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Global singularity theory for the Gauss curvature equation

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Abstract. We study the structure of the singularity sets of weak solutions to the Plateau problem for gaussian curvature. We show that these sets consist of convex hulls of subsets of the boundary. This allows us to take a geometric approach to the construction of smooth solutions.

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Chapter 1

Introduction

1.1 Singularities and the Plateau problem

The theory of singularities of solutions of totally non-linear partial differential equations presents a vast and fascinating field of mathematics about which much remains to be learnt. In this text, we study the singularities of otherwise smooth solutions of operators of Hessian type. Although little is known in the general case, when the operator is also of convex type and the ambient space is flat, a complete and satisfying description of the singularity set of any solution becomes possible. Indeed, this set decomposes as a union of convex hulls, thereby presenting a nice analogy with the linear case, where Hörmander showed (c.f. [10]) that the wave-front set of any solution of a linear partial differential equation is a union of complete bicharacteristic orbits of the Hamiltonian vector field of its principal symbol.

The case of gaussian curvature presents a nice geometric framework within which to present this theory, though it should be borne in mind that the techniques developed in the sequel apply equally well in a far wider context. We first recall some basic definitions of riemannian geometry (c.f. [7]). Let S be a smooth, oriented, embedded hypersurface in $\mathbb{R}^{n+1}$. Let N be the unit normal vector field over S which is compatible with the orientation. Let A be its **shape operator** (also known as the **Weingarten operator**), which is defined to be the derivative of N. That is, for all $x \in S$, and for every tangent vector V to S at x,

$$A(x)V := D\mathsf{N}(x)V.$$

We recall that, for all x, the linear map $A(x)$ sends the tangent space of S at x to itself. In particular, it always has a well-defined determinant, and so we define the function $\kappa : S \to \mathbb{R}$ by

$$\kappa(x) := \mathrm{Det}(A(x)).$$

We call this function the **gaussian curvature** (or **extrinsic curvature**) of S. It is one of an immense family of possible scalar curvature functions which includes the mean curvature, the so-called "scalar curvature", and so on. Within this family, the gaussian curvature itself is of particular interest since, after the mean curvature, it is often the most analytically tractable.

We will study the Plateau problem for gaussian curvature, which asks for constant curvature hypersurfaces with prescribed boundary. Before stating the result, we consider it worth reviewing certain geometric features of the gaussian curvature. The first concerns its relationship with convexity. Let Symm denote the space of real, symmetric $n \times n$ matrices and consider the determinant function $\mathrm{Det} : \mathrm{Symm} \to \mathbb{R}$. The set $Z := \mathrm{Det}^{-1}(\{0\})$ divides Symm into $n+1$ connected components. Indeed, for $0 \leq k \leq n$, denote by Symm_k the subset of Symm consisting of all invertible, symmetric matrices with exactly k positive eigenvalues. The complement of Z coincides with the union of all the Symm_k, each of which is connected. Significantly, Symm_n coincides with the set of positive-definite matrices. In particular, if $\mathrm{Det}(A)$ is positive and if A lies in the correct connected component of the complement of Z, then A is positive definite.

Now consider the embedded hypersurface S. We recall that S is said to be **strictly convex** whenever the matrix $A(x)$ is positive definite at every point. However, if the gaussian curvature of S is everywhere strictly positive, then it follows by connectedness that S is strictly convex whenever $A(x)$ is an element of Symm_n for one single x. In other words, when the gaussian curvature is strictly positive, the condition of strict convexity of S reduces to a single topological datum which may take one of only $n+1$ possible values.

Now suppose that $(S_m)_{m\in\mathbb{N}}$ is a sequence of embedded hypersurfaces converging smoothly (in some reasonable sense) to S. If S_m is strictly convex for all m, then S will also be convex, though not necessarily strictly so. However, if the gaussian curvature of S is everywhere strictly positive, then S will also be strictly convex. That is, strict convexity, which is a-priori an open condition, becomes also a closed condition, provided again that the gaussian curvature is assumed to be strictly positive.

This describes the relationship between gaussian curvature and strict convexity. It is of particular significance to us as strict convexity plays an important role in the development of the singularity theory presented in the sequel. These properties are particular to the gaussian curvature and are not possessed, for example, by the mean curvature, nor by the so-called "scalar curvature". Nonetheless, there is an important class of scalar curvatures which do possess this property, which we refer to as the class of curvatures of convex type. We will not discuss this further here, but we refer the interested reader to [25] for a complete treatment.

The second feature of the Plateau problem for gaussian curvature is the importance of outer barriers. This more subtle feature arises from the totally non-linear nature of the problem. The situation is best illustrated by the case of a circle, C, of unit radius in the plane. We furnish C with the canonical orientation, and for $k > 0$, we look for compact, oriented, embedded surfaces, S, in $\mathbb{R}^3$ of constant gaussian curvature k^2, with boundary C, and which lie locally to the right of this boundary curve. For $k \in]0,1[$, it is easy to construct two distinct solutions to this problem. Indeed, there are exactly two spheres of radius $1/k$ containing C, the first, which we denote by S_k^+, lying mostly above the plane, and second, which we denote by S_k^-, lying mostly below it. If H now denotes the closed lower half-space $\{(x,y,z) \mid z \leq 0\}$, then the intersections $S_k^\pm \cap H$ are the desired solutions.

Observe that, as k tends to 1, the pair $S_k^\pm$ degenerates to the single sphere S_1, which is the unique sphere with equator C. For $k < 1$, the two solutions are then distinguished geometrically by their position with respect to $S_1 \cap H$. Indeed, the "small solution", $S_k^+ \cap H$, lies on the inside, that is to say, the concave side, of $S_1 \cap H$, whilst the "big solution", $S_k^- \cap H$, lies on the outside, that is to say, the convex side, of this surface. In technical terms, we say that $S_1 \cap H$ serves as an **outer barrier** for $S_k^- \cap H$, but not for $S_k^+ \cap H$.

This qualitative difference influences in a fundamental manner the behaviour of nearby solutions which follow perturbations of the boundary curve. Indeed, if $(C_t)_{|t|<\epsilon}$ is a smooth family of curves such that $C_0 = C$, and if $(S_{k,t}^\pm)_{|t|<\epsilon}$ are smooth families of surfaces such that $S_{k,0}^\pm = S_k^\pm \cap H$, and that, for all t, $S_{k,t}^\pm$ is a solution to the Plateau problem with gaussian curvature equal to k and boundary curve C_t, then, although the existence of an outer barrier makes it relatively easy to ensure that the family $(S_{k,t}^+)_{|t|<\epsilon}$ of small solutions remains within some fixed compact set, the same cannot be said for the family $(S_{k,t}^-)_{|t|<\epsilon}$ of big solutions. Indeed, depending on $(C_t)_{|t|<\epsilon}$, they may diverge in arbitrarily short time. It is for this reason that the assumption of existence of an outer barrier is often indispensable in the statement of our results.

We are now in a position to state the main result of this text. Here the role of the outer barrier is played by the boundary, ∂K, of the convex set, K.

Theorem 1.1 (Existence and Singularities). *Choose $k > 0$. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ with non-trivial interior. Let X be a closed subset of ∂K whose convex hull also has non-trivial interior. If ∂K is smooth with gaussian curvature greater than k at every point of $(\partial K) \setminus X$, then there exists a compact, convex subset $K_0 \subseteq K$ with non-*

trivial interior such that

(1) $K_0 \cap \partial K = X$*; and*

(2) $\partial K_0 \cap K^o$ *has constant gaussian curvature equal to* k *in the viscosity sense.*

Furthermore, if we denote by $\text{Sing}(K_0)$ *the set of all points in* ∂K_0 *near which* ∂K_0 *is not smooth, then there exists a family* $(X_\alpha)_{\alpha \in A}$ *of subsets of* X *such that*

$$\text{Sing}(K_0) = \mathop{\cup}_{\alpha \in A} \text{Conv}(X_\alpha),$$

where $\text{Conv}(Y)$ *here denotes the convex hull of* Y *for any set* Y*.*

When the set X has more structure, straightforward geometric arguments may often be applied to ensure that the singularity set is empty. For example, we obtain the following more classical version of the Plateau problem (c.f. [12] and [26]).

Theorem 1.2. *Choose* $k > 0$*. Let* K *be a compact, convex subset of* $\mathbb{R}^{n+1}$ *with smooth boundary. Let* X *be a closed subset of* ∂K *with* C^2 *boundary* $C := \partial X$*. If* ∂K *has gaussian curvature bounded below by* k *at every point of* $(\partial K) \setminus X$*, then there exists a compact, strictly convex,* $C^{0,1}$ *embedded hypersurface* $S \subseteq \mathbb{R}^{n+1}$ *such that*

(1) $S \subseteq K$*;*

(2) $\partial S = C$*; and*

(3) $S \setminus C$ *is smooth and has constant gaussian curvature equal to* k*.*

Remark: *In fact, if* ∂X *is smooth, then the techniques of Chapter 2 may readily be adapted to show that* S *is smooth up to the boundary (and not just over its interior). We shall not study this here, although we refer the interested reader to [24] for a proof of this result following a slightly different approach.*

1.2 Overview and acknowledgements

The proof of Theorem 1.1 leads us on a grand tour of various geometric and analytic aspects of the theory of non-linear partial differential equations of Hessian type. First, in Chapters 2 and 3, we prove the existence of solutions to the Plateau problem for the classical case of graphs over compact, convex subsets of $\mathbb{R}^n$ with smooth boundary. We carry this out in two stages. First, in Chapter 2, we obtain compactness results for families of smooth solutions to the gaussian curvature equation, which we achieve using the technique of Caffarelli, Nirenberg and Spruck (c.f. [2],

[3] and [4]). This general technique applies to solutions of any non-linear partial differential equation of Hessian type, and presents a fascinating object of study in its own right. With this compactness result in hand, we then use a differential-topological argument to prove existence. The formal development of this argument, which requires a considerable detour through basic functional analysis, forms the content of Chapter 3.

The local theory of singularities is introduced in Chapter 4. We describe the possible singularities that appear in C^0 limits of families of those solutions to the classical Plateau problem which we constructed in Chapter 3. To this end, we first present an in-depth study of the elementary geometry of convex subsets of $\mathbb{R}^{n+1}$. With a firm understanding of this theory in hand, we then apply a straightforward barrier argument dating back to the work [17] of Pogorelov to show that singularities always lie along open straight-line segments contained within the limiting surface. This property, which we call the local geodesic property, directly implies the global structure of singularities described in Theorem 1.1.

In order to apply this singularity theory, we introduce in Chapter 6 a concept of weak supersolutions to the gaussian curvature equation, which we refer to as weak barriers. This concept, which is a more sophisticated variant of that of viscosity supersolutions, requires considerable technical work in order to establish its basic properties. This forms the content of Sections 6.1 to 6.5 inclusive, which also makes use of additional properties of convex sets studied in the parenthetical Chapter 5. Once fully developed, however, this theory allows us to easily construct weak solutions to the Plateau problem via the Perron method, that is, by constructing a unique minimiser of a certain functional - in this case, the volume functional - amongst the set of weak barriers. Finally, in Section 6.8, we show that weak solutions are viscosity solutions, and, using the existence result of Chapter 3, which here serves as a local regularising operation, together with the local singularity theory developed in Chapter 4, we obtain the complete description of the structure of the singular parts of these solutions, thereby completing the proof of Theorem 1.1.

A brief overview of the notations and terminology used throughout the text is presented in the appendix. This text is an expanded and revised version of a mini-course presented to students in the XVII Escola de Geometria Diferencial, held in July 2012, in Manaus, Brazil. The text was written whilst the author was benefitting from a Marie Curie postdoctoral fellowship at the Centre de Recerca Matemàtica, Barcelona, Spain. The author is grateful to Lucio Rodriguez and Olivier Druet for many helpful suggestions and comments.

Chapter 2

The CNS Method

The Caffarelli-Nirenberg-Spruck (CNS) method yields a-priori second order bounds for smooth solutions of non-linear, Hessian-type PDEs given the existence of an upper barrier. It constitutes the main step towards proving the main result of this chapter, namely Theorem 2.26, which yields compactness in the C^∞ sense for families of smooth solutions to such PDEs, and, in particular, for families of smooth functions whose graphs have constant gaussian curvature.

The CNS method reduces to a barrier argument (c.f. Chapter 3 of [11]). That is, bounds are obtained by applying the maximum principal to certain, carefully chosen superharmonic functions, which are constructed so as to have suitable properties along the boundary. There is always a certain art to the construction of barrier functions, and the CNS technique is no exception. Consequently, in order to attain a deeper understanding of their approach, we consider it worthwhile to focus on three specific points which we believe stand out. The first is convexity, which is used repeatedly throughout the chapter as a source of positivity. The second is the correct choice of generalised Laplacian with respect to which the superharmonicity of functions will be determined, and which turns out to be the linearisation of the non-linear PDE in question around the function being studied. Finally, the third concerns superharmonicity itself, more precisely, which relations constitute useful upper bounds for intermediate functions and which do not. This is a rather subtle point which will hopefully become clear in Section 2.4, below.

In all situations where the CNS method may be applied, once a-priori second-order bounds have been obtained, higher order bounds follow from general results. The Krylov technique (c.f. Theorem 2.25) yields a-priori $C^{2+\alpha}$ bounds for solutions, then the Schauder technique (c.f. Theorem 2.24) then yields a-priori C^k bounds for all k, and compactness follows by the classical Arzelà-Ascoli theorem. Since detailed proofs of Theorems

2.24 and 2.25 would take us too far afield, we refer the interested reader to [3] and [11] for a complete treatment.

2.1 The framework

Let $\overline{\Omega}$ be a compact, convex subset of $\mathbb{R}^n$ with smooth boundary. Let $f \in C^\infty(\overline{\Omega})$ be a smooth, strictly convex function which vanishes along $\partial\Omega$. Let $\kappa : \overline{\Omega} \to]0, \infty[$ be such that, for all x, $\kappa(x)$ is equal to the gaussian curvature of the graph of f at the point $(x, f(x))$. It is a straightforward exercise to show that

$$\mathrm{Det}(D^2 f(x)) = \kappa(x)(1 + \|Df(x)\|^2)^{\frac{n+2}{2}}. \tag{A}$$

We prefer to consider a more general setting which we believe better illustrates the main concepts of the CNS method without introducing excessive complexity. Thus, let Symm := Symm$(2, \mathbb{R}^n)$ be the space of real-valued, symmetric matrices of order n and let $\Gamma \subseteq$ Symm be the open cone of positive-definite, symmetric matrices. We define the function $F : \Gamma \to]0, \infty[$ by

$$F(A) := \mathrm{Det}(A)^{\frac{1}{n}}.$$

Let $G : \mathbb{R}^n \to [0, \infty[$ be a smooth, convex function bounded below by 1. For any smooth function $\phi \in C^\infty(\overline{\Omega},]0, \infty[)$, we now consider smooth, strictly convex functions $f \in C^\infty(\overline{\Omega})$ which satisfy the following non-linear PDE with boundary condition.

$$F(D^2 f) = \phi G(Df), \qquad f|_{\partial\Omega} = 0. \tag{B}$$

We leave the reader to verify that (A) presents a special case of this problem.

We are interested in studying the problem given the existence of a **lower barrier**, which is defined to be a smooth, strictly convex function $\hat{f} \in C^\infty(\overline{\Omega})$ which satisfies the following non-linear partial differential inequation with boundary condition.

$$F(D^2 \hat{f}) > \phi G(D\hat{f}), \qquad \hat{f}|_{\partial\Omega} = 0. \tag{C}$$

In particular, we define

$$\delta(\hat{f}) := \operatorname*{Inf}_{x \in \overline{\Omega}} \Big(F(D^2 \hat{f}(x)) - \phi(x) G(D\hat{f}(x)) \Big). \tag{D}$$

This quantity will be of use in the sequel.

Given the lower barrier, we are interested in solutions f of (B) such that $f \geq \hat{f}$. These will be obtained using degree theory, which is a generalisation of the continuity method and which will be discussed in

Section 3.4. This techniques requires in particular compactness results for families of solutions of (B) which are bounded below by corresponding families of lower barriers. Furthermore, by the classical Arzelà-Ascoli theorem, such compactness results are equivalent to a-priori bounds for the norms of the k'th derivatives of solutions for all k. Obtaining such bounds is therefore our main aim in this chapter. A-priori C^0 and C^1 bounds follow without further ado.

Lemma 2.1. *If $f \geq \hat{f}$, then*

$$\begin{aligned} \|f\|_0 &\leq \|\hat{f}\|_0, \\ \|Df\|_0 &\leq \|D\hat{f}\|_0. \end{aligned}$$

Proof. Let Δ be the standard Laplacian on $\mathbb{R}^n$. Since f is smooth and strictly convex $\Delta f > 0$, and so, by the maximum principal,

$$\operatorname*{Sup}_{x\in\overline{\Omega}} f(x) = \operatorname*{Sup}_{x\in\partial\Omega} f(x) = 0.$$

Since $f \geq \hat{f}$, it follows that $\|f\|_0 \leq \|\hat{f}\|_0$, as desired. We now claim that

$$\|Df\|_0 = \operatorname*{Sup}_{x\in\partial\Omega} \|Df(x)\|.$$

Indeed, let x be any point of Ω. Denote $V = Df(x)/\|Df(x)\|$ and define $\gamma : \mathbb{R} \to \mathbb{R}^n$ by $\gamma(t) := x + tV$. Since Ω is compact and convex, $\gamma^{-1}(\overline{\Omega})$ is a closed interval, $[a, b]$ say, containing 0. Define $g : [a, b] \to \mathbb{R}$ by $g(t) := (f \circ \gamma)(t) = f(x+tV)$. Since f is convex, so too is g. In particular, g' is monotone, and, without loss of generality we may therefore assume that $g'(b) \geq g'(0) = \|Df(x)\|$. However, using the Cauchy/Schwarz inequality, we obtain

$$\begin{aligned} \|Df(x)\| &\leq g'(b) \\ &= \langle Df(\gamma(b)), V\rangle \\ &\leq \|Df(\gamma(b))\|\|V\| \\ &= \|Df(\gamma(b))\| \\ &\leq \operatorname*{Sup}_{y\in\partial\Omega} \|Df(y)\|, \end{aligned}$$

and since $x \in \Omega$ is arbitrary, the assertion follows. However, since $\hat{f} \leq f \leq 0$, and since $\hat{f} = f = 0$ along $\partial\Omega$, it follows that, for all $y \in \partial\Omega$,

$$\|Df(y)\| \leq \|D\hat{f}(y)\|,$$

so that $\|Df\|_0 \leq \|D\hat{f}\|_0$, as desired. This completes the proof. $\square$

Remark. *The approach described here and in the sequel extends to a far more general framework (c.f. [4]). Indeed, first let* $\mathrm{O}(n)$ *be the group of orthogonal matrices of order* n. *Recall that* $\mathrm{O}(n)$ *acts on* Symm *by conjugation. That is, for* $M \in \mathrm{O}(n)$ *and for* $A \in \mathrm{Symm}$,

$$M(A) := M^{-1}AM \in \mathrm{Symm}.$$

Now denote by Γ_0 *the open cone of positive-definite matrices in* Symm, *and observe that every element of* $\mathrm{O}(n)$ *maps* Γ_0 *bijectively to itself. We then consider any other cone* $\Gamma \subseteq \mathrm{Symm}$ *centered on the origen which is convex, invariant under the action of* $\mathrm{O}(n)$ *on* Symm, *and which, in addition, has the property that for all* $x \in \Gamma$, $x + \Gamma_0 \subseteq \Gamma$. *Given such a* Γ, *the theory of Caffarelli, Nirenberg and Spruck applies to a large family of functions* $F \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ *which are concave, homogeneous of order* 1, *invariant under the action of* $\mathrm{O}(n)$ *on* Symm, *which vanish along the boundary of* Γ, *and which satisfy the property that for all* $A \in \Gamma$ *and for all* $B \in \overline{\Gamma}_0 \setminus \{0\}$, $DF(A)(B) > 0$.

This may appear very abstract. However, consider the symmetric polynomials $(\sigma_k)_{0 \leq k \leq n} : \mathrm{Symm} \to \mathbb{R}$ *defined uniquely by the relation*

$$\mathrm{Det}(\mathrm{Id} + tA) =: \sum_{i=0}^{n} t^i \sigma_i(A),$$

for all $A \in \mathrm{Symm}$ *and for all* $t \in \mathbb{R}$. *For* $0 \leq k \leq n$, *define* $\Gamma_k \subseteq \mathrm{Symm}$ *by*

$$\Gamma_k := \{A \mid \sigma_0(A), ..., \sigma_k(A) > 0\},$$

and define $F_k : \Gamma_k \to [0, \infty[$ *by*

$$F_k(A) := (\sigma_k(A))^{1/k}.$$

The pair (Γ_k, F_k) *possesses the properties described above. In particular, when* $k = n$, $F_k = \mathrm{Det}^{1/n}$ *and when* $k = 1$, $F_k = \mathrm{Tr}$, *so that the Monge-Ampère operator (studied here) and the Laplacian are in fact both covered by this framework.*

2.2 Basic properties of F

Higher order a-priori bounds require a deeper understanding of the differential properties of the function F. Let $\mathrm{End}(n)$ be the space of linear endomorphisms of $\mathbb{R}^n$. Recall that the canonical inner product of $\mathrm{End}(n)$ can be written in the form

$$\langle A, B \rangle = \mathrm{Tr}(A^t B),$$

and observe that for $A, B \in \mathrm{Symm}$, this becomes

$$\langle A, B\rangle = \mathrm{Tr}(AB).$$

We identify every linear map $\alpha : \mathrm{End}(n) \to \mathbb{R}$ with a matrix $A \in \mathrm{End}(n)$ via this inner product. We readily obtain

Lemma 2.2. *For all $A \in \Gamma$,*

$$DF(A) = \frac{1}{n}F(A)A^{-1}.$$

In particular, this yields

Corollary 2.3. *For all $A \in \Gamma$,*

$$DF(A)(A) = F(A).$$

Remark. *This relation in fact follows directly from the homogeneity of F.*

Lemma 2.4. *Suppose that $A \in \Gamma$ is diagonal and let $0 < \lambda_1 \leq ... \leq \lambda_n$ be its eigenvalues. Then, for all $B \in \mathrm{Symm}$,*

$$D^2F(A)(B,B) \leq -\frac{1}{n}F(A)\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}B_{ij}B_{ij}.$$

Remark. *An analogous relation may be deduced for more general F using the properties of concavity and ellipticity (c.f. [21] for details).*

Proof. Differentiating Lemma 2.2 yields

$$D^2F(A)(B,B) = \frac{1}{n}F(A)\left(\frac{1}{n}\mathrm{Tr}(A^{-1}B)^2 - \mathrm{Tr}(A^{-1}BA^{-1}B)\right).$$

Since A is diagonal, this yields

$$D^2F(A)(B,B) = \frac{1}{n}F(A)\left(\frac{1}{n}\sum_{i,j}\frac{1}{\lambda_i\lambda_j}B_{ii}B_{jj} - \sum_{i,j}\frac{1}{\lambda_i\lambda_j}B_{ij}^2\right).$$

However, by the Cauchy/Schwarz inequality,

$$n\sum_{i=1}^n\frac{1}{\lambda_i^2}B_{ii}^2 = \left(\sum_{i=1}^n 1^2\right)\left(\sum_{i=1}^n\frac{1}{\lambda_i^2}B_{ii}^2\right) \geq \left(\sum_{i=1}^n\frac{1}{\lambda_i}B_{ii}\right)^2.$$

The result follows by combining this with the preceeding relation. □

In particular, this yields

Corollary 2.5. *F is concave over Γ.*

We invite the reader to observe the frequency with which the concavity of F and the convexity of G will be used throughout the sequel to remove awkward terms. We recall that this is the first key point of the CNS technique. Furthermore, of these two properties, the concavity of F is perhaps more fundamental, as it is used to eliminate third order terms, wheras the convexity of G only eliminates second-order terms.

Remark. *In fact, in the more general framework described at the end of the previous section, an explicit formula for DF is not necessary. The results obtained in the sequel can be deduced from more general relations derived from the properties of concavity, homogeneity, ellipticity and* $\mathrm{O}(n)$*-invariance. See [4] for details.*

Finally, the concavity of F yields the following lower estimate.

Lemma 2.6. *For all A in Γ,*

$$\frac{1}{n}\mathrm{Tr}(A) \geq F(A).$$

Proof. By concavity,

$$DF(\mathrm{Id})(A - \mathrm{Id}) \geq F(A) - F(\mathrm{Id}).$$

Thus, using Lemma 2.2 and the fact that $F(\mathrm{Id}) = 1$, we obtain

$$\begin{array}{rr} & \frac{1}{n}\mathrm{Tr}(A - \mathrm{Id}) \geq F(A) - 1 \\ \Rightarrow & \frac{1}{n}\mathrm{Tr}(A) \geq F(A), \end{array}$$

as desired. □

2.3 Linearisation

Since estimates are obtained using the maximum principal, we will be interested in proving the superharmonicity of various functions. Importantly, however, the concept of “superharmonicity” depends implicitely on the choice of generalised Laplacian used, and thus constitutes the second key point of the CNS technique. We thus define $\mathcal{L}_f : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ by

$$\mathcal{L}_f g := DF(D^2 f)(D^2 g) - \phi DG(Df)(Dg).$$

The informed reader will notice that this is precisely the linearisation around f of the partial differential operator Φ given by $\Phi(Dg, D^2 g) = F(D^2 g) - \phi G(Dg)$ (c.f. Section 3.4).

Lemma 2.7. *$\mathcal{L}_f$ is a second-order, linear, elliptic, partial differential operator.*

Proof. By definition, $\mathcal{L}_f$ is a second-order, linear, partial differential operator. It thus remains to show ellipticity. Let $\sigma_2(\mathcal{L}_f)$ be its principle symbol (c.f. [10]). We have to show that $\sigma_2(\mathcal{L}_f)$ is everywhere positive definite. However, using Lemma 2.2, we obtain

$$\begin{aligned}\sigma_2(\mathcal{L}_f)(\xi) &= DF(D^2 f)(\xi \otimes \xi)\\ &= \frac{1}{n}F(D^2 f)\mathrm{Tr}((D^2 f)^{-1}(\xi \otimes \xi))\\ &= \frac{1}{n}F(D^2 f)\langle \xi, (D^2 f)^{-1}\xi\rangle.\end{aligned}$$

Since f is strictly convex, $D^2 f$ is positive definite at every point, and therefore so too is its inverse. The principle symbol of $\mathcal{L}_f$ is therefore everywhere positive definite, and this completes the proof. □

The following result provides an important source of superharmonic functions to be used in the sequel.

Theorem 2.8. *If $\delta \geq 0$ is a non-negative real number and if $g, h \in C^\infty(\overline{\Omega})$ are smooth, strictly convex functions such that*

$$F(D^2 g) = \phi G(Dg), \qquad F(D^2 h) \geq \phi G(Dh) + \delta,$$

then

$$\mathcal{L}_g(g - h) \leq -\delta.$$

Proof. By definition,

$$F(D^2 g) - \phi G(Dg) = 0 \leq F(D^2 h) - \phi G(Dh) - \delta.$$

By concavity of F,

$$DF(D^2 g)(D^2 g - D^2 h) \leq F(D^2 g) - F(D^2 h).$$

By positivity of ϕ and convexity of G,

$$\phi DG(Dg)(Dh - Dg) \leq \phi G(Dh) - \phi G(Dg).$$

Combining the above relations and recalling the definition of $\mathcal{L}_g$ yields

$$\mathcal{L}_g(g - h) \leq -\delta,$$

as desired. □

This yields in particular the strong maximum principal in the non-linear setting.

Lemma 2.9. *Let $g, h \in C^\infty(\Omega)$ be strictly convex functions such that $F(D^2h)/G(Dh) \geq F(D^2g)/G(Dg)$ and let p be a point in $\overline{\Omega}$ where $g - h$ attains its minimum value. If $D(g - h) = 0$ at p, then $(g - h)$ is constant.*

Remark. *The reader unfamiliar with this formulation of the maximum principal may observe that if $(g - h)$ attains its minimum value at an interior point, p, say, then $D(g - h) = 0$ at this point, and so $g - h$ is constant (c.f. [11]).*

Proof. Suppose the contrary, so that $g - h$ is non-constant. Define $\psi := F(D^2g)/G(Dg) > 0$. In particular,

$$F(D^2h) - \psi G(Dh) \geq 0 = F(D^2g) - \psi G(Dg).$$

Thus, by Theorem 2.8 applied with $\phi = \psi$,

$$\mathcal{L}_g(g - h) \leq 0.$$

It follows from Hopf's maximum principle (c.f. Lemma 3.4 of [11]) that $D(g - h) \neq 0$ at p. This is absurd by hypothesis, and we conclude that $g - h$ is constant, as desired. □

2.4 The CNS technique

Let X be a smooth vector field in $\mathbb{R}^n$ tangent to $\partial\Omega$. In particular, and importantly, the function $Xf = Df(X)$ vanishes along the boundary. This is the function that we aim to control using the maximum principal. To this end, we first obtain a-priori bounds for $\mathcal{L}_f(Xf)$. Absolute bounds, however, do not exist. Instead, $\mathcal{L}_f(Xf)$ is controlled by a certain function which depends on the data. This is the third key point of the CNS technique: understanding that the terms that are really useful are precisely those that are bounded by this function. We thus define

$$B^{ij} = \frac{1}{n} F(D^2f)(D^2f^{-1})^{ij}, \tag{E}$$

so that B is the matrix of $DF(D^2f)$, and we define

$$\Lambda(f) := DF(D^2f)(\mathrm{Id}) = B^{ij}\delta_{ij}. \tag{F}$$

Fixed multiples of $\Lambda(f)$ also bound terms that are already known to be bounded by constants. Indeed, we have

Lemma 2.10. *For all f,*

$$\Lambda(f) \geq 1.$$

Remark. *Observe that the proof is valid for any concave F homogeneous of order 1 such that $F(\mathrm{Id}) = 1$ (c.f. the remark following Corollary 2.3).*

Proof. Indeed, by Corollary 2.3, for all $A \in \Gamma$,

$$DF(A)(A) = F(A).$$

However, since F is concave,

$$DF(A)(\mathrm{Id} - A) \geq F(\mathrm{Id}) - F(A) = 1 - F(A).$$

Thus, by linearity,

$$\begin{aligned} \mathrm{Tr}(DF(A)) &= DF(A)(\mathrm{Id}) \\ &= DF(A)(\mathrm{Id} - A) + DF(A)(A) \\ &\geq 1 - F(A) + F(A) \\ &= 1, \end{aligned}$$

as desired. □

We now control $\mathcal{L}_f(Xf)$.

Lemma 2.11. *There exists $C > 0$ which only depends on $\|\phi\|_1$, $\|X\|_2$ and $\|f\|_1$ such that,*

$$|\mathcal{L}_f(Xf)| \leq C\Lambda(f).$$

Remark. *Observe that the proof only uses the homogeneity of F. Indeed, the idea is that since f is a solution of* (B)*, any derivative of f should satisfy the linearisation of* (B) *(c.f. Section 3.4) modulo lower order terms. There is a problem, however, since $\mathcal{L}_f(Xf)$ actually involves terms which are of second order in f and for which we have not yet obtained a-priori bounds. These are nonetheless readily removed using Corollary 2.3, which, as remarked previously, only really uses homogeneity.*

Proof. For all i, differentiate (B) in the direction of e_i. Since $D^3 f$ is symmetric, by definition of $\mathcal{L}_f$, this yields, for all i,

$$\mathcal{L}_f f_i = \phi_i G(Df).$$

Thus, using the summation convention,

$$X^i \mathcal{L}_f f_i = X^i \phi_i G(Df).$$

There therefore exists $C_1 > 0$ which only depends on $\|\phi\|_1$, $\|X\|_0$ and $\|f\|_1$ such that

$$\left|X^i \mathcal{L}_f f_i\right| \leq C_1.$$

We aim to move X^i to the other side of $\mathcal{L}_f$. To this end, we define the operator $\mathcal{L}_f^1$ by

$$\mathcal{L}_f^1 g := -\phi DG(Df)(Dg).$$

That is, $\mathcal{L}_f^1$ is the first order component of $\mathcal{L}_f$. By the chain rule,

$$\mathcal{L}_f^1(X^i f_i) - X^i \mathcal{L}_f^1(f_i) = f_i \mathcal{L}_f^1(X^i).$$

There therefore exists $C_2 > 0$ which only depends on $\|\phi\|_0$, $\|f\|_1$ and $\|X\|_1$ such that,

$$\left|\mathcal{L}_f^1(X^i f_i) - X^i \mathcal{L}_f^1(f_i)\right| \leq C_2.$$

Furthermore, by Lemma 2.2,

$$\begin{aligned} &X^i DF(D^2 f)(D^2 f_i) - DF(D^2 f)(D^2(Xf)) \\ =&X^i B^{pq} f_{ipq} - B^{pq}(X^i f_i)_{pq} \\ =&- B^{pq} X^i{}_p f_{iq} - B^{pq} X^i{}_q f_{ip} - B^{pq} X^i{}_{pq} f_i. \end{aligned}$$

However, since that f is a solution of (B),

$$B^{pq} f_{ip} = \frac{1}{n} F(D^2 f)\delta^q{}_i = \frac{1}{n}\phi G(Df)\delta^q{}_i.$$

This eliminates the terms on the right hand side which are of second-order in f. We conclude that there exists $C_3 > 0$, which only depends on $\|\phi\|_0$, $\|X\|_2$ and $\|f\|_1$ such that

$$\left|X^i DF(D^2 f)(D^2 f_i) - DF(D^2 f)(D^2(Xf))\right| \leq C_3 \Lambda(f).$$

Combining the above relations, and using Lemma 2.10 along with the triangle inequality, we obtain

$$\begin{aligned} |\mathcal{L}_f(Xf)| &\leq \left|DF(D^2 f)(D^2(Xf)) - X^i DF(D^2 f)(D^2 f_i)\right| \\ &\qquad + \left|\mathcal{L}_f^1(X^i f_i) - X^i \mathcal{L}_f^1(f_i)\right| + \left|X^i(\mathcal{L}_f f_i)\right| \\ &\leq C_1 + C_2 + C_3\Lambda(f) \\ &\leq (C_1 + C_2 + C_3)\Lambda(f), \end{aligned}$$

as required. □

We now introduce the first component of the barrier function which the CNS technique uses to provide a-priori bounds for Xf. Consider the function $f - \hat{f}$. This function is non-negative, and by Theorem 2.8, is superharmonic with respect to $\mathcal{L}_f$. We aim to perturb it so as to be strictly negative away from a given boundary point without losing superharmonicity. Thus, for all $p \in \partial\Omega$, and for all $\epsilon > 0$, we define

$$\hat{f}_{p,\epsilon}(x) := \hat{f}(x) - \epsilon\|x - p\|^2. \qquad \text{(G)}$$

Lemma 2.12. *There exists $\epsilon_0 > 0$, which only depends on $\|\phi\|_0$, $\delta(\hat{f})$ and $\|\hat{f}\|_2$ such that, for all $p \in \partial\Omega$, and for all $\epsilon < \epsilon_0$,*

$$\mathcal{L}_f(f - \hat{f}_{p,\epsilon}) \leq 0.$$

Proof. Indeed, by compactness, there exists $\epsilon_0 > 0$ which only depends on $\|\phi\|_0$, $\delta(\hat{f})$ and $\|\hat{f}\|_2$ such that for all $p \in \partial\Omega$ and for all $\epsilon < \epsilon_0$,

$$F(D^2\hat{f}_{p,\epsilon}) \geq \phi G(D\hat{f}_{p,\epsilon}).$$

The result now follows by Theorem 2.8. □

For all $p \in \partial\Omega$, we define the function $d_p : \Omega \to \mathbb{R}$ by

$$d_p(x) := \|x - p\|. \tag{H}$$

This is the second component of the CNS barrier function.

