{12.7}$$

Remark 12.14. For $\theta = \pi/3$, the rhombus can be divided into two triangles, and the dual of this graph is the hexagonal lattice. Now, the loop $O(n)$-model on the hexagonal lattice can be rewritten as a model of plaquets on the square lattice, see Fig. 12.4, with

$$\begin{cases} u_1/t &= x, \\ u_2/t &= x^2, \\ v/t &= x^2, \\ w_1/t &= x^2, \\ w_2/t &= 0. \end{cases}$$

We therefore deduce that the solution of (12.7) for $n \in [0,1]$, $\theta = \pi/3$ and $\sigma = 1 - \frac{3}{4\pi}\arccos(-n/2)$ is $u_1 = tx_c(n)$, $u_2 = v = w_1 = tx_c(n)^2$, and $w_2 = 0$.

Glazman [Gla13] also observes that for $n = 0$ and $\theta \in [\pi/3, \pi/2]$, the weights correspond to a self-repulsive non-self-crossing edge-avoiding random walk on the square lattice. He uses the observable to determine the critical point for this model. Even though this model is not the classical self-avoiding walk (walks of fixed lengths can have different weights), the result is one of the only generalizations of [DCS12b]. Let us mention another generalization.

Theorem 12.15 (Jensen, Guttmann [JG98])**.** *The connective constant $\mu(\mathbb{L})$ of the 3.12^2 lattice (see Fig. 12.4) $\mathbb{L}$ is the positive root of*

$$x^3 - \sqrt{2+\sqrt{2}}\,x = \sqrt{2+\sqrt{2}}.$$

In particular, it is algebraic of order 12, and it can be computed explicitly.

Note that the result is anterior to the computation of the connective constant of the hexagonal lattice. The reason is that [JG98] is a conditional result which is based on the assumption that $\mu_c(\mathbb{H}) = \sqrt{2+\sqrt{2}}$.

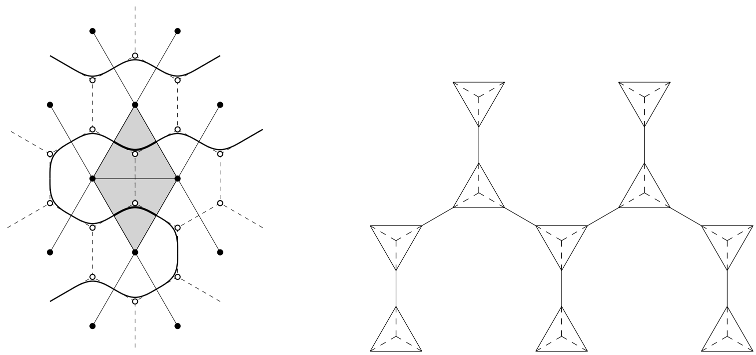

Figure 12.4: **Left.** A configuration of a loop $O(n)$-model on the hexagonal lattice and one associated plaquet when the model is seen as a loop $O(n)$-model on the square lattice with $\theta = \pi/3$. **Right.** The 3.12^2 lattice $\mathbb{L}$.

Proof. Set $G_{\mathbb{H}}$ for the partition function of self-avoiding walks starting from mid-edges on the hexagonal lattice. Call a vertex v of a self-avoiding walk ω in $\mathbb{L}$ *pivot* if it is the center of an edge not in a triangle (in other words, if it is the center of an edge of $\mathbb{H}$). The sequence of pivots forms a self-avoiding walk on the mid-edges of the hexagonal lattice. Moreover, the possibilities between two pivots are limited: the part of the walk is either composed of two edges forming the geodesic between the two pivots, or it contains three edges. Therefore, the partition function $G_{\mathbb{L}}$ of self-avoiding walks in $\mathbb{L}$ satisfies

$$G_{\mathbb{L}}(z) = G_{\mathbb{H}}(z^2 + z^3)$$

which implies that $\mu(\mathbb{L})^{-3} + \mu(\mathbb{L})^{-2} = \mu(\mathbb{H})^{-1}$. □

12.5.1 Discrete observables and Yang-Baxter's equation

The study of parafermionic observables can be generalized to a variety of lattice models, see the work of Cardy, Ikhlef, Riva, Rajabpour [RC06, RC07, IC09]. Unfortunately, the observable satisfies only some of the discrete Cauchy-Riemann equations except for the Ising case. Interestingly, weights for which there exists a "half-holomorphic" observable which is not degenerate in the scaling limit always correspond to weights for which the famous Yang-Baxter relation (we refer to [Bax89] for details on this relation) holds. A perfect example is the system of solutions given in the previous section for the $O(n)$-model on the square lattice. This Yang-Baxter relation is a crucial tool when computing the free energy of a model[7]. The existence of this link between integrability and discrete holomorphicity was explored in [IWWZJ13], where the construction of [BL91] is used to recover the parafermionic observables from the construction of so-called conserved currents using Quantum Groups. This approach may lead to new observables in the models described in this book, as well as new observables in other models.