Lemma 2.13. *There exists $r > 0$, which only depends on $\|\phi\|_0$ and $\|f\|_1$ such that, for all $p \in \partial\Omega$,*

$$\mathcal{L}_f d_p^2 \geq \Lambda(f),$$

over $\Omega \cap B_r(p)$.

Proof. By definition, for all g,

$$\mathcal{L}_f g = DF(D^2 f)(D^2 g) - \phi DG(Df)(Dg).$$

However, when $g = d_p^2$, for all x,

$$\|Dg(x)\| = 2d_p(x).$$

Thus, bearing in mind Lemma 2.10, there exists $r > 0$ which only depends on $\|\phi\|_0$ and $\|f\|_1$ such that, for $d_p(x) < r$,

$$|\phi(x)DG(Df(x))(Dg(x))| \leq 1 \leq \Lambda(f)(x).$$

However, for all x,

$$DF(D^2 f(x))(D^2 d_p^2(x)) = DF(D^2 f(x))(2\mathrm{Id}) = 2\Lambda(f)(x).$$

The result now follows by subtracting these two relations. □

The following result lies at the heart of the CNS technique.

Lemma 2.14. *There exists $C > 0$, which only depends on $\|\phi\|_1$, $\|X\|_2$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$ such that if $f \geq \hat{f}$, then for all $p \in \partial\Omega$,*

$$\|D(Xf)(p)\| \leq C.$$

Proof. Let C be as in Lemma 2.11 and let r be as in Lemma 2.13. For all $p \in \partial\Omega$, throughout $\Omega \cap B_r(p)$,

$$-\mathcal{L}_f(Cd_p^2) \leq \mathcal{L}_f(Xf) \leq \mathcal{L}_f(Cd_p^2).$$

Now let $\epsilon < \epsilon_0$ be as in Lemma 2.12. For all $A > 0$, and for all $p \in \partial\Omega$, throughout $\Omega \cap B_r(p)$,

$$\mathcal{L}_f(A(f - \hat{f}_{p,\epsilon}) - Cd_p^2) \leq \mathcal{L}_f(Xf) \leq \mathcal{L}_f(-A(f - \hat{f}_{p,\epsilon}) + Cd_p^2).$$

For $p \in \partial\Omega$, $\partial(\Omega \cap B_r(p))$ consists of two components, namely $\partial\Omega \cap B_r(p)$ and $\Omega \cap \partial B_r(p)$. Furthermore, for all $p \in \partial\Omega$,

$$f - \hat{f}_{p,\epsilon} \geq \epsilon d_p^2.$$

Setting $A > C\epsilon^{-1} + \|X\|_0\|f\|_1\epsilon^{-1}r^{-2}$, we obtain, for all $p \in \partial\Omega$, and for all $x \in \partial\Omega \cap B_r(p)$,

$$(A(f - \hat{f}_{p,\epsilon}) - Cd_p^2)(x) \geq 0 = |(Xf)(x)|.$$

Likewise, for all $p \in \partial\Omega$, and for all $x \in \Omega \cap \partial B_r(p)$,

$$(A(f - \hat{f}_{p,\epsilon}) - Cd_p^2)(x) \geq |(Xf)(x)|.$$

That is, for all $p \in \partial\Omega$, and for all $x \in \partial(\Omega \cap B_r(p))$,

$$(A(f - \hat{f}_{p,\epsilon}) - Cd_p^2)(x) \geq (Xf)(x) \geq -(A(f - \hat{f}_{p,\epsilon}) - Cd_p^2)(x).$$

It follows by the maximum principal that for all $p \in \partial\Omega$,

$$A(f - \hat{f}_{p,\epsilon}) - Cd_p^2 \geq Xf \geq -A(f - \hat{f}_{p,\epsilon}) + Cd_p^2$$

throughout $\Omega \cap B_r(p)$. However, by definition, these three functions are all equal to 0 at p, so that

$$\begin{aligned} \|D(Xf)(p)\| &\leq A\|D(f - \hat{f}_{p,\epsilon_0})(p)\| \\ &= A\|D(f - \hat{f})(p)\| \\ &\leq A(\|f\|_{C^1} + \|\hat{f}\|_{C^1}), \end{aligned}$$

as desired. □

Now fix $p \in \partial\Omega$. Upon applying an isometry of $\mathbb{R}^n$, we may suppose that $p = 0$ and that the tangent space to $\partial\Omega$ at p is spanned by the vectors $e_1, ..., e_{n-1}$. For all r, let $B_r'(0)$ be the ball of radius r about 0 in $\mathbb{R}^{n-1}$. For sufficiently small r, there exists a smooth function $\omega : B_r'(0) \to]-r, r[$ whose graph coincides with $\partial\Omega \cap (B_r'(0) \times]-r, r[)$. Using this construction, Lemma 2.14 is now expressed as follows.

Corollary 2.15. *There exists $C > 0$ which only depends on $\|\phi\|_1$, $\delta(\hat{f})$, $\|\hat{f}\|_2$, $\|f\|_1$ and $\|\omega\|_3$ such that if $f \geq \hat{f}$ then, for all $(i,j) \neq (n,n)$,*

$$|f_{ij}(0)| \leq C.$$

Proof. Since $D^2 f$ is symmetric, we may suppose that $j < n$. Let $\chi \in C_0^\infty(B_r'(0)\times]-r,r[)$ be a smooth function of compact support equal to 1 near 0, and define the vector field X_j by

$$X_j(x',t) := \chi(x',t)(e_j, \omega_j(x')).$$

Observe that X_j is tangent to $\partial\Omega$ and that $\|X_j\|_2$ is controlled by $\|\omega\|_3$. Moreover, $(D\omega)(0) = 0$ and so $X_j(0,0) = e_j$. Thus, for all $1 \leq i \leq n$,

$$|f_{ij}(0)| \leq \|D(X_j f)(0)\| + |f(0)(\partial_i X_j)(0)|,$$

and the result now follows by Lemma 2.14. □

2.5 The double normal derivative

By Corollary 2.15, it only remains to control the second derivative in the double normal direction. In more general applications of the CNS technique, this can present a serious difficulty, often requiring a further, lengthy barrier argument (c.f. [3]). In the current case, however, this term is controlled by a straightforward ad-hoc argument which we now describe. We continue to use the notation introduced at the end of Section 2.4, and we thus aim to control $|f_{nn}(0)|$.

Lemma 2.16. *For all $B > 0$, there exists $C > 0$ with the property that if M is a symmetric $n \times n$ matrix such that*

(1) $|M_{ij}| < B$ *for all* $(i,j) \neq (n,n)$*;*

(2) $|\mathrm{Det}(M)| < B^n$*; and*

(3) $|\mathrm{Det}(M')| > B^{-1}$,

where M' is the $(n-1)\times(n-1)$ matrix given by the upper-left hand corner of M, then,

$$|M_{nn}| \leq C.$$

Proof. Indeed, by hypothesis,

$$B^n > |\mathrm{Det}(M)| > |M_{nn}|\,|\mathrm{Det}(M')| - (n-1)(n-1)!B^n,$$

and the result follows with $C := (1 + (n-1)(n-1)!)B^{n+1}$. □

It thus suffices to obtain lower bounds for the absolute value of the determinant of $(f_{ij}(0))_{1\leq i,j\leq(n-1)}$.

Lemma 2.17. *For all $\epsilon > 0$, there exists $\delta > 0$, which only depends on ϵ and the geometry of Ω such that if $f_n(0) < -\epsilon$, then*

$$\left|\mathrm{Det}((f_{ij}(0))_{1\leq i,j\leq(n-1)})\right| > \delta.$$

Proof. By definition, $D\omega(0)=0$ and since Ω is strictly convex, so too is ω. Now observe that the function $x'\mapsto f(x',\omega(x'))$ vanishes identically. Thus, by the chain rule, for all $1\leq i,j\leq(n-1)$,

$$f_{ij}+f_n\omega_{ij}=0.$$

In particular, by definition of ϵ,

$$\left|\mathrm{Det}((f_{ij}(0))_{1\leq i,j\leq(n-1)})\right|>\epsilon^{n-1}\left|\mathrm{Det}(\omega_{ij}(0))\right|,$$

and the result follows. □

Lemma 2.18. *There exists $\delta>0$ which only depends on $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that $f_n(p)<-\delta$.*

Proof. We continue to consider $\mathbb{R}^n$ as the product $\mathbb{R}^{n-1}\times\mathbb{R}$. Since $\partial\Omega$ is smooth, there exists $r>0$, which only depends on the geometry of Ω, such that $B_r((0,r))$ is contained within Ω. For all $\delta>0$, define $h_\delta\in C^\infty(B_r((0,r)))$ by

$$h_\delta(x):=\delta\|x-(0,r)\|^2-\delta r^2.$$

Observe that $F(D^2h_\delta)=2\delta$ and so, for $2\delta<\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$,

$$F(D^2h_\delta)-\phi G(Dh_\delta)\leq 0=F(D^2f)-\phi G(Df).$$

However, for all $x\in\partial B_r((0,r))$,

$$(h_\delta-f)(x)=-f(x)\geq 0.$$

It thus follows by the maximum principal (Lemma 2.9) that $h_\delta-f\geq 0$ throughout $B_r((0,r))$. Since h_δ and f coincide at 0, this yields

$$f_n(0)\leq(\partial_nh_\delta)(0)=-2\delta r,$$

as desired. □

This yields the desired a-priori second order bounds for f at every boundary point of Ω.

Theorem 2.19. *There exists $C>0$ which only depends on $\|\phi\|_1$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$ such that if $f\geq\hat{f}$ then, for all $x\in\partial\Omega$,*

$$\|D^2f(x)\|\leq C.$$

Proof. By Corollary 2.15, there exists $C_1>0$ which only depends on $\|\phi\|_1$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$ such that if $f\geq\hat{f}$ then, for all $(i,j)\neq(n,n)$,

$$|f_{ij}(0)|\leq C_1.$$

By Lemma 2.18, there exists $\delta_1 > 0$ such that

$$f_n(0) \leq -\delta_1.$$

By Lemma 2.17 there exists $\delta_2 > 0$, which only depends on δ_1 and C_1 such that

$$\left|\mathrm{Det}((f_{ij}(0))_{1\leq i,j\leq(n-1)})\right| \geq \delta_2.$$

Thus, by Lemma 2.16, there exists $C_2 > C_1$, which only depends on δ_2 and C_1, such that

$$|f_{nn}(0)| \leq C_2,$$

and we conclude that $\|Df(0)\| \leq C_2$, as desired. □

2.6 Boundary to global

We have obtained a-priori second-order bounds for f at every boundary point of Ω, and we now use the maximum principal to extend these to global second-order bounds over the whole of Ω. Since the problem is non-linear, this is not wholly trivial, and we are required to use an auxiliary superharmonic function to obtain these results. In the present case, the function $f - \hat{f}$ serves our purposes perfectly satisfactorily, and is also trivially well-defined over the whole of Ω. However, in more general settings, suitable auxiliary functions do not always exist, so that this can yield another condition for determining whether or not the Plateau problem can be solved over a given domain.

Let $\lambda_1, ..., \lambda_n : \Omega \to \mathbb{R}$ be such that, for all x, $0 < \lambda_1(x) \leq ... \leq \lambda_n(x)$ are the eigenvalues of $D^2 f(x)$, and, for all $1 \leq i \leq n$, define $\mu_i : \Omega \to \mathbb{R}$ by $\mu_i := \mathrm{Log}(\lambda_i)$. Observe that, although these functions are continuous, they are not necessarily smooth. It is therefore useful to introduce the following definition.

Definition 2.20. Let $U \subseteq \mathbb{R}^n$ be an open set, let L be a second-order, linear, partial differential operator defined over U, let $f : U \to \mathbb{R}$ be a continuous function and let $g : U \times \mathbb{R}^n \to \mathbb{R}$ be any other function. We say that $(Lf)(x) > g(x, Df(x))$ in the weak sense whenever, for all $x \in U$, there exists a smooth function α such that

(1) $\alpha(x) = f(x)$;

(2) $\alpha \leq f$; and

(3) $(L\alpha)(x) > g(x, D\alpha(x))$.

Remark. *The informed reader will notice similarities with the concept of viscosity supersolutions (c.f. [8]). Definition 2.20 however yields a stronger property, since the definition of viscosity solutions does not require the existence of smooth test functions at every point.*

We now recall the matrix B^{ij} defined in Section 2.4.

Lemma 2.21. *If $g : \overline{\Omega} \to \mathbb{R}$ is a smooth, positive function, then*

$$\mathcal{L}_f \mathrm{Log}(g) = \frac{1}{g}\mathcal{L}_f g - B^{ij}\partial_i \mathrm{Log}(g)\partial_j \mathrm{Log}(g).$$

Proof. Indeed, by the chain rule,

$$\begin{aligned} D\mathrm{Log}(g) &= \frac{1}{g}Dg, \\ (D^2\mathrm{Log}(g))_{ij} &= \frac{1}{g}(D^2 g)_{ij} - (D\mathrm{Log}(g))_i (D\mathrm{Log}(g))_j. \end{aligned}$$

Since $DF(D^2 f)$ and $DG(Df)$ are linear, this yields

$$\begin{aligned} \mathcal{L}_f \mathrm{Log}(g) =& DF(D^2 f)(D^2 \mathrm{Log}(g)) - \phi DG(Df)(D\mathrm{Log}(g)) \\ =& \frac{1}{g}DF(D^2 f)(D^2 g) - \frac{1}{g}\phi DG(Df)(Dg) \\ & - DF(D^2 f)^{ij}(D\mathrm{Log}(g))_i (D\mathrm{Log}(g))_j \\ =& \frac{1}{g}\mathcal{L}_f g - DF(D^2 f)^{ij}(D\mathrm{Log}(g))_i (D\mathrm{Log}(g))_j, \end{aligned}$$

as desired. □

Lemma 2.22. *There exists $C > 0$, which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that if $x \in \overline{\Omega}$ and if e_n coincides with the eigenvector of $D^2 f(x)$ corresponding to the greatest eigenvalue λ_n, then, at x,*

$$\mathcal{L}_f \mu_n + \frac{C}{\lambda_n}(\partial_n \mu_n)^2 \geq -C + B^{ij}(\partial_i \mu_n)(\partial_j \mu_n)$$

in the weak sense.

Remark. *Observe that the coefficient of the second term on the left-hand side tends to zero as λ_n tends to infinity. Furthermore, although the coefficient of the second term on the right-hand side in Lemma 2.21 is negative, the coefficient of the corresponding term in the above formula is positive. This phenomenon, which plays an important role in the sequel is a consequence of Lemma 2.4, and thus continues to hold even for more general functions F which are both convex and elliptic.*

Proof. Choose $x \in \Omega$. By applying an isometry, we may assume that $e_1, ..., e_n$ are the eigenvectors of $D^2 f(x)$ corresponding to the eigenvalues $\lambda_1(f)(x), ..., \lambda_n(f)(x)$ respectively. Define $\alpha(x) := \mathrm{Log}(f_{nn}(x))$. Observe that α is smooth, $\alpha \leq \mu_n(f)$ and $\alpha(x) = \mu_n(f)(x)$. It thus suffices to prove the desired relation for α at x. Differentiating (B) twice in the e_n direction at x yields

$$\begin{aligned}&DF(D^2f(x))(\partial_n\partial_nD^2f(x))+D^2F(D^2f(x))(\partial_nD^2f(x),\partial_nD^2f(x))\\=&\phi_{nn}(x)G(Df(x))+2\phi_n(x)DG(Df(x))(\partial_nDf(x))\\&+\phi(x)DG(Df(x))(\partial_n\partial_nDf(x))+\phi(x)D^2g(Df(x))(\partial_nDf(x),\partial_nDf(x)).\end{aligned}$$

Since the derivatives of f are symmetric, since G is convex, since ϕ is positive and recalling the definition of $\mathcal{L}_f$, this simplifies to

$$\begin{aligned}(\mathcal{L}_ff_{nn})(x)\geq\ &\phi_{nn}(x)G(Df(x))+2\phi_n(x)DG(Df(x))(Df_n(x))\\&-D^2F(D^2f(x))(D^2f_n(x),D^2f_n(x)).\end{aligned}$$

Since $\|D^2f(x)\|\leq\lambda_n(x)=f_{nn}(x)$, there exists $C_1>0$ which only depends on $\|\phi\|_2$ and $\|f\|_1$ such that

$$(\mathcal{L}_ff_{nn})(x)\geq-C_1-C_1\lambda_n(x)-D^2F(D^2f(x))(D^2f_n(x),D^2f_n(x)).$$

However, by Lemma 2.4 and the positivity of $D^2f(x)$,

$$\begin{aligned}-(D^2F)(D^2f(x))(D^2f_n(x),D^2f_n(x))\geq&\frac{1}{n}F(D^2f(x))\sum_{i\neq j}\frac{1}{\lambda_i(x)\lambda_j(x)}(f_{ijn})(x)^2\\\geq&\frac{2}{n}F(D^2f(x))\lambda_n(x)\sum_{i=1}^{n-1}\frac{1}{\lambda_i(x)\lambda_n^2(x)}(f_{inn})(x)^2\\=&2\lambda_n(x)B^{ij}(\partial_i\mathrm{Log}(f_{nn})(x))(\partial_j\mathrm{Log}(f_{nn})(x))\\&-\frac{2}{n}\phi(x)G(Df(x))(\partial_n\mathrm{Log}(f_{nn})(x))^2\\=&2\lambda_n(x)B^{ij}(\partial_i\alpha)(x)(\partial_j\alpha)(x)\\&-\frac{2}{n}\phi(x)G(Df(x))(\partial_n\alpha)(x)^2.\end{aligned}$$

Thus, upon increasing C_1 if necessary,

$$(\mathcal{L}_ff_{nn})(x)\geq-C_1-C_1\lambda_n(x)-C_1(\partial_n\alpha)(x)^2+2\lambda_n(x)B^{ij}(x)(\partial_i\alpha)(x)(\partial_j\alpha)(x),$$

and so, by Lemma 2.21,

$$(\mathcal{L}_f\alpha)(x)\geq-\frac{C_1}{\lambda_n(x)}-C_1-\frac{C_1}{\lambda_n(x)}(\partial_n\alpha)(x)^2+B^{ij}(x)(\partial_i\alpha(x))(\partial_j\alpha(x)).$$

Finally, by (B) and Lemma 2.6,

$$\lambda_n(x)\geq\frac{1}{n}\mathrm{Tr}(D^2f(x))\geq F(D^2f(x))=\phi(x)G(Df(x))\geq\inf_{y\in\overline{\Omega}}\phi(y)>0,$$

and the result follows. □

The final term in Lemma 2.22 makes this relation insufficient in itself for a direct application of the maximum principal. It is for this reason that the auxiliary function is required in the proof of the following theorem.

Theorem 2.23. *There exists* $C > 0$ *which only depends on* $\|\phi\|_2$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ *and* $\|f\|_1$ *such that if* $f \geq \hat{f}$ *then,*

$$\|f\|_2 \leq C.$$

Proof. For $A > 0$, consider the function $g_A : \overline{\Omega} \to \mathbb{R}$ given by:

$$g_A := \mu_n - A(f - \hat{f}).$$

It suffices to prove that $g_A \leq C$ for some constants A and C which both depend only on $\|\phi\|_2$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$, $\delta(\hat{f})$, $\|\hat{f}\|_2$ and $\|f\|_1$. However, by compactness of $\overline{\Omega}$, g_A assumes its maximum at some point $x \in \overline{\Omega}$, say. First suppose that x is a boundary point of Ω. Let C_1 be as in Theorem 2.19, so that C_1 only depends on $\|\phi\|_1$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$, $\delta(\hat{f})$, $\|f\|_2$ and $\|f\|_1$ and

$$g_A(x) = \mu_n(x) = \mathrm{Log}(\|D^2 f(x)\|) \leq \mathrm{Log}(C_1),$$

as desired.

Now suppose that x is an interior point of Ω. Let C_2 be as in Lemma 2.22 and fix $A := (C_2 + 1)\delta(\hat{f})^{-1}$. By Theorem 2.8,

$$(\mathcal{L}_f g_A)(x) \geq 1 - \frac{C_2}{\lambda_n(x)}(\partial_n(g_A + A(f - \hat{f})))(x)^2,$$

in the weak sense. Now let $\alpha : \Omega \to \mathbb{R}$ be such that $\alpha \leq g_A$ and $\alpha(x) = g_A(x)$. In particular, α attains its maximum at x, so that $(\partial_n\alpha)(x) = 0$. Using the fact that $f - \hat{f} \geq 0$, this yields

$$\begin{aligned}(\mathcal{L}_f \alpha)(x) &\geq 1 - \frac{C_2 A^2}{\lambda_n(x)}(\partial_n(f - \hat{f}))(x)^2 \\ &\geq 1 - C_2 A^2(\|f\|_1 + \|\hat{f}\|_1)e^{-\alpha(x)}.\end{aligned}$$

However, by the maximum principal,

$$(\mathcal{L}_f \alpha)(x) \leq 0,$$

so that

$$g_A(x) = \alpha(x) \leq \mathrm{Log}\left(C_2 A^2(\|f\|_1 + \|\hat{f}\|_1)\right),$$

as desired. This completes the proof. □

2.7 Higher order bounds

We have so far obtained a-priori C^2 bounds for solutions of (B) satisfying $f \geq \hat{f}$. We now review in this section the general principals required to obtain a-priori bounds of arbitrary order.

We first describe how a-priori C^k bounds are obtained for all k provided we have already obtained a-priori $C^{2+\alpha}$ bounds for some $\alpha > 0$ (Hölder spaces and Hölder norms will be introduced and discussed in more detail in Section 3.4). Let M be a compact manifold with boundary, let U be an open subset of $\oplus_{i=0}^2\mathrm{Symm}(i,\mathbb{R}^n)$ and let $\Phi : M\times\overline{\Omega}\times U\to\mathbb{R}$ be a smooth function. We consider the manifold M as the parameter space for a smooth family of functions from $\overline{\Omega}\times U$ into $\mathbb{R}$. For all $\xi\in\oplus_{i=0}^1\mathrm{Symm}(i,\mathbb{R}^n)$, we define $U_\xi\subseteq\mathrm{Symm}(2,\mathbb{R}^n)$ by

$$U_\xi := \{A\in\mathrm{Symm}(2,\mathbb{R}^n)\ |\ (\xi,A)\in U\},$$

and for all $(p,x,\xi)\in M\times\overline{\Omega}\times\oplus_{i=0}^1\mathrm{Symm}(i,\mathbb{R}^n)$, we define $\Phi_{p,x,\xi}:U_\xi\to\mathbb{R}$ by $\Phi_{p,x,\xi}(A) := \Phi(p,x,\xi,A)$. As in Section 2.2, for all $(p,x,\xi,A)\in M\times\overline{\Omega}\times U$, we identify $D\Phi_{p,x,\xi}(A)$ with an element of $\mathrm{Symm}(2,\mathbb{R}^n)$. We say that Φ is **elliptic** whenever $D\Phi_{x,p,\xi}(A)$ is positive-definite for all $(p,x,\xi,A)\in M\times\overline{\Omega}\times U$. The following result encapsulates much of classical Schauder theory (c.f. Chapter 6 of [11]).

Theorem 2.24. *If Φ is elliptic, then for every compact subset $K\subseteq U$, for all $\alpha\in]0,1[$, for all $k\in\mathbb{N}$ and for all $B>0$, there exists $C>0$ such that if p is a point in M and if $g:\overline{\Omega}\to\mathbb{R}$ is a smooth function with the properties that*

(1) $J^2(g)(x)\in K$ for all $x\in\overline{\Omega}$;

(2) $\Phi(p,x,J^2g(x))=0$ for all $x\in\overline{\Omega}$;

(3) $\|g\|_{2+\alpha}\leq B$; and

(4) $g|_{\partial\Omega}=0$,

then

$$\|g\|_k\leq C.$$

A-priori $C^{2+\alpha}$ bounds are obtained from a-priori C^2 bounds using the following result. Let M be a compact manifold with boundary, let U be an open subset of $\oplus_{i=0}^1\mathrm{Symm}(i,\mathbb{R}^n)$, let $\Gamma\subseteq\mathrm{Symm}$ be the open cone of positive-definite, symmetric matrices, and let $\Phi : M\times\overline{\Omega}\times U\times\Gamma\to\mathbb{R}$ be a smooth function. As before, for all $(p,x,\xi)\in M\times\overline{\Omega}\times U$, we define $\Phi_{p,x,\xi}:\Gamma\to\mathbb{R}$ by $\Phi_{p,x,\xi}(A) := \Phi(p,x,\xi,A)$. The following result is a special case of Theorem 1 of [3].

Theorem 2.25. *Suppose that Φ is elliptic and that for all $(p,x,\xi)\in M\times\overline{\Omega}\times U$, $\Phi_{p,x,\xi}$ is concave. Then, for every compact $K\subseteq U$ and for all $B>0$, there exists $\alpha\in]0,1[$ and $C>0$ such that if p is a point in M and if $g:\overline{\Omega}\to\mathbb{R}$ is a smooth function with the properties that*

(1) $J^2g(x) \in K \times \Gamma$ *for all* $x \in \overline{\Omega}$*;*

(2) $\Phi(p, x, J^2g(x)) = 0$ *for all* $x \in \overline{\Omega}$*;*

(3) $\|g\|_2 \leq B$*; and*

(4) $g|_{\partial\Omega} = 0$*,*

then

$$\|g\|_{2+\alpha} \leq C.$$

We now return to the case where

$$\Phi(p, x, (t, \xi, A)) = F(A) - \phi(p, x)G(p, \xi),$$

where $\phi > 0$, $G \geq 1$ and $\xi \mapsto G(p, \xi)$ is concave for all p.

Theorem 2.26. *Let* $(p_m)_{m\in\mathbb{N}}$ *be a sequence of points in* M *and let* $(f_m)_{m\in\mathbb{N}}, (\hat{f}_m)_{m\in\mathbb{N}} \in C_0^\infty(\overline{\Omega})$ *be strictly convex functions such that for all* m*,* $f_m \geq \hat{f}_m$ *and, for all* $x \in \overline{\Omega}$*,*

$$\Phi(p_m, x, J^2\hat{f}_m) \geq 0 = \Phi(p_m, x, J^2 f_m).$$

If there exists $x_0 \in M$ *towards which* $(x_m)_{m\in\mathbb{N}}$ *converges and* $\hat{f}_\infty \in C_0^\infty(\overline{\Omega})$ *towards which* $(\hat{f}_m)_{m\in\mathbb{N}}$ *converges in the* C^∞ *sense, then there exists* $f_\infty \in C_0^\infty(\overline{\Omega})$ *towards which* $(f_m)_{m\in\mathbb{N}}$ *subconverges in the* C^∞ *sense.*

Proof. By Lemma 2.1, there exists $C_1 > 0$ such that for all m, $\|f_m\|_1 \leq C_1$. By Theorem 2.19, there exists $C_2 > 0$ such that, for all m, $\|f_m\|_2 \leq C_2$. By Corollary 2.5, $\Phi_{(p,x,(t,\xi))}$ is concave for all $(p, x, (t, \xi))$. Thus, by Theorem 2.25, there exists $\alpha > 0$ and $C_{2+\alpha} > 0$ such that, for all m, $\|f_m\|_{2+\alpha} \leq C_{2+\alpha}$. By Theorem 2.24, for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that, for all m, $\|f_m\|_k \leq C_k$. It now follows by the Arzelà-Ascoli theorem (c.f. Theorem 11.28 of [19]) that there exists $f_\infty \in C_0^\infty(\overline{\Omega})$ towards which $(f_m)_{m\in\mathbb{N}}$ subconverges, and this completes the proof. □

Chapter 3

Degree Theory

We develop a differential-topological degree for smooth mappings between open subsets of Banach spaces. Together with Theorem 2.26, this yields the main result of this chapter, namely Theorem 3.16, which proves the existence of unique solutions to the classical Plateau problem for gaussian curvature in the case of graphs. This existence result constitutes an important component of the proof of Theorem 1.2, and we will see in Chapter 6, below, how it is used construct a local regularisation operation for weak barriers.

The topological degree theory we use dates back to Smale's infinite-dimensional adaptation (c.f. [23]) of the classical finite-dimensional theory (c.f. [13] and [15]) and requires a fairly in-depth detour into functional analysis. A complete exposition of the required background material would take us too far afield, and we therefore quote a number of results without proof. We hope that this will not obscure too much the main ideas, and we refer the interested reader to the numerous excellent introductions to functional analysis (c.f. for example [1], [16], [19] and [20]) for more information.

The key result is Theorem 3.4, which constructs a $\mathbb{Z}_2$-valued differential-topological degree for the zero set of a given smooth function between Banach spaces. The main step in our argument, encapsulated in Lemma 3.15, uses the classical Sard Theorem together with finite-dimensional reduction. In particular, even though we essentially follow Smale's reasoning (c.f. [23]), we do not directly use the Sard-Smale Theorem. We hope that this approach will be of use to the novice reader, partly as we believe it clarifies the main ideas of Smale's result, but also because Smale's result is sometimes too specific as stated to be applied in many settings of interest in present-day mathematics.

The content of this chapter is independent of the rest of the text, and the reader only interested in understanding the theory of singularities of the Gauss curvature equation may skip it if he so wishes.

3.1 Smooth mappings and differential operators

Let E and F be normed vector spaces. Denote by $\mathrm{Lin}(E,F)$ the space of *bounded* linear maps from E into F. Observe that $\mathrm{Lin}(E,F)$ is also a normed vector space with norm given by

$$\|A\| := \mathop{\mathrm{Sup}}_{x\in E\setminus\{0\}} \frac{\|Ax\|}{\|x\|}.$$

Let U be an open subset of E and let Φ be a mapping from U into F. For $x \in U$, we say that Φ is **differentiable** at x whenever there exists a *bounded* linear map $A : E \to F$ such that

$$\mathop{\mathrm{Lim}}_{y\to 0} \frac{1}{\|y\|}\|\Phi(x+y) - \Phi(x) - A(y)\| = 0,$$

where y varies over all vectors in $E\setminus\{0\}$ having the property that $x+y \in U$. We refer to A as the **derivative** of Φ at x. Whenever it exists, the derivative is unique, and we denote it by $D\Phi(x)$.

Remark. *This definition only differs from the finite-dimensional version by the requirement - unnecessary in the finite-dimensional case - that the derivative be a bounded operator. Importantly, since* $\mathrm{Lin}(E,F)$ *is itself a normed vector space, the concept of differentiability iterates, and derivatives of arbitrary order may therefore be defined.*

We say that a function $\Phi : U \to F$ is C^1 whenever $D\Phi$ exists at every point of U and defines a continuous function from U into $\mathrm{Lin}(E,F)$. We define inductively the notion of higher order differentiability, and we say that Φ is C^k whenever $D\Phi$ exists at every point of U and defines a C^{k-1} function from U into $\mathrm{Lin}(E,F)$. We say that Φ is **smooth** whenever it is C^k for all $k \in \mathbb{N}$.

In order to make use of this concept, we require elementary rules for the construction of smooth functions over normed spaces. Theorem 3.8, below, will provide an important tool for the construction of a large family of smooth functions, which, in particular, includes almost every function that arises in geometry. At this stage, however, we recall the following three elementary rules, which are derived in exactly the same manner as in the finite-dimensional case.

Chain Rule: Let E_1, E_2 and E_3 be normed vector spaces. Let U_1 and U_2 be open subsets of E_1 and E_2 respectively, and let $\Phi : U_1 \to U_2$ and $\Psi : U_2 \to E_3$ be smooth mappings. The composition $\Psi \circ \Phi$ is smooth, and its first derivative is given by

$$D(\Psi\circ\Phi)(x) = D\Psi(\Phi(x))D\Phi(x).$$

Direct sums: Let E, F_1,...,F_n be normed vector spaces. Let U be an open subset of E, and for $1 \leq i \leq n$, let $\Phi_i : U \to F_i$ be a smooth mapping. The function $\Phi := (\Phi_1, ..., \Phi_n)$ defines a smooth mapping from E into $F_1 \oplus ... \oplus F_n$, and its first derivative is given by

$$D\Phi(x) = (D\Phi_1(x), ..., D\Phi_n(x)).$$

Multilinear forms: Let $E_1, ..., E_n$ and F be normed vector spaces. Let $\Phi : E_1 \oplus ... \oplus E_n \to F$ be a bounded, multilinear map. Φ is smooth, and its first derivative is given by

$$D\Phi(x_1, ..., x_n)(V_1, ..., V_n) = \Phi(V_1, x_2, ..., x_n) + ... + \Phi(x_1, ..., x_{n-1}, V_n).$$

Remark. *In particular, the product rule constitutes a special case of the above results.*

3.2 Banach spaces

Let E be a normed vector space. We say that E is a **Banach space** whenever it is complete. With this extra hypothesis, we have the inverse function theorem (c.f. [19]).

Theorem 3.1. *Let E and F be Banach spaces. Let U be an open subset of E and let Φ be a smooth mapping from U to F. If $D\Phi(x)$ is invertible at some point $x \in U$, then there exist neighbourhoods V of x in U, W of $\Phi(x)$ in F and a smooth mapping $\Psi : W \to V$ such that $W = \Phi(V)$ and*

$$\Psi \circ \Phi = \mathrm{Id}, \qquad \Phi \circ \Psi = \mathrm{Id}.$$

Let E be a Banach space. Let X be a subset of E. For $n \in \mathbb{N}$, we say that X is an n-dimensional **submanifold** of E whenever there exists a Banach space F with the property that for all $x \in X$, there exist neighbourhoods U of x in E and V of $(0, 0)$ in $\mathbb{R}^n \times F$ and a smooth mapping $\Phi : U \to V$ with smooth inverse such that $\Phi(X \cap U) = (\mathbb{R}^n \times \{0\}) \cap V$. We refer to the triplet (Φ, U, V) as a **trivialising chart** of X about x. Recall that an abstract manifold is a separable metrisable space furnished with an atlas of charts all of whose transition maps are smooth.