[7] When the free energy can be computed explicitly, the model is said to be *integrable*.

Chapter 13

Many questions and a few answers

This chapter presents a few open problems in this area. We gathered them in three categories: open questions related to the Ising model, then to the random-cluster model, and finally to loop $O(n)$-models.

13.1 Ising model

13.1.1 Universality of the Ising model with respect to the lattice

Until now, the Ising model was considered on the square lattice or on isoradial graphs with critical weights. Nevertheless, the renormalization group (see the introduction for more details) predicts that the scaling limit should be universal, meaning that it does not depend on the fine definition of the model (in particular the graph). In other words, the limit of critical Ising models on planar graphs should always be the same.

Let us define a large class of lattices for which we would like to prove universality. We focus on planar locally-finite doubly periodic weighted graphs $(\mathcal{G}, J)$ (here $\mathcal{G}$ is the graph and $J = (J_e)_{e \in E_{\mathcal{G}}}$ is the set of *non-negative* weights), i.e. weighted graphs which are invariant under the action of some lattice $\Lambda \simeq \mathbb{Z} \oplus \mathbb{Z}$. In such case, $\mathcal{G}/\Lambda =: G$ is a finite graph embedded in the torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$. By convention $\mathcal{G}$ will always denote a doubly periodic graph embedded in the plane, while G will denote a graph embedded in the torus.

We further assume that $\mathcal{G}$ (or equivalently G) is *non-degenerate*, i.e. that the complement of the edges is the union of topological discs[1].

[1] It guarantees that $\mathcal{G}$ itself is not the union of one-dimensional of graphs but really a two-dimensional lattice.

Ising probability measures can be constructed on $\mathcal{G}$ as limits of finite volume probability measures.

Recall that $\mathcal{E}_G$ is the set of even subgraphs of G, that is, the set of subgraphs γ of G such that every vertex of G is adjacent to an even number of edges of γ. Let $\mathcal{E}_G^0$ denote the set of even subgraphs of G that wind around each of the two directions of the torus an even number of times. Set $\mathcal{E}_G^1 = \mathcal{E}_G \setminus \mathcal{E}_G^0$.

Theorem 13.1 (Cimasoni, Duminil-Copin [CDC13], Li [Li12]). *The critical inverse temperature β_c for the Ising model on the weighted graph $(\mathcal{G}, J)$ is the unique solution $0 < \beta < \infty$ to the equation*

$$\sum_{\gamma \in \mathcal{E}_G^0} x(\gamma) = \sum_{\gamma \in \mathcal{E}_G^1} x(\gamma), \tag{13.1}$$

where $x(\gamma) = \prod_{e \in \gamma} x_e$ and $x_e = \tanh(\beta J_e)$.

Let us briefly mention that the proof is based on the so-called *Kac-Ward matrix* associated to the weighted graph (G, J) and to a pair of non-vanishing complex numbers (z, w). This matrix is related to s-holomorphicity and fermionic observables; see [Cim13, Lis13]. Therefore, this identification of the critical inverse temperature using the Kac-Ward matrices opens a way for understanding conformal invariance for the Ising model on arbitrary doubly periodic graphs.

Question 13.2. *Prove that there always exists a periodic embedding of $\mathcal{G}$ such that the Ising model on $\mathcal{G}$ is conformally invariant.*

Remark 13.3. When formulating the previous question, one should be careful about the way the graph is drawn in the plane. For instance, the isotropic spin Ising model of Chapter 7, when considered on a stretched square lattice (every square is replaced by a rectangle with aspect ratio not equal to 1) is not conformally invariant (it is not invariant under the rotation by $\pi/2$). Isoradial graphs mentioned in Chapter 8 form a large family of graphs possessing a natural embedding on which a critical Ising model is expected to be conformally invariant, but for more general lattices, one may have to apply a non-trivial transformation of the lattice[2].

In another direction, universality could also be proved for specific non-planar lattices which are in some sense a slight modification of a planar

[2] Let us give a trivial example. The isotropic Ising model on the stretched square lattice where every face is a rectangle of width 1 and height λ is not conformally invariant. One needs to apply the transformation $(x, y) \mapsto (x, y/\lambda)$ to obtain something conformally invariant (here the stretched lattice is simply mapped back to the standard square lattice).

lattice. There, interfaces do not make sense anymore, but we can focus on the energy and spin fields. Impressive advances have been achieved in this direction over the past few years. We propose to describe one of them to illustrate the possible extensions.

Consider the Ising model on $\mathbb{Z}^2$ with Hamiltonian

$$H_\lambda(\sigma) = -\sum_{x,y\in\mathbb{Z}^2:\,x\sim y}\sigma_x\sigma_y - \lambda\sum_{x,y\in\mathbb{Z}^2}v(y-x)\sigma_x\sigma_y,$$

where $v:\mathbb{Z}^2\to\mathbb{R}_+$ is invariant under the rotation by an angle $\pi/2$, and $v(x)=0$ for $|x|\ge R$, where R is called the *range* of the interaction. For $\beta>0$, consider the measure $\mu_{\beta,\lambda}$ to be the Gibbs measure for this Ising model. While the case $\lambda=0$ is the nearest neighbor Ising model, whose free energy was computed by Onsager as mentioned several times in this book already, the case $\lambda\ne 0$ is not integrable, in the sense that the free energy cannot be computed explicitly. Nevertheless, for $\lambda\ll 1$ this Ising model can be studied using a rigorous approach to the renormalization group. Let us say a few words about this. Pinson and Spencer [PS00, Spe00] used as a starting point the fact that the partition function of certain non-integrable Ising models can be expressed in terms of a non-Gaussian Grassman integral by generalizing a mapping between the nearest-neighbor Ising model with a fermionic system discovered by Lieb, Mattis and Schultz [LSM64]. While the nearest-neighbor Ising model is equivalent to a system of non-interacting fermions, the general model is equivalent to a system of interacting fermions. Nevertheless, one may use techniques coming from Quantum Field Theory to relate the two models when the interaction is weak enough. Using this method, Pinson and Spencer [PS00] and Spencer [Spe00] showed that the exponent governing the energy density (see Section 9.3.1) is the same for some next-to-nearest neighbor interactions (there, they used the fact that the mappings provide a perturbative expansion of the energy density). Later, Giuliani, Greenblatt and Mastropietro [GGM12] extended these techniques in two directions. First, they treated a more general class of interactions, namely any finite range model with λ small enough, and second, they were able to show that the scaling limit of n-point correlations of the energy field in the full plane is the same as for $\lambda=0$. A natural question there is to do the same analysis for the spin field. One may also ask what happens in finite volume, even though this problem seems out of reach of today's techniques.

Remark 13.4. Note that the question of universality is not restricted to the Ising case and could be asked for all the models described in this book. We will not expand on this here.

13.1.2 Universality of the Ising model in the regime $\beta < \beta_c$

The site percolation on the triangular lattice is critical for $p = 1/2$ and is conformally invariant as shown in [Smi01]. Now, the Ising model at high temperature on the triangular lattice resembles the site percolation quite a lot (see [BCM10] for more details). Each spin is either +1 or −1 with probability 1/2 (in this case there is only one infinite-volume measure, hence the symmetry $+1/-1$). Moreover, the correlations between spins decay exponentially fast (the random-cluster representation obtained via the Edwards-Sokal coupling is subcritical). The scaling-limit of the interfaces $+1/-1$ should be conformally invariant, and should satisfy the so-called locality property (see [Law05, Section 6.3] for additional details). Schramm observed that SLE(6) is the only SLE(κ) satisfying this property and it is therefore natural to conjecture that interfaces in the high-temperature Ising model converge to SLE(6). Note that this model interpolates between critical Ising and percolation on the triangular lattice, two models for which conformal invariance is known.

Question 13.5. *Prove conformal invariance of the Ising model at high-temperature ($\beta < \beta_c$) on the triangular lattice.*

An answer to this question would provide another universality result, namely universality in the parameter β instead of the lattice.

Remark 13.6. One could also consider the Ising model on more general triangulations (in such case, site percolation is critical for $p = 1/2$). On the contrary, interfaces would not be conformally invariant on the square lattice, since the site percolation on the square lattice is not critical at $p = 1/2$.

13.1.3 Full scaling limit of the critical Ising model

It has been proved in [CDCH$^+$13] that the scaling limit of Ising interfaces in Dobrushin domains is SLE(3). The next question is to understand the full scaling limit of the interfaces. This question raises interesting technical problems. Consider the Ising model with free boundary conditions. Interfaces now form a family of loops. By consistency, each macroscopic loop should look like a SLE(3). Sheffield and Werner [SW10, SW12] introduced a one-parameter family of processes of non-intersecting loops which are conformally invariant – called the Conformal Loop Ensembles CLE(κ) for $\kappa > 8/3$. Not-surprisingly, loops of CLE(κ) are locally similar to SLE(κ) and these processes are natural candidates for the scaling limits of planar models of statistical physics.

Question 13.7. *Prove that the family of loops in the critical Ising model on the square lattice converges to CLE(3).*

We believe that the strong form of crossing estimates proved in Chapter 10 would be useful to study these interfaces. In [HK11], Hongler and Kytolä made one step towards the complete picture by studying interfaces with $+1/-1/$free boundary conditions.

13.2 Random-cluster model with cluster-weight $q \geq 0$

We gather some of the conjectures listed below in the phase diagram Fig. 13.1.

13.2.1 Case $q > 4$: discontinuity of the phase transition

As mentioned before, the phase transition is conjectured to be discontinuous (first order) for $q > 4$. In particular, the critical random-cluster model with wired boundary conditions should possess an infinite cluster almost surely while the critical random-cluster model with free boundary conditions should not (in this case, the connectivity probabilities should even decay exponentially fast by **P4** of Theorem 5.24). This result is known only for $q \geq 25.72$ (see [Gri06, LMR86, LMMS⁺91] and references therein).