Lemma 3.2. *Let E be a Banach space. Let X be a finite-dimensional submanifold of E and let $e : X \to E$ be the canonical embedding. If X is separable, then X is a smooth, finite-dimensional manifold, and $e : X \to E$ is a smooth mapping.*

Proof. By hypothesis, X is separable, and as it is a subset of a normed space, it is also metrisable. It thus suffices to construct a smooth atlas of charts for X. Choose $x \in X$ and let $(\Phi, \tilde{U}, \tilde{V})$ be a trivialising chart of X

about x. Denote $\phi := \Phi|_{X\cap\tilde{U}}$, $U := X\cap\tilde{U}$ and $V := (\mathbb{R}^n\times\{0\})\cap\tilde{V}$. We see that (ϕ,U,V) defines a homeomorphism from an open subset of X to an open subset of $\mathbb{R}^n$. We claim that the family of all such charts constitutes a smooth atlas for X. Indeed, fix $x'\in X$. Let $(\Phi',\tilde{U}',\tilde{V}')$ be another trivialising chart of X about x' and denote $\phi' := \Phi'|_{X\cap\tilde{U}'}$, $U' := X\cap\tilde{U}'$ and $V' := (\mathbb{R}^n\times\{0\})\cap\tilde{V}'$. Observe that $U\cap U' = \tilde{U}\cap\tilde{U}'\cap X$ and

$$\phi'\circ(\phi^{-1})|_{\phi(U\cap U')} = \Phi'\circ(\Phi^{-1})|_{\Phi(\tilde{U}\cap\tilde{U}'\cap X)}.$$

In particular, the transition map is smooth. Since $x,x'\in X$ are arbitrary, it follows that the set of all such charts constitutes a smooth atlas, as desired. Finally, in the chart (ϕ,U,V), the canonical immersion coincides with ϕ^{-1}, and since

$$\phi^{-1} = \Phi^{-1},$$

it follows that this map is smooth. This completes the proof. □

Let E and F be two Banach spaces. Recall that a bounded linear mapping $A\in\mathrm{Lin}(E,F)$ is said to be **Fredholm** whenever it has closed image and both its kernel and cokernel are finite-dimensional. We define the **index** of a Fredholm mapping by

$$\mathrm{Ind}(A) := \mathrm{Dim}(\mathrm{Ker}(A)) - \mathrm{Dim}(\mathrm{Coker}(A)).$$

Let U be an open subset of E, and let Φ be a smooth mapping from U into F. We say that Φ is **Fredholm** whenever $D\Phi(x)$ is a Fredholm mapping for all $x\in U$. Recall that the space of linear Fredholm mappings constitutes an open subset of $\mathrm{Lin}(E,F)$ and that two linear Fredholm mappings in the same connected component have the same index. It follows that if U is connected, then $\mathrm{Ind}(D\Phi(x))$ is independent of $x\in U$, and we therefore refer to it as the index of the mapping Φ. In addition, recall that the set of surjective, linear Fredholm mappings also constitutes an open subset of $\mathrm{Lin}(E,F)$. This is relevant to situations where we apply the following submersion theorem.

Theorem 3.3. *Let E and F be two Banach spaces. Let U be an open subset of E and let $\Phi : U\to F$ be a smooth, Fredholm map. If $D\Phi$ is surjective for all $x\in\Phi^{-1}(\{0\})$, then $\Phi^{-1}(\{0\})$ is a smooth* $\mathrm{Ind}(\Phi)$*-dimensional submanifold of E.*

Proof. Indeed, choose $x_0\in\Phi^{-1}(\{0\})$. Let $\mathrm{Ker}(D\Phi(x_0))$ be the kernel of $D\Phi(x_0)$. Since $D\Phi(x_0)$ is Fredholm and surjective, $\mathrm{Dim}(\mathrm{Ker}(D\Phi(x_0)))$ is equal to $\mathrm{Ind}(\Phi)$. By the Hahn-Banach theorem (Theorem 5.16 of [19]), the identity map $\mathrm{Id} : \mathrm{Ker}(D\Phi(x_0))\to\mathrm{Ker}(D\Phi(x_0))$ extends to a bounded, linear projection π from E onto $\mathrm{Ker}(D\Phi(x_0))$. Define the mapping $\hat{\Phi} : U\to\mathrm{Ker}(D\Phi)(x_0)\times F$ by $\hat{\Phi}(x) := (\pi(x-x_0),\Phi(x))$. This function is smooth and, for all vectors $y\in E$, $D\hat{\Phi}(x_0)(y) = (\pi(y),D\Phi(x_0)(y))$. In particular, $D\hat{\Phi}(x_0)$ is bijective. It follows by the closed graph theorem (c.f. Theorems 5.9 and 5.10 of [19]), that $D\hat{\Phi}(x_0)$ is invertible with bounded, linear inverse, and so, by Theorem 3.1 that there exist neighbourhoods U of x_0 in E and V of $(0,0)$ in $\mathrm{Ker}(D\Phi(x_0))\times F$, and a smooth mapping $\Psi : V\rightarrow U$ such that $\hat{\Phi}(U)=V$, $\hat{\Phi}\circ\Psi=\mathrm{Id}$ and $\Psi\circ\hat{\Phi}=\mathrm{Id}$. We readily verify that $\hat{\Phi}(X\cap U)$ coincides with $(\mathrm{Ker}(D\Phi(x_0))\times\{0\})\cap V$, we conclude that $\Phi^{-1}(\{0\})$ is smooth $\mathrm{Ind}(\Phi)$-dimensional submanifold of E, as desired. □

3.3 Degree theory

Let M be a finite-dimensional manifold and let E and F be Banach spaces. Since the results of this section are local, we may suppose that M is an open subset of some finite-dimensional vector space. Let U be an open subset of E and let $\Phi : M\times U\rightarrow F$ be a smooth Fredholm mapping of index equal to the dimension of M. We define the **solution space** of Φ by

$$\mathcal{Z} := \{(p,x)\in M\times E \mid \Phi(p,x)=0\}\,.$$

Observe that if $D\Phi(p,x)$ is surjective for all $(p,x)\in\mathcal{Z}$ then, by Theorem 3.3, $\mathcal{Z}$ is a smooth, finite-dimensional submanifold of $M\times E$ of dimension equal to $\mathrm{Ind}(\Phi)=\mathrm{Dim}(M)$. Let $\Pi : M\times E\rightarrow M$ be the projection onto onto the first factor, and let $\Pi_{\mathcal{Z}}$ be its restriction to $\mathcal{Z}$. If we consider Φ as a smooth family of non-linear operators from U into F parametrised by M, then, for each p, $\Pi_{\mathcal{Z}}^{-1}(\{p\})$ is the set of solutions of the operator $\Phi(p,\cdot)$. In particular, the topological degree of the projection $\Pi_{\mathcal{Z}}$ yields information about the size of the solution set of each $\Phi(p,\cdot)$.

Theorem 3.4. *If $D\Phi(p,x)$ is surjective for all $(p,x)\in\mathcal{Z}$, and if $\Pi_{\mathcal{Z}}$ is proper, then there exists an open, dense subset $M'\subseteq M$ with the property that, for all $p\in M'$, $\Pi_{\mathcal{Z}}^{-1}(\{p\})$ is finite and for all $p,q\in M'$,*

$$\left|\Pi_{\mathcal{Z}}^{-1}(\{p\})\right| = \left|\Pi_{\mathcal{Z}}^{-1}(\{q\})\right| \;\; \mathit{Mod}\ 2.$$

Proof. Denote $n=\mathrm{Dim}(M)=\mathrm{Ind}(\Phi)$. Since Φ is smooth and Fredholm, and since $D\Phi$ is surjective at every point of $\mathcal{Z}=\Phi^{-1}(\{0\})$, by Theorem 3.3, $\mathcal{Z}=\Phi^{-1}(\{0\})$ is a smooth, n-dimensional submanifold of $M\times E$. Since M is a finite-dimensional manifold, in particular, it is separable, and so, since $\Pi_{\mathcal{Z}} : \mathcal{Z}\rightarrow M$ is proper, $\mathcal{Z}$ is also separable. It follows by Lemma 3.2, that $\mathcal{Z}$ is a smooth, n-dimensional manifold and that the canonical embedding $e : \mathcal{Z}\rightarrow M\times E$ is a smooth mapping. In particular, $\Pi_{\mathcal{Z}}=\Pi\circ e$ is also smooth.

We now apply standard differential topological techniques to $\Pi_{\mathcal{Z}}$, using the terminology of [13]. Define M' to be the set of regular values of $\Pi_{\mathcal{Z}}$.

If $p \in M'$, then $\Pi_{\mathcal{Z}}^{-1}(\{p\})$ is discrete, and since $\Pi_{\mathcal{Z}}$ is proper, this set is compact and therefore finite. Moreover, since $\Pi_{\mathcal{Z}}$ is a smooth, proper mapping, by Sard's Theorem (c.f. [13]), M' is open and dense, and the first assertion follows.

Now choose $p, q \in M'$ and let $\gamma : [0,1] \to M$ be any smooth, embedded curve such that $\gamma(0) = x$ and $\gamma(1) = y$. By genericity (c.f. [13]), there exists another smooth, embedded curve $\tilde{\gamma} : [0,1] \to M$ which we may choose as close to γ as we wish in the C^∞ sense with the property that $\tilde{\gamma}(0) = \gamma(0) = p$, $\tilde{\gamma}(1) = \gamma(1) = q$ and $\tilde{\gamma}$ is transverse to $\Pi_{\mathcal{Z}}$. If we denote by $\tilde{\Gamma} \subseteq M$ the image of $\tilde{\gamma}$, then, by transversality, $\Pi_{\mathcal{Z}}^{-1}(\tilde{\Gamma})$ is a smooth, 1-dimensional, embedded, submanifold of $\mathcal{Z}$ with boundary given by

$$\partial\Pi_{\mathcal{Z}}^{-1}(\Gamma') = \Pi_{\mathcal{Z}}^{-1}(\{p\}) \cup \Pi_{\mathcal{Z}}^{-1}(\{q\}).$$

Since $\Pi_{\mathcal{Z}}^{-1}$ is proper, $\Pi_{\mathcal{Z}}^{-1}(\tilde{\Gamma})$ is compact, and therefore has an even number of boundary points, so that

$$\left|\Pi_{\mathcal{Z}}^{-1}(\{p\})\right| + \left|\Pi_{\mathcal{Z}}^{-1}(\{q\})\right| = \left|\partial\Pi_{\mathcal{Z}}^{-1}(\Gamma')\right| = 0 \text{ Mod } 2,$$

as desired. □

3.4 Hölder spaces and Hölder norms

Let E be a finite-dimensional normed vector space. For $\alpha \in]0,1]$, we denote by $[\cdot]_\alpha$ the **Hölder semi-norm** over $C^0(\overline{\Omega}, E)$ of order α. That is, for all $f \in C^0(\overline{\Omega}, E)$,

$$[f]_\alpha := \mathrm{Sup}_{x \neq y \in \overline{\Omega}} \frac{\|f(x) - f(y)\|}{\|x - y\|^\alpha}.$$

We readily obtain

Lemma 3.5. *If $\overline{\Omega}$ is convex, then, for all continuously differentiable $f \in C^0(\overline{\Omega}, E)$,*

$$[f]_1 = \|Df\|_0.$$

Remark. *In general, for a compact set Ω with rectifiable boundary, $[f]_1 < C(\Omega)\|Df\|_0$, where $C \geq 1$ depends on the geometry of Ω. In fact, convex sets are characterised amongst all compact sets with rectifiable boundary by the property that $C(\overline{\Omega}) = 1$.*

For all $\lambda = k + \alpha \in]0, \infty[$, where $k \in \mathbb{N}$ and $\alpha \in]0,1]$, we denote by $\|\cdot\|_\lambda$ the **Hölder norm** over $C^k(\overline{\Omega})$ of order λ. That is, for all $f \in C^k(\overline{\Omega})$,

$$\|f\|_\lambda := \sum_{i=0}^{k} \|D^i f\|_0 + [D^k f]_\alpha.$$

For all such λ, we denote by $C^\lambda(\overline{\Omega})$ the space of all functions $f \in C^k(\overline{\Omega})$ such that $\|f\|_\lambda < \infty$. We refer to $C^\lambda(\overline{\Omega})$ as the space of λ-times Hölder differentiable functions over $\overline{\Omega}$.

We restate the classical Arzelà-Ascoli theorem in the following form (c.f. [19]).

Theorem 3.6. *Choose $\lambda \in]0,\infty[$ and let $(f_m)_{m\in\mathbb{N}}$ be a sequence of functions in $C^\lambda(\overline{\Omega})$. If there exists $B > 0$ such that $\|f_m\|_\lambda \leq B$ for all m, then there exists $f_\infty \in C^\lambda(\overline{\Omega})$ such that $\|f_\infty\|_\lambda \leqslant B$ and $(f_m)_{n\in\mathbb{N}}$ subconverges to f_∞ in the C^μ norm for all $\mu < \lambda$.*

In particular, this yields

Lemma 3.7. *For all $\lambda \in]0,\infty[$, $(C^\lambda(\overline{\Omega}), \|\cdot\|_\lambda)$ is a Banach space.*

Proof. Choose $\lambda > 0$ and let $(f_m)_{m\in\mathbb{N}}$ be a Cauchy sequence of functions in $C^\lambda(\overline{\Omega})$. We need to show that $(f_m)_{n\in\mathbb{N}}$ converges in $C^\lambda(\overline{\Omega})$. For all $i \in \mathbb{N}$, define the subset F_i of $C^\lambda(\overline{\Omega})$ by

$$F_i := \{f_m \mid m \geq i\},$$

and define d_i to be its diameter. Since $(f_m)_{m\in\mathbb{N}}$ is a Cauchy sequence, the sequence $(d_i)_{i\in\mathbb{N}}$ converges to 0.

By Theorem 3.6, there exists a subsequence $(m_i)_{i\in\mathbb{N}}$ and a function f_∞ in $C^\lambda(\overline{\Omega})$ such that $(f_{m_i})_{i\in\mathbb{N}}$ converges to f_∞ in the C^μ norm for all $\mu < \lambda$. Moreover, we may assume that $m_i \geq i$ for all i. Consequently, for all i and for all $j > i$, $f_{m_j} \in F_i$ and so

$$\|f_{m_j} - f_i\| \leq d_i.$$

Choose $i \in \mathbb{N}$. By Theorem 3.6 again, there exists $g \in C^\lambda(\overline{\Omega})$ such that $\|g\|_\lambda \leq d_i$ and $(f_{m_j} - f_i)_{j\in\mathbb{N}}$ subconverges to g in the C^μ norm for all $\mu < \lambda$. However, since $(f_{m_j} - f_i)_{j\in\mathbb{N}}$ also converges to $(f_\infty - f_i)$, it follows that $g = f_\infty - f_i$. In particular, $\|f_\infty - f_i\|_\lambda = \|g\|_\lambda \leq d_i$ and we conclude that $(f_i)_{i\in\mathbb{N}}$ converges to f_∞ as desired. □

For all $\lambda \geqslant 1$, denote by $C^\lambda_0(\overline{\Omega})$ the linear subspace of $C^\lambda(\overline{\Omega})$ consisting of those functions which vanish along the boundary. Observe that, for all λ, $C^\lambda_0(\overline{\Omega})$ is a closed subspace of $C^\lambda(\overline{\Omega})$ and in particular, is a Banach space in its own right.

We leave the reader to verify that for all $\lambda \leq \mu$, $C^\mu(\overline{\Omega})$ (resp. $C^\mu_0(\overline{\Omega})$) canonically embeds as a subspace of $C^\lambda(\overline{\Omega})$ (resp. $C^\lambda_0(\overline{\Omega})$). Moreover, this embedding is continuous and if $\lambda < \mu$, it is also a compact mapping. In addition,

$$C^\infty(\overline{\Omega}) = \bigcap_{\lambda>0} C^\lambda(\overline{\Omega}), \qquad C^\infty_0(\overline{\Omega}) = \bigcap_{\lambda>0} C^\lambda_0(\overline{\Omega}),$$

and, moreover, a sequence $(f_m)_{m\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ (resp. $C^\infty_0(\overline{\Omega})$) converges to a limit f_∞ in $C^\infty(\overline{\Omega})$ (resp. $C^\infty_0(\overline{\Omega})$) if and only if it converges to f_∞ in the C^λ-norm for all $\lambda > 0$.

3.5 Smooth mappings of Hölder spaces

Let E be a finite-dimensional vector space and let U be an open subset of E. For all $\lambda \geqslant 1$, we define $C^\lambda(\overline{\Omega}, U)$ to be the open subset of $C^\lambda(\overline{\Omega}, E)$ consisting of all functions g such that $g(x) \in U$ for all x. Let F be another finite dimensional vector space and let $\phi : \overline{\Omega} \times U \to F$ be a smooth mapping. We define the mapping $\Phi : C^0(\overline{\Omega}, U) \to C^0(\overline{\Omega})$ by $\Phi(f)(x) := \phi(x, f(x))$. Together with the three construction rules already presented in Section 3.1, the following result allows us to apply the techniques of the preceeding sections to almost every function that we will encounter.

Theorem 3.8. *For all $\lambda > 0$ and for all $g \in C^\lambda(\overline{\Omega}, U)$, $\Phi(g) \in C^\lambda(\overline{\Omega})$. Moreover, Φ defines a smooth mapping from $C^\lambda(\overline{\Omega}, U)$ into $C^\lambda(\overline{\Omega})$, and for all $h \in C^\lambda(\overline{\Omega}, E)$,*

$$(D\Phi(g)h)(x) = D_2\phi(g(x))h(x),$$

where $D_2\phi$ is the partial derivative of ϕ with respect to the second component.

Let M be a finite dimensional manifold. Let U be an open subset of $\oplus_{i=0}^{2}\mathrm{Symm}(i, \mathbb{R}^n)$. Let $F : M \times \overline{\Omega} \times U \to \mathbb{R}$ be a smooth function. For all $\lambda \geq 2$, we define $\mathcal{U}_0^\lambda(\overline{\Omega})$ to be the set of all functions g in $C_0^\lambda(\overline{\Omega})$ such that $J^2g(x) \in U$ for all x. We define the mapping $\mathcal{F} : M \times \mathcal{U}_0^2(\overline{\Omega}) \to C^0(\overline{\Omega})$ by $\mathcal{F}(p, g)(x) = F(p, x, J^2g(x))$.

Lemma 3.9. *For all $\lambda \in]0, \infty[$ and for all $(p, g) \in M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, $\mathcal{F}(p, g) \in C^\lambda(\overline{\Omega})$. Moreover, $\mathcal{F}$ defines a smooth mapping from $M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$ and its partial derivative with respect to the second component is given by*

$$(D_2\mathcal{F}(p, g)h)(x) = D_3F(p, x, J^2g(x))J^2h(x),$$

where D_3F is the partial derivative of F with respect to the third component.

Proof. Choose $(p, g) \in M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$. Then $(p, J^2g) \in M \times C^\lambda(\overline{\Omega}, \oplus_{i=0}^{2}\mathrm{Symm}(i, \mathbb{R}^n))$ and, by Theorem 3.8, $\mathcal{F}(p, g) = F(p, x, J^2g)$ is an element of $C^\lambda(\overline{\Omega})$. Furthermore, since the mapping $g \mapsto J^2g$ defines a bounded linear map from $C^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$, in particular it is smooth, and so, by Theorem 3.8 and the chain rule, $\mathcal{F}$ defines a smooth mapping from $M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$ and, for all $h \in C_0^{\lambda+2}(\overline{\Omega})$,

$$(D_2\mathcal{F}(p, g)h)(x) = D_3F(p, x, J^2g(x))J^2h(x),$$

as desired. □

As before, for all $\lambda \geqslant 1$, we think of $\mathcal{F}$ as a smooth family of mappings parametrised by M which send $\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. In particular, we think of $\mathcal{F}$ as a smooth family of operators of the type studied in Section 3.2. Observe that for all $(p,g) \in M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, $D_2\mathcal{F}(p,g)$ is a second-order, linear, partial differential operator from $C_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. Now, as in Section 2.7, for all $\xi \in \oplus_{i=0}^1 \mathrm{Symm}(i,\mathbb{R}^n)$, we define $U_\xi \subseteq \mathrm{Symm}(2,\mathbb{R}^n)$ by

$$U_\xi := \{A \in \mathrm{Symm}(2,\mathbb{R}^n) \mid (\xi, A) \in U\},$$

and for all $(p,x,\xi) \in M \times \overline{\Omega} \times \oplus_{i=0}^1 \mathrm{Symm}(i,\mathbb{R}^n)$, we define $F_{p,x,\xi} : U_\xi \to \mathbb{R}$ by

$$F_{p,x,\xi}(A) := F(p,x,\xi,A).$$

We say that F is an **elliptic function** whenever the derivative $DF_{p,x,\xi}(A)$ is a positive-definite matrix for all $(p,x,\xi,A) \in M \times \overline{\Omega} \times U$.

Lemma 3.10. *If F is an elliptic function, then $D_2\mathcal{F}(p,g)$ is an elliptic operator for all $(p,g) \in M \times \mathcal{U}_0^2(\overline{\Omega})$.*

Proof. Denote by D_3F the partial derivative of F with respect to the third factor. By Lemma 3.9, for $(p,g) \in M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, for $h \in C_0^{\lambda+2}(\overline{\Omega})$ and for $x \in \overline{\Omega}$,

$$(D_2\mathcal{F}(p,g)h)(x) = D_3F(p,x,J^2g(x))J^2h(x).$$

If $\sigma_2(D_2\mathcal{F}(p,g))(x)$ denotes the principal symbol of this operator at the point x (c.f. [10]), then, for all $\xi \in \mathbb{R}^n$,

$$\sigma_2(D_2\mathcal{F}(p,g))(x)(\xi,\xi) = DF_{p,x,J^1g(x)}(D^2g(x))^{ij}\xi_i\xi_j.$$

Since F is elliptic, this is positive for all ξ, and since $x \in \overline{\Omega}$ is arbitrary, we conclude that $D_2\mathcal{F}(p,g)$ is an elliptic operator, as desired. □

Observe that the compactness result of the previous chapter only applies to smooth functions. In particular, it does not necessarily apply to functions which are only known to be Hölder differentiable of some finite order. However, the following regularity result, derived by inductively applying the classical Schauder estimates to difference quotients (c.f. Section 6.4 of [11]) makes this distinction irrelevant.

Theorem 3.11. *Choose $\lambda \geqslant 1$ and $(p,g) \in M \times \mathcal{U}^{2+\lambda}(\overline{\Omega})$. If F is an elliptic function and if $\mathcal{F}(p,g) = 0$, then g is a smooth function.*

In order to apply the degree theory described in Sections 3.1, 3.2 and 3.3, we require that $\mathcal{F}$ be a Fredholm mapping. However, this readily follows from classical elliptic theory (c.f. [11]).

Lemma 3.12. *If F is an elliptic function, then for all $\lambda \notin \mathbb{N}$, $\mathcal{F}$ defines a Fredholm mapping from $M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. Moreover, $\mathrm{Ind}(\mathcal{F}) = \mathrm{Dim}(M)$.*

Remark. *First, observe that $\mathcal{F}$ is only Fredholm for non-integer values of λ, which is a significant limitation of elliptic theory in Hölder spaces. Second, we draw the reader's attention to the fact that the calculation of the index follows from general considerations. Indeed, if an elliptic operator sends sections of a bundle E_1 into sections of a bundle E_2, then the index of the operator only depends on the topology of the bundle $\mathrm{Lin}(E_1, E_2)$. In particular, in the case at hand, we may show that any elliptic operator sending $C_0^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$ has index 0. We refer the interested reader to [1], [11] and [20] for more details.*

Proof. Let $D_1\mathcal{F}$ and $D_2\mathcal{F}$ be the partial derivatives of $\mathcal{F}$ with respect to the first and second components respectively. By Lemma 3.10, for all $(p,g) \in M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, $D_2\mathcal{F}(p,g)$ is an elliptic operator. By classical elliptic theory, for all such (p,g), $D_2\mathcal{F}(p,g)$ is a Fredholm operator from $C_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. Since $D_2\mathcal{F}(p,g)$ acts on real valued functions, $\mathrm{Ind}(D_2\mathcal{F}(p,g)) = 0$. Let π_1 and π_2 be the canonical projections of $M \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ onto the first and second factors respectively. Trivially, $D\pi_2$ is Fredholm of index $\mathrm{Dim}(M)$. Since the composition of two Fredholm operators is Fredholm of index equal to the sum of the indices of each component, $D_2\mathcal{F}(p,g) \circ D\pi_2$ is also Fredholm of index equal to $\mathrm{Dim}(M)$. Since M has finite dimension, $D\pi_1$ has finite rank, and therefore so too does $D_1\mathcal{F}(p,g) \circ D\pi_1$. Since the sum of a Fredholm operator and a finite rank operator is also a Fredholm operator of the same index, it follows that $D\mathcal{F}(p,g) = D_1\mathcal{F}(p,g) \circ D\pi_1 + D_2\mathcal{F}(p,g) \circ D\pi_2$ is also Fredholm of index equal to $\mathrm{Dim}(M)$, as desired. □

3.6 Existence

We now recall the construction of Section 2. Let $\Gamma \subseteq \mathrm{Symm}(2, \mathbb{R}^n)$ be the open cone of positive-definite, symmetric matrices, and denote $U = (\oplus_{i=0}^1 \mathrm{Symm}(i, \mathbb{R}^n)) \times \Gamma$. Let $G : \mathbb{R}^n \to \mathbb{R}$ be a smooth, convex function bounded below by 1. Let $\phi \in C^\infty([0,1] \times \overline{\Omega},]0, \infty[)$ be a smooth family of smooth, positive functions over $\overline{\Omega}$. Let X be a finite dimensional subspace of $C^\infty(\overline{\Omega})$ and let $r > 0$ be a positive number, both of which we will chose presently (c.f. Lemma 3.15). We denote by B_r the ball of radius r about 0 in X and we define $F : [0,1] \times B_r \times \overline{\Omega} \times U \to \mathbb{R}$ by

$$F(s, g, x, (t, \xi, A)) := \mathrm{Det}(A) - (\phi_s(x) + g(x))(sG(\xi) + (1-s)).$$

Observe that there exists $r > 0$ such that for all $(s,g) \in [0,1] \times B_r$, $\phi_s + g > 0$. Furthermore, by Lemma 2.2, F is an elliptic function. For all λ, we now define the mapping $\mathcal{F} : [0,1] \times B_r \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega}) \to C^\lambda(\overline{\Omega})$ by

$$\mathcal{F}(s, g, h)(x) := F(s, g, x, J^2 h(x)).$$

Lemma 3.13. *If $\lambda \notin \mathbb{N}$, then $\mathcal{F}$ defines a smooth Fredholm mapping from $[0,1] \times B_r \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$. Moreover,* $\mathrm{Ind}(\mathcal{F}) = \mathrm{Dim}(X) + 1$.

Proof. By Lemma 3.9, $\mathcal{F}$ defines a smooth mapping from $[0,1] \times B_r \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ into $C^\lambda(\overline{\Omega})$, and, by Lemma 3.12, this mapping is Fredholm of Fredholm index equal to $\mathrm{Dim}(X) + 1$. This completes the proof. □

We now apply the degree theory of Section 3.3 to $\mathcal{F}$. Define the **solution space** $\mathcal{Z} \subseteq [0,1] \times B_r \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ by

$$\mathcal{Z} := \{(s, g, f) \mid \mathcal{F}(s, g, f) = 0\}.$$

Let $\Pi : [0,1] \times B_r \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega}) \to [0,1] \times B_r$ be the projection into the first two factors, and let $\Pi_{\mathcal{Z}}$ be the restriction of Π to $\mathcal{Z}$. Let $\hat{f} \in C^\infty([0,1] \times \overline{\Omega})$ be a smooth family of strictly convex functions such that, for all s and for all $x \in \partial\Omega$,

$$\hat{f}_s(x) = 0,$$

and for all s and for all $x \in \overline{\Omega}$,

$$\mathcal{F}(s, 0, \hat{f}_s)(x) > 0.$$

By compactness, upon reducing r if necessary, we may suppose that for all $(s, g) \in [0,1] \times B_r$, and for all $x \in \overline{\Omega}$,

$$\mathcal{F}(s, g, \hat{f}_s)(x) > 0.$$

Lemma 3.14. *Suppose that for all $(s, g) \in [0,1] \times B_r$ and for all $x \in \overline{\Omega}$, $\phi_s(x) - g(x) > 0$ and $\mathcal{F}(s, g, \hat{f}_s)(x) > 0$. If $\lambda \notin \mathbb{N}$, then $\Pi_{\mathcal{Z}}$ is a proper mapping.*

Proof. Let $(s_m, g_m)_{m\in\mathbb{N}}$ be sequence in $[0,1] \times B_r$ converging to the limit, (s_∞, g_∞), say in $[0,1] \times B_r$. Let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that for all m, $(s_m, g_m, f_m) \in \mathcal{Z}$. That is $\mathcal{F}(s_m, g_m, f_m) = 0$. Since F is an elliptic function, by Theorem 3.11, for all m, f_m is smooth. By Lemma 2.9, for all m, $f_m \geq \hat{f}_m$. Thus, by Theorem 2.26, there exists $f_\infty \in C^\infty(\overline{\Omega}) \subseteq C_0^{\lambda+2}(\overline{\Omega})$ towards which $(f_m)_{m\in\mathbb{N}}$ subconverges in the C^∞ sense, and, in particular, $\mathcal{F}(s_\infty, g_\infty, f_\infty) = 0$. It remains to show that $f_\infty \in \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$, that is, that f_∞ is strictly convex. Indeed, suppose the contrary. Since f_∞ is a limit of a sequence of convex functions, it is convex. Since it is not strictly convex, there exists a point $x \in \overline{\Omega}$ at which $D^2 f_\infty$ is degenerate. However, at this point, $\mathrm{Det}(D^2 f_\infty(x)) = 0$, and so $F(s_\infty, g_\infty, x, J^2 f_\infty)(x) < 0$. This is absurd, and we conclude that $f_\infty \in \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ as asserted. In particular, $(s_\infty, g_\infty, f_\infty) \in \mathcal{Z}$ and compactness follows. □

Both X and r are now chosen to ensure surjectivity.

Theorem 3.15. *If $\lambda \notin \mathbb{N}$, then there exists a finite dimensional subspace $X \subseteq C^\infty(\overline{\Omega})$ and $r > 0$ such that for all $(s, g, f) \in \mathcal{Z}$, $D\mathcal{F}(s, g, f)$ is surjective.*

Proof. Choose $\lambda \notin \mathbb{N}$. Define $\mathcal{F}_0 : [0,1] \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega}) \to C^\lambda(\overline{\Omega})$ by

$$\mathcal{F}_0(s, f)(x) := F(s, 0, x, J^2 h(x)) = \mathrm{Det}(D^2 f(x)) - \phi_s(x)(sG(Df(x)) + (1-s)).$$

Choose $(s_1, f_1) \in [0,1] \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that $\mathcal{F}_0(s_1, f_1) = 0$. By Lemma 3.13, $D\mathcal{F}_0(s_1, f_1)$ is a Fredholm operator. In particular, its cokernel is finite-dimensional. Let X_1 be the orthogonal complement to $\mathrm{Im}(D\mathcal{F}_0(s_1, f_1))$ in $C^\lambda(\overline{\Omega})$ with respect to the L^2 inner product. X is finite-dimensional and

$$C^\lambda(\overline{\Omega}) = \mathrm{Im}(D\mathcal{F}_0(s_1, f_1)) \oplus X_1.$$

Since $C^\infty(\overline{\Omega})$ is dense as a subset of $C^\lambda(\overline{\Omega})$ with respect to the L^2 norm, we may perturb X_1 to a subspace X_1' of $C^\infty(\overline{\Omega})$ such that $C^\lambda(\overline{\Omega}) = \mathrm{Im}(D\mathcal{F}_0(s_1, f_1)) \oplus X_1'$. Since surjectivity of Fredholm mappings is an open property, there exists a neighbourhood U_1 of (s_1, f_1) in $[0,1] \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that, for all $(s, f) \in U_1$,

$$C^\lambda(\overline{\Omega}) = \mathrm{Im}(D\mathcal{F}_0(s_1, f_1)) \oplus X_1'.$$

By Lemma 3.14, there exist finitely many points $(s_i, f_i)_{1 \leq i \leq n}$ in $[0,1] \times \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that $\mathcal{F}_0^{-1}(\{0\})$ is contained in the union of the collection $(U_i)_{1 \leq i \leq n}$. We therefore choose $X = X_1' + ... + X_n'$, and for all $(s, f) \in \mathcal{F}_0^{-1}(\{0\})$, we obtain,

$$C^\lambda(\overline{\Omega}) = \mathrm{Im}(D\mathcal{F}_0(s_1, f_1)) + X \subseteq \mathrm{Im}(D\mathcal{F}(s_1, 0, f_1)).$$

Since surjectivity of Fredholm mappings is an open property, by Lemma 3.14 again, there exists $r > 0$ such that if $g \in B_r$ and if $(s, g, f) \in \mathcal{Z}_r$, then $D\mathcal{F}(s, g, f)$ is surjective, and this completes the proof. □

Theorem 3.16. *Let $\overline{\Omega}$ be a compact, convex subset of $\mathbb{R}^n$ with smooth boundary and non-trivial interior. Let $\phi \in C^\infty(\overline{\Omega})$ be a smooth, positive function. If there exists a strictly convex function $\hat{f} \in C_0^\infty(\overline{\Omega})$ such that*

$$F(D^2 \hat{f}) > \phi G(D\hat{f}), \qquad \hat{f}|_{\partial\Omega} = 0,$$

then there exists a unique strictly convex function $f \in C_0^\infty(\overline{\Omega})$ such that

$$F(D^2 f) = \phi G(Df), \qquad f|_{\partial\Omega} = 0.$$

Proof. First, if $f, f' \in C_0^\infty(\overline{\Omega})$ are both solutions, then, by Lemma 2.9, both $f - f'$ and $f' - f$ attain their minimum values along the boundary, so that $f = f'$, and uniqueness follows.

For all $t \in [0,1]$, define $G_t := tG + (1-t)$. Now fix $\alpha \in]0,1[$. Since $\hat{f}$ is strictly convex, so too is $\alpha\hat{f}$. Denote $\phi_0 := F(D^2(\alpha\hat{f}))$. Observe that $D^2\hat{f} > D^2(\alpha\hat{f})$, and so $F(D^2\hat{f}) > F(D^2(\alpha\hat{f})) = \phi_0$. For $\delta > 0$ and $t \in [0,1]$, define ϕ_t by

$$\phi_t := \mathrm{Max}((1-t/\delta)\phi_0, (1-(1-t)/\delta)\phi, \delta) > 0.$$

For sufficiently small δ, $\mathcal{F}(t, 0, \hat{f}) > 0$ for all $t \in [0,1]$. In addition, upon perturbing ϕ_t slightly, we may suppose that this function is smooth. By Lemma 3.15, there exists a finite-dimensional subspace $X \subseteq C^\infty(\overline{\Omega})$ and $r > 0$ such that $D\mathcal{F}$ is surjective at every point of $\mathcal{Z}$. Finally, upon reducing r further if necessary, we may suppose in addition that $\mathcal{F}(t, g, \hat{f}) > 0$ for all $(t, g) \in [0,1] \times B_r(0)$.