Question 13.8. *Prove that there exists an infinite cluster for $\phi^1_{p_c,q}$ whenever $q > 4$.*

A weaker result was obtained in [DCST13] where it is shown that there is an infinite cluster on $\mathbb{U}$ (see Chapter 6 for the definition of $\mathbb{U}$) for $\phi^1_{p_c,q,\mathbb{U}}$ when $q > 4$. Bootstrapping this result from $\mathbb{U}$ to $\mathbb{Z}^2$ would allow one to answer Question 13.8.

13.2.2 Conformal invariance for $q \leq 4$

The vertex parafermionic observable[3] is now used to predict the critical behavior for general $q \leq 4$. For $q \neq 2$, this function is not s-holomorphic or even discrete holomorphic: the relations given in Chapter 6 provide partial information on Cauchy-Riemann equations, which justifies the heuristic claim that we possess only half of the information. In particular, they do not determine the observable from its boundary conditions, a property which was crucial in the proof of conformal invariance for $q = 2$. Nevertheless, the fact that discrete contour integrals vanish

[3] The vertex parafermionic observable at a medial-vertex not on the boundary is equal to $\frac{1}{2}$ times the sum of the edge parafermionic observable at incident medial-edges.

(for the edge parafermionic observable), together with the fact that the complex argument of the edge parafermionic observable is determined on the boundary suggest that the only possible limit (if any), is given by the solution to a certain Riemann-Hilbert boundary value problem. In order to justify this convergence, it could be useful to find complementary local relations for this observable allowing to prove discrete holomorphicity, but it is unclear whether such relations exist at the discrete level (the edge-observable seems to be a divergence free operator whose curl vanishes in the scaling limit only). In any case, we obtain the following conjecture.

Conjecture 13.9 (Smirnov [Smi06]). *Let $q < 4$ and (Ω, a, b) be a simply connected domain with two points on its boundary. Consider a family of Dobrushin domains $(\Omega_\delta, a_\delta, b_\delta)$ converging to (Ω, a, b) in the Carathéodory sense. Then,*

$$\frac{1}{(2\delta)^\sigma} F_\delta(z) \longrightarrow \phi'(z)^\sigma$$

uniformly on every compact subset of Ω, where $\sigma = \frac{2}{\pi}\arcsin(\sqrt{q}/2)$, F_δ is the vertex-observable at $p_c(q)$ in $(\Omega_\delta, a_\delta, b_\delta)$ with spin σ, and ϕ is any conformal map from Ω to $\mathbb{R} \times (0,1)$ sending a to $-\infty$ and b to ∞.

For $q = 4$, a similar conjecture can be formulated using the specific parafermionic observable available at this point.

Then, one may wish to prove conformal invariance of the exploration path by implementing the same program as in Chapter 9:

- Prove conformal invariance of the parafermionic observable (Conjecture 13.9).
- Show that (γ_δ) is precompact and that any sub-sequential limit of the exploration path (γ_δ) is a Loewner chain driven by a random continuous process W_t.
- Identify W_t using Lévy's theorem and the martingale argument presented in Chapter 9.

When $1 \le q \le 4$, the second step follows from Theorem 9.15. The last step is similar to the $q = 2$ case, except that the expansion of $\phi'(z)^\sigma$ leads to (we forget the stopping time present in the proof of Proposition 9.20 in the following computation):

$$\begin{aligned}
\sqrt{\pi} \cdot \mathbb{E}[M_t^z | \mathcal{G}_s] &= \mathbb{E}\left[\left(\frac{1 - 2t/z^2 + O(1/z^3)}{z - W_t + 2t/z + O(1/z^2)}\right)^\sigma \ \Big|\ \mathcal{G}_s\right] \\
&= \frac{1}{z^\sigma}\ \mathbb{E}\left[1 + \sigma W_t/z + \left(\frac{\sigma(\sigma+1)}{2} W_t^2 - 4\sigma t\right)/z^2 + O\left(1/z^3\right) \ \Big|\ \mathcal{G}_s\right] \\
&= \frac{1}{z^\sigma}\left(1 + \sigma \mathbb{E}[W_t | \mathcal{G}_s]/z + \mathbb{E}\Big[\frac{\sigma(\sigma+1)}{2} W_t^2 - 4\sigma t \Big| \mathcal{G}_s\Big]/z^2 + O\left(1/z^3\right)\right).
\end{aligned}$$

Taking $s = t$ yields

$$\sqrt{\pi} \cdot M_s^z = \frac{1}{z^\sigma}\Big(1 + \sigma W_s/z + \Big[\frac{\sigma(\sigma+1)}{2} W_s^2 - 4\sigma s\Big]/z^2 + O(1/z^3)\Big).$$

Therefore, we obtain that for $1 \le q \le 4$, Conjecture 13.9 implies the following conjecture by Schramm.