We define $f_0 = \alpha\hat{f}$. By construction, $f_0|_{\partial\Omega} = 0$, f_0 is strictly convex, and $\mathcal{F}(0,0,f_0) = 0$. By uniqueness, it is the only function with these properties, so that $\Pi_{\mathcal{Z}}^{-1}(\{(0,0)\}) = \{(0,0,f_0)\}$. We now claim that $(0,0)$ is a regular value of $\Pi_{\mathcal{Z}}$. Since $(0,0,f_0)$ is the only element of $\Pi_{\mathcal{Z}}^{-1}(\{(0,0)\})$, it suffices to show that $D\Pi_{\mathcal{Z}}$ is surjective at this point. However, let $L := D_3\mathcal{F}(0,0,f_0)$ be the partial derivative of $\mathcal{F}$ with respect to the third component at $(0,0,f_0)$. We first claim that L is invertible. Indeed, by Lemmas 2.2 and 3.9,

$$Lg = \frac{1}{n}F(D^2 f_0)(D^2 f_0^{-1})^{ij} g_{ij}.$$

By classical elliptic theory, L is Fredholm of index 0. Furthermore, if $g \in \mathrm{Ker}(L)$ then, by the maximum principal, g attains its maximum and minimum values along $\partial\Omega$. However, since g is also an element of $C_0^{\lambda+2}(\overline{\Omega})$, it vanishes along the boundary and therefore vanishes uniquely. The kernel of $D_2\mathcal{F}(0,0,f_0)$ is therefore trivial, and we conclude that L is invertible, as asserted.

Now let (t,g) be any vector in $\mathbb{R} \times X$. By invertibility, there exists $h \in C_0^{\lambda+2}(\overline{\Omega})$ such that $D_2\mathcal{F}(0,0,f_0)h = Lh = -D\mathcal{F}(0,0,f_0)(t,g,0)$. In particular, $D\mathcal{F}(0,0,f_0)(t,g,h) = 0$, so that (t,g,h) is a tangent vector to $\mathcal{Z}$ at $(0,0,f_0)$. However,

$$D\Pi_{\mathcal{Z}}(0,0,f_0)(t,g,h) = (t,g),$$

and since $(t,g) \in \mathbb{R} \times E$ is arbitrary, we conclude that $D\Pi_{\mathcal{Z}}$ is surjective at this point, as desired. It follows that $(0,0)$ is a regular value of $\Pi_{\mathcal{Z}}$ and, in particular, the degree of $\Pi_{\mathcal{Z}}$ is equal to 1 modulo 2. By Theorem 3.4, there exists a sequence $(t_m, g_m)_{m\in\mathbb{N}}$ of regular values of $\Pi_{\mathcal{Z}}$ in $[0,1] \times B_r$ which converges to $(1,0)$. Since the degree of $\Pi_{\mathcal{Z}}$ is nonzero modulo 2, for all m, there exists a function $f_m \in \mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that $\mathcal{F}(t_m, g_m, f_m) = 0$. By Lemma 3.14, $(f_m)_{m\in\mathbb{N}}$ converges to a limit, f_∞, say, in $\mathcal{U}_0^{\lambda+2}(\overline{\Omega})$ such that $\mathcal{F}(1, 0, f_\infty) = 0$. In other words, $f_\infty|_{\partial\Omega} = 0$, f_∞ is strictly convex, and

$$F(D^2 f_\infty) = \phi G(D f_\infty).$$

Finally, by Theorem 3.11, f_∞ is smooth, and this completes the proof. □

Chapter 4

Singularities

We study the singularities that arise in Hausdorff limits of smooth hypersurfaces of constant gaussian curvature. We show that the singular set of any such limit comprises precisely those points which posses the local geodesic property (defined in Section 4.5). This result, which forms the content of Theorem 4.28, immediately yields a global geometric characterisation of the singular set. Indeed, by Theorem 4.18 it is contained in the convex hull of some subset of the boundary.

The analytic content of Theorem 4.28 follows from interior a-priori estimates obtained using a technique dating back to Pogorelov (c.f. [17], but see also [5] and [21]), and which bears some similarities to that already used in Section 2.6 to derive global second-order bounds from second-order bounds along the boundary. However, in order to derive the full geometric consequences of these estimates, we require a thorough understanding of the elementary geometry of convex subsets of euclidean space, and this forms the content of Sections 4.1 to 4.5 inclusive. Although some readers may find these sections elementary, we have attempted to include detailed, and hopefully clear, proofs of certain fundamental results which, to our knowledge, are not readily available elsewhere in the literature. Indeed, in Theorems 4.18 and 4.19, for example, we prove that the local geodesic property characterises convex hulls in euclidean space. Likewise, in Theorem 4.12, we provide a straightforward proof of the well known fact that every convex set with non-trivial interior is locally the graph of a convex, Lipschitz continuous function.

4.1 The Hausdorff topology

Let X and Y be two non-empty compact subsets of $\mathbb{R}^{n+1}$, we recall that $d_H(X,Y)$, the **Hausdorff distance** between X and Y, is defined by

$$d_H(X,Y) := \operatorname*{Sup}_{x\in X}\operatorname*{Inf}_{y\in Y}\|x-y\| + \operatorname*{Sup}_{y\in Y}\operatorname*{Inf}_{x\in X}\|x-y\|.$$

We readily verify that d_H defines a metric on the set of non-empty compact subsets of $\mathbb{R}^{n+1}$.

Lemma 4.1. *Let $K_1 \supseteq K_2 \supseteq \ldots$ be a nested sequence of non-empty, compact subsets of $\mathbb{R}^{n+1}$ and denote*

$$K_\infty := \bigcap_{m\in\mathbb{N}} K_m.$$

Then K_∞ is non-empty and $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense.

Proof. Since K_∞ is the intersection of a countable, nested family of non-empty, compact sets, it is also non-empty and compact. Now suppose that $(K_m)_{m\in\mathbb{N}}$ does not converge to K_∞ in the Hausdorff sense. Upon extracting a subsequence, we may suppose that there exists $\epsilon > 0$ and a sequence $(x_m)_{m\in\mathbb{N}}$ such that for all m, $x_m \in K_m$ and $\|x_m - y\| \geq \epsilon$ for all $y \in K_\infty$. Observe that for all m and for all $n \geqslant m$, $x_n \in K_m$. Thus, by compactness, we may suppose that there exists x_∞ towards which $(x_m)_{m\in\mathbb{N}}$ converges and that $x_\infty \in K_m$ for all m. In particular, $x_\infty \in K_\infty$ and so $\|x_m - x_\infty\| \geqslant \epsilon$ for all m. This is absurd, and the result follows. □

Theorem 4.2. *For all $R > 0$, the set of non-empty, compact subsets of $\mathbb{R}^{n+1}$ contained in $\overline{B}_R(0)$ is compact in the Hausdorff topology.*

Remark. *We leave the reader to verify that the result generalises to the set of non-empty, compact subsets of any given compact metric space.*

Proof. Choose $R > 0$ and let $(X_m)_{m\in\mathbb{N}} \subseteq \overline{B}_R(0)$ be a sequence of non-empty, compact sets. For all $k \in \mathbb{N}$, denote by $Q_k \subseteq \mathbb{R}^{n+1}$ the closed cube of side length $1/2^k$ based on the origin. That is,

$$Q_k := [0, 2^{-k}]^{n+1}.$$

For every vector $\alpha \in \mathbb{Z}^{n+1}$, define

$$Q_{k,\alpha} := Q_k + 2^{-k}\alpha.$$

For all m and for all k, define

$$X_{m,k} := \bigcup_{Q_{k,\alpha}\cap X_m \neq \emptyset} Q_{k,\alpha}.$$

For all k, the sequence $(X_{m,k})_{m\in\mathbb{N}}$ contains a subsequence converging in the Hausdorff sense to a compact limit, $X_{\infty,k}$, say, in $\mathbb{R}^{n+1}$. By a diagonal argument, we may suppose that, for all k, the whole sequence $(X_{m,k})_{m\in\mathbb{N}}$ converges to this limit. Define

$$X_\infty := \bigcap_{k\in\mathbb{N}} X_{\infty,k}.$$

For all k and for all m, $X_{m,k+1} \subseteq X_{m,k}$. Thus, upon taking limits, we obtain $X_{\infty,k+1} \subseteq X_{\infty,k}$, and it follows by Lemma 4.1 that X_∞ is non-empty and compact and that $(X_{\infty,k})_{k\in\mathbb{N}}$ converges to X_∞ in the Hausdorff sense.

It remains to show that $(X_m)_{m\in\mathbb{N}}$ converges to X_∞ in the Hausdorff sense. However, for all k and for all m,

$$d_H(X_m, X_{m,k}) \leq 2^{-k}\sqrt{n+1}.$$

Likewise, for all $k \leq l$ and for all m,

$$d_H(X_{m,k}, X_{m,l}) \leq 2^{-k}\sqrt{n+1}.$$

Taking limits yields, for all $k \leq l$,

$$d_H(X_{\infty,k}, X_{\infty,l}) \leq 2^{-k}\sqrt{n+1}.$$

Letting l tend to infinity yields, for all k,

$$d_H(X_{\infty,k}, X_\infty) \leq 2^{-k}\sqrt{n+1}.$$

Now choose $\delta > 0$ and $k > 0$ such that $2^{-k}\sqrt{n+1} < \delta/3$ and let $M > 0$ be such that for $m \geq M$, $d_H(X_{m,k}, X_{\infty,k}) < \delta/3$. Then, for $m \geq M$,

$$d_H(X_m, X_\infty) \leq d_H(X_m, X_{m,k}) + d_H(X_{m,k}, X_{\infty,k}) + d_H(X_{\infty,k}, X_\infty) < \delta.$$

Since δ may be chosen arbitrarily small, we conclude that $(X_m)_{m\in\mathbb{N}}$ converges to X_∞ in the Hausdorff sense, as desired. □

Lemma 4.3. *For all $R > 0$, the set of non-empty, compact, convex subsets of $\overline{B}_R(0)$ is a closed subset of the set of compact subsets of $\overline{B}_R(0)$ with respect to the Hausdorff topology. In particular, this set is also compact in the Hausdorff topology.*

Proof. Let $(K_m)_{m\in\mathbb{N}}$ be a sequence of non-empty, compact, convex subsets of $\overline{B}_R(0)$ converging to a compact limit $K_\infty \subseteq \overline{B}_R(0)$ in the Hausdorff sense. Choose two points $x_\infty, y_\infty \in K_\infty$. There exist sequences $(x_m)_{m\in\mathbb{N}}$ and $(y_m)_{m\in\mathbb{N}}$ converging to x_∞ and y_∞ respectively such that for all m, the points x_m and y_m are elements of K_m. Choose $t \in [0,1]$. By convexity, for all m, $(1-t)x_m + ty_m \in K_m$ and taking limits yields $(1-t)x_\infty + ty_\infty \in K_\infty$. Since $x_\infty, y_\infty \in K_\infty$ and $t \in [0,1]$ are arbitary, we conclude that K_∞ is convex, as desired. □

4.2 Supporting normals

Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere. We recall that if K is a convex subset of $\mathbb{R}^{n+1}$, if x is a point of K, and if N is a vector in Σ^n, then N is said to be a **supporting normal** to K at x whenever every other $y \in K$ satisfies

$$\langle y - x, \mathsf{N} \rangle \leq 0.$$

Observe that supporting normals only exist at boundary points of K. However, when they do exist, they need not be unique. Hence, for any boundary point x of K, we denote its set of supporting normals by $\mathcal{N}(x; K)$, and when there is no ambiguity concerning K, we denote this set merely by $\mathcal{N}(x)$. We now show that $\mathcal{N}(x)$ is non-empty and compact for any boundary point x of K, and, moreover, that this set varies semi-continuously with x in a sense that will be made clear presently. We first prove a straightforward closure result.

Lemma 4.4. *Let $(K_m)_{m\in\mathbb{N}}$ and K_∞ be compact, convex subsets of $\mathbb{R}^{n+1}$ such that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. For all finite m, let x_m be a boundary point of K_m and let N_m be a supporting normal to K_m at x_m. If $(x_m)_{m\in\mathbb{N}}$ and $(\mathsf{N}_m)_{m\in\mathbb{N}}$ converge to x_∞ and N_∞ respectively, then x_∞ is a boundary point of K_∞ and N_∞ is a supporting normal to K_∞ at x_∞.*

Proof. Indeed, choose $y \in K_\infty$. There exists a sequence $(y_m)_{m\in\mathbb{N}}$ in $\mathbb{R}^{n+1}$ converging to y such that y_m is an element of K_m for all m. For all m, since N_m is a supporting normal to K_m at x_m,

$$\langle y_m - x_m, \mathsf{N}_m \rangle \leq 0.$$

Taking limits therefore yields

$$\langle y - x_\infty, \mathsf{N}_\infty \rangle \leq 0.$$

Since $y \in K_\infty$ is arbitrary, we conclude that $x_\infty \in \partial K_\infty$ and that N_∞ is a supporting normal to K_∞ at x_∞, as desired. □

Corollary 4.5. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. If x is a boundary point of K then $\mathcal{N}(x)$ is compact.*

The supporting normals characterise the closest point to any exterior point of a given convex set in the following sense.

Lemma 4.6. *Let K be a closed, convex subset of $\mathbb{R}^{n+1}$. Let x be a point in the complement of K. Then $y \in K$ minimises distance to x if and only if y is a boundary point of K and $(x-y)/\|x-y\|$ is a supporting normal to K at y.*

Proof. Suppose that y is a boundary point and that $(x-y)/\|x-y\|$ is a supporting normal to K at y. Then, for all other $z \in K$,

$$\begin{aligned}\|z-x\|^2 &= \|(z-y)-(x-y)\|^2\\ &= \|z-y\|^2 - 2\langle z-y, x-y\rangle + \|y-x\|^2\\ &\geq \|y-x\|^2.\end{aligned}$$

Since $z \in K$ is arbitrary, we conclude that y minimises distance to x in K, as desired.

Conversely, suppose that $y \in K$ minimises distance to x. We claim that $\langle z-y, x-y\rangle \leq 0$ for all other $z \in K$. Indeed, suppose the contrary, so that there exists $z \in K$ such that $\langle z-y, x-y\rangle > 0$. For all $t \in [0,1]$, denote $z_t = (1-t)y + tz$. By convexity, $z_t \in K$ for all t. However

$$\partial_t \|z_t - x\|^2|_{t=0} = 2\langle z-y, y-x\rangle < 0.$$

Thus, for sufficiently small t, $\|z_t - x\|^2 < \|y-x\|^2$, which is absurd, and the assertion follows. We conclude that y is a boundary point of K and that $(x-y)/\|x-y\|$ is a supporting normal to K at y, as desired. □

Theorem 4.7. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. For every boundary point x of K there exists a supporting normal N to K at x. That is, for all $x \in \partial K$, $\mathcal{N}(x) \neq \emptyset$.*

Proof. Indeed, choose $x \in \partial K$. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence of points in the complement of K converging to x. By compactness, for all m, there exists a point y_m in K minimising distance to x_m. In particular, for all m, $d(x_m, y_m) \leq d(x_m, x)$ and so

$$d(x, y_m) \leq d(x, x_m) + d(x_m, y_m) \leq 2d(x_m, x),$$

so that $(y_m)_{m\in\mathbb{N}}$ therefore also converges to x. For all m, we define $\mathsf{N}_m \in \Sigma^n$ by

$$\mathsf{N}_m := \frac{x_m - y_m}{\|x_m - y_m\|}.$$

By Lemma 4.6, for all m, y_m is a boundary point of K and N_m is a supporting normal to K at y_m. After extracting a subsequence we may suppose that $(\mathsf{N}_m)_{m\in\mathbb{N}}$ converges to a unit vector N, say. By Lemma 4.4, N is a supporting normal to K at x, as desired. □

We conclude by studying various properties of $\mathcal{N}(x)$. We first examine the semi-continuous dependence of these sets on x. Thus, for two non-empty subsets X and Y of Σ^n, we define

$$\delta(X,Y) := \underset{y\in Y}{\mathrm{Sup}}\ \underset{x\in X}{\mathrm{Inf}}\ d_\Sigma(x,y) = \underset{y\in Y}{\mathrm{Sup}}\ d_\Sigma(X,y).$$

Observe, that δ is not symmetric. However, by definition,

$$d_{H,\Sigma}(X,Y)=\delta(X,Y)+\delta(Y,X)\geq\delta(X,Y).$$

In particular, δ is continuous with respect to Hausdorff distance in the sphere.

Lemma 4.8. *Let $(K_m)_{m\in\mathbb{N}}$ and K_∞ be compact, convex subsets of $\mathbb{R}^{n+1}$ such that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. For all m, let x_m be a boundary point of K_m, and suppose that the sequence $(x_m)_{m\in\mathbb{N}}$ converges to x_∞, say, in K_∞. For all $\epsilon>0$ there exists $M\in\mathbb{N}$ such that for $m\geq M$, $\delta(\mathcal{N}(x_\infty),\mathcal{N}(x_m))<\epsilon$.*

Proof. Suppose the contrary. Upon extracting a subsequence, we may suppose that there exists $\epsilon>0$ such that $\delta(\mathcal{N}(x_\infty),\mathcal{N}(x_m))\geq\epsilon$ for all m. For all m, there therefore exists $\mathsf{N}_m\in\mathcal{N}(x_m)$ such that $d(\mathcal{N}(x_\infty),\mathsf{N}_m)\geq\epsilon$. By compactness of the sphere, we may suppose that there exists N_∞ towards which $(\mathsf{N}_m)_{m\in\mathbb{N}}$ converges, and taking limits yields $d(\mathcal{N}(x_\infty),\mathsf{N}_\infty)\geq\epsilon$. However, by Lemma 4.4, $\mathsf{N}_\infty\in\mathcal{N}(x_\infty)$. This is absurd, and the result follows. □

We now show that $\mathcal{N}(x)$ is in fact defined locally. Indeed, recall that if K and L are compact, convex sets, then so too is their intersection.

Lemma 4.9. *Let K and L be compact, convex subsets of $\mathbb{R}^{n+1}$. Let x be a boundary point of $K\cap L$ and let $\mathsf{N}\in\Sigma^n$ be a supporting normal to $K\cap L$ at this point. If $x\in L^o$, then x is also a boundary point of K and N is also a supporting normal to K at x. That is, for all such x, $\mathcal{N}(x;K)=\mathcal{N}(x;K\cap L)$.*

Proof. It suffices to show that for all $y\in K$,

$$\langle y-x,\mathsf{N}\rangle\leq 0.$$

However, suppose the contrary. There exists $y\in K$ such that $\langle y-x,\mathsf{N}\rangle>0$. For all $t\in[0,1]$, denote $y_t:=(1-t)x+ty$. Then, for all t,

$$\langle y_t-x,\mathsf{N}\rangle=t\langle y-x,\mathsf{N}\rangle>0.$$

However, by convexity, y_t is an element of K for all t. Furthermore, since x is an interior point of L, for sufficiently small t, y_t is also an element of L. That is, for sufficiently small t, y_t is an element of $K\cap L$, so that $\langle y_t-x,\mathsf{N}\rangle\leq 0$. This is absurd, and the result follows. □

Finally, we include non-compact convex subsets of $\mathbb{R}^{n+1}$ into this framework.

Lemma 4.10. *Let K be a closed, convex subset of $\mathbb{R}^{n+1}$. For every boundary point x of K, there exists a supporting normal N to K at x. That is, for all $x\in\partial K$, $\mathcal{N}(x)\neq\emptyset$.*

Proof. Indeed, let x be a boundary point of K. For all $r > 0$, denote $K_r := K \cap \overline{B}_r(x)$, and observe that K_r is compact and convex. Now fix $r > 0$. Trivially, $\mathcal{N}(x;K) \subseteq \mathcal{N}(x;K_r)$. Conversely, by Lemma 4.9, for all $s > r$, $\mathcal{N}(x;K_s) = \mathcal{N}(x;K_r)$. In particular, if $\mathsf{N} \in \mathcal{N}(x,K_r)$ and if $y \in K$, then since $y \in K_s$ for some $s > r$, $\langle y-x, \mathsf{N}\rangle \leq 0$. We conclude that $\mathcal{N}(x;K_r) \subseteq \mathcal{N}(x;K)$, and the two sets therefore coincide. In particular, by Theorem 4.7, $\mathcal{N}(x;K) = \mathcal{N}(x;K_r)$ is non-empty, and the result follows. □

4.3 Convex sets as graphs

Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a boundary point of K and let N be a supporting normal to K at x. Upon applying an affine isometry, we may suppose that $x = 0$ and that N is any given unit vector in the sphere so that the results which follow are completely general. We decompose $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$ and we use the notation outlined in the appendix. For $C, r > 0$, we say that ∂K is a C-Lipschitz **graph** over a radius r near 0 whenever there exists a C-Lipschitz function $f : B'_r(0) \to]-2Cr, 2Cr[$ such that the intersection of ∂K with $B'_r(0) \times]-2Cr, 2Cr[$ coincides with the graph of f over $B'_r(0)$.

Lemma 4.11. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ and suppose that 0 is a boundary point of K. Choose $\theta \in [0, \pi/2[$ and $r > 0$ and suppose that for all $x \in \partial K \cap B_r(0)$, and for every supporting normal N to K at x,*

$$\langle \mathsf{N}, -e_{n+1}\rangle \geq \cos(\theta).$$

Then, for all $(x', s), (y', t) \in \partial K \cap B_r(0)$,

$$|s - t| \leq \tan(\theta)\|x' - y'\|.$$

Proof. Indeed, let N be a supporting normal to K at x and let N' be its orthogonal projection onto $\mathbb{R}^n$. In particular,

$$\|\mathsf{N}'\|^2 = 1 - \langle \mathsf{N}, e_{n+1}\rangle^2 \leq 1 - \cos^2(\theta) = \sin^2(\theta).$$

Now denote $x := (x', s)$ and $y := (y', t)$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\langle y - x, \mathsf{N}\rangle &= \langle y - x, \langle \mathsf{N}, e_{n+1}\rangle e_{n+1}\rangle + \langle y - x, \mathsf{N}'\rangle \\ &= \langle \mathsf{N}, -e_{n+1}\rangle (s - t) + \langle y' - x', \mathsf{N}'\rangle \\ &\geq \langle \mathsf{N}, -e_{n+1}\rangle (s - t) - \|x' - y'\| \sin(\theta).\end{aligned}$$

However, by definition of the supporting normal, $\langle y - x, \mathsf{N}\rangle \leq 0$, and so

$$(s - t) \leq \|x' - y'\| \frac{\sin(\theta)}{\langle \mathsf{N}, -e_{n+1}\rangle} \leq \tan(\theta)\|x' - y'\|.$$

By symmetry, we conclude that

$$|s-t| \leq \tan(\theta)\|x'-y'\|,$$

as desired. ☐

Theorem 4.12. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ and suppose that 0 is a boundary point of K. Choose $\theta \in [0,\pi/2[$ and $r>0$ and suppose that for all $x \in \partial K \cap B_r(0)$, and for every supporting normal N to K at x,*

$$\langle \mathsf{N}, -e_{n+1}\rangle \geq \cos(\theta).$$

Then, denoting $C := \tan(\theta)$ and $\rho := \frac{r}{\sqrt{1+4C^2}}$, there exists a unique function $f : B'_\rho(0) \to]-C\rho, C\rho[$ such that

(1) $f(0)=0$;

(2) f is convex and C-Lipschitz; and

(3) $(\partial K)\cap(B'_\rho(0)\times]-2C\rho,2C\rho[)$ coincides with the graph of f.

Remark. *In other words, ∂K is a C-Lipschitz graph over a radius ρ near 0.*

Proof. Observe that $B'_\rho(0)\times]-2C\rho,2C\rho[\subseteq B_r(0)$. For all $x'\in B'_\rho(0)$, denote $L_{x'} := \{x'\}\times]-2C\rho,2C\rho[$ and consider the set $L_{x'}\cap\partial K$. First, if $s,t\in]-2C\rho,2C\rho[$ are such that $(x',s),(x',t)\in\partial K$, then, by Lemma 4.11, $|s-t|=0$, and so $s=t$. It follows that $L_{x'}\cap\partial K$ contains at most one point.

We now prove existence. For all $t\in]-2C\rho,2C\rho[$, denote $B'_t := B'_\rho(0)\times\{t\}$. We first claim that $B'_{\pm C\rho}$ does not intersect ∂K. Indeed, otherwise, there exists $x'\in B'_\rho(0)$ such that $(x',\pm C\rho)\in\partial K$. However, since $(0,0)\in\partial K$, by Lemma 4.11,

$$C\rho \leq C\|x'\| < C\rho.$$

This is absurd, and the assertion follows. In particular, by connectedness, $B'_{\pm C\rho}$ is entirely contained either in the interior of K, or in $\mathbb{R}^{n+1}\setminus K$. However, for any supporting normal N to K at $(0,0)$,

$$\langle(0,-C\rho)-(0,0),\mathsf{N}\rangle = 2C\rho\langle -e_{n+1},\mathsf{N}\rangle \geq 2C\rho\cos(\theta) > 0,$$

so that the point $(0,-C\rho)$ does not lie in K. In particular, $B'_{-C\rho}$ intersects $\mathbb{R}^{n+1}\setminus K$ non-trivially, and is therefore entirely contained in $\mathbb{R}^{n+1}\setminus K$.

We now show that $B'_{C\rho}$ is entirely contained in the interior of K. By hypothesis, $(0,0)$ lies in $L_0\cap\partial K$, and, by uniqueness, $L_0\cap\partial K$ contains no other point. However, by compactness and convexity, $L_0\cap K$ is a relatively closed, connected subset of L_0. Since, furthermore, the relative boundary of $L_0 \cap K$ in L_0 is contained in $L_0 \cap \partial K$, it follows that $L_0 \cap K$ coincides with one of $\{(0,0)\}$, $\{0\} \times [0, 2C\rho]$, $\{0\} \times [-2C\rho, 0]$ or $\{0\} \times]-2C\rho, 2C\rho[$. The last two are excluded since $(0, -C\rho) \notin K$ and it thus remains to show that $L_0 \cap K \neq \{(0,0)\}$. However, suppose the contrary. For $0 < \delta < \mathrm{Min}(1, C)\rho$, the point $x := (0, \delta)$ does not lie in K. Let $y := (y', t)$ be the closest point in K to x. By Lemma 4.6, $\mathsf{N} := (x - y)/\|x - y\|$ is a supporting normal to K at x. However, $t \leq \delta$, since otherwise, by convexity, the point $(\delta y'/t, \delta)$ also lies in K, but is closer to x than y. In particular, $\langle \mathsf{N}, -e_{n+1} \rangle = t - \delta < 0$. This is absurd, and we conclude that $L_0 \cap K$ coincides with $\{0\} \times [0, 2C\rho]$. In particular, the set $B'_{C\rho}$ intersects K non-trivially, and is therefore entirely contained in the interior of K, as desired.

Since $B'_{-C\rho}$ is contained in $\mathbb{R}^{n+1} \setminus K$ and since $B'_{C\rho}$ is contained in the interior of K, it follows that for all $x' \in B'_\rho(0)$, there exists a unique point $f(x') \in]-C\rho, C\rho[$ such that $(x', f(x')) \in \partial K$. In particular, by Lemma 4.11, for all $x', y' \in B'_\rho(0)$,

$$|f(x') - f(y')| \leq C\|x' - y'\|,$$

so that the function f is C-Lipschitz.

Finally, denote $\mathrm{Cyl} := B'_\rho(0) \times]-2C\rho, 2C\rho[$ and $\hat{K} := K \cap \mathrm{Cyl}$, and denote the graph of f by $\mathrm{Gr}(f)$. Observe that $\mathrm{Cyl} \setminus \mathrm{Gr}(f)$ consists of two connected components. Furthermore, since $\partial K \cap \mathrm{Cyl} = \mathrm{Gr}(f)$, the set $K \cap (\mathrm{Cyl} \setminus \mathrm{Gr}(f))$ is both open and closed in $\mathrm{Cyl} \setminus \mathrm{Gr}(f)$. Thus, since $B_{C\rho} \subseteq \hat{K}$ and $B_{-C\rho} \subseteq \mathrm{Cyl} \setminus \hat{K}$, it follows that $\hat{K}$ coincides with the closure of the connected component of $\mathrm{Cyl} \setminus \mathrm{Gr}(f)$ lying above $\mathrm{Gr}(f)$. That is,

$$\hat{K} = \{(x', t) \mid t \geq f(x')\}.$$

Now choose $x', y' \in B'_r(0)$ and, for all $s \in [0,1]$, denote $x'_s := (1-s)x' + sy'$ and $t_s := (1-s)f(x') + sf(y')$. By convexity, since $(x', f(x'))$ and $(y', f(y'))$ are both elements of $\hat{K}$, for all $s \in [0,1]$, so too is (x'_s, t_s), so that

$$f((1-s)x' + sy') \leq (1-s)f(x') + sf(y').$$

Since $x', y' \in B'_r(0)$ are arbitrary, we conclude that f is convex, and this completes the proof. $\square$

4.4 Convex hulls

Let X be a subset of $\mathbb{R}^{n+1}$. We define the **convex hull** of X to be the intersection of all open, convex subsets of $\mathbb{R}^{n+1}$ containing X. We denote this set by $\mathrm{Conv}(X)$. Observe, in particular, that $\mathrm{Conv}(X)$ is also convex.

Lemma 4.13. *If K is a convex set, then $\overline{K}$ and K^o are also convex.*

Proof. Choose $x, y \in \overline{K}$. Let $(x_m)_{m\in\mathbb{N}}$, $(y_m)_{m\in\mathbb{N}}$ be sequences of points in K converging to x and y respectively. Choose $t \in [0,1]$. By convexity, $(1-t)x_m+ty_m \in K$ for all m. Taking limits, it follows that $(1-t)x+ty \in \overline{K}$. Since $x, y \in \overline{K}$ and $t \in [0,1]$ are arbitrary, we conclude that $\overline{K}$ is convex, as desired.

Now choose $x, y \in K^o$, choose $\delta > 0$ such that $B_\delta(x), B_\delta(y) \subseteq K$ and choose $t \in [0,1]$. By convexity, for all $z \in B_\delta(0)$,

$$(1-t)x + ty + z = (1-t)(x+z) + t(y+z) \in K.$$

It follows that $B_\delta((1-t)x+ty) \subseteq K$ and so $(1-t)x+ty \in K^o$. Since $x, y \in K^o$ and $t \in [0,1]$ are arbitrary, we conclude that K^o is convex, as desired. □

Lemma 4.14. *If X is compact, then* $\mathrm{Conv}(X)$ *is compact.*

Proof. We first show that $\partial\mathrm{Conv}(X) \subseteq \mathrm{Conv}(X)$. Indeed, suppose the contrary, and choose $x \in \partial\mathrm{Conv}(X) \setminus \mathrm{Conv}(X)$. By definition, there exists an open, convex set K such that $X \subseteq K$ but $x \notin K$. However, since $\mathrm{Conv}(X) \subseteq K$, in particular, $\partial\mathrm{Conv}(X) \subseteq \overline{K}$, so that $x \in \partial K$. By Lemma 4.13, $\overline{K}$ is convex, and so, by Lemma 4.10, there exists a supporting normal N to $\overline{K}$ at x. By definition of supporting normals, for all $y \in X \subseteq K \subseteq \overline{K}$, $\langle y-x, \mathsf{N}\rangle < 0$. Thus, by compactness, there exists $\delta > 0$ such that for all $y \in X$, $\langle y-x, \mathsf{N}\rangle < -\delta$. Now define

$$K' := \{y \in K \mid \langle y-x, \mathsf{N}\rangle < -\delta\}.$$

Since K' is open and convex, and since $X \subseteq K'$, it follows that $\mathrm{Conv}(X) \subseteq K'$. However, $x \notin \overline{K}' \supseteq \partial\mathrm{Conv}(X)$. This is absurd, and it follows that $\partial\mathrm{Conv}(X) \subseteq \mathrm{Conv}(X)$, as desired. In particular, $\mathrm{Conv}(X)$ is closed. Finally, since X is bounded, there exists $R > 0$ such that $X \subseteq B_R(0)$. Since $B_R(0)$ is open and convex, it follows that $\mathrm{Conv}(X) \subseteq B_R(0)$, and is therefore also bounded. It follows by the Heine-Borel Theorem that $\mathrm{Conv}(X)$ is compact, as desired. □

Lemma 4.15. *If X is open, then* $\mathrm{Conv}(X)$ *is open.*

Proof. Choose $x \in \partial\mathrm{Conv}(X)$. By Lemma 4.13, $\overline{\mathrm{Conv}}(X)$ is convex. By Lemma 4.10, there exists a supporting normal N to $\overline{\mathrm{Conv}}(X)$ at x. Since X is open, for all $y \in X \subseteq \mathrm{Conv}(X) \subseteq \overline{\mathrm{Conv}}(X)$, $\langle y-x, \mathsf{N}\rangle < 0$. Thus, if we define

$$K := \left\{y \in \mathbb{R}^{n+1} \mid \langle y-x, \mathsf{N}\rangle < 0\right\},$$

then $X \subseteq K$ and since K is open and convex, $\mathrm{Conv}(X) \subseteq K$. In particular, $x \in \mathbb{R}^{n+1} \setminus K \subseteq \mathbb{R}^{n+1} \setminus \mathrm{Conv}(X)$, and since $x \in \partial\mathrm{Conv}(X)$ is arbitrary, we conclude that $\partial\mathrm{Conv}(X) \cap \mathrm{Conv}(X) = \emptyset$, so that $\mathrm{Conv}(X)$ is open, as desired. □

Lemma 4.16. *If K is compact and convex, then* $\mathrm{Conv}(K) = K$.