Conjecture 13.10 (Schramm [Sch07]). *Consider the critical random-cluster model with cluster-weight $q \le 4$. The exploration path $(\gamma_\delta)_{\delta>0}$ converges weakly to the Schramm-Loewner Evolution with parameter* $\kappa = 8/(\sigma+1) = 4\pi/[\pi - \arccos(\sqrt{q}/2)]$.

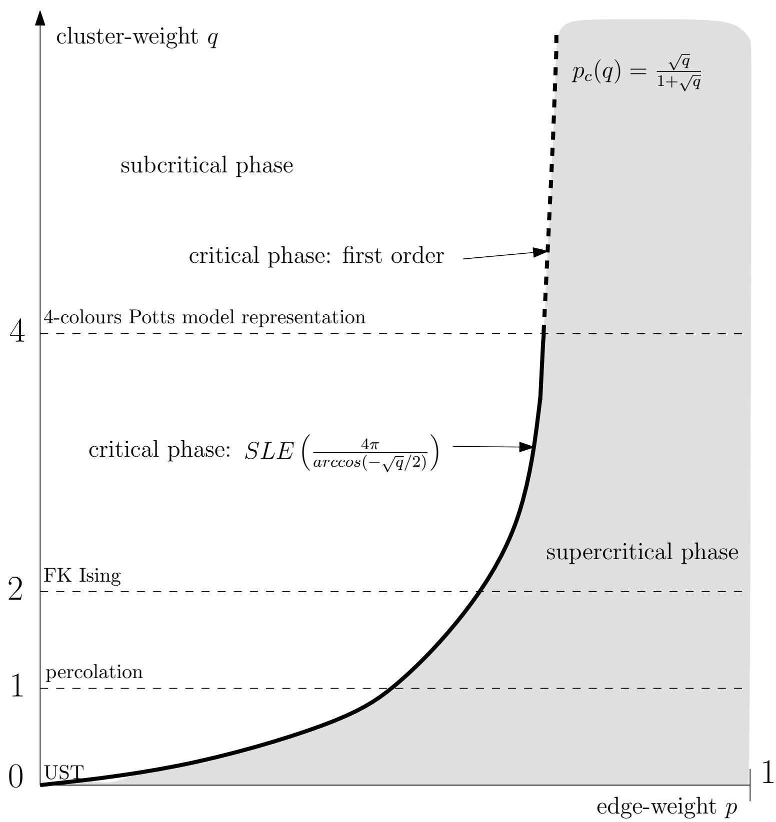

Figure 13.1: The phase diagram of the random-cluster model on the square lattice.

The conjecture was proved by Lawler, Schramm and Werner [LSW11] for $q = 0$, when they showed that the perimeter curve of the uniform spanning tree converges to SLE(8). Note that the loop representation with Dobrushin boundary conditions still makes sense for $q = 0$ (more precisely for the model obtained by letting $q \to 0$ and $p/q \to 0$). Then, configurations have no loops, just a curve running from a to b (which then necessarily passes through all the edges), with all configurations being equally probable. The $q = 2$ case corresponds to Theorem 9.14. All other cases are open. The $q = 1$ case is particularly interesting, since it is bond percolation on the square lattice.

13.2.3 Scaling relations for $q \le 4$

We are ultimately interested in the rigorous computation of critical exponents for the random-cluster model with $q \in [0,4]$. Let us start by presenting several critical exponents together with their predicted values. In the next table, σ is the spin involved in the definition of the observable. Recall that it is equal to

$$\sigma := \frac{2}{\pi} \arcsin\left(\tfrac{\sqrt{q}}{2}\right).$$

Exp.	**Definition**	**Prediction**
α	$\frac{\mathrm{d}}{\mathrm{d}p}\phi^0_{p,q}(\omega(e)=1) = \lvert p - p_c(q)\rvert^{-\alpha+o(1)}$ (as $p \to p_c$)	$\alpha = \frac{2(2\sigma-1)}{3\sigma}$ *
β	$\theta(p,q) = (p-p_c)^{\beta+o(1)}$ (as $p \searrow p_c$)	$\beta = \frac{2-\sigma}{12}$
γ	$\chi^0(p,q) = (p_c - p)^{-\gamma+o(1)}$ (as $p \nearrow p_c$)	$\gamma = \frac{\sigma^2+2\sigma+4}{6\sigma}$
δ	$\phi^0_{p_c,q,h}(0 \longleftrightarrow g) = h^{1/\delta+o(1)}$ (as $h \to 0$)	$\delta = \frac{(2+\sigma)(4+\sigma)}{\sigma(2-\sigma)}$
ν	$\xi(p,q) = (p_c - p)^{-\nu+o(1)}$ (as $p \nearrow p_c$)	$\nu = \frac{1+\sigma}{3\sigma}$
η	$\phi^0_{p_c,q}(0 \longleftrightarrow x) = \lvert x\rvert^{-\eta+o(1)}$ (as $\lvert x\rvert \to \infty$)	$\eta = \frac{\sigma(2-\sigma)}{2(1+\sigma)}$
ξ_1	$\phi^0_{p_c,q}(0 \longleftrightarrow \partial\Lambda_n) = n^{-\xi_1+o(1)}$ (as $n \to \infty$)	$\xi_1 = \frac{\sigma(2-\sigma)}{4(1+\sigma)}$
ξ_{1010}	$\phi^0_{p_c,q}(A_{1010}(0,n)) = n^{-\xi_{1010}+o(1)}$ (as $n \to \infty$)	$\xi_{1010} = \frac{3\sigma^2+10\sigma+3}{4(1+\sigma)}$

Let us say a few words on the quantities involved in the previous table:

- The quantity $\frac{\mathrm{d}}{\mathrm{d}p}\phi^0_{p,q}(\omega(e)=1)$ is the derivative of the energy-density (the probability of being open). For $q \ge 2$, this quantity is related to the *specific heat* of the model. The symbol * means that the formula is valid only for $q \ge 2$. For $q < 2$, this derivative remains of order 1 when approaching criticality. In this context, a critical exponent α can still be introduced but the definition involves the specific heat directly (and it is not equal to 0). We do not discuss this subtlety here.

- The second, third and fifth lines hardly require any explanation since the quantities involved are simply the cluster-density, the susceptibility and the correlation length.
- The fourth line requires a little bit more explanations. Add a vertex called the *ghost vertex* g outside $\mathbb{Z}^2$ which is connected to each vertex of $\mathbb{Z}^2$ by an edge. The random-cluster measure with edge-weight p_c for edges of $\mathbb{Z}^2$ and $1 - e^{-2h}$ for edges having g as an end-point is denoted by $\phi^0_{p_c,q,h}$. The probability that 0 is connected to *ghost* has an interpretation in terms of spin models with a magnetic field. For $q = 2$ for instance, the probability of this event is equal to the spontaneous magnetization with an external field h. In the Ising model, the exponent δ is therefore telling us how the critical spontaneous magnetization behaves with respect to h. A similar interpretation also holds for the 3-state and 4-state Potts models.
- The last three lines contain critical exponents describing the fractal properties of the critical phase. The two first quantities are trivial to interprete. The event $A_{1010}(0, n)$ can be related to the event that the edge between the origin and $(1, 0)$ is pivotal for crossing events. We already mentioned how useful this quantity is for percolation in Chapter 11.

The prediction for most of the previous critical exponents can be found, for example, in [Wu82], except for the exponents ξ_1 and ξ_{1010} which can be predicted using the (conjectured) convergence towards the Schramm-Loewner Evolution.

These exponents are not independent of each other: they are related by *scaling* and *hyperscaling* relations. These relations are expected to hold for different universality classes. Proving such relations reduces drastically the number of exponents to compute, and therefore would be of great interest. Let us mention some of them.

R1 $\eta = 2\xi_1$,
R2 $2\beta = \nu\eta$,
R3 $(2 - \eta)\nu = \gamma$,
R4 $\alpha + 2\beta + \gamma = 2$,
R5 $\gamma = \beta(\delta - 1)$.

Let us mention that many alternative and equivalent relations can be found between the different critical exponents. While **R1** is straightforward using property **P5** of Corollary 6.16, the other four relations can be easily derived heuristically but are difficult to prove rigorously. In the case of percolation, **R2**, **R3** and **R5** are known (we refer to [Gri99]) but **R4** is not verified rigorously.

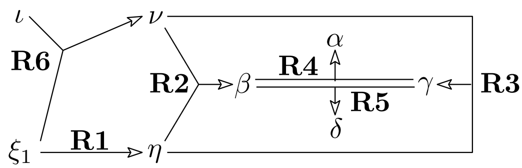

Figure 13.2: How exponents would follow from ξ_1 and ι.

Question 13.11. *Prove* **R2**, **R3**, **R4** *and* **R5** *for* $1 < q \le 4$.

Since conformal invariance holds true only at criticality, it is convenient to express all the exponents in terms of quantities which can potentially be expressed using the critical configuration. In particular, for percolation, a sixth relation $1 = \nu(2 - \xi_{1010})$ is useful in order to relate all the exponents to only two of them (see Fig. 13.2). The main challenge in proving this relation lies in proving the following *stability result*: for $p < p_c(q)$ and n below the correlation length, the random-cluster with parameters p and p_c in Λ_n should "look similar". Such a statement was proved in [Kes87] for $q = 1$. It led to a good understanding of these relations for Bernoulli percolation.

Anyway, in Chapter 11, this relation was shown to be violated for $q = 2$. In fact, the probability of being pivotal should be replaced by the influence (see Chapter 5). This leads to the relation

$$\textbf{R6}\ 1 = \nu(2 - \iota),$$

where ι is defined as the exponent governing the derivative of crossing probabilities, namely,

$$\Big(\frac{d}{dp}\phi^0_{p_c,q}[\mathcal{C}_h(\Lambda_n)]\Big)(p_c) = n^{2-\iota+o(1)} \text{ as } n \to \infty.$$

Russo's formula implies that $\iota = \xi_{1010}$ for $q = 1$, but for $q > 1$, the value of ι is expected to be $\frac{1+\sigma}{2-\sigma}$, which differs from the conjectured value for ξ_{1010}.