Remark. *Observe that the analogous result for open convex sets follows immediately from our definition of the convex hull.*

Proof. Choose $x \in \mathbb{R}^{n+1} \setminus K$. Let $y \in K$ be a point minimising distance to x, and define $\mathsf{N} = (x-y)/\|x-y\|$. By Lemma 4.6, N is a supporting normal to K at y. By definition of supporting normals, for all $z \in K$, $\langle z-y, \mathsf{N}\rangle \leq 0$, and so

$$\langle z-x, \mathsf{N}\rangle = \langle (z-y)+(y-x), \mathsf{N}\rangle \leq -\|x-y\| < 0.$$

Define

$$K' := \left\{ z \in \mathbb{R}^{n+1} \mid \langle z-x, \mathsf{N}\rangle < 0 \right\}.$$

Since K' is open and convex and since $K \subseteq K'$, it follows that $\mathrm{Conv}(K) \subseteq K'$. In particular, $x \notin \mathrm{Conv}(K)$. Since $x \in \mathbb{R}^{n+1} \setminus K$ is arbitrary, we conclude that $\mathbb{R}^{n+1} \setminus K \subseteq \mathbb{R}^{n+1} \setminus \mathrm{Conv}(K)$, so that $\mathrm{Conv}(K) \subseteq K$, and since K is trivially contained in $\mathrm{Conv}(K)$, we conclude that the two sets coincide, as desired. □

4.5 The local geodesic property

We define an open straight-line segment in $\mathbb{R}^{n+1}$ to be any set Γ of the form

$$\Gamma = \{x + ty \mid a < t < b\},$$

where x is a point in $\mathbb{R}^{n+1}$, y is a non-zero vector and $a < b$ are real numbers. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ and let x be any point of K. We say that K satisfies the **local geodesic property** at x whenever there exists an open straight-line segment Γ such that $x \in \Gamma \subseteq K$. Every interior point of K trivially satisfies the local geodesic property.

Lemma 4.17. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a boundary point of K and let Γ be an open straight-line segment such that $x \in \Gamma \subseteq K$. If N is a supporting normal to K at x then Γ is contained in the hyperplane passing through x normal to N. In particular, Γ is contained in the boundary of K.*

Proof. By definition, there exists $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and real numbers $a < 0 < b$ such that

$$\Gamma = \{x + ty \mid a < t < b\}.$$

By definition of supporting normals, for all $t \in]a, b[$,

$$t\langle y, \mathsf{N}\rangle = \langle (x+ty) - x, \mathsf{N}\rangle \leq 0.$$

It follows that $\langle y, \mathsf{N}\rangle = 0$, and so $\langle (x+ty)-x, \mathsf{N}\rangle = 0$ for all $t \in]a,b[$ as desired. $\square$

The local geodesic property characterises convex hulls in the following sense.

Theorem 4.18. *Let K be a compact, convex set, let X be a subset of ∂K, and let Y be the set of all points of ∂K satisfying the local geodesic property. If $X \cup Y$ is closed, then $Y \subseteq \mathrm{Conv}(X)$.*

Proof. We prove this by induction on the dimension of the ambient space. First suppose that $n = 1$. In particular, $K =: [a,b]$ is a compact interval. We claim that $\{a,b\} \subseteq X$. Indeed, observe that $]a,b[\subseteq Y$. Thus, since $X \cup Y$ is closed, it follows that $[a,b] \subseteq X \cup Y$. Since neither a nor b is an element of Y, it follows that $\{a,b\} \subseteq X$, as asserted. In particular, $K = [a,b] = \mathrm{Conv}(X)$, as desired.

Now consider an ambient space of arbitrary dimension greater than 1. Choose $y \in Y$ and let H be a supporting hyperplane to K at y. Denote $K' := K \cap H$, $Y' := Y \cap \partial K'$ and $X' := X \cap \partial K'$. Observe that K' is a compact, convex subset of H and that $X' \cup Y'$ is closed. We claim that Y' coincides with the set of all boundary points of K' satisfying the local geodesic property. Indeed, if $z \in Y'$, then there exists an open straight-line segment Γ such that $z \in \Gamma \subseteq K$. Since H is a supporting hyperplane to K, by Lemma 4.17, $\Gamma \subseteq H$. In particular, $\Gamma \subseteq K'$ and so K' also satisfies the local geodesic property at z. Conversely, if z is a boundary point of K' and if K' satisfies the local geodesic property at z, then z is also a boundary point of K and K also satisfies the local geodesic property at z. The assertion follows and we conclude by the inductive hypothesis that $Y' \subseteq \mathrm{Conv}(X') \subseteq \mathrm{Conv}(X)$.

If $y \in Y'$, then $y \in \mathrm{Conv}(X)$, and we are done. Otherwise, suppose that $y \in Y \setminus Y'$. That is, y lies in the interior of K'. Let V be any vector in H. Define $\gamma : \mathbb{R} \to H$ by $\gamma(t) = y + tV$. Denote $I = \gamma^{-1}(K')$. Since K' is compact and convex, I is a compact interval. Trivially, for all $t \in I^o$, K satisfies the local geodesic property at $\gamma(t)$. That is, $I^o \subseteq \gamma^{-1}(Y) \subseteq \gamma^{-1}(Y \cup X)$, so that taking closures yields, $I \subseteq \gamma^{-1}(Y \cup X)$. Since, in addition, $\partial I \subseteq \gamma^{-1}(\partial K')$, we have $\partial I \subseteq \gamma^{-1}(Y' \cup X') \subseteq \gamma^{-1}(\mathrm{Conv}(X'))$. It follows that $y \in \mathrm{Conv}(X')$ and since $y \in Y$ is arbitrary, we conclude that $Y' \subseteq \mathrm{Conv}(X')$, and the result now follows by induction. $\square$

Conversely, we have

Theorem 4.19. *If X is a compact subset of $\mathbb{R}^{n+1}$, then $\mathrm{Conv}(X)$ satisfies the local geodesic property at every point of $\mathrm{Conv}(X) \setminus X$.*

Proof. We prove this by induction on the dimension. The result trivially holds when the ambient space is 1-dimensional. Now choose $x \in \mathrm{Conv}(X) \setminus X$. If x is an interior point of $\mathrm{Conv}(X)$, then we are done. We therefore assume that x is a boundary point of $\mathrm{Conv}(X)$. By Lemma 4.7, there exists a supporting normal N to $\mathrm{Conv}(X)$ at x. Let H be the hyperplane normal to N passing through x, and denote $X' := H \cap X$. We claim that $\mathrm{Conv}(X') = \mathrm{Conv}(X) \cap H$. Indeed, since X' is compact, by Lemma 4.14, so too is $\mathrm{Conv}(X')$. Choose $x' \in H \setminus \mathrm{Conv}(X')$ and let $y' \in H$ be a point in $\mathrm{Conv}(X')$ minimising distance to x'. Denote $\mathsf{N}' := (x' - y')/\|x' - y'\|$. By Lemma 4.6, N' is a supporting normal to $\mathrm{Conv}(X')$ at y'. In particular, for all $z' \in X' \subseteq \mathrm{Conv}(X')$,

$$\langle z' - y', \mathsf{N}' \rangle \leq 0,$$

so that

$$\langle z' - x', \mathsf{N}' \rangle = \langle (z' - y') + (y' - x'), \mathsf{N}' \rangle \leq -\|x' - y'\| < 0.$$

However, by definition of N and H, for all $z \in X \setminus X' = X \setminus H \subseteq \mathrm{Conv}(X) \setminus H$,

$$\langle z - x', \mathsf{N} \rangle = \langle z - x, \mathsf{N} \rangle + \langle x - x', \mathsf{N} \rangle < 0.$$

Combining these relations yields, for sufficiently large λ and for all $z \in X$,

$$\langle z - x', (\lambda \mathsf{N} + \mathsf{N}') \rangle < 0.$$

Define $K \subseteq \mathbb{R}^{n+1}$ by

$$K := \{ z \mid \langle z - x', (\lambda \mathsf{N} + \mathsf{N}') \rangle < 0 \}.$$

Since K is open and convex and since $X \subseteq K$, it follows that $\mathrm{Conv}(X) \subseteq K$. In particular, since x' is not an element of K, it is not an element of $\mathrm{Conv}(X)$ either. Since $x' \in H \setminus \mathrm{Conv}(X')$ is arbitrary, we conclude that $\mathrm{Conv}(X) \cap H \subseteq \mathrm{Conv}(X')$. Conversely, let K be an open, convex set containing X. Then $K \cap H$ is also convex and relatively open. Since $K \cap H$ contains X', by definition, it also contains $\mathrm{Conv}(X')$. Upon taking the intersection over all such open sets, we conclude that $\mathrm{Conv}(X') \subseteq \mathrm{Conv}(X) \cap H$, and the two sets therefore coincide as asserted. It follows by the inductive hypothesis that $\mathrm{Conv}(X) \cap H = \mathrm{Conv}(X')$ satisfies the local geodesic property at x and therefore so too does $\mathrm{Conv}(X)$. This completes the proof. $\square$

We introduce an alternative characterisation of the local geodesic property which will be of use in the sequel. Let $X \subseteq \Sigma^n$ be any closed subset. We say that X is **strictly contained in a hemisphere** whenever there exists a unit vector $\mathsf{N} \in \Sigma^n$ such that for all $\mathsf{N}' \in X$,

$$\langle \mathsf{N}, \mathsf{N}' \rangle < 0.$$

Lemma 4.20. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a point in K and suppose that K satisfies the local geodesic property at x. Then, for all sufficiently small $r > 0$, $K \cap \partial B_r(x)$ is not strictly contained in a hemisphere.*

Proof. Let Γ be the open straight-line segment passing through x contained in K. By definition, there exists a non-zero vector $y \in \mathbb{R}^{n+1}$ and real numbers $a < 0 < b$ such that

$$\Gamma = \{x + ty \; | a < t < b\} .$$

Choose $r < \mathrm{Min}(-a, b)$. Then $x \pm ry \in K \cap \partial B_r(x)$. In particular, if there exists $\mathsf{N} \in \Sigma^n$ such that $\langle (x \pm ry) - x, \mathsf{N}\rangle < 0$, then, $\pm\langle y, \mathsf{N}\rangle < 0$. This is absurd, and the assertion follows. □

Lemma 4.21. *If $X \subseteq \Sigma^n$ is a closed subset not strictly contained in a hemisphere, then 0 is an element of $\mathrm{Conv}(X)$.*

Proof. Suppose the contrary. By Lemma 4.14, $\mathrm{Conv}(X)$ is compact. Let $x \in \mathrm{Conv}(X)$ be the point minimising distance in $\mathrm{Conv}(X)$ to 0 and denote $\mathsf{N} := -x/\|x\|$. By Lemma 4.6, N is a supporting normal to $\mathrm{Conv}(X)$ at x. Thus, for all $y \in X \subseteq \mathrm{Conv}(X)$, $\langle y - x, \mathsf{N}\rangle \leq 0$, and so

$$\langle y, \mathsf{N}\rangle = \langle (y - x) + x, \mathsf{N}\rangle \leq -\|x\| < 0.$$

Since $y \in X$ is arbitrary, we conclude that X is strictly contained in a hemisphere. This is absurd, and so 0 is an element of $\mathrm{Conv}(X)$ as desired. □

We obtain the following converse to Lemma 4.20.

Lemma 4.22. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a boundary point of K and let N be a supporting normal to K at x. Suppose that K does not satisfy the local geodesic property at x. Then, for all $r > 0$ there exists N', which we may choose as close to N as we wish, with the property that for all $y \in K \cap (B_r(x))^c$,*

$$\langle y - x, \mathsf{N}'\rangle < 0.$$

Remark. *In particular, using the terminology of links which we introduce in Section 5.4, below, for all $r > 0$, the closure of $\mathcal{L}_r(x; K)$ is strictly contained in a hemisphere.*

Proof. Upon applying an affine isometry, we may suppose that $x = 0$. Choose $r > 0$. Since K does not satisfy the local geodesic property at 0, by Lemma 4.19, 0 does not lie in the convex hull of $K \cap \partial B_r(0)$. By Lemma 4.21, $K \cap \partial B_r(0)$ is strictly contained in a hemisphere. There therefore exists a unit vector $\mathsf{N}' \in \Sigma^n$ such that for all $y \in K \cap \partial B_r(0)$,

$$\langle y, \mathsf{N}'\rangle < 0.$$

For all $\epsilon > 0$, denote $\mathsf{N}_\epsilon := (\mathsf{N} + \epsilon\mathsf{N}')/\|\mathsf{N} + \epsilon\mathsf{N}'\|$. We claim that N_ϵ has the desired properties for all $\epsilon > 0$. Indeed, for all $\epsilon > 0$, and for all $y \in K \cap \partial B_r(0)$, we obtain

$$\|\mathsf{N} + \epsilon\mathsf{N}'\|\langle y, \mathsf{N}_\epsilon\rangle = \langle y, \mathsf{N}\rangle + \epsilon\langle y, \mathsf{N}'\rangle \leq \epsilon\langle y, \mathsf{N}'\rangle < 0.$$

However, if $z \in K \cap (B_r(x))^c$, then, by convexity, $z = sy$ for some $y \in K \cap \partial B_r(0)$ and some $s \geq 1$, so that

$$\langle z, \mathsf{N}_\epsilon \rangle = s\langle y, \mathsf{N}_\epsilon \rangle < 0,$$

as desired. □

4.6 Interior a-priori bounds

We now return to the framework of Section 2, and consider smooth, convex functions $f : \overline{\Omega} \to]-\infty, 0]$ which are solutions of (B). We develop a-priori estimates that will allow us in the following section to describe the local geometric structure of singularities of uniform limits of sequences of such functions. We achieve this by once again using the maximum principal via an argument that dates back to Pogorelov (c.f. [17], but see also [5] and [21]). We will use the notation of Section 2.6, and we first complement Lemma 2.22 with the following two estimates.

Lemma 4.23. *There exists $C > 0$ which only depends on $\|\phi\|_0$ such that*

$$\mathcal{L}_f \mathrm{Log}(-f) \geq -C(-f)^{-1} - B^{ij}(\partial_i \mathrm{Log}(-f))(\partial_j \mathrm{Log}(-f)).$$

Proof. By Corollary 2.3,

$$DF(D^2 f)(D^2 f) = F(D^2 f).$$

Furthermore, since ϕ is positive and since G is convex,

$$\phi DG(Df)(Df) \geq \phi G(Df) - \phi G(0).$$

Thus, by definition of $\mathcal{L}_f$,

$$\mathcal{L}_f f \leq F(D^2 f) - \phi G(Df) + \phi G(0) = \phi G(0).$$

There therefore exists $C > 0$ which only depends on $\|\phi\|_0$ such that,

$$\mathcal{L}_f(-f) \geq -C.$$

Thus, by Lemma 2.21,

$$\mathcal{L}_f \mathrm{Log}(-f) \geq -C(-f)^{-1} - B^{ij}(\partial_i \mathrm{Log}(-f))(\partial_j \mathrm{Log}(-f)),$$

as desired. □

Lemma 4.24. *There exists $C > 0$ which only depends on $\|\phi\|_1$ and $\|f\|_1$ such that*

$$\mathcal{L}_f \|Df\|^2 \geq -C + \frac{2\phi}{n}\lambda_n(f).$$

Proof. By the product rule, for all i, using the summation convention,

$$\begin{aligned}\partial_i\|Df\|^2 &= 2f_{ik}f_k\\ \partial_i\partial_j\|Df\|^2 &= 2f_{ijk}f_k + 2f_{ik}f_{jk}.\end{aligned}$$

Thus, by definition of B^{ij}, and using the symmetry of the derivatives of f,

$$\begin{aligned}\mathcal{L}_f\|Df\|^2 &= 2f_kDF(D^2f)(D^2f_k) + 2B^{ij}f_{ik}f_{jk} - 2f_k\phi DG(Df)(Df_k)\\ &= 2f_k\mathcal{L}_ff_k + \frac{2}{n}F(D^2f)f_{kk}.\end{aligned}$$

Since f satisfies (B), since ϕ is positive, since $G \geq 1$ and since D^2f is positive definite,

$$\frac{2}{n}F(D^2f)f_{kk} = \frac{2\phi}{n}G(Df)\mathrm{Tr}(D^2f) \geq \frac{2\phi}{n}\lambda_n(f).$$

On the other hand, differentiating (B) once in the e_k direction yields

$$\mathcal{L}_ff_k = \phi_kG(Df).$$

Combining these relations, we obtain

$$\mathcal{L}_f\|Df\|^2 \geq 2f_k\phi_kG(Df) + \frac{2\phi}{n}\lambda_n(f).$$

There therefore exists $C > 0$ which only depends on $\|\phi\|_1$ and $\|f\|_1$ such that

$$\mathcal{L}_f\|Df\|^2 \geq -C + \frac{2\phi}{n}\lambda_n(f),$$

as desired. □

Theorem 4.25. *There exists $C > 0$ which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that*

$$\mathrm{Sup}_{x\in\overline{\Omega}}\,|f(x)|^2\,\|D^2f(x)\| \leq C.$$

Proof. For $\epsilon \in]0,1[$, define the function $\psi_\epsilon : \Omega \to \mathbb{R}$ by

$$\psi_\epsilon := \mu_n(f) + 2\mathrm{Log}(-f) + \epsilon\|Df\|^2.$$

It suffices to obtain a-priori bounds for ψ_ϵ for some $\epsilon > 0$. Fix ϵ and observe that ψ_ϵ is continuous and tends to $-\infty$ near the boundary of Ω. It therefore attains its maximum at some interior point $x \in \Omega$. Upon applying an affine isometry, we may suppose that $x = 0$ and that $e_1, ..., e_n$ are the eigenvectors of $D^2f(0)$ corresponding to the eigenvalues $\lambda_1, ..., \lambda_n$

respectively. By Lemmas 2.22, 4.23 and 4.24, there exists $C_1 > 0$, which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that

$$\begin{aligned}\mathcal{L}_f\psi_\epsilon \geq &-\frac{C_1}{(-f)^2} + \frac{2\epsilon\phi}{n}\lambda_n(f) - \frac{C_1}{\lambda_n}(\partial_n(\psi_\epsilon - 2\mathrm{Log}(-f) - \epsilon\|Df\|^2))^2\\ &+ B^{ij}\partial_i(\psi_\epsilon - 2\mathrm{Log}(-f) - \epsilon\|Df\|^2)\partial_j(\psi_\epsilon - 2\mathrm{Log}(-f) - \epsilon\|Df\|^2)\\ &- 2B^{ij}\partial_i\mathrm{Log}(-f)\partial_j\mathrm{Log}(-f)\end{aligned}$$

in the weak sense. Let $\alpha : \Omega \to \mathbb{R}$ be such that $\alpha \leq \psi_\epsilon$ and $\alpha(0) = \psi_\epsilon(0)$. In particular, α attains its maximum at 0, and so $(\partial_n\alpha)(0) = 0$, so that, at the origin,

$$\begin{aligned}\mathcal{L}_f\alpha \geq &-\frac{C_1}{(-f)^2} + \frac{2\epsilon\phi}{n}\lambda_n(f) - \frac{C_1}{\lambda_n}(2\partial_n\mathrm{Log}(-f) + \epsilon\partial_n\|Df\|^2)^2\\ &+ B^{ij}\partial_i(2\mathrm{Log}(-f) + \epsilon\|Df\|^2)\partial_j(2\mathrm{Log}(-f) + \epsilon\|Df\|^2)\\ &- 2B^{ij}\partial_i\mathrm{Log}(-f)\partial_j\mathrm{Log}(-f).\end{aligned}$$

Let A_1 and A_2 denote respectively the third term and the sum of the last two terms on the right-hand side of the above equation. By definition of λ_n,

$$A_1 = -\frac{C_1}{\lambda_n}\left(\frac{2}{(-f)}(-f_n) + 2\epsilon\lambda_n f_n\right)^2.$$

However, by Lemma 2.6, by definition of G, and since f solves (B),

$$\lambda_n \geq \frac{1}{n}\mathrm{Tr}(D^2f) \geq F(D^2f) = \phi G(Df) \geq \phi.$$

There therefore exists $C_2 > 0$, which only depends on $\|f\|_1$ and $\inf_{x\in\overline{\Omega}}\phi(x)$ such that

$$A_1 \geq -\frac{C_2}{(-f)^2} - C_2\epsilon^2\lambda_n.$$

Since B^{ij} is symmetric and positive definite,

$$A_2 \geq 4\epsilon B^{ij}(\partial_i\mathrm{Log}(-f))(\partial_j\|Df\|^2).$$

Thus, by definition of B^{ij}, and since f solves (B),

$$A_2 \geq \frac{-8\epsilon}{n(-f)}F(D^2f)\|Df\|^2 = \frac{-8\epsilon\phi}{n(-f)}G(Df)\|Df\|^2 \geq \frac{-8\epsilon\phi}{n(-f)}\|Df\|^2.$$

In particular, there exists $C_3 > 0$, which only depends on $\|\phi\|_0$ and $\|f\|_1$ such that

$$A_2 \geq -\frac{C_3}{(-f)^2}.$$

Combining these relations, we conclude that there exists $C_4 > 0$, which only depends on $\|\phi\|_2$, $\|f\|_1$ and $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ such that, at the origin,

$$\mathcal{L}_f\alpha \geq -\frac{C_4}{(-f)^2} + \epsilon\left(\frac{2\phi}{n} - C_4\epsilon\right)\lambda_n.$$

In particular, for $\epsilon > 0$ sufficiently small, there exists $C_5 > 0$ which only depends on $\|\phi\|_2$, $\|f\|_1$, $\mathrm{Inf}_{x\in\overline{\Omega}}\,\phi(x)$ and ϵ such that, at the origin,

$$\mathcal{L}_f\alpha \geq \frac{1}{(-f)^2}\left(-C_5 + \frac{1}{C_5}e^{\alpha}\right).$$

However, since α attains its maximum at the origin, $\mathcal{L}_f\alpha \leq 0$, so that

$$\psi_\epsilon(0) = \alpha(0) \leq 2\mathrm{Log}(C_5),$$

as desired. □

4.7 The structure of singularities

We now describe the local structure of singularities that arise upon taking limits. We first require some preliminary results.

Lemma 4.26. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. If K has non-trivial interior, then $\mathcal{N}(x)$ is strictly contained in a hemisphere for every boundary point x of K.*

Proof. Let y be an interior point of K. Let x be a boundary point of K. Denote $\mathsf{N} = (y-x)/\|y-x\|$. Choose $\mathsf{M} \in \mathcal{N}(x)$. Since y is an interior point of K, $\langle \mathsf{N}, \mathsf{M}\rangle < 0$, and since $\mathsf{M} \in \mathcal{N}(x)$ is arbitrary, we conclude that $\mathcal{N}(x)$ is strictly contained in a hemisphere as desired. □

Lemma 4.27. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let x be a boundary point of K. If $\mathcal{N}(x)$ is strictly contained in a hemisphere, then there exists a supporting normal N to K at x such that $\langle \mathsf{N}, \mathsf{M}\rangle > 0$ for all $\mathsf{M} \in \mathcal{N}(x)$.*

Proof. Upon applying a linear isometry, we may suppose that $\langle \mathsf{N}, e_{n+1}\rangle < 0$ for all $\mathsf{N} \in \mathcal{N}(x)$. Observe that $\mathcal{N}(x)$ is closed. If $-e_{n+1} \in \mathcal{N}(x)$, then we are done. Otherwise, suppose that $-e_{n+1} \notin \mathcal{N}(x)$. Define

$$\mathrm{Cone}(\mathcal{N}(x)) := \{t\mathsf{N} \mid t \in [0,\infty[,\ \mathsf{N} \in \mathcal{N}(x)\}.$$

Observe that $\mathrm{Cone}(\mathcal{N}(x))$ is closed and convex. Moreover, $-e_{n+1} \notin \mathrm{Cone}(\mathcal{N}(x))$. Let $y \in \mathrm{Cone}(\mathcal{N}(x))$ minimise distance in $\mathrm{Cone}(\mathcal{N}(x))$ to $-e_{n+1}$. We claim that $\mathsf{N} := y/\|y\|$ has the desired property. Indeed, denote

$\hat{N} := -(e_{n+1}+y)/\|e_{n+1}+y\|$. By Lemma 4.6, $\hat{N}$ is a supporting normal to $\mathrm{Cone}(\mathcal{N}(x))$ at y. Thus, for all $t\in[0,\infty[$, since $ty\in\mathrm{Cone}(\mathcal{N}(x))$,

$$\langle ty-y,\hat{\mathsf{N}}\rangle\leq 0,$$

and differentiating this relation at $t=1$ yields $\langle y,\hat{\mathsf{N}}\rangle=0$. Now choose $\mathsf{N}\in\mathcal{N}(x)\subseteq\mathrm{Cone}(\mathcal{N}(x))$. Using the fact that $\langle\mathsf{M},-e_{n+1}\rangle>0$, we have

$$\begin{aligned}0&\geq\langle\mathsf{M}-y,\hat{\mathsf{N}}\rangle\\&=\langle\mathsf{M},\hat{\mathsf{N}}\rangle\\&=\|e_n+y\|^{-1}\langle\mathsf{M},-e_{n+1}-y\rangle\\&>\|e_n+y\|^{-1}\|y\|\langle\mathsf{M},-y/\|y\|\rangle,\end{aligned}$$

so that $\langle\mathsf{M},\mathsf{N}\rangle>0$, and since $\mathsf{M}\in\mathcal{N}(x)$ is arbitrary, the result follows. □

Theorem 4.28. *Let $(K_m)_{m\in\mathbb{N}}$ and K_∞ be compact, convex subsets of $\mathbb{R}^{n+1}$. Suppose that K_∞ has non-trivial interior and that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. Let $k>0$ be a real number, let U be an open subset of $\mathbb{R}^{n+1}$ and suppose that for all m, $(\partial K_m)\cap U$ is smooth with constant gaussian curvature equal to k. If $y\in(\partial K_\infty)\cap U$ then either*

(1) there exists $r>0$ such that $(\partial K_\infty)\cap B_r(y)$ is smooth with constant gaussian curvature equal to k; or

(2) K_∞ satisfies the local geodesic property at y.

Proof. Choose $y\in(\partial K_\infty)\cap U$ and suppose that K_∞ does not satisfy the local geodesic property at y. Let $(y_m)_{m\in\mathbb{N}}$ be a sequence converging to y such that $y_m\in\partial K_m$ for all m. Upon applying a convergent sequence of affine isometries, we may suppose that $y_m=0$ for all m. Since K_∞ has non-trivial interior, by Lemma 4.26, $\mathcal{N}(0;K_\infty)$ is strictly contained in a hemisphere. In particular, by Lemma 4.27, we may suppose that $-e_{n+1}\in\mathcal{N}(0;K_\infty)$ and that $\langle\mathsf{N},-e_{n+1}\rangle>3\cos(\theta)$ for all other $\mathsf{N}\in\mathcal{N}(0;K_\infty)$ and for some $\theta\in[0,\pi/2[$. Denote $C:=\tan(\theta)$.

By Lemma 4.4, upon extracting a subsequence, we may suppose that there exists $r>0$ such that $B_r(0)\subseteq U$ and for all m, for all $x\in(\partial K_m)\cap B_r(0)$, and for all $\mathsf{N}\in\mathcal{N}(x;K_m)$, $\langle\mathsf{N},-e_{n+1}\rangle>2\cos(\theta)$. Denote $\rho:=r/\sqrt{1+4C^2}$. By Lemma 4.22, upon applying a small rotation, we may suppose that for all $x\in K_\infty\setminus B_{\rho/2}(0)$, $\langle x,-e_{n+1}\rangle<0$. Choosing this rotation sufficiently small, we may continue to assume that for all m, for all $x\in(\partial K_m)\cap B_r(0)$ and for all $\mathsf{N}\in\mathcal{N}(x;K_m)$, $\langle\mathsf{N},-e_{n+1}\rangle>\cos(\theta)$.

By Theorem 4.12, for all m, there exists a convex, C-Lipschitz function, $f_m:B'_\rho(0)\to]-C\rho,C\rho[$ such that $f_m(0)=0$ and $(\partial K_m)\cap(B'_\rho(0)\times]-2C\rho,2C\rho[)$ coincides with the graph of f_m over $B'_\rho(0)$. By the Arzelà-Ascoli theorem, every subsequence of $(f_m)_{m\in\mathbb{N}}$ has a subsubsequence converging in the local uniform sense over $B'_\rho(0)$ to some limit f'_∞ say. Since $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense, it follows that $f'_\infty = f_\infty$, and we conclude that $(f_m)_{m\in\mathbb{N}}$ converges in the local uniform sense over $B'_\rho(0)$ to f_∞.

Observe that f_m is smooth for all $m<\infty$. Furthermore, there exists $\delta>0$ such that, $f_\infty(x')>4\delta$ for all $x'\in\partial B'_{\rho/2}(0)$. Since $(f_m)_{m\in\mathbb{N}}$ converges locally uniformly to f_∞ over $B'_\rho(0)$, we may suppose that, for all m and for all $x'\in B'_{\rho/2}(0)$, $f_m(x')>2\delta$. For all $m<\infty$, since f_m is C-Lipschitz,

$$\|f_m|_{\overline{B}'_{\rho/2}(0)}\|_1 \leq C(1+\rho/2).$$

Thus, by Theorem 4.25, there exists $C_2>0$ such that for all $m<\infty$, and for all $x\in B'_{\rho/2}(0)$,

$$|2\delta - f_m(x)|^2\,\|D^2 f_m(x)\| \leq C_2.$$

Since $f_\infty(0)=0$, by continuity, there exists $s\in]0,\rho/2[$ such that $f_\infty(x')\leq\delta/2$ for all $x'\in\overline{B}'_s(0)$. Since $(f_m)_{m\in\mathbb{N}}$ converges locally uniformly to f_∞, we may suppose that for all $m<\infty$ and for all $x\in\overline{B}'_s(0)$, $f_m(x')\leq\delta$, so that

$$\|D^2 f_m(x)\| \leq C_2/\delta^2.$$

By the Krylov estimates (c.f Theorem 2.25) and the Schauder estimates (c.f. Theorem 2.24), for all $k\in\mathbb{N}$, there exists $C_k>0$ such that for all $m<\infty$,

$$\|f_m|_{\overline{B}'_{s/2}(0)}\|_k \leq C_k.$$

By the Arzelà-Ascoli theorem, every subsequence of $(f_m)_{m\in\mathbb{N}}$ has a subsubsequence which converges in the C^∞ sense over $\overline{B}'_{s/4}(0)$ to some limit f'_∞ say. Since $(f_m)_{m\in\mathbb{N}}$ converges uniformly to f_∞, it follows that $f'_\infty=f_\infty$. We conclude that $(f_m)_{m\in\mathbb{N}}$ converges to f_∞ in the C^∞ sense over $\overline{B}'_{s/4}(0)$. In particular, f_∞ is smooth over $B'_{s/4}(0)$ and its graph has constant gaussian curvature equal to k. In other words, $(\partial K_\infty)\cap(B'_{s/4}(0)\times]-2C\rho,2C\rho[)$ is smooth and has constant gaussian curvature equal to k, as desired. □

Chapter 5

Duality of Convex Sets

Before analysing the general Plateau problem, it will be useful to continue our study of the elementary geometry of convex sets. In particular, we review the concept of duality for subsets of the sphere, showing how it is closely related to the concept of the convex hull, which we introduced in the preceeding chapter. We introduce the infinitesimal link of a given boundary point of a given compact, convex subset of $\mathbb{R}^{n+1}$ with non-trivial interior. This is defined to be an open subset of the sphere, and we show that it coincides with the dual of the set of supporting normal vectors to the convex set at that point. This allows us to prove the most important result of this chapter, namely Theorem 5.20, which determines the supporting normal set of the intersection of two given convex sets at any point on the boundary of this intersection.

Although the results of this chapter are of use in the sequel, they are only tangential to the main flow of this text. In particular, Theorem 5.20, although interesting, may be substituted by ad-hoc arguments in the relatively straightforward cases where it will be applied.

5.1 Open half spaces and convex hulls

Let N be a unit vector and let $t > 0$ be a positive real number (possibly $+\infty$). We define the subset $H(\mathsf{N}, t)$ of $\mathbb{R}^{n+1}$ by

$$H(\mathsf{N}, t) := \{x \mid \langle x, \mathsf{N}\rangle < t\},$$

and we refer to this set as the **open half-space** normal to N of height t. Observe that this definition incorporates the degenerate case $\mathbb{R}^{n+1} = H(\mathsf{N}, \infty)$.

Lemma 5.1. *If K is an open, convex subset of $\mathbb{R}^{n+1}$, then $K = (\overline{K})^o$.*

Proof. Since K is open, $K \subseteq (\overline{K})^o$. We now show that $(\overline{K})^o \subseteq K$. By Lemma 4.13, $\overline{K}$ is convex. Choose $x \in (\overline{K})^o$. Without loss of generality, we may suppose that $x = 0$. Choose $\delta > 0$ such that $B_\delta(0) \subseteq \overline{K}$. Then $K \cap B_\delta(0)$ is a dense subset of $B_\delta(0)$. Upon applying a homothety, we may suppose that $\delta = 2$. Choose $x_1, ..., x_k \in B_1(0)$ such that

$$\partial B_1(0) \subseteq \bigcup_{i=1}^{k} B_1(x_i).$$

Since $K \cap B_\delta(0)$ is dense, upon perturbing $x_1, ..., x_k$ if necessary, we may suppose that $x_i \in K$ for all i. Let L be the convex hull of $\{x_1, ..., x_k\}$. In particular, L is a compact, convex subset of K. We claim that 0 is an element of L. Indeed, suppose the contrary. Let $y \in L$ be the point minimising distance to 0 and denote $\mathsf{N} := -y/\|y\|$. By Lemma 4.6, N is a supporting normal to L at y. Thus, for all $z \in L$, $\langle z - y, \mathsf{N}\rangle \leq 0$. In particular, for all $1 \leq i \leq k$,

$$\langle x_i, \mathsf{N}\rangle = \langle x_i - y, \mathsf{N}\rangle + \langle y, \mathsf{N}\rangle \leq -\|y\| < 0,$$

so that, for all i,

$$\|x_i - \mathsf{N}\|^2 = \|x_i\|^2 - 2\langle x_i, \mathsf{N}\rangle + \|\mathsf{N}\|^2 > 2.$$

However, by definition of $x_1, ..., x_k$, there exists $1 \leq i \leq k$ such that $\|x_i - \mathsf{N}\|^2 < 1$. This is absurd, and we conclude that $0 \in L \subseteq K$ as asserted. Since $x \in (\overline{K})^o$ is arbitrary, it follows that $(\overline{K})^o \subseteq K$, and the two sets therefore coincide, as desired. □

Theorem 5.2. *For any subset X of $\mathbb{R}^{n+1}$, the convex hull of X coincides with the intersection of all open half-spaces containing X.*

Proof. Denote by $\hat{X}$ the intersection of all open half-spaces containing X. Since every open half-space is also convex, by definition of the convex hull, $\mathrm{Conv}(X) \subseteq \hat{X}$. We now show that $\hat{X} \subseteq \mathrm{Conv}(X)$. Indeed, choose $x \in \mathbb{R}^{n+1} \setminus \mathrm{Conv}(X)$. Let K be an open, convex set such that $X \subseteq K$ and $x \notin K$. By Lemma 4.13, $\overline{K}$ is also convex. We now have two cases to consider. Suppose first that $x \in \mathbb{R}^{n+1} \setminus \overline{K}$. Let y be a point in $\overline{K}$ minimising distance to x and denote $\mathsf{N} := (x - y)/\|x - y\|$. By Lemma 4.10, N is a supporting normal to $\overline{K}$ at y, and so, for all $z \in K \subseteq \overline{K}$,

$$\langle z - x, \mathsf{N}\rangle = \langle (z - y) + (y - x), \mathsf{N}\rangle \leq -\|x - y\| < 0.$$

Now suppose that $x \in \overline{K} \setminus K$. By Lemma 5.1, $K = (\overline{K})^o$, and so $x \in \overline{K} \setminus (\overline{K})^o = \partial\overline{K}$. Let N be a supporting normal to $\overline{K}$ at x. For all $z \in K$, $\langle z - x, \mathsf{N}\rangle \leq 0$, and, since K is open, $\langle z - x, \mathsf{N}\rangle < 0$. In both cases, we conclude that $K \subseteq H(\mathsf{N}, \langle \mathsf{N}, x\rangle)$ so that, by definition, $\hat{X} \subseteq H(\mathsf{N}, \langle \mathsf{N}, x\rangle)$. In particular, $x \notin \hat{X}$, and since $x \notin \mathrm{Conv}(X)$ is arbitrary, we conclude that

$\mathbb{R}^{n+1} \setminus \mathrm{Conv}(X) \subseteq \mathbb{R}^{n+1} \setminus \hat{X}$. Taking complements yields $\hat{X} \subseteq \mathrm{Conv}(X)$, and the two sets therefore coincide, as desired. □

We also have the following complement of Lemma 5.1.