Let us conclude this section by mentioning that it is unclear how ι can be computed using conformal invariance.

Question 13.12. *Assuming conformal invariance, find a way of computing* ι *and prove* **R6** *for* $1 < q \le 4$.

13.2.4 Case $q < 1$: existence of a phase transition

The study of the random-cluster model with $q < 1$ is very challenging, due to the fact that the model does not satisfy the Fortuin-Kasteleyn-Ginibre (FKG) inequality (4.9). In this case, the model is expected to be negatively correlated; see [Gri06, Section 3.9]. As a consequence, the following question is still completely open.

Question 13.13. *Fix $q < 1$. Prove that there exists $p_c(q) \in (0,1)$ such that for any infinite-volume random-cluster measure $\phi_{p,q}$ with edge-weight p and cluster-weight q, $\phi_{p,q}[0 \longleftrightarrow \infty] > 0$ if $p > p_c(q)$ and $\phi_{p,q}[0 \longleftrightarrow \infty] = 0$ if $p < p_c(q)$.*

Answering this question would require to develop a theory of negatively correlated models, whose scope would exceed the theory of percolation models.

13.3 Loop $O(n)$-model on the hexagonal lattice with $n \in [0,2]$

The loop $O(n)$-model is expected to exhibit one critical behavior at $x_c(n)$ and another on the interval $(x_c(n), +\infty)$, both being conformally invariant in the sense that the interface should converge to Schramm-Loewner Evolutions. In this sense, both regimes are critical yet different since the parameter κ in the scaling limit is not the same: one regime corresponds to the so-called "dilute" phase and the other one to the "dense" one (when in the scaling limit the loops are simple and non-simple correspondingly), see Fig. 13.3.

It would be of great interest to show that a phase transition indeed occurs at $x_c(n)$. Unfortunately, no obvious monotonicity exists in the model, and the existence of a critical point separating the two regimes remains a mystery. Recent progress suggest that for large values of n, a mathematical proof of the absence of phase transition can be implemented (this is the analogue of the fact that spin $O(n)$-models are conjectured not to have a phase transition for $n > 2$; see the previous chapter).

Recall the definition of the parafermionic observables from Section 12.4.3. The local relation of Proposition 13.17 can be seen as a discrete version of the Cauchy-Riemann equation on the triangular lattice and the discrete contour integrals of the observable vanish yet again. Once again, the relations do not determine the observable for a general n. Nonetheless, it could possibly be used to determine $x_c(n)$, as illustrated by the example of the self-avoiding walk.

Question 13.14. *Prove that the critical point of the loop $O(n)$-model with $n \in [0,2]$ is $1/\sqrt{2+\sqrt{2-n}}$.*

Similarly to the random-cluster case, we are led to the following conjecture.

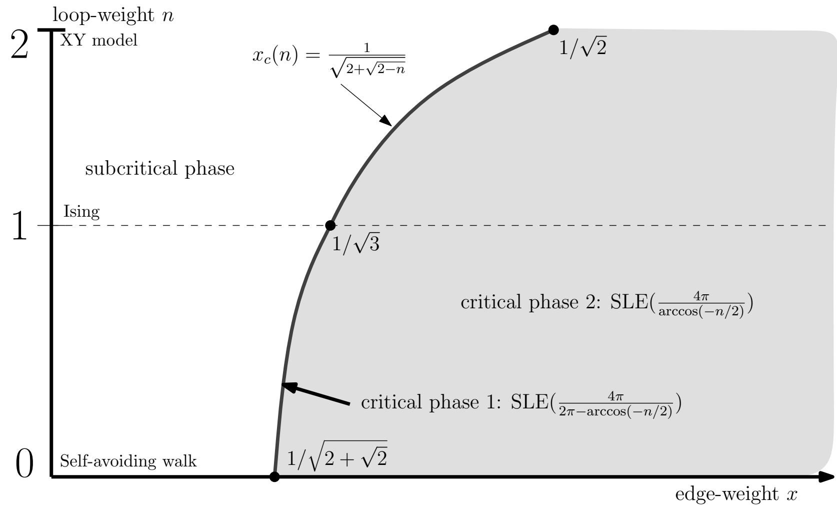

Figure 13.3: The phase diagram of the loop $O(n)$-model on the hexagonal lattice.

Conjecture 13.15 (Smirnov). *Let $n \in [0,2]$ and (Ω, a, b) be a simply connected domain with two points on the boundary. For $x = x_c(n)$,*

$$F_\delta(z) \longrightarrow \left(\frac{\psi'(z)}{\psi'(b)}\right)^\sigma \tag{13.2}$$

uniformly on every compact subset of Ω, where $\sigma = 1 - \frac{3}{4\pi}\arccos(-n/2)$, F_δ is the observable in the discrete domain with spin σ and ψ is any conformal map from Ω to the upper half plane sending a to ∞ and b to 0.