Lemma 5.3. *If K is a compact, convex subset of $\mathbb{R}^{n+1}$ with non-trivial interior, then $K = \overline{K^o}$.*

Proof. Since $K^o \subseteq K$ and since K is closed, $\overline{K^o} \subseteq K$. Conversely, choose $x \in K$. Since K has non-trivial interior, there exists $y \in K^o$. Choose $\delta > 0$ such that $B_\delta(y) \subseteq K$. Bearing in mind that K is convex, for all $t \in]0,1]$ and for all $z \in B_{t\delta}(0)$,

$$(1-t)x + ty + z = (1-t)x + t(y + t^{-1}z) \in K.$$

In other words, for all $t \in]0,1]$, $B_{t\delta}((1-t)x + ty) \subseteq K$, so that $(1-t)x + ty \in K^o$. It follows that $x \in \overline{K^o}$, and since $x \in K$ is arbitrary, we conclude that $K \subseteq \overline{K^o}$, and the two sets therefore coincide, as desired. □

5.2 Convex subsets of the sphere

Let $\mathsf{N}_0, \mathsf{N}_1 \in \Sigma^n$ be points in the sphere. We say that N_0 and N_1 are **non-antipodal** whenever $\mathsf{N}_0 + \mathsf{N}_1 \neq 0$. In this case, we define the curve $\mathsf{N} : [0,1] \to \Sigma^n$ by

$$\mathsf{N}(s) := \frac{(1-s)\mathsf{N}_0 + s\mathsf{N}_1}{\|(1-s)\mathsf{N}_0 + s\mathsf{N}_1\|}.$$

We refer to N as the **great-circular arc** joining N_0 to N_1. This terminology is justified by the following result.

Lemma 5.4. *If $\mathsf{N}_0, \mathsf{N}_1 \in \Sigma^n$ are distinct, non-antipodal points of the sphere, then there exists a unique great circle C passing through N_0 and N_1. Moreover, if N is the great-circular arc joining N_0 to N_1 then, for all $s \in [0,1]$, $\mathsf{N}(s)$ is an element of C.*

Proof. Observe that every great circle in Σ^n coincides with the intersection of Σ^n with a plane in $\mathbb{R}^{n+1}$ containing the origin. Conversely, the intersection of any such plane with Σ^n is a great circle. Now let N_0 and N_1 be distinct, non-antipodal points. In particular, they are linearly independent. There therefore exists a unique plane, $E \subseteq \mathbb{R}^{n+1}$, which passes through N_0, N_1 and the origin, and the intersection $C := E \cap \Sigma^n$ is therefore the unique great circle passing through these two points. Finally, for all s, $\mathsf{N}(s) \in E$ and $\mathsf{N}(s) \in \Sigma^n$, so that $\mathsf{N}(s) \in E \cap \Sigma^n = C$, as desired. □

Let X be a subset of Σ^n which is strictly contained in a hemisphere. In particular, no two points of X are antipodal and so there is a well defined great-circular arc joining any two of them. We say that X is **convex** whenever it has in addition the property that for all $\mathsf{N}_0, \mathsf{N}_1 \in X$, the great-circular arc joining N_0 and N_1 is also contained in K.

For all $\mathsf{N} \in \Sigma^n$, we define the subset $\Sigma^n_-(\mathsf{N})$ of Σ^n by

$$\Sigma^n_-(\mathsf{N}) := \{x \mid \langle x, \mathsf{N}\rangle < 0\}.$$

We refer to $\Sigma^n_-(\mathsf{N})$ is the **open hemisphere** defined by N. In particular, when $\mathsf{N} = e_{n+1}$, we define the **southern hemisphere** of Σ^n by $\Sigma^n_- := \Sigma_-(e_{n+1})$. We now identify $\mathbb{R}^{n+1}$ with the product $\mathbb{R}^n \times \mathbb{R}$. Observe that Σ^n_- then coincides with the intersection of Σ with $\mathbb{R}^n \times]-\infty, 0[$. We define the mapping $P : \Sigma^n_- \to \mathbb{R}^n$ by

$$P(x', t) := -x'/t,$$

and we refer to P as the **affine projection** of Σ^n_- onto $\mathbb{R}^n$.

Lemma 5.5. *P defines a smooth diffeomorphism from Σ^n_- onto $\mathbb{R}^n$.*

Proof. Define $\hat{P} : \mathbb{R}^n \times]-\infty, 0[\to \mathbb{R}^n$ by $\hat{P}(x', t) := -x'/t$. Since $\hat{P}$ is smooth, and since P coincides with its restriction to Σ^n_-, P is also smooth. Now define $Q : \mathbb{R}^n \to \Sigma^n_-$ by $Q(x') := (x', -1)/\sqrt{1 + \|x'\|^2}$. Observe that Q is smooth. Moreover, for all $(x', t) \in \Sigma^n_-$, bearing in mind that $\|x'\|^2 + t^2 = 1$,

$$(Q \circ P)(x', t) = Q(-x'/t) = (-x'/t, -1)/\sqrt{1 + x^2/t^2} = (x', t).$$

Conversely, for all $x' \in \mathbb{R}^n$,

$$(P \circ Q)(x') = P((x', -1)/\sqrt{1 + \|x'\|^2}) = x'.$$

We conclude that P is a smooth diffeomorphism with inverse Q as desired.
□

Lemma 5.6. *P maps the set of great-circular arcs in Σ^n_- bijectively onto the set of straight-line segments in $\mathbb{R}^n$.*

Proof. Since straight-line segments and great-circular arcs are uniquely defined by their end points, it suffices to show that for any two distinct points $\mathsf{N}_0, \mathsf{N}_1 \in \Sigma^n_-$, if $\mathsf{N} : [0, 1] \to \Sigma^n_-$ is the great-circular arc joining these two points, then $P \circ \mathsf{N}$ is, up to reparametrisation, the straight-line segment joining $P(\mathsf{N}_0)$ to $P(\mathsf{N}_1)$. However, for all t,

$$(P \circ \mathsf{N})(t) = (1 - \tau(s))P(\mathsf{N}_0) + \tau(s)P(\mathsf{N}_1),$$

where

$$\tau(s) := \frac{s\langle \mathsf{N}_1, e_{n+1}\rangle}{s\langle \mathsf{N}_1, e_{n+1}\rangle + (1 - s)\langle \mathsf{N}_0, e_{n+1}\rangle}.$$

Furthermore, $\tau(0) = 0$, $\tau(1) = 1$, and

$$\tau'(s) = \frac{\langle \mathsf{N}_0, e_{n+1}\rangle\langle \mathsf{N}_1, e_{n+1}\rangle}{s\langle \mathsf{N}_1, e_{n+1}\rangle + (1-s)\langle \mathsf{N}_0, e_{n+1}\rangle} > 0.$$

The function τ is therefore a reparametrisation of the unit interval, and so the image of N under P is a reparametrised straight-line segment from $P(\mathsf{N}_0)$ to $P(\mathsf{N}_1)$, as desired. □

In technical terms, Lemma 5.6 means that Σ^n_- is affine equivalent to $\mathbb{R}^n$. In particular, this immediately yields

Theorem 5.7. *If X is a subset of Σ^n_-, then X is convex if and only if $P(X)$ is convex.*

Convex subsets of Σ^n_- therefore possess all the properties of convex subsets of $\mathbb{R}^{n+1}$ studied in Chapter 4. In particular, if X is a subset of Σ^n_- which is strictly contained in a hemisphere, then we define the convex hull of X to be the intersection of all open convex sets in Σ^n containing X. We denote this set by $\mathrm{Conv}(X)$.

Lemma 5.8. *Let X and K be subsets of Σ^n which are strictly contained in hemispheres. Then,*

(1) if X is compact, then $\mathrm{Conv}(X)$ *is compact;*

(2) if X is open, then $\mathrm{Conv}(X)$ *is open;*

(3) if K is compact and convex, then $\mathrm{Conv}(K) = K$*; and*

(4) if K is open and convex, then $\mathrm{Conv}(K) = K$.

Proof. Upon applying rotations, we may suppose that $X, K \subseteq \Sigma^n_-$. Observe that if K' is an open convex subset of Σ^n_- containing X, then so too is $K' \cap \Sigma^n_-$. It follows that $\mathrm{Conv}(X)$ coincides with the intersection of all open, convex subsets of Σ^n_- containing X. Thus, since P is a diffeomorphism mapping convex sets to convex sets,

$$P(\mathrm{Conv}(X)) = \mathrm{Conv}(P(X)).$$

(1) and (2) now follow by Lemmas 4.14 and 4.15. (3) follows by Lemma 4.16. (4) is trivial, and this completes the proof. □

Lemma 5.9. *P maps the set of open hemispheres in Σ^n bijectively onto the set of open half-spaces in $\mathbb{R}^n$.*

Remark. *We include here the empty set as the trivial open half-space.*

Proof. The operation of intersection defines a bijection between the set of open linear half-spaces in $\mathbb{R}^{n+1}$ and the set of open hemispheres in Σ^n. Now identify $\mathbb{R}^n$ with the affine hyperplane $R := \mathbb{R}^n \times \{-1\}$ in $\mathbb{R}^{n+1}$. The operation of intersection defines a bijection between the set of open linear half-spaces in $\mathbb{R}^{n+1}$ and the set of open half-spaces in R. However, for every open linear half-space H in $\mathbb{R}^{n+1}$,

$$P(H \cap \Sigma^n_-) = H \cap R,$$

and the result follows. □

This yields an alternative characterisation of the convex hull of a subset of the sphere.

Theorem 5.10. *Let X be a subset of Σ^n. If X is strictly contained in a hemisphere, then* $\mathrm{Conv}(X)$ *coincides with the intersection of all open hemispheres containing X.*

Proof. This follows from Theorem 5.2 and Lemma 5.9. □

5.3 Duality

Let X be a subset of Σ^n. We define the **dual** subset X^* to X by

$$X^* := \{\mathsf{M} \in \Sigma^n \mid \langle \mathsf{N}, \mathsf{M} \rangle < 0 \ \forall\ \mathsf{N} \in X\} = \bigcap_{\mathsf{N} \in X} \Sigma_-(\mathsf{N}).$$

Lemma 5.11. *X^* is non-empty if and only if X is strictly contained in a hemisphere.*

Proof. Suppose X^* is non-empty. Choose $\mathsf{N} \in X^*$. For all $\mathsf{M} \in X$, $\langle \mathsf{M}, \mathsf{N} \rangle < 0$ and so X is strictly contained in a hemisphere, as desired. Conversely, suppose that X is strictly contained in a hemisphere. Let $\mathsf{M} \in \Sigma^n$ be such that $\langle \mathsf{N}, \mathsf{M} \rangle < 0$ for all $\mathsf{N} \in X$. By definition, $M \in X^*$ and so X^* is non-empty, as desired. □

Lemma 5.12. *If X is non-empty, then X^* is convex.*

Proof. By definition, X^* is the intersection of a family of convex sets, and the result follows. □

Lemma 5.13. *If X is closed, then X^* is open. If X is open, then X^* is closed.*

Proof. Suppose that X is closed. Choose $\mathsf{M} \in X^*$. By compactness of X, there exists $\epsilon > 0$ such that $\langle \mathsf{N}, \mathsf{M} \rangle \leq -\epsilon$ for all N in X. If

$\mathsf{M}' \in B_\epsilon(\mathsf{M}) \cap \Sigma^n$, then, using the Cauchy-Schwarz inequality, we obtain, for all $\mathsf{N} \in X$,

$$\begin{aligned}\langle \mathsf{N}, \mathsf{M}' \rangle &= \langle \mathsf{N}, \mathsf{M}' - \mathsf{M} \rangle + \langle \mathsf{N}, \mathsf{M} \rangle \\ &\leq \|\mathsf{N}\| \|\mathsf{M}' - \mathsf{M}\| + \langle \mathsf{N}, \mathsf{M} \rangle \\ &< \epsilon - \epsilon \\ &= 0.\end{aligned}$$

Since $\mathsf{M}' \in B_\epsilon(\mathsf{M}) \cap \Sigma^n$ is arbitrary, we conclude that $B_\epsilon(\mathsf{M}) \cap \Sigma^n \subseteq X^*$, and since $\mathsf{M} \in X^*$ is arbitrary, we conclude that X^* is open, as desired.

Now suppose that X is open. Choose $\mathsf{M} \in \Sigma^n \setminus X^*$. There exists $\mathsf{N} \in X$ such that $\langle \mathsf{N}, \mathsf{M} \rangle \geq 0$. For all $s > 0$, denote $\mathsf{N}_s := (\mathsf{N} + s\mathsf{M})/\|\mathsf{N} + s\mathsf{M}\|$. For all $s > 0$, we have

$$\begin{aligned}\langle \mathsf{N}_s, \mathsf{M} \rangle &= \frac{1}{\|\mathsf{N} + s\mathsf{N}\|} \langle \mathsf{N} + s\mathsf{M}, \mathsf{M} \rangle \\ &\geq \frac{s}{\|\mathsf{N} + s\mathsf{N}\|} \\ &> 0.\end{aligned}$$

Since X is open, for sufficiently small s, $\mathsf{N}_s \in X$. Thus, upon replacing N with N_s, we may suppose that $\langle \mathsf{N}, \mathsf{M} \rangle =: \epsilon > 0$. If $\mathsf{M}' \in B_\epsilon(\mathsf{M}) \cap \Sigma^n$, then, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\langle \mathsf{N}, \mathsf{M}' \rangle &= \langle \mathsf{N}, \mathsf{M}' - \mathsf{M} \rangle + \langle \mathsf{N}, \mathsf{M} \rangle \\ &\geq -\|\mathsf{N}\| \|\mathsf{M}' - \mathsf{M}\| + \langle \mathsf{N}, \mathsf{M} \rangle \\ &\geq -\epsilon + \epsilon \\ &= 0.\end{aligned}$$

Since $\mathsf{M}' \in B_\epsilon(\mathsf{N}) \cap \Sigma^n$ is arbitrary, we conclude that $B_\epsilon(\mathsf{M}) \cap \Sigma^n \subseteq \Sigma^n \setminus X^*$, and since $\mathsf{M} \in \Sigma^n \setminus X^*$ is arbtrary, we conclude that $\Sigma^n \setminus X^*$ is open, so that X^* is closed, as desired. □

Lemma 5.14. *If X is strictly contained in a hemisphere, then $X^{**} = \mathrm{Conv}(X)$.*

Proof. By Lemma 5.10, $\mathrm{Conv}(X)$ is the intersection of all open hemispheres containing X. However, by definition, M is an element of X^* if and only if X is contained in $\Sigma_-(\mathsf{M})$. That is, X^* parametrises the set of open hemispheres containing X. Thus,

$$X^{**} = \bigcap_{\mathsf{M} \in X^*} \Sigma_-(\mathsf{M}) = \mathrm{Conv}(X),$$

as desired. □

Lemma 5.15. *If X_1 and X_2 are both strictly contained in the same hemisphere, then $(X_1 \cup X_2)^* = X_1^* \cap X_2^*$.*

Proof. Suppose $\mathsf{M} \in (X_1 \cup X_2)^*$. Then $\langle \mathsf{N}, \mathsf{M} \rangle < 0$ for all $\mathsf{N} \in X_1 \cup X_2$ and so $\mathsf{M} \in X_1^* \cap X_2^*$. Conversely, if $\mathsf{M} \in X_1^* \cap X_2^*$, then $\langle \mathsf{N}, \mathsf{M} \rangle < 0$ for all $\mathsf{N} \in X_1 \cup X_2$ and so $\mathsf{M} \in (X_1 \cup X_2)^*$. These two sets therefore coincide, as desired. □

Lemma 5.16. *Let X_1 and X_2 be convex subsets of Σ^n which are both strictly contained in a hemisphere. If X_1 and X_2 are either both open or both closed, and if $X_1 \cap X_2$ is non-empty, then $(X_1 \cap X_2)^* = \mathrm{Conv}(X_1^* \cup X_2^*)$.*

Proof. Suppose that X_1 and X_2 are open (resp. closed). Since they are both convex, by Lemmas 5.8 and 5.14, $X_1 = \mathrm{Conv}(X_1) = X_1^{**}$ and $X_2 = \mathrm{Conv}(X_2) = X_2^{**}$. We denote $Y_1 = X_1^*$ and $Y_2 = X_2^*$. Observe that, if $\mathsf{N} \in X_1 \cap X_2$, then $Y_1 = X_1^* \in \Sigma_-(\mathsf{N})$ and $Y_2 = X_2^* \subseteq \Sigma_-(\mathsf{N})$. That is, since $X_1 \cap X_2$ is non-empty, Y_1 and Y_2 are both strictly contained in the same hemisphere. Lemma 5.15 therefore yields

$$(X_1^* \cup X_2^*)^* = (Y_1 \cup Y_2)^* = Y_1^* \cap Y_2^* = X_1 \cap X_2,$$

so that, by Lemma 5.14,

$$\mathrm{Conv}(X_1^* \cup X_2^*) = (X_1^* \cup X_2^*)^{**} = (X_1 \cap X_2)^*,$$

as desired. □

5.4 Links

Let K be a compact, convex set with non-trivial interior. Let x be a boundary point of K. For $r > 0$, we define $\mathcal{L}_r(x; K) \subseteq \Sigma^n$, the **link** of K of radius r about x by,

$$\mathcal{L}_r(x; K) := \{\mathsf{N} \mid x + r\mathsf{N} \in K^o\}.$$

When there is no ambiguity concerning K, we denote $\mathcal{L}_r(x) = \mathcal{L}_r(x; K)$.

Lemma 5.17. *For every boundary point x of K and for all $r < s$, $\mathcal{L}_s(x) \subseteq \mathcal{L}_r(x)$.*

Proof. Indeed, choose $\mathsf{N} \in \mathcal{L}_s(x)$. Then $x + s\mathsf{N} \in K^o$. Viewing N as an element of $\mathbb{R}^{n+1}$, there exists $\delta > 0$ such that for all $V \in B_\delta(0)$, $x + s\mathsf{N} + V \in K$. Thus, by convexity, for all $V \in B_{r\delta/s}(0)$,

$$x + r\mathsf{N} + V = (1 - \frac{r}{s})x + \frac{r}{s}(x + s\mathsf{N} + \frac{s}{r}V) \in K.$$

It follows that $x + r\mathsf{N} \in K^o$, and so $\mathsf{N} \in \mathcal{L}_r(x)$. Since $\mathsf{N} \in \mathcal{L}_s(x)$ is arbitrary, we conclude that $\mathcal{L}_s(x) \subseteq \mathcal{L}_r(x)$ as desired. □

$(\mathcal{L}_r(x))_{r>0}$ therefore constitutes an increasing, nested family of open sets. We define $\mathcal{L}(x;K) \subseteq \Sigma^n$, the **link** of K at x by,

$$\mathcal{L}(x;K) := \underset{r>0}{\cup} \mathcal{L}_r(x;K).$$

When there is no ambiguity concerning K, we denote $\mathcal{L}(x) = \mathcal{L}(x;K)$. Since it is the union of a family of open sets, $\mathcal{L}(x)$ is also open.

Lemma 5.18. *Let K be a compact, convex set with non-trivial interior. Then for every boundary point x of K, $\mathcal{N}(x;K) = \mathcal{L}(x;K)^*$.*

Proof. Suppose that $\mathsf{N} \in \mathcal{N}(x;K)$. For all $z \in K$, $\langle z - x, \mathsf{N}\rangle \leq 0$, and so, for all $z \in K^o$, $\langle z - x, \mathsf{N}\rangle < 0$. Choose $\mathsf{M} \in \mathcal{L}(x;K)$. Choose $r > 0$ such that $\mathsf{M} \in \mathcal{L}_r(x;K)$. Then $x + r\mathsf{M} \in K^o$, and so

$$\langle \mathsf{M}, \mathsf{N}\rangle = \frac{1}{r}\langle (x + r\mathsf{M}) - x, \mathsf{N}\rangle < 0.$$

Since $\mathsf{M} \in \mathcal{L}(x;K)$ is arbitrary, we conclude that $\mathsf{N} \in \mathcal{L}(x;K)^*$, and since $\mathsf{N} \in \mathcal{N}(x;K)$ is arbitrary, we conclude that $\mathcal{N}(x;K) \subseteq \mathcal{L}(x;K)^*$.

Conversely, choose $\mathsf{N} \in \mathcal{L}(x;K)^*$. Since K has non-trivial interior, by Lemma 5.3, $K = \overline{K^o}$. Choose $y \in K^o$. Denote $r := \|y - x\|$. Then $(y - x)/r \in \mathcal{L}_r(x;K)$, and so

$$\langle \mathsf{N}, y - x\rangle = r\langle \mathsf{N}, (y - x)/r\rangle < 0,$$

Thus, by continuity, for all $y \in \overline{K^o} = K$,

$$\langle \mathsf{N}, y - x\rangle \leq 0,$$

so that $\mathsf{N} \in \mathcal{N}(x;K)$. Since $\mathsf{N} \in \mathcal{L}(x;K)^*$ is arbitrary, we conclude that $\mathcal{L}(x;K)^* \subseteq \mathcal{N}(x;K)$, and the two sets therefore coincide, as desired. □

Lemma 5.19. *For every compact, convex set K with non-trivial interior, and for every boundary point x of K, $\mathcal{N}(x;K)$ is closed, convex and strictly contained in a hemisphere.*

Proof. By Lemma 5.18, $\mathcal{N}(x;K) = \mathcal{L}(x;K)^*$. Since K has non-trivial interior, $\mathcal{L}(x;K)$ is non-empty, and so, by Lemma 5.11, $\mathcal{N}(x;K)$ is strictly contained in a hemisphere. Since $\mathcal{L}(x;K)$ is open, by Lemma 5.12, $\mathcal{N}(x;K)$ is closed. Finally, by Lemma 5.12, $\mathcal{N}(x;K)$ is convex, and this completes the proof. □

Theorem 5.20. *Let K_1 and K_2 be two compact, convex sets whose intersection has non-trivial interior. Choose $x \in \partial(K_1 \cap K_2)$. Then,* *(1) if* $x \in (\partial K_1) \cap K_2^o$, *then* $\mathcal{N}(x; K_1 \cap K_2) = \mathcal{N}(x; K_1)$*;*

(2) if $x \in K_1^o \cap (\partial K_2)$, *then* $\mathcal{N}(x; K_1 \cap K_2) = \mathcal{N}(x; K_2)$*; and*

(3) if $x \in (\partial K_1) \cap (\partial K_2)$, *then* $\mathcal{N}(x; K_1 \cap K_2) = \mathrm{Conv}(\mathcal{N}(x; K_1) \cup \mathcal{N}(x; K_2))$.

Proof. Cases (1) and (2) follow from Lemma 4.9. However, using Lemmas 5.16 and 5.18, for $x \in (\partial K_1) \cap (\partial K_2)$, we obtain

$$\begin{aligned}
\mathcal{N}(x; K_1 \cap K_2) &= \mathcal{L}(x; K_1 \cap K_2)^* \\
&= (\mathcal{L}(x; K_1) \cap \mathcal{L}(x; K_2))^* \\
&= \mathrm{Conv}(\mathcal{L}(x; K_1)^* \cup \mathcal{L}(x; K_2)^*) \\
&= \mathrm{Conv}(\mathcal{N}(x; K_1) \cup \mathcal{N}(x; K_2)),
\end{aligned}$$

as desired. □

Chapter 6

Weak Barriers

For k a positive real number, the set of weak barriers of gaussian curvature at least k will be defined to be essentially the closure in the Hausdorff topology of the set of compact, convex sets with smooth boundary of gaussian curvature at least k. This concept will allow us to solve the Plateau problem in euclidean space for very general data. Once solutions have been found, the theory developed in Section 4 is then applied to identify their singular sets. In particular, under suitable conditions on the boundary, these are shown to be empty, so that the solutions are actually smooth.

Existence is proven via the Perron method. The main requirement for the application of this technique is the closure of the family of weak barriers under the operation of intersection. That is, if K_1 and K_2 are weak barriers, then so too is $K_1 \cap K_2$. The proof of this result, which is encapsulated in Theorem 6.34 is rather technical, and forms the content of Sections 6.1 to 6.5 inclusive. The techniques used are mostly elementary, although some experience of the theory of distributions will be required, and we refer the reader to [10] for a clear and straightforward introduction.

The experienced reader will notice that weak barriers are always viscosity supersolutions of the Gauss curvature equation (c.f. [8]). Like the space of weak barriers, the space of viscosity supersolutions is closed with respect to the Hausdorff topology and is closed under finite intersections. Furthermore, in contrast to weak barriers, these properties for viscosity supersolutions are almost trivial. However, viscosity supersolutions, on the other hand, do not obviously possess the properties required for us to apply Theorem 3.16 as the regularising operation in the application of the Perron method. It is precisely for this reason that the more technical notion of weak barriers is required.

Finally, the results of Sections 6.1 to 6.5 inclusive are very general, and are useful for constructing convex barriers in a wide range of settings. In particular, we leave the enthusiastic reader to verify that they remain valid in any riemannian manifold.

6.1 Distance functions

Let K be a closed, convex subset of $\mathbb{R}^{n+1}$. Let $d_K : \mathbb{R}^{n+1} \to [0,\infty[$ be the distance in $\mathbb{R}^{n+1}$ to K. That is,

$$d_K(x) := \inf_{y\in K} \|x-y\|.$$

Since it is the infimum of a family of convex functions, d_K is also convex. We now consider the closest point projection from $\mathbb{R}^{n+1}\setminus K$ onto K. First, we prove

Lemma 6.1. *Let K be a closed, convex subset of $\mathbb{R}^{n+1}$. Choose $x \in \mathbb{R}^{n+1} \setminus K$. There is at most one point y in the boundary of K with the property that $x = y + t\mathsf{N}$ for some $t > 0$ and for some supporting normal N to K at y.*

Proof. Suppose the contrary. Let y and y' be two such boundary points. Let N and N' be supporting normals to K at y and y' respectively and let $t, t' > 0$ be such that $x = y + t\mathsf{N} = y' + t'\mathsf{N}'$. By definition of the supporting normal,

$$\langle y'-y,\mathsf{N}\rangle,\ \langle y-y',\mathsf{N}'\rangle \leq 0.$$

In particular,

$$\langle y'-y,x-y\rangle,\ \langle y-y',x-y'\rangle \leq 0.$$

Summing these two relations yields $\|y'-y\|^2 \leq 0$, so that $\|y'-y\| = 0$, and so $y' = y$, as desired. □

Lemma 6.2. *Let K be a closed, convex subset of $\mathbb{R}^{n+1}$. For all $x \in \mathbb{R}^{n+1}$, the point $y \in K$ minimising distance to x is unique.*

Proof. Choose $x \in \mathbb{R}^{n+1}$. Let $y \in K$ minimise distance to x. If $x \in K$, then $y = x$ is unique, as desired. Otherwise, denote $\mathsf{N} := (x-y)/\|x-y\|$. By Lemma 4.6, y is a boundary point of K and N is a supporting normal to K at y. In particular $x = y + \|x-y\|\mathsf{N}$, and by Lemma 6.1, y is unique, as desired. This completes the proof. □

We define $\Pi_K : \mathbb{R}^{n+1} \to K$ to be the closest point projection. We now relate Π_K to the derivative of d_K.

Lemma 6.3. *If K as a closed, convex subset of $\mathbb{R}^{n+1}$, then d_K is differentiable at every point x in $\mathbb{R}^{n+1} \setminus K$ and, for all such x,*

$$\Pi_K(x) = x - d_K(x)Dd_K(x).$$

Proof. Choose $x \in \mathbb{R}^{n+1} \setminus K$. Denote $\mathsf{N} = (x - \Pi_K(x))/\|x - \Pi_K(x)\|$. By Lemma 4.6, N is a supporting normal to K at $\Pi_K(x)$. Since the image of Π_K is contained in K, it follows that, for all $y \in \mathbb{R}^{n+1}$, $\langle \Pi_K(y) - \Pi_K(x), \mathsf{N}\rangle \leq 0$. Using the Cauchy-Schwarz inequality and the fact that N has unit length, we therefore obtain, for all y,

$$\begin{aligned} d_K(y) &= \|y - \Pi_K(y)\| \\ &\geq \langle y - \Pi_K(y), \mathsf{N}\rangle \\ &= \langle y - \Pi_K(x), \mathsf{N}\rangle + \langle \Pi_K(x) - \Pi_K(y), \mathsf{N}\rangle \\ &\geq \langle y - \Pi_K(x), \mathsf{N}\rangle. \end{aligned}$$

On the other hand, $d_K(y) \leq d(y, \Pi_K(x))$, so that

$$\langle y - \Pi_K(x), \mathsf{N}\rangle \leq d_K(y) \leq d(y, \Pi_K(x)).$$

The first and the last functions in this inequality are smooth at x. Moreover, the coincide up to order 1 at this point, with derivative equal to N. It follows that d_K is differentiable at x and $Dd_K(x) = \mathsf{N}$. In particular,

$$\Pi_K(x) = x - \|x - \Pi_K(x)\|\mathsf{N} = x - d_K(x)Dd_K(x),$$

as desired. □

Lemma 6.4. *Let K be a closed, convex subset of $\mathbb{R}^{n+1}$. If x is a point of $\mathbb{R}^{n+1} \setminus K$, then d_K is twice differentiable at x if and only if Π_K is differentiable at x. Moreover, at any such point, for all vectors V and W,*

$$\langle D\Pi_K(x)V, W\rangle = \langle \pi(V), \pi(W)\rangle - d_K(x)D^2 d_K(x)(V, W),$$

where π is the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\langle Dd_K(x)\rangle^\perp$.

Proof. By Lemma 6.3, for all $x \in \mathbb{R}^{n+1} \setminus K$, d_K is differentiable at x and $\Pi_K(x) = x - d_K(x)Dd_K(x)$. Since $d_K(x) > 0$, it follows by the product and quotient rules that Dd_K is differentiable at x if and only if Π_K is. Furthermore, at any such point

$$\begin{aligned} \langle D\Pi_K(x)V, W\rangle &= \langle V, W\rangle - \langle V, Dd_K(x)\rangle\langle W, Dd_K(x)\rangle - d_K(x)D^2 d_K(x)(V, W) \\ &= \langle \pi(V), \pi(W)\rangle - d_K(x)D^2 d_K(x)(V, W), \end{aligned}$$

as desired. □

We now consider the regularity of Π_K.

Lemma 6.5. *If K is a closed, convex subset of $\mathbb{R}^{n+1}$, then Π_K is 1-Lipschitz.*

Proof. If $x, x' \in K$, then $\Pi_K(x) = x$ and $\Pi_K(x') = x'$. In particular, $\|\Pi_K(x) - \Pi_K(x')\| = \|x - x'\|$, as desired. If $x \in K$ and if $x' \in \mathbb{R}^{n+1} \setminus K$, then $\pi_K(x) = x$. Define $y' := \Pi_K(x')$. By Lemma 4.6, $(x' - y')/\|x' - y'\|$ is a supporting normal to K at y'. In particular, $\langle x - y', x' - y' \rangle \leq 0$, and so

$$\begin{aligned}\|x - x'\|^2 &= \|(x - y') - (x' - y')\|^2 \\ &= \|x - y'\|^2 - 2\langle x - y', x' - y' \rangle + \|x' - y'\|^2 \\ &\geq \|x - y'\|^2 + \|x' - y'\|^2 \\ &\geq \|x - y'\|^2,\end{aligned}$$

so that $\|\Pi_K(x) - \Pi_K(x')\| \leq \|x - x'\|$, as desired. Finally, choose $x, x' \in \mathbb{R}^{n+1} \setminus K$. Denote $y := \Pi_K(x)$, $y' := \Pi_K(x')$. By Lemma 4.6, $(x - y)/\|x - y\|$ and $(x' - y')/\|x' - y'\|$ are supporting normals to K at y and y' respectively. In particular,

$$\langle y' - y, x - y \rangle, \langle y - y', x' - y' \rangle \leq 0.$$

Consequently,

$$\langle x - x', y - y' \rangle = \langle x - y, y - y' \rangle + \langle y - y', y - y' \rangle + \langle y' - x', y - y' \rangle \geq \|y - y'\|^2.$$

Using the Cauchy-Schwarz inequality, this yields

$$\|y - y'\|^2 \leq \langle x - x', y - y' \rangle \leq \|x - x'\| \|y - y'\|,$$

so that,

$$\|y - y'\|(\|x - x'\| - \|y - y'\|) \geq 0,$$

and we conclude that $\|y - y'\| \leq \|x - x'\|$, as desired. □

Lemma 6.6. *If K is a closed, convex subset of $\mathbb{R}^{n+1}$, then Π_K is differentiable almost everywhere. Moreover, the pointwise derivative of Π_K coincides with its distributional derivative, and $\|D\Pi_K(x)\|_{L^\infty} \leq 1$.*

Proof. Since Π_K is Lipschitz, it follows from Rademacher's Theorem (c.f. Theorem 5.2 of [22]) that Π_K is differentiable almost everywhere and, moreover, that its pointwise derivative coincides with its distributional derivative. Furthermore, since Π_K is 1-Lipschitz, it follows that $\|D\Pi_K\|_{L^\infty} \leq 1$, and this completes the proof. □

Lemma 6.7. *If K is a closed, convex subset of $\mathbb{R}^{n+1}$, then d_K is twice differentiable almost everywhere in $\mathbb{R}^{n+1} \setminus K$. Moreover, the pointwise second derivative of d_K coincides with its second-order distributional derivative, and if d_K is twice differentiable at $x \in \mathbb{R}^{n+1} \setminus K$, then $\|D^2 d_K(x)\| \leq 2/d_K(x)$.*

Proof. By Lemma 6.4, d_K is twice differentiable wherever Π_K is differentiable, and so, by Lemma 6.6, d_K is twice differentiable almost everywhere. By Lemma 6.3, for all $x \in \mathbb{R}^{n+1} \setminus K$,

$$Dd_K(x) = (x - \Pi_K(x))/d_K(x).$$

Since $d_K(x)$ never vanishes over this set, using the quotient rules for pointwise derivatives and for distributional derivatives, it follows from Lemma 6.6 again that the pointwise second-order derivative of d_K coincides with its second-order distributional derivative. Furthermore, at any point x where d_K is twice differentiable, for all vectors V and W,

$$D^2 d_K(V, W) = \frac{1}{d_K(x)}(\langle \pi(V), \pi(W) \rangle - \langle D\Pi_K(x)V, W \rangle),$$

where π is the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\langle Dd_K(x) \rangle^\perp$. In particular, since both π and $D\Pi_K(x)$ have norm 1,

$$\left| D^2 d_K(x)(V, W) \right| \leq \frac{2}{d_K(x)} \|V\| \|W\|,$$

so that $\|D^2 d_K(x)\| \leq 2/d_K(x)$, as desired. □

We also show that the second derivatives of d_K are almost everywhere symmetric.

Lemma 6.8. *For almost all $x \in \mathbb{R}^{n+1} \setminus K$, d_K is twice differentiable at x and its second derivative is symmetric at that point.*

Proof. By Lemma 6.7, d_K has L^∞_{loc} second-order, distributional derivatives over $\mathbb{R}^{n+1} \setminus K$. Denote this second-order distributional derivative by A. Then, for any $\phi \in C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus K)$, and for all $1 \leq i, j \leq n$,

$$\begin{aligned}
\int_{\mathbb{R}^{n+1} \setminus K} A(x)(\partial_i, \partial_j)\phi(x)\mathrm{dVol}_x &= \int_{\mathbb{R}^{n+1} \setminus K} d_K(x) D^2\phi(x)(\partial_j, \partial_i)\mathrm{dVol}_x \\
&= \int_{\mathbb{R}^{n+1} \setminus K} d_K(x) D^2\phi(x)(\partial_i, \partial_j)\mathrm{dVol}_x \\
&= \int_{\mathbb{R}^{n+1} \setminus K} A(x)(\partial_j, \partial_i)\phi(x)\mathrm{dVol}_x.
\end{aligned}$$

Since $\phi \in C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus K)$ is arbitrary, we conclude that $A(x)(\partial_i, \partial_j) = A(x)(\partial_j, \partial_i)$ for almost all $x \in \mathbb{R}^{n+1} \setminus K$, and since $1 \leq i, j \leq n$ are arbitrary, we conclude that $A(x)$ is symmetric for almost all $x \in \mathbb{R}^{n+1} \setminus K$. However, by Lemma 6.7, again, for almost all $x \in \mathbb{R}^{n+1} \setminus K$, d_K is twice differentiable at x in the classical sense and $D^2 d_K(x) = A(x)$, so that $D^2 d_K(x)$ is almost everywhere defined and symmetric, as desired. □

6.2 Convex sets with smooth boundary

Let K be a closed, convex subset of $\mathbb{R}^{n+1}$. Let U be an open subset of $\mathbb{R}^{n+1}$. We denote $U(K) = U \cap (\partial K)$, and we suppose that $U(K)$ is smooth. We now use the terminology of riemannian geometry (c.f. [7]). Let $\mathsf{N} : U(K) \to \Sigma^n$ be the outward-pointing, unit, **normal** vector field over $U(K)$. Let A be the **shape operator** of $U(K)$ associated to this normal. That is, for all $x \in U(K)$ and for any vector V tangent to $U(K)$ at x, $A(x)V = D\mathsf{N}(x)V$.

If $M \in \mathrm{Symm}(2, \mathbb{R}^{n+1})$ is a symmetric matrix over $\mathbb{R}^{n+1}$, and if E is any subspace of $\mathbb{R}^{n+1}$, we denote by $\mathrm{Det}(M; E)$ the determinant of the restriction of M to E. We are interested in estimating $\mathrm{Det}(D^2 d_K; \langle D d_K \rangle^{\perp})$ near $U(K)$. This quantity will be used in the sequel to estimate the gaussian curvature of smooth hypersurfaces approximating K.

In this section, we study the functions d_K and Π_K over the set $\Pi_K^{-1}(U(K))$. We define $\Phi : U(K) \times [0, \infty[\to \mathbb{R}^{n+1}$ by $\Phi(x, t) = x + t\mathsf{N}(x)$.

Lemma 6.9. *Φ defines a smooth diffeomorphism from $U(K) \times [0, \infty[$ onto $\Pi_K^{-1}(U(K))$.*

Proof. We first show that $\mathrm{Im}(\Phi) = \Pi_K^{-1}(U(K))$. Indeed, choose $(x, t) \in U(K) \times [0, \infty[$. Then $\Phi(x, t) = x + t\mathsf{N}(x)$. By Lemma 4.6, x minimises distance to $\Phi(x, t)$ in K so that $x = (\Pi_K \circ \Phi)(x, t)$, and, in particular, $\Phi(x, t) \in \Pi_K^{-1}(U(K))$. Since $(x, t) \in U(K) \times [0, \infty[$ is arbitrary, we conclude that $\mathrm{Im}(\Phi) \subseteq \Pi_K^{-1}(U(K))$. Conversely, choose $y \in \Pi_K^{-1}(U(K))$. Denote $x = \Pi_K(y) \in U(K)$. By definition, x minimises distance in K to y. There are two cases to consider. First, if $y \in K$, then $y = x = \Phi(x, 0)$, so that $y \in \mathrm{Im}(\Phi)$. Second, if $y \in \mathbb{R}^{n+1} \setminus K$, then, by Lemma 4.6, there exists $t > 0$ such that $y = x + t\mathsf{N}(x) = \Phi(x, t)$, so that $y \in \mathrm{Im}(\Phi)$ in this case also. Since $y \in \Pi_K^{-1}(U(K))$ is arbitrary, we conclude that $\Pi_K^{-1}(U(K)) \subseteq \mathrm{Im}(\Phi)$, and the two sets therefore coincide, as desired.

If $x, x' \in U(K)$ and $t, t' \in [0, \infty[$ are such that $x + t\mathsf{N}(x) = x' + t'\mathsf{N}(x')$, then, by Lemma 6.1, $x = x'$ and $t = t'$, and it follows that Φ is injective. It remains to show that Φ is smooth with smooth inverse. Choose $(x, t) \in U(K) \times [0, \infty[$. Denote by ∂_t the unit vector in the t direction. Observe that

$$\begin{aligned} & D\Phi(x,t)(0,\partial_t) = \mathsf{N}(x) \\ \Rightarrow \quad & \|D\Phi(x,t)(0,\partial_t)\|^2 = 1. \end{aligned}$$

Let X be a tangent vector to ∂K at x. Then,

$$\begin{aligned} D\Phi(x)(X,0) &= X + tD\mathsf{N}(x)X \\ &= X + tA(x)X \\ \Rightarrow \quad \|D\Phi(x)(X,0)\|^2 &= \|(\mathrm{Id} + tA(x))(X)\|^2. \end{aligned}$$

However, by convexity, $A(x)$ is non-negative definite, and so,

$$\|D\Phi(x)(X,0)\|^2 = \|(\mathrm{Id} + tA(x))(X)\|^2 \geq \|X\|^2.$$

Finally, bearing in mind that $\langle A(x)X, \mathsf{N}(x)\rangle = 0$,

$$\langle D\Phi(x,t)(X,0), D\Phi(x,t)(0,\partial_t)\rangle = \langle X + tA(x)X, \mathsf{N}(x)\rangle = 0.$$

It follows that $\|D\Phi(x,t)(V)\|^2 > 0$ for all non-zero V and so $D\Phi(x,t)$ is invertible. Since $(x,t) \in U \times [0,\infty[$ is arbitrary, we conclude from the inverse function theorem that Φ is everywhere a smooth local diffeomorphism. By injectivity, it is a smooth global diffeomorphism, and this completes the proof. $\square$

Lemma 6.10. *Π_K and d_K define smooth functions over $\Pi_K^{-1}(U(K)) \setminus K$. Moreover, for all vectors V and W,*

$$\begin{aligned} Dd_K(x)(V) &= \langle \mathsf{N}(\Pi_K(x)), V\rangle, \\ D^2 d_K(x)(V,W) &= \langle A(\Pi_K(x)) D\Pi_K(x)V, W\rangle. \end{aligned}$$

Proof. Choose $(x,t) \in U(K)\times]0,\infty[$. Since $\Phi(x,t) = x + t\mathsf{N}(x)$, by Lemma 4.6, x minimises distance in K to $\Phi(x,t)$. It follows that $(d_K \circ \Phi)(x,t) = t$ and $(\Pi_K \circ \Phi)(x,t) = x$. In particular, $\Pi_K \circ \Phi$ and $d_K \circ \Phi$ are both smooth, and, composing with Φ^{-1}, we conclude that d_K and Π_K are also both smooth, as desired. Now choose $x \in \Pi_K^{-1}(U(K)) \setminus K$. Observe that $\mathsf{N}(\Pi_K(x))$ is the unique supporting normal to K at $\Pi_K(x)$. Thus, by Lemma 6.3,

$$Dd_K(x) = \frac{1}{d_K(x)}\left(x - \Pi_K(x)\right) = \mathsf{N}(\Pi_K(x)).$$

The formula for the second derivative of d_K follows by differentiating this relation, and this completes the proof. $\square$

Lemma 6.11. *For every compact subset X of U, there exists $C > 0$ such that for all $x \in \Pi_K^{-1}(X \cap U(K)) \setminus K$,*

$$\|D\Pi_K(x) - \pi\| \leq C d_K(x),$$

where π is the orthogonal projection onto $\langle Dd_K(x)\rangle^{\perp}$.

Proof. Let C be such that $\|A(y)\| \leq C$ for all $y \in X \cap U(K)$. By Lemma 6.6, $\|D\Pi_K(x)\| \leq 1$. Thus, by Lemma 6.10, for all vectors V and W,

$$\begin{aligned} \left|D^2 d_K(x)(V,W)\right| &= |\langle A(\Pi_K(x)) D\Pi_K(x)V, W\rangle| \\ &\leq \|A(\Pi_K(x))\| \|V\| \|W\| \\ &\leq C\|V\|\|W\|. \end{aligned}$$

However, by Lemma 6.4, for all vectors V and W,

$$\langle D\Pi_K(x)V, W\rangle = \langle \pi(V), \pi(W)\rangle - d_K(x)D^2d_K(x)(V,W),$$

where π is the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\langle Dd_K(x)\rangle^\perp$. Since $\langle\pi(V),\pi(W)\rangle = \langle\pi(V), W\rangle$, it follows that,

$$|\langle D\Pi_K(x)V - \pi(V), W\rangle| \leq d_K(x)\left|D^2d_K(x)(V,W)\right| \leq Cd_K(x)\|V\|\|W\|,$$

so that $\|D\Pi_K(x) - \pi\| \leq Cd_K(x)$, as desired. □

Lemma 6.12. *Choose $k > 0$ and suppose that $U(K)$ has gaussian curvature everywhere at least k. For every compact subset X of U and for all $\epsilon > 0$, there exists $r > 0$ such that for all $x \in \Pi_K^{-1}(X \cap U(K)) \setminus K$, if $d_K(x) < r$, then $\mathrm{Det}(D^2d_K(x); \langle Dd_K(x)\rangle^\perp) \geq (k-\epsilon)^n$.*

Proof. By compactness, there exists $\delta > 0$ such that for all $x \in X \cap U(K)$ and for all $M \in B_\delta(A(x))$, $\mathrm{Det}(M; \langle Dd_K(x)\rangle^\perp) \geq (k-\epsilon)^n$. Let C_1 be such that for all $y \in X \cap U(K)$, $\|A(y)\| \leq C_1$. Let C_2 be as in Lemma 6.11. If $x \in \Pi_K^{-1}(X \cap U(K))$ is such that $d_K(x) < \delta/C_1C_2$, then, for all vectors V and W in $\langle Dd_K(x)\rangle^\perp$,

$$\begin{aligned}\left|D^2d_K(x)(V,W) - \langle A(\Pi_K(x))V, W\rangle\right| &= |\langle A(\Pi_K(x))(D\Pi_K(x)(V) - V), W\rangle| \\ &< \delta\|V\|\|W\|,\end{aligned}$$

so that $\mathrm{Det}(D^2d_K(x); \langle Dd_K(x)\rangle^\perp) \geq (k-\epsilon)^n$, as desired. □

6.3 Intersecting convex sets

Let K_1 and K_2 be compact, convex subsets of $\mathbb{R}^{n+1}$ whose intersection has non-trivial interior. Let U be an open subset of $\mathbb{R}^{n+1}$ and suppose that $U(K_1)$ and $U(K_2)$ are both smooth of gaussian curvature at least k. We denote $K := K_1 \cap K_2$, we denote by N_1 and N_2 the outward-pointing, unit, normal vector fields over $K_1(U)$ and $K_2(U)$ respectively and we denote by A_1 and A_2 their respective shape operators. Moreover, we denote $d := d_{K_1\cap K_2}$, $d_1 := d_{K_1}$ and $d_2 := d_{K_2}$, and $\Pi := \Pi_{K_1\cap K_2}$, $\Pi_1 := \Pi_{K_1}$ and $\Pi_2 := \Pi_{K_2}$. We recall by Lemma 6.8 that d is almost everywhere twice differentiable with symmetric second derivative. We are now interested in estimating lower bounds for $\mathrm{Det}(D^2d; \langle Dd\rangle^\perp)$. There are four different cases to consider.

Lemma 6.13 (Case 1). *If $x \in \Pi^{-1}(U(K) \cap (\partial K_1) \cap K_2^o) \setminus K$ then $d = d_1$ and $\Pi = \Pi_1$.*

Remark. *Observe that this set is open, and so $Dd = Dd_1$ and $D^2d = D^2d_1$ over this set.*

Proof. Denote $y := \Pi(x)$. Denote $\mathsf{N} := (x-y)/\|x-y\|$. By Lemma 4.6, N is a supporting normal to K at y. Since $y \in \partial K_1 \cap K_2^o$, By Lemma 4.9, N is also a supporting normal to K_1 at y. By Lemma 4.6, y minimises distance in K_1 to x. In particular, $d_1(x) = \|x-y\| = d(x)$, and $\Pi_1(x) = y = \Pi(x)$, as desired. □

Lemma 6.14 (Case 2). *If $x \in \Pi^{-1}(U(K) \cap (\partial K_2) \cap K_1^o) \setminus K$ then $d = d_2$ and $\Pi = \Pi_2$.*

Proof. Denote $y := \Pi(x)$. Denote $\mathsf{N} := (x-y)/\|x-y\|$. By Lemma 4.6, N is a supporting normal to K at y. Since $y \in \partial K_2 \cap K_1^o$, by Lemma 4.9, N is also a supporting normal to K_2 at y. By Lemma 4.6, y minimises distance in K_2 to x. In particular, $d_2(x) = \|x-y\| = d(x)$, and $\Pi_2(x) = y = \Pi(x)$, as desired. □

Lemma 6.15 (Case 3). *If $x \in \Pi^{-1}(U(K) \cap (\partial K_1) \cap (\partial K_2)) \setminus K$, if $(\mathsf{N}_1 \circ \Pi)(x) = (\mathsf{N}_2 \circ \Pi)(x)$, and if d is twice differentiable at x, then, for every vector V,*

$$D^2 d(x)(V,V) \geq \mathrm{Max}(D^2 d_1(x)(V,V), D^2 d_2(x)(V,V)).$$

Proof. Denote $y := \Pi(x)$. By Lemma 4.6, $Dd(x)$ is a supporting normal to K at y. By Theorem 5.20, the set of supporting normals to K at y is the convex hull of $\{\mathsf{N}_1(y), \mathsf{N}_2(y)\}$. Since these two points coincide, this convex hull consists of a single point, and so $Dd(x) = \mathsf{N}_1(y) = \mathsf{N}_2(y)$. In particular, $Dd(x)$ is also a supporting normal to both K_1 and K_2 at y. It follows from Lemma 4.6 that y minimises distance in K_1 to x. In particular, $d(x) = d_1(x)$. However, since $K \subseteq K_1$, for all y,

$$d(y) = \mathop{\mathrm{Inf}}_{z \in K} \|y - z\| \geq \mathop{\mathrm{Inf}}_{z \in K_1} \|y - z\| = d_1(y).$$

Thus, by differentiating, for all vectors V,

$$D^2 d(x)(V,V) \geq D^2 d_1(x)(V,V).$$

In like manner, we show that $D^2 d(x)(V,V) \geq D^2 d_2(x)(V,V)$, and so

$$D^2 d(x)(V,V) \geq \mathrm{Max}(D^2 d_1(V,V), D^2 d_2(V,V)),$$

as desired. □

Before treating the fourth case, we require the following preliminary result.

Lemma 6.16. *If $x \in \Pi^{-1}(U(K) \cap (\partial K_1) \cap (\partial K_2)) \setminus K$ and if $(\mathsf{N}_1 \circ \Pi)(x) \neq (\mathsf{N}_2 \circ \Pi)(x)$, then there exists a unique $s \in [0,1]$ such that*

$$Dd(x) = \frac{1-s}{l}(\mathsf{N}_1 \circ \Pi)(x) + \frac{s}{l}(\mathsf{N}_2 \circ \Pi)(x),$$

where

$$l = \|(1-s)(\mathsf{N}_1 \circ \Pi)(x) + s(\mathsf{N}_2 \circ \Pi)(x)\|.$$

In particular,

$$l \geq \frac{1}{2}\|(\mathsf{N}_1 \circ \Pi)(x) + (\mathsf{N}_2 \circ \Pi)(x)\|.$$

Proof. Denote $y := \Pi(x)$. By Lemma 4.6, $Dd(x)$ is a supporting normal to K at $\Pi(x)$. By Theorem 5.20, the set of supporting normals to K at $\Pi(x)$ is the convex hull of $\{\mathsf{N}_1(y), \mathsf{N}_2(y)\}$. This coincides with the great-circular arc joining $\mathsf{N}_1(y)$ to $\mathsf{N}_2(y)$ (c.f. Section 5.2), and the first assertion follows. Now observe that the vectors $\mathsf{N}_1(y) + \mathsf{N}_2(y)$ and $\mathsf{N}_1(y) - \mathsf{N}_2(y)$ are orthogonal. Thus

$$\begin{aligned} l^2 &= \|(1-s)\mathsf{N}_1(y) + s\mathsf{N}_2(y)\|^2 \\ &= \|\frac{1}{2}(\mathsf{N}_1(y) + \mathsf{N}_2(y)) + \frac{(1-2s)}{2}(\mathsf{N}_1(y) - \mathsf{N}_2(y))\|^2 \\ &= \frac{1}{4}\|\mathsf{N}_1(y) + \mathsf{N}_2(y)\|^2 + \frac{(1-2s)^2}{4}\|\mathsf{N}_1(y) - \mathsf{N}_2(y)\|^2 \\ &\geq \frac{1}{4}\|\mathsf{N}_1(y) + \mathsf{N}_2(y)\|^2, \end{aligned}$$

and the second assertion follows. This completes the proof. □

Lemma 6.17 (Case 4a). *If $x \in \Pi^{-1}(U(K)\cap(\partial K_1)\cap(\partial K_2)) \setminus K$, if $(\mathsf{N}_1\circ\Pi)(x) \neq (\mathsf{N}_2\circ\Pi)(x)$, and if d is twice differentiable at x, then for every vector V and for every vector W which is orthogonal to both $(\mathsf{N}_1 \circ \Pi)(x)$ and $(\mathsf{N}_2 \circ \Pi)(x)$,*

$$D^2d(x)(V,W) = \frac{1-s}{l}\langle A_1(\Pi(x))D\Pi(x)V, W\rangle + \frac{s}{l}\langle A_2(\Pi(x))D\Pi(x)V, W\rangle,$$

where s and l are as in Lemma 6.16.

Remark. *Upon applying an isometry, we may suppose that the linear span of $\{e_n, e_{n+1}\}$ coincides with that of $\{(\mathsf{N}_1 \circ \Pi)(x), (\mathsf{N}_2 \circ \Pi)(x)\}$. Consequently, when $D^2d(x)$ is symmetric, this result determines every component of $D^2d(x)$ except $D^2d(x)(e_i, e_j)$ for $(i,j) \in \{n, n+1\}^2$.*

Proof. Denote $y := \Pi(x)$. Let V be a vector in $\mathbb{R}^{n+1}$. Define $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ by $\gamma(t) = x + tV$. Let $(t_m)_{m\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}$ converging to 0. For all m, denote $x_m := \gamma(t_m)$ and $y_m := (\Pi \circ \gamma)(t_m)$. Upon extracting a subsequence, we may suppose that one of the following holds.
1: $x_m \in \Pi^{-1}(U(K)\cap(\partial K_1)\cap K_2^o) \setminus K$ for all m. By Lemma 6.13, for all m, $Dd(x_m) = Dd_1(x_m)$. Taking limits and bearing in mind Lemma 6.10, it follows that

$$Dd(x) = Dd_1(x) = (\mathsf{N}_1 \circ \Pi)(x) = \mathsf{N}_1(y).$$

In particular, $s=0$ and $l=1$. Moreover, for all vectors W and for all m,

$$\frac{1}{t_m}\langle Dd(x_m)-Dd(x),W\rangle=\frac{1}{t_m}\langle \mathsf{N}_1(y_m)-\mathsf{N}_1(y),W\rangle,$$

so that, by the chain rule, upon taking limits, we obtain

$$D^2d(x)(V,W)=\langle A_1(y)D\Pi(x)V,W\rangle,$$

as desired.
2: $x_m\in\Pi^{-1}(U(K)\cap(\partial K_2)\cap K_1^o)\setminus K$ for all m. As in Step (1), we show that $s=1$, $l=1$ and

$$D^2d(x)(V,W)=\langle A_2(y)D\Pi(x)V,W\rangle,$$

as desired.
3: $x_m\in\Pi^{-1}(U(K)\cap(\partial K_1)\cap(\partial K_2))\setminus K$ for all m. For all m, denote $\mathsf{N}_{1,m}:=\mathsf{N}_1(y_m)$ and $\mathsf{N}_{2,m}:=\mathsf{N}_2(y_m)$. Observe that, for sufficiently large m, $\mathsf{N}_{1,m}\neq\mathsf{N}_{2,m}$. Thus, by Lemma 6.16, for all $m\in\mathbb{N}$, there exists a unique $s_m\in[0,1]$ such that

$$Dd(x_m)=\frac{1-s_m}{l_m}\mathsf{N}_{1,m}+\frac{s_m}{l_m}\mathsf{N}_{2,m},$$

where $l_m:=\|(1-s_m)\mathsf{N}_{1,m}+s_m\mathsf{N}_{2,m}\|$. Since N_1, N_2, Dd and Π are continuous, $(s_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ converge to the limits s_∞ and l_∞ respectively. Let W be a vector normal to both $\mathsf{N}_1(y)$ and $\mathsf{N}_2(y)$. In particular W is normal to $Dd(x)$. For all m,

$$\begin{aligned}&\frac{1}{t_m}\langle Dd(x_m)-Dd(x),W\rangle\\=&\frac{1}{t_m}\langle Dd(x_m),W\rangle\\=&\frac{1-s_m}{l_mt_m}\langle\mathsf{N}_{1,m},W\rangle+\frac{1-s_m}{l_mt_m}\langle\mathsf{N}_{2,m},W\rangle\\=&\frac{1-s_m}{l_mt_m}\langle\mathsf{N}_{1,m}-\mathsf{N}_1(y),W\rangle+\frac{1-s_m}{l_mt_m}\langle\mathsf{N}_{2,m}-\mathsf{N}_2(y),W\rangle.\end{aligned}$$

By the chain rule, upon taking limits, we obtain

$$D^2d(x)(V,W)=\frac{1-s}{l}\langle A_1(y)D\Pi(x)V,W\rangle+\frac{s}{l}\langle A_2(y)D\Pi(x)V,W\rangle,$$

as desired. □

Lemma 6.18 (Case 4b). *If $x\in\Pi^{-1}(U(K)\cap(\partial K_1)\cap(\partial K_2))\setminus K$, if $(\mathsf{N}_1\circ\Pi)(x)\neq(\mathsf{N}_2\circ\Pi)(x)$, and if d is twice differentiable at x, then for every vector V and for every vector W which is tangent to the linear span of $\{(\mathsf{N}_1\circ\Pi)(x),(\mathsf{N}_2\circ\Pi)(x)\}$ and normal to $Dd(x)$,*

$$D^2d(x)(V,W)=\frac{1}{d(x)}\langle V,W\rangle.$$

Remark. *Upon applying an isometry, we may suppose that* $e_{n+1} = Dd(x)$ *and that the linear span of* $\{e_n, e_{n+1}\}$ *coincides with that of* $\{(\mathsf{N}_1 \circ \Pi)(x), (\mathsf{N}_2 \circ \Pi)(x)\}$*. Consequently when* $D^2d(x)$ *is symmetric, this result along with Lemma 6.17 determines every component of* $D^2d(x)$ *except for* $D^2d(x)(e_{n+1}, e_{n+1})$*. In fact, we readily show that* $D^2d(x)(e_{n+1}, e_{n+1}) = 0$*, but since this is not necessary for our purposes, we leave it as an easy exercise for the interested reader.*

Proof. Denote $y := \Pi(x)$. By Theorem 5.20, the set of supporting normals to K at y coincides with the convex hull of $\{\mathsf{N}_1(y), \mathsf{N}_2(y)\}$, which in turn coincides with the great-circular arc joining $\mathsf{N}_1(y)$ to $\mathsf{N}_2(y)$. Denote this great-circular arc by $\mathsf{N} : [0, 1] \to \Sigma$. In particular, for all r, $\mathsf{N}(r)$ is a supporting normal to K at y. We define $\gamma : [0, 1] \to \mathbb{R}^{n+1}$ by $\gamma(r) = y + d(x)\mathsf{N}(r)$. By Lemma 4.6, for all t, y minimises distance in K to $\gamma(r)$. In particular, by Lemma 6.3, for all r, $(Dd \circ \gamma)(r) = \mathsf{N}(r)$. Let s be as in Lemma 6.16. Since W lies in the plane spanned by $\mathsf{N}_1(y)$ and $\mathsf{N}_2(y)$ but is normal to $Dd(x)$, upon multiplying by a scalar factor, we may suppose that $W = (\partial_r \gamma)(s)$. Thus

$$\begin{aligned} D^2d(x)(W, V) &= \langle \partial_r(Dd \circ \gamma)(s), V \rangle \\ &= \langle (\partial_r \mathsf{N})(s), V \rangle \\ &= \frac{1}{d(x)} \langle (\partial_r \gamma)(s), V \rangle \\ &= \frac{1}{d(x)} \langle W, V \rangle, \end{aligned}$$

as desired. □

Lemma 6.19. *For every compact subset* X *of* U *there exists* $C > 0$ *with the property that for every* $x \in (\partial K_1) \cap (\partial K_2) \cap X$*,*

$$\|\mathsf{N}_1(x) + \mathsf{N}_2(x)\| \geq \frac{1}{C}.$$

Proof. Suppose the contrary. By compactness, there exists $x \in (\partial K_1) \cap (\partial K_2) \cap X$ such that $\mathsf{N}_1(x) + \mathsf{N}_2(x) = 0$. By definition of supporting normals, for all $y \in K_1 \cap K_2$, $\langle y - x, \mathsf{N}_1(x) \rangle \leq 0$ and $\langle y - x, \mathsf{N}_2(x) \rangle \leq 0$. Since $\mathsf{N}_1(x) = -\mathsf{N}_2(x)$, it follows that for all $y \in K_1 \cap K_2$, $\langle y - x, \mathsf{N}_1 \rangle = \langle y - x, \mathsf{N}_2 \rangle = 0$. In other words $K_1 \cap K_2$ is contained in the hyperplane normal to $\mathsf{N}_1 = -\mathsf{N}_2$ passing through x, and therefore has trivial interior. This is absurd, and the result follows. □

Lemma 6.20. *For every compact subset* X *of* U *there exists* $C > 0$ *with the property that if* $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_1) \cap (\partial K_2)) \setminus K$*, if* $(\mathsf{N}_1 \circ \Pi)(x) \neq (\mathsf{N}_2 \circ \Pi)(x)$*, if* d *is twice differentiable at* x*, and if* $D^2d(x)$ *is symmetric, then*

$$\|D\Pi(x) - \pi^{1,2}\| \leq Cd(x),$$

where $\pi^{1,2}$ *is the orthonogonal projection from* $\mathbb{R}^{n+1}$ *onto* $\langle(\mathsf{N}_1\circ\Pi)(x),(\mathsf{N}_2\circ\Pi)(x)\rangle^\perp$.

Proof. Denote $y:=\Pi(x)$. Let $C_1>0$ be such that $\|A_1(z)\|\leq C_1$ and $\|A_2(z)\|\leq C_1$ for all z in $X\cap U(K_1)$ and $X\cap U(K_2)$ respectively. Let $C_2>0$ be as in Lemma 6.19. By Lemma 6.4, Π is differentiable at x and, for all vectors V and W,

$$\langle D\Pi(x)V,W\rangle=\langle\pi(V),\pi(W)\rangle-d(x)D^2d(x)(V,W),$$

where π is the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\langle Dd(x)\rangle^\perp$. Observe, in particular, that since D^2d is symmetric, so too is $D\Pi$. Let V be any vector in $\mathbb{R}^{n+1}$. Define $\gamma:\mathbb{R}\to\mathbb{R}^{n+1}$ by $\gamma(t):=x+tV$. Since $(\Pi\circ\gamma)(t)\in K$ for all t, it follows that for each $\mathsf{N}\in\{\mathsf{N}_1(y),\mathsf{N}_2(y)\}$ and for all t,

$$\langle(\Pi\circ\gamma)(t)-y,\mathsf{N}\rangle\leq 0.$$

By the chain rule, differentiating this relation yields

$$\langle D\Pi(x)V,\mathsf{N}\rangle=0.$$

Thus, by linearity and symmetry, for any vector W in the linear span of $\{\mathsf{N}_1(y),\mathsf{N}_2(y)\}$,

$$\langle D\Pi(x)W,V\rangle=\langle D\Pi(x)(V),W\rangle=0.$$

Now let V and W both be orthogonal to $\langle\mathsf{N}_1(y),\mathsf{N}_2(y)\rangle$. In particular, V and W are both orthogonal to $Dd(x)$. Thus, by Lemma 6.4,

$$\langle D\Pi(x)V,W\rangle=\langle V,W\rangle-d(x)D^2d(x)(V,W).$$

Let s and l be as in Lemma 6.16. Then, by Lemma 6.17 and bearing in mind Lemma 6.6,

$$\begin{aligned}\left|D^2d(x)(V,W)\right|&=\left|\frac{1-s}{l}\langle A_1(y)D\Pi(x)V,W\rangle+\frac{s}{l}\langle A_2(y)D\Pi(x)V,W)\rangle\right|\\&\leq\frac{1-s}{l}C_1\|V\|\|W\|+\frac{s}{l}C_1\|V\|\|W\|\\&\leq 2C_1C_2\|V\|\|W\|.\end{aligned}$$

Thus

$$|\langle D\Pi(x)V,W\rangle-\langle V,W\rangle|\leq 2C_1C_2d(x)\|V\|\|W\|.$$

Combining these relations, we conclude that $\|D\Pi(x)-\pi^{1,2}\|\leq 2C_1C_2d(x)$, as desired. $\square$

Lemma 6.21. *For every compact subset X of U and for all $\epsilon > 0$, there exists $r > 0$ with the property that if $x \in \Pi^{-1}(X \cap U(K)) \setminus K$, if $d(x) < r$ and if $D^2 d(x)$ is defined and is symmetric, then*

$$\mathrm{Det}(D^2 d(x); \langle Dd(x)\rangle^{\perp}) \geq (k-\epsilon)^n.$$

Proof. We consider the following cases.
1: Suppose that $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_1) \cap K_2^o) \setminus K$. By Lemma 6.13, $D^2 d(x) = D^2 d_1(x)$, and the result follows by Lemma 6.12.
2: Suppose that $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_2) \cap K_1^o) \setminus K$. By Lemma 6.14, $D^2 d(x) = D^2 d_2(x)$, and the result follows by Lemma 6.12.
3: Suppose that $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_1) \cap (\partial K_2)) \setminus K$ and $\mathsf{N}_1(\Pi(x)) = \mathsf{N}_2(\Pi(x))$. By Lemma 6.15, for all vectors $V \in \mathbb{R}^{n+1}$,

$$D^2 d(x)(V,V) \geq \mathrm{Max}(D^2 d_1(x)(V,V), D^2 d_2(x)(V,V)).$$

In particular, bearing in mind that $Dd(x) = Dd_1(x) = Dd_2(x)$,

$$\mathrm{Det}(D^2 d(x); \langle Dd(x)\rangle^{\perp}) \geq \mathrm{Det}(D^2 d_1(x); \langle Dd_1(x)\rangle^{\perp}), \mathrm{Det}(D^2 d_2(x); \langle Dd_2(x)\rangle^{\perp}),$$

and the result now follows by Lemma 6.12.
4: Suppose that $x \in \Pi^{-1}(X \cap U(K) \cap (\partial K_1) \cap (\partial K_2)) \setminus K$ and that $\mathsf{N}_1(\Pi(x)) \neq \mathsf{N}_2(\Pi(x))$. Denote $y := \Pi(x)$. Let s and l be as in Lemma 6.16. Let $C_1 > 1$ be such that $(1/C_1)\mathrm{Id} \leq A_1(y) \leq C_1\mathrm{Id}$ and $(1/C_1)\mathrm{Id} \leq A_2(y) \leq C_1\mathrm{Id}$ for all y in $X \cap U(K_1)$ and $X \cap U(K_2)$ respectively. Let $C_2 \geq 0$ be as in Lemma 6.20. Define $r := 1/(2C_1^2 C_2)$. Then if $d(x) < r$, for all vectors V normal to $\mathsf{N}_1(y)$ and $\mathsf{N}_2(y)$,

$$\begin{aligned}\langle A_1(y)D\Pi(x)V, V\rangle &= \langle A_1(y)V, V\rangle + \langle A_1(y)(D\Pi(x) - \pi^{1,2})(V), V\rangle \\ &\geq \frac{1}{C_1}\|V\|^2 - \frac{1}{2C_1}\|V\|^2 \\ &= \frac{1}{2C_1}\|V\|^2.\end{aligned}$$

Likewise, for all such x and V,

$$\langle A_2(y)D\Pi(x)V, V\rangle \geq \frac{1}{2C_1}\|V\|^2.$$

Thus, if s and l are as in Lemma 6.16, by Lemma 6.17, for all such x and V,

$$\begin{aligned}D^2 d(x)(V,V) &= \frac{1-s}{l}\langle A_1(y)D\Pi(x)V, V\rangle + \frac{s}{l}\langle A_2(y)D\Pi(x)V, V\rangle \\ &\geq \frac{1-s}{2C_1 l}\|V\|^2 + \frac{s}{2C_1 l}\|V\|^2 \\ &\geq \frac{1}{2C_1}\|V\|^2.\end{aligned}$$

Upon applying an isometry, we may suppose that the plane spanned by e_n and e_{n+1} coincides with the plane spanned by $\mathsf{N}_1(y)$ and $\mathsf{N}_2(y)$ and furthermore that $e_{n+1} = Dd(x)$. We denote by M the restriction of $D^2d(x)$ to $\langle e_1, ..., e_{n-1}\rangle$. By the preceeding discussion, $M \geq (1/2C_1)\mathrm{Id}$. However, by Lemma 6.18, for all i,

$$D^2d(x)(e_i, e_n) = D^2d(x)(e_n, e_i) = \frac{1}{d(x)}\delta_{in}.$$

Reducing r if necessary, we may suppose that $r < (2C_1)^{1-n}(k-\epsilon)^{-n}$ so that, if $d(x) < r$, then

$$\mathrm{Det}(D^2d(x), \langle Dd(x)\rangle^\perp) \geq (k-\epsilon)^n,$$

as desired. □

6.4 Smoothing functions and convexity

Let $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ be a smooth, non-negative function such that $\chi = 0$ outside the unit ball $B_1(0)$, and

$$\int_{\mathbb{R}^{n+1}} \chi(x)\mathrm{dVol}_x = 1$$

For all $s > 0$, we define $\chi_s \in C_0^\infty(\mathbb{R}^{n+1})$ by

$$\chi_s(x) := s^{-(n+1)}\chi(x/s).$$

Let E be a finite-dimensional vector space. For any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}, E)$, and for all $s > 0$, we define the function $f_s : \mathbb{R}^{n+1} \to E$ by,

$$f_s(x) := \int_{\mathbb{R}^{n+1}} f(x-y)\chi_s(y)\mathrm{dVol}_y.$$

We recall the following properties of smoothing functions.