A conjecture on the scaling limit for the interface from a to b in the $O(n)$-model can be deduced from these considerations (using the same program as for the Ising model).

Conjecture 13.16 (see e.g. Kager, Nienhuis [KN04]). *For $n \in [0,2)$ and $x_c(n) = 1/\sqrt{2+\sqrt{2-n}}$, as the lattice step goes to zero, the law of $O(n)$ interfaces converges to the Schramm-Loewner Evolution with parameter $\kappa = 4\pi/(2\pi - \arccos(-n/2))$.*

This conjecture is only proved in the case $n = 1$ (simply adapt the proof of Theorem 9.21 to the hexagonal lattice). The other cases are open. The case $n = 0$ is especially interesting since it corresponds to self-avoiding walks. Proving the conjecture in this case would pave the way to a deep understanding of geometric properties of the self-avoiding walk (for instance, one could compute the mean-square displacement exponent mentioned in the introduction), see [LSW04] for further details on this problem. Note that up to now, almost nothing is rigorously known on the geometric properties of the two-dimensional self-avoiding walk (see [DCH13, DCGHM13] for very partial results).

The phase $x < x_c(n)$ is subcritical and not conformally invariant (the interface converges to the shortest curve between a and b). The critical phase $x \in (x_c(n), \infty)$ should be conformally invariant, and universality is predicted: the interfaces are expected to converge to the same Schramm-Loewner Evolution. The edge-weight $\tilde{x}_c(n) = 1/\sqrt{2-\sqrt{2-n}}$, which appears in Nienhuis's works [Nie82, Nie84], seems to play a specific role in this phase. Interestingly, a parafermionic observable with a well-chosen spin $\tilde{\sigma}(n) \neq \sigma(n)$ satisfies local relations at $\tilde{x}_c(n)$.

Proposition 13.17. *If $x = \tilde{x}_c(n)$, let F be the parafermionic observable with spin $\tilde{\sigma} = \tilde{\sigma}(n) = -\frac{1}{2} - \frac{3}{4\pi}\arccos(-n/2)$, then*

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0 \tag{13.3}$$

where p, q and r are the three mid-edges adjacent to a vertex v.

A convergence statement corresponding to Conjecture 13.15 for the observable with spin $\tilde{\sigma}$ enables us to predict the value of κ for $\tilde{x}_c(n)$, and thus for every $x > x_c(n)$ thanks to universality.

Conjecture 13.18 (see e.g. Kager, Nienhuis [KN04])**.** *For $n \in [0,2)$ and $x \in (x_c(n), \infty)$, as the lattice step goes to zero, the law of $O(n)$ interfaces converges to the Schramm-Loewner Evolution with parameter $\kappa = 4\pi/\arccos(-n/2)$.*

The case $n = 1$ corresponds to the high-temperature expansion of the Ising model at low temperature on the hexagonal lattice, which also corresponds to the low-temperature Ising model on the triangular lattice via Kramers-Wannier duality. The interfaces should converge to SLE(6). In the case $n = 0$, the scaling limit should be SLE(8) which is space-filling. For both cases, a (slightly different) model is known to converge to the corresponding SLE (site percolation on the triangular lattice for SLE(6) [Smi01], and the perimeter curve of the uniform spanning tree for SLE(8) [LSW11]). Yet, known proofs do not extend to this context. As mentioned above, proving that the whole critical phase $(x_c(n), \infty)$ has the same scaling limit would be an important example of universality (not on the graph, but on the parameter this time).

Remark 13.19. Let us finish by a remark on the general strategy presented in this book. The two previous sections presented a program to prove convergence of discrete curves towards the Schramm-Loewner Evolution. It was based on discrete martingales converging to continuous SLE martingales. One can study directly SLE martingales (*i.e.* with respect to the filtration $\mathcal{F}_t = \sigma(\gamma[0,t])$). In particular, $g_t'(z)^\alpha[g_t(z)-W_t]^\beta$ is a martingale for SLE(κ) where $\kappa = 4(\alpha-\beta)/[\beta(\beta-1)]$. All the limits in these notes are of the previous forms. In other words, the parafermionic observables are discretizations of very simple SLE martingales. Can new preholomorphic observables be found by looking at discretizations of more complicated SLE martingales? Conversely, in [SS05], the harmonic explorer is constructed in such a way that a natural discretization of a SLE(4) martingale is a martingale of the discrete curve. This fact implied the convergence of the harmonic explorer to SLE(4). Can this reverse engineering be done for other values of κ in order to find discrete models converging to SLE?

Remark 13.20. It would also be nice to find applications of parafermionic observables to more general loop $O(n)$-models. Glazman's result [Gla13] provides us with a good example of such an application but one should be able to find many others.

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