Lemma 6.22. *For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and for all $s > 0$, f_s is continuous.*

Remark: *In fact, as is well known, f_s is smooth.*

Proof. Choose $x \in \mathbb{R}^{n+1}$ and $s > 0$. By local uniform continuity, there exists $\delta > 0$ such that if $\|y\| < s$ and $\|z\| < \delta$, then $|\chi_s(y+z) - \chi_s(y)| < \epsilon$.

Thus, if $\|z-x\|<\delta$, using a change of variable, we obtain

$$\begin{aligned}\|f_s(z)-f_s(x)\| &= \|\int_{\mathbb{R}^{n+1}} f(z-y)\chi_s(y)-f(x-y)\chi_s(y)\mathrm{dVol}_y\|\\ &= \|\int_{\mathbb{R}^{n+1}} f(x-y)(\chi_s(y+(z-x))-\chi_s(y))\mathrm{dVol}_y\|\\ &\leq \int_{\mathbb{R}^{n+1}} \|f(x-y)\|\,|\chi_s(y+(z-x))-\chi_s(y)|\,\mathrm{dVol}_y\\ &\leq \epsilon\int_{B_{s+\delta}(x)} \|f(x-y)\|\mathrm{dVol}_y.\end{aligned}$$

Since ϵ may be chosen arbitrarily small, continuity of f_s at x follows. Since $x\in\mathbb{R}^{n+1}$ is arbitrary, it follows that f_s is continuous, as desired. □

Lemma 6.23. *Choose $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. If f has L^1_{loc} distributional derivatives, then, for all $s>0$, f_s is differentiable and $D(f_s)=(Df)_s$.*

Proof. Choose $x\in\mathbb{R}^{n+1}$ and $s>0$. Choose $\epsilon>0$. Since χ_s is smooth, there exists $\eta>0$ with the property that for all y and for all vectors V such that $\|V\|\leq\eta$,

$$|\chi_s(y+V)-\chi_s(y)-D\chi_s(y)V|\leq\epsilon\|V\|.$$

Thus, using the definition of the distributional derivative and a change of variable, for all V such that $\|V\|\leq\eta$, we obtain

$$\begin{aligned}&\|f_s(x+V)-f_s(x)-(Df)_s(x)V\|\\ =&\|\int_{\mathbb{R}^{n+1}} f(x+V-y)\chi_s(y)-f(x-y)\chi_s(y)-Df(x-y)V\chi_s(y)\mathrm{dVol}_y\|\\ =&\|\int_{\mathbb{R}^{n+1}} f(x-y)(\chi_s(y+V)-\chi_s(y)-D\chi_s(y)V)\mathrm{dVol}_y\|\\ \leq&\epsilon\|V\|\int_{B_s(x)}\|f(x-y)\|\mathrm{dVol}_y.\end{aligned}$$

Since V is arbitrary, and since ϵ may be chosen arbitrarily small, we conclude that f_s is differentiable at x with derivative equal to $(Df)_s(x)$, as desired. □

Lemma 6.24. *Choose $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. If f has L^1_{loc} distributional derivatives up to order k, then, for all $s>0$, f_s is C^k and $J^k(f_s)=(J^kf)_s$.*

Proof. We prove this by induction on k. By Lemma 6.23, the result holds for $k=1$. Suppose that f has L^1_{loc} distributional derivatives up to order $k+1$. In particular, J^kf is L^1_{loc} and has L^1_{loc} distributional derivatives. By the induction hypothesis, f_s is k-times differentiable and

$J^k(f_s) = (J^k f)_s$. However, by Lemma 6.23 again, $(J^k f)_s$ is differentiable, and $J^1(J^k f)_s = (J^1 J^k f)_s = (J^{k+1} f)_s$. It follows that f_s is $(k+1)$-times differentiable and $J^{k+1}(f_s) = J^1 J^k(f_s) = J^1(J^k f)_s = (J^{k+1} f)_s$. The result now follows by induction. □

Importantly, the smoothing operation preserves preserves convexity.

Lemma 6.25. *If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is convex, then so too is f_s for all $s > 0$.*

Proof. Fix $s > 0$. Using the convexity of f and the positivity of χ_s, for all $x, y \in \mathbb{R}^{n+1}$ and for all $t \in [0,1]$, we obtain

$$\begin{aligned} f_s(tx + (1-t)y) &= \int_{B_s(0)} f(tx + (1-t)y + z)\chi_s(z)\mathrm{dVol}_z \\ &= \int_{B_s(0)} f(t(x+z) + (1-t)(y+z))\chi_s(z)\mathrm{dVol}_z \\ &\geq \int_{B_s(0)} \left(tf(x+z) + (1-t)f(y+z)\right)\chi_s(z)\mathrm{dVol}_z \\ &= tf_s(x) + (1-t)f_s(y), \end{aligned}$$

so that f_s is convex, as desired. □

Often when smoothing functions are defined, it is not necessary to suppose non-negativity. However, when χ is taken to be non-negative, Lemma 6.25 can be refined to yield

Lemma 6.26. *Let E be a finite dimensional vector space. Let K be a closed convex subset of E. Let U be an open subset of $\mathbb{R}^{n+1}$. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ be such that for almost all $x \in U$, $f(x) \in K$. Then for all $s > 0$ and for all x with the property that $B_s(x) \subseteq U$, we have $f_s(x) \in K$.*

Proof. We use the terminology of Section 5. Let $H(\mathsf{N}, t)$ be an open half-space of E containing K. In particular, $\langle z, \mathsf{N}\rangle < t$ for all $z \in K$. Choose $s > 0$ and $x \in \mathbb{R}^{n+1}$ such that $B_s(x) \subseteq U$. Then, bearing in mind that χ is positive,

$$\begin{aligned} \langle f_s(x), \mathsf{N}\rangle &= \langle \int_{B_s(0)} f(x-y)\chi_s(y)\mathrm{dVol}_y, \mathsf{N}\rangle \\ &= \int_{B_s(0)} \langle f(x-y), \mathsf{N}\rangle \chi_s(y)\mathrm{dVol}_y \\ &< \int_{B_s(0)} t\chi_s(y)\mathrm{dVol}_y \\ &= t, \end{aligned}$$

so that $f_s(x) \in H(\mathsf{N}, t)$. Since $H(\mathsf{N}, t)$ is an arbitrary open half-space containing K, we conclude that $f_s(x) \in \mathrm{Conv}(K) = K$, and the result follows. □

In particular, this allows us to take a different angle on other well-known properties of smooth functions.

Lemma 6.27. *Choose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. If f is continuous, then $(f_s)_{s>0}$ converges to f locally uniformly as r tends to 0.*

Proof. Choose $R > 0$ and $\epsilon > 0$. By uniform continuity, there exists $\delta > 0$ such that if $\|x\| < R$ and if $\|y\| < \delta$, then $f(x+y) \in [f(x) - \epsilon, f(x) + \epsilon]$. Thus, by Lemma 6.26, for $\|x\| < R$ and for $s < \delta$, $f_s(x) \in [f(x) - \epsilon, f(x) + \epsilon]$, that is, $|f_s(x) - f(x)| < \epsilon$. Since $R, \epsilon > 0$ are arbitrary, we conclude that f_s converges locally uniformly to f as s tends to 0, as desired. □

Corollary 6.28. *Choose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. If f is C^k, then $(f_s)_{s>0}$ converges to f in the C^k_{loc} sense as s tends to 0.*

Proof. Since $J^k f$ is continuous, by Lemmas 6.24 and 6.27, $(J^k(f_s))_{s>0} = ((J_k f)_s)_{s>0}$ converges locally uniformly to $J^k f$ as s tends to 0. The result follows. □

Finally, a useful variant of Lemma 6.27 is

Lemma 6.29. *If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is 1-Lipschitz, then for all $s > 0$,*

$$\|f - f_s\|_0 \leq s.$$

Proof. Choose $x \in \mathbb{R}^{n+1}$. Since f is 1-Lipschitz, for all $y \in B_s(x)$, $f(y) \in [f(x) - s, f(x) + s]$. Since this interval is convex, by Lemma 6.26, $f_s(x) \in [f(x) - s, f(x) + s]$, so that $|f(x) - f_s(x)| \leq s$. Since $x \in \mathbb{R}^{n+1}$ is arbitrary, it follows that $\|f - f_s\|_0 \leq s$, as desired. □

6.5 Smoothing the intersection

We return to the situation discussed in Section 6.3. Thus, let K_1 and K_2 be compact, convex subsets of $\mathbb{R}^{n+1}$ whose intersection has non-trivial interior. Let U be an open subset of $\mathbb{R}^{n+1}$ and suppose that both $U(K_1)$ and $U(K_2)$ are smooth of gaussian curvature at least k. As before, we denote $K := K_1 \cap K_2$ and we denote $d := d_{K_1 \cap K_2}$, $d_1 := d_{K_1}$ and $d_2 := d_{K_2}$. We recall the following version of the submersion theorem.

Lemma 6.30. *Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $f : U \to \mathbb{R}$ be a smooth mapping and denote $\Sigma = f^{-1}(\{0\})$. If 0 is a regular value of f, then Σ is a smooth, embedded submanifold. Moreover, for all $x \in \Sigma$, $Df(x)/\|Df(x)\|$ is a unit, normal vector field over Σ, and if we denote by A the shape* *operator of* Σ *with respect to this normal, then for all* $x \in \Sigma$ *and for all* X, Y *tangent to* Σ *at* x*,*

$$\langle A(x)X, Y\rangle = \frac{1}{\|Df(x)\|}D^2f(x)(X,Y).$$

Proof. If 0 is a regular value of f, then it follows by the submersion theorem (c.f. [13]) that Σ is a smooth, embedded submanifold of U. Choose $x \in U$ and let X be a tangent vector to Σ at x. Let $\gamma :]-\epsilon, \epsilon[\rightarrow \Sigma$ be a smooth curve such that $\gamma(0) = x$ and $\gamma'(0) = X$. In particular, $(f \circ \gamma)(t) = 0$ for all t. Thus, by the chain rule,

$$\langle Df(x), X\rangle = \langle Df(x), \gamma'(0)\rangle = (f\circ\gamma)'(0) = 0.$$

Since X is an arbitrary vector tangent to Σ at x, it follows that $Df(x)$ is normal to Σ at x. Since, furthermore, $\|Df(x)\| \neq 0$, we conclude that $Df(x)/\|Df(x)\|$ is a unit normal vector to Σ at x, as desired. Now let X and Y be tangent vectors to Σ at x. We denote $\mathsf{N} = Df/\|Df\|$. By definition of A, and using the chain and product rules,

$$\begin{aligned}\langle A(x)X, Y\rangle &= \langle D\mathsf{N}(x)X, Y\rangle \\ &= \langle D(Df/\|Df\|)(x)X, Y\rangle \\ &= \frac{1}{\|Df(x)\|}D^2f(x)(X,Y) - \frac{1}{\|Df(x)\|^3}\langle Df(x), Y\rangle\langle Df(x), X\rangle.\end{aligned}$$

However, by the previous discussion, $Df(x)$ is normal to Σ at x, and so,

$$\langle A(x)X, Y\rangle = \frac{1}{\|Df(x)\|}D^2f(x)(X,Y),$$

as desired. □

For all $k, B > 0$, and for all $\mathsf{N} \in \Sigma^n$, we define the set $\kappa(k, B, \mathsf{N}) \subseteq \mathrm{Symm}(2, \mathbb{R}^{n+1})$ by

$$\kappa(k, B, \mathsf{N}) := \left\{A \mid \|A\| \leq B,\ A \geq 0,\ \mathrm{Det}(A; \langle\mathsf{N}\rangle^{\perp}) \geq k^n\right\}.$$

Lemma 6.31. *For all* $k, B > 0$ *and for all* $\mathsf{N} \in \Sigma^n$*,* $\kappa(k, B, \mathsf{N})$ *is compact and convex.*

Proof. The set of all matrices of norm no greater than B is compact. Since $\kappa(k, B, \mathsf{N})$ is a closed subset of this set, it too is compact. Observe that the space of positive-definite matrices is convex. Furthermore, by Lemma 2.5, the function $\mathrm{Det}(M; \langle\mathsf{N}\rangle^{\perp})^{\frac{1}{n}}$ is convex over this space. Since the norm is also convex, we conclude that $\kappa(k, B, N)$ is convex, as desired. □

Lemma 6.32. *For every compact subset* X *of* U *and for all* $\epsilon > 0$*, there exists* $\rho > 0$ *with the property that if* $x \in \Pi^{-1}(X \cap U(K)) \setminus K$*, if* $d(x) < \rho$ *and if* $D^2d(x)$ *is defined and is symmetric, then*

$$D^2d(x) \in \kappa(k - \epsilon, 2/d(x), Dd(x)).$$

Proof. By Lemma 6.7, $\|D^2d(x)\| \leq 2/d(x)$ and the result follows by Lemma 6.21. □

We now consider smoothings of d as described in Section 6.4.

Lemma 6.33. *For every compact subset X of U and for all $\epsilon > 0$, there exists $\rho > 0$ with the property that for all $r < \rho$, there exists $\sigma > 0$ such that if $s < \sigma$, if $x \in X$ and if $d_s(x) = r$, then $0 < \|Dd_s(x)\| \leq 1$ and*

$$D^2d_s(x) \in \kappa(k - \epsilon, 4/r, Dd_s(x)/\|Dd_s(x)\|).$$

Proof. Choose $\sigma_1 > 0$ such that $X_1 := \overline{B}_{\sigma_1}(X) \subseteq U$. Since X is compact, so too is X_1. We first claim that there exists a compact subset X_2 of U and $\rho_1 > 0$ such that

$$X_1 \cap d^{-1}(]0, \rho_1[) \subseteq \Pi^{-1}(X_2 \cap U(K)) \setminus K.$$

Indeed, suppose the contrary. There exists a sequence $(x_m)_{m\in\mathbb{N}}$ in X_1 with the properties that $d(x_m) > 0$ for all m, $(d(x_m))_{m\in\mathbb{N}}$ converges to 0 and $(\Pi(x_m))_{m\in\mathbb{N}}$ is not contained in any compact subset of U. For all m, denote $y_m := \Pi(x_m)$ and $\mathsf{N}_m := Dd(x_m)$. Since K is compact, there exists $y_\infty \in K$ towards which $(y_m)_{m\in\mathbb{N}}$ subconverges. By hypothesis, y_∞ lies in the boundary of U. By Lemma 6.3, for all m, $x_m = y_m + d(x_m)\mathsf{N}_m$. In particular, since $(d(x_m))_{m\in\mathbb{N}}$ converges to 0 and since N_m has unit length for all m, it follows that $(x_m)_{m\in\mathbb{N}}$ also subconverges to y_∞. By compactness, y_∞ is also an element of X_1, which is absurd, and the assertion follows.

By Lemma 6.32, there exists $\rho_2 < \rho_1$ with the property that if $x \in \Pi^{-1}(X_2 \cap U(K)) \setminus K$, if $d(x) < \rho_2$ and if $D^2d(x)$ is defined and is symmetric, then

$$D^2d(x) \in \kappa(k - \epsilon/4, 2/d(x), Dd(x)).$$

Choose $r \in [0, \rho_2[$. Let $\delta \in]0, 1[$ be such that if N is any vector in Σ^n and if $V \in \overline{B}_\delta(\mathsf{N})$, then

$$\kappa(k - \epsilon/4, 4/r, V/\|V\|) \subseteq \kappa(k - \epsilon/2, 4/r, \mathsf{N}) \subseteq \kappa(k - \epsilon, 4/r, V/\|V\|).$$

Fix $\eta > 0$ such that $2\eta < \mathrm{Min}(r/2, \rho_2 - r)$. Since K is compact, so too is $d^{-1}([r - \eta, r + \eta])$. By continuity, there therefore exists $\sigma_2 < \mathrm{Min}(\eta, \sigma_1)$ such that if $x \in d^{-1}([r - \eta, r + \eta])$ and if $y \in B_{\sigma_2}(x)$, then $Dd(y) \subseteq \overline{B}_\delta(Dd(x))$. Now choose $s < \sigma_2$. Fix $x \in d_s^{-1}(\{r\}) \cap X$. By Lemma 6.29, $x \in d^{-1}([r - \eta, r + \eta])$. Thus, if $y \in B_{\sigma_2}(x)$, then $d(y) \in]r/2, \rho_2[$ and $Dd(y) \in \overline{B}_\delta(Dd(x))$. Furthermore, every such y is an element of $X_1 \cap d^{-1}(]0, \rho_2[) \subseteq \Pi^{-1}(X_2 \cap U(K)) \setminus K$, so that if $D^2d(y)$ is defined and symmetric, then

$$D^2d(y) \in \kappa(k - \epsilon/4, 4/r, Dd(y)) \subseteq \kappa(k - \epsilon/2, 4/r, Dd(x)).$$

Since $\overline{B}_\delta(Dd(x))$ is compact and convex, it follows by Lemma 6.26 that

$$Dd_s(x) \in \overline{B}_\delta(Dd(x)),$$

and, in particular, $Dd_s(x) \neq 0$. Likewise, since $\kappa(k - \epsilon/4, 4/r, Dd(x))$ is compact and convex, by Lemma 6.26 again,

$$D^2 d_s(x) \in \kappa(k - \epsilon/2, 4/r, Dd(x)) \subseteq \kappa(k - \epsilon, 4/r, Dd_s(x)/\|Dd_s(x)\|).$$

Finally, since $Dd(y) \in \overline{B}_1(0)$ at every point where it is defined, and since $\overline{B}_1(0)$ is closed and convex, it follows by Lemma 6.26 again that $\|Dd_s(x)\| \leq 1$, and this completes the proof. □

Theorem 6.34. *For every compact subset X of U and for all $\epsilon > 0$, there exists $\rho > 0$ with the property that for all $r < \rho$, there exists $\sigma > 0$ such that if $s < \sigma$, if $x \in X$ and if $d_s(x) = r$, then $d_s^{-1}(\{r\})$ is smooth near x and has gaussian curvature at least $k - \epsilon$ at x.*

Proof. Let ρ be as in Lemma 6.33. Choose $r < \rho$. Let σ be as in Lemma 6.33. Choose $s < \sigma$. We denote $\Sigma_{r,s} = d_s^{-1}(\{r\})$. Choose $x \in X \cap \Sigma_{r,s}$. By Lemma 6.33, $Dd_s(x) \neq 0$ and $\|Dd_s(x)\| \leq 1$. Thus, by Lemma 6.30, $\Sigma_{r,s}$ is smooth near x and $Dd_s(x)/\|Dd_s(x)\|$ is the normal to $\Sigma_{r,s}$ at x. Moreover, if we denote by $A(x)$ the shape operator of $\Sigma_{r,s}$ at x with respect to this normal, then, for all vectors X and Y tangent to $\Sigma_{r,s}$ at x,

$$A(x)(X,Y) = \frac{1}{\|Dd_s(x)\|} D^2 d_s(x)(X,Y).$$

Thus, bearing in mind that $\|Dd_s(x)\| \leq 1$, if we denote by $\kappa(x)$ the gaussian curvature of Σ at x, then

$$\kappa(x) = \mathrm{Det}(A(x))^{1/n} \geq \mathrm{Det}(D^2 d(x); \langle Dd_s(x)\rangle^\perp)^{1/n}.$$

However, by Lemma 6.33,

$$D^2 d(x) \in \kappa(k - \epsilon, 4/r, Dd_s(x)/\|Dd_s(x)\|),$$

so that $\kappa(x) \geq k - \epsilon$, as desired. □

6.6 Weak barriers

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $k > 0$ be a positive real number. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. We say that K is a **strong barrier** of gaussian curvature at least k inside U whenever $(\partial K) \cap U$ is smooth and has gaussian curvature at least k at every point. We say that K is a **weak barrier** of gaussian curvature at least k inside U whenever there exists a sequence $(\epsilon_m)_{m\in\mathbb{N}} > 0$ converging to 0, an increasing sequence $(V_m)_{m\in\mathbb{N}}$ of open sets and a sequence $(K_m)_{m\in\mathbb{N}}$ of convex sets converging to K in the Hausdorff sense with the properties that $U = \cup_{m\in\mathbb{N}} V_m$ and, for all m, K_m is a strong barrier of gaussian curvature at least $k - \epsilon_m$ inside V_m. That is, weak barriers are Hausdorff limits of strong barriers.

We first show that the set of weak barriers is closed in the Hausdorff topology.

Lemma 6.35. *Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $k > 0$ be a positive real number. Let $(U_m)_{m\in\mathbb{N}}$ be an increasing sequence of open sets such that $U = \cup_{m\in\mathbb{N}} U_m$. Let $(k_m)_{m\in\mathbb{N}}$ be a sequence of positive real numbers converging to k. Let $(K_m)_{m\in\mathbb{N}}, K_\infty$ be compact, convex subsets of $\mathbb{R}^{n+1}$ and suppose that $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. If K_m is a weak barrier of gaussian curvature at least k_m inside U_m for all m, then K_∞ is a weak barrier of gaussian curvature at least k inside U.*

Proof. Upon extracting a subsequence, we may suppose that for all m, $d_H(K_m, K_\infty) \leq 1/m$ and that $k_m \geq k - 1/m$. For all m, let $(\epsilon_{m,p})_{p\in\mathbb{N}} > 0$ be a sequence converging to 0, let $(V_{m,p})_{p\in\mathbb{N}}$ be an increasing sequence of open subsets of U_m such that $U_m = \cup_{p\in\mathbb{N}} V_{m,p}$ and let $(K_{m,p})_{p\in\mathbb{N}}$ be a sequence of convex sets converging to K_m in the Hausdorff sense such that, for all m, $K_{m,p}$ is a strong barrier of gaussian curvature at least $k_m - \epsilon_{m,p}$ inside $V_{m,p}$. Upon extracting subsequences, we may suppose, in addition, that for all m and for all p, $\epsilon_{m,p} \leq 1/p$ and $d_H(K_{m,p}, K_m) \leq 1/p$. Let $(V_m)_{m\in\mathbb{N}}$ be an increasing sequence of relatively compact open subsets of U such that $U = \cup_{m\in\mathbb{N}} V_m$. We may suppose that $V_p \subseteq V_{m,p}$ for all m and for all p. For all m, define $K'_m := K_{m,m}$. Then, for all m, $d_H(K'_m, K_\infty) \leq 2/m$ and K'_m is a strong barrier of gaussian curvature at least $k_m - 1/m \geq k - 2/m$ inside V_m. In particular, $(K'_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense and we conclude that K_∞ is a weak barrier of gaussian curvature at least k over U as desired. □

We now show that the set of weak barriers in closed under intersection.

Lemma 6.36. *Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $k > 0$ be a positive real number. Let K_1 and K_2 be compact, convex subsets of $\mathbb{R}^{n+1}$. If K_1 and K_2 are both weak barriers of gaussian curvature at least k inside U, and if $K_1 \cap K_2$ has non-trivial interior, then $K_1 \cap K_2$ is also a weak barrier of gaussian curvature at least k inside U.*

Proof. By definition, for each $i \in \{1, 2\}$, there exists an increasing sequence $(V_{i,m})_{m\in\mathbb{N}}$ of open subsets of U, a sequence $(\epsilon_{i,m})_{m\in\mathbb{N}}$ of positive real numbers converging to 0, and a sequence $(K_{i,m})_{m\in\mathbb{N}}$ of compact, convex subsets of $\mathbb{R}^{n+1}$ converging to K_i in the Hausdorff sense with the properties that $U = \cup_{m\in\mathbb{N}} V_{i,m}$ and, for all m and for all $x \in (\partial K_{i,m}) \cap V_{i,m}$, $(\partial K_{i,m})$ is smooth near x and has gaussian curvature at least $k - \epsilon_{i,m}$ at x. We may suppose that there exists a sequence, $(W_m)_{m\in\mathbb{N}}$, of relatively compact open subsets of U such that $U=\cup_{m\in\mathbb{N}}W_m$ and that, for all m, $\overline{W}_m\subseteq V_{1,m}\cap V_{2,m}$. For all m, denote $K_m:=K_{1,m}\cap K_{2,m}$. Observe that $(K_m)_{m\in\mathbb{N}}$ converges to $K_1\cap K_2$ in the Hausdorff sense, and we may therefore suppose that $d_H(K_m,K_1\cap K_2)\leq 1/m$ for all m. Choose $m\in\mathbb{N}$. Denote by d_m the distance in $\mathbb{R}^{n+1}$ to K_m. By Theorem 6.34, there exists $r<1/2m$ and $\sigma<r$ such that if $s<\sigma$, if $x\in\overline{W}_m$ and if $d_{m,s}(x)=r$, then $d_{m,s}^{-1}(\{r\})$ is smooth near x with guassian curvature at least $k-2\epsilon_m$ at x. In particular, if we denote $K_m':=d_{m,s}^{-1}(]-\infty,r])$ then for all m, K_m' is a strong barrier of gaussian curvature at least $k-2\epsilon_m$ in W_m.

By Lemma 6.29, for all $s<\sigma<r$, $\|d_{m,s}-d_m\|_0\leq r$, and so

$$K_m=d_m^{-1}(]-\infty,0])\subseteq d_{m,s}^{-1}(]-\infty,r])=K_m',$$

and

$$K_m'=d_{m,s}^{-1}(]-\infty,r])\subseteq d_m^{-1}(]-\infty,2r[)=\overline{B}_{2r}(K_m),$$

so that $d_H(K_m,K_m')\leq 2r<1/m$. It follows that $d_H(K_m',K_1\cap K_2)<2/m$, so that $(K_m')_{m\in\mathbb{N}}$ converges to $K_1\cap K_2$ in the Hausdorff sense, and $K_1\cap K_2$ is therefore a weak barrier of gaussian curvature at least k in U, as desired. □

We refine Lemma 6.36 in order to construct a local excision operation which allows us to obtain regularity for extremal weak barriers, as we shall see presently.

Lemma 6.37. *Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let V be an open, convex subset of $\mathbb{R}^{n+1}$. Let L be a compact, convex subset of $\overline{V}$. If $K\cap(\partial V)\subseteq L$, then $(K\setminus\overline{V})\cup(K\cap L)$ is compact and convex.*

Proof. Denote $K':=(K\setminus\overline{V})\cup(K\cap L)$. Since $K\cap(\partial V)\subseteq L$, $K'=(K\setminus V)\cup(K\cap L)$, and since both $K\setminus V$ and $K\cap L$ are compact, so too is K'. Choose $x,x'\in K'$. For all $t\in[0,1]$, denote $x_t:=(1-t)x+tx'$. We claim that $x_t\in K'$ for all t. Indeed, since x and x' are both elements of K, by convexity, $x_t\in K$ for all t. Let I be the set of all t such that $x_t\in\overline{V}$. Observe that I is a closed subinterval of $[0,1]$. Consider $t\in\partial I$. If $t\in]0,1[$, then x_t is an element of $K\cap\partial V\subseteq L$. Otherwise, if $t\in\{0,1\}$, then $x_t\in K'\cap\overline{V}=K\cap L\subseteq L$. In each case, $x_t\in L$ for each $t\in\partial I$, and so, by convexity, $x_t\in L$ for all $t\in I$. That is, for all such t, $x_t\in K\cap L\subseteq K'$. However, for all $t\in[0,1]\setminus I$, $x_t\in K\setminus\overline{V}\subseteq K'$, so that $x_t\in K'$ for all t. Since $x,x'\in K'$ are arbitrary, we conclude that K' is convex, as desired. □

Lemma 6.38. *Choose $k>0$. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$. Let U be an open subset of $\mathbb{R}^{n+1}$ and suppose that K is a strong barrier of gaussian curvature at least k in U. Let V be an open, convex subset of $\mathbb{R}^{n+1}$ whose closure is contained in U, and let L be a compact, convex subset of $\overline{V}$. If L is a strong barrier of gaussian curvature at least k in*

V and if $K\cap(\partial V)$ is contained in the relative interior of L in $\overline{V}$, then $(K\setminus\overline{V})\cup(K\cap L)$ is a weak barrier of gaussian curvature at least k in U.

Proof. Denote $K':=(K\setminus\overline{V})\cup(K\cap L)$. Let d', d_K and d_L be the respective distances in $\mathbb{R}^{n+1}$ to K', K and L. Likewise, let Π', Π_K and Π_L be their respective closest point projections. Since $K\setminus V$ is compact, and since $K\cap(\partial V)$ is contained in the relative interior of L in $\overline{V}$, there exists $\delta>0$ such that for all $x\in K\setminus V$, $(\overline{B}_\delta(x)\cap\overline{V})\subseteq L$. We claim that for all $x\in K\setminus V$,

$$K\cap\overline{B}_\delta(x)=K'\cap\overline{B}_\delta(x). \tag{I}$$

Indeed, for all $x\in K\setminus V$,

$$(K\cap\overline{V})\cap\overline{B}_\delta(x)\subseteq K\cap L\cap\overline{V}\cap\overline{B}_\delta(x)=(K'\cap\overline{V})\cap\overline{B}_\delta(x),$$

However, for all x,

$$(K\setminus\overline{V})\cap\overline{B}_\delta(x)=(K'\setminus\overline{V})\cap\overline{B}_\delta(x),$$

so that, for all $x\in K\setminus V$,

$$K\cap\overline{B}_\delta(x)\subseteq K'\cap\overline{B}_\delta(x).$$

On the other hand, since $K'\subseteq K$, $K'\cap\overline{B}_\delta(x)\subseteq K\cap\overline{B}_\delta(x)$, and the two sets therefore coincide, as desired.

Define

$$X:=K'\setminus\bigcup_{x\in K\setminus V}B_\delta(x),$$

and observe that X is a compact subset of V. Choose $\rho_1>0$ such that $X_1:=\overline{B}_{2\rho_1}(X)\subseteq V$. We now claim that for all $x\in\mathbb{R}^{n+1}\setminus\overline{B}_{\rho_1}(X)$ such that $d'(x)<\rho_1$,

$$d'(x)=d_K(x).$$

Indeed, for such an x, denote $y:=\Pi'(x)$ and $\mathsf{N}:=(x-y)/\|x-y\|$. In particular, y is the closest point in K' to x and by Lemma 4.6, N is a supporting normal to K' at this point. However, since $y\notin X$, there exists $y'\in K\setminus V=K'\setminus V$ such that $y\in B_\delta(y')$. In particular, N is a supporting normal to $\overline{B}_\delta(y')\cap K'$ at y. However, by (I), $\overline{B}_\delta(y')\cap K'=\overline{B}_\delta(y')\cap K$, so that, by Lemma 4.9, N is a supporting normal to K at y. In particular, by Lemma 4.6 again, y is also the closest point in K to x, so that

$$d_K(x)=\|y-z\|=d'(x),$$

as asserted.

We now claim that for all $r<\rho_1$, there exists $\sigma_1:=\sigma_1(r)<r$ such that if $s<\sigma_1$, if $x\notin X_1$ and if $d'_s(x)=r$, then for all y near x,

$$d'_s(y)=d_{K,s}(y).$$

Indeed, choose $r < \rho_1$. Fix $\eta > 0$ such that $3\eta < \mathrm{Min}(r, \rho_1 - r)$ and fix $\sigma_1 < \eta$. Choose $s < \sigma_1$ and $y \notin X_1$ such that $d'_s(y) \in [r-\eta, r+\eta]$. By Lemma 6.29, $d'(y) \in [r-2\eta, r+2\eta]$. Thus, if $z \in B_{\sigma_1}(y)$, then $d'(z) \in]0, \rho_1[$ and $z \notin \overline{B}_{\rho_1}(X)$, so that, by the discussion of the preceeding paragraph, $d'(z) = d_K(z)$, and it follows that $d'_s(y) = d_{K,s}(y)$, as desired.

Let $(W_m)_{m\in\mathbb{N}}$ be an increasing family of relatively compact open subsets of U with closure contained in U such that $U = \cup_{m\in\mathbb{N}} W_m$. Suppose furthermore that $X_1 \subseteq W_m$ for all m. Fix $m \in \mathbb{N}$. Choose $R > 0$ such that $K \subseteq \overline{B}_R(0)$. By Theorem 6.34 (with $K_2 = \overline{B}_R(0)$), there exists $\rho_2 < \mathrm{Min}(\rho_1, \frac{1}{2n})$ with the property that for all $r < \rho_2$, there exists $\sigma_2 := \sigma_2(r) < \sigma_1(r)$ such that if $s < \sigma_2$, if $x \in \overline{W}_m$ and if $d_{K,s}(x) = r$, then $(d_{K,s})^{-1}(\{r\})$ is smooth near x and has gaussian curvature at least $k - 1/m$ at x. In particular, by the discussion of the preceeding paragraph, if $r < \rho_2$, if $s < \sigma_2$ and if $x \in \overline{W}_m \setminus X_1$ is such that $d'_s(x) = r$, then $(d'_s)^{-1}(\{r\})$ is also smooth near x and has gaussian curvature at least $k - 1/m$ at x. On the other hand, by Theorem 6.34 again, there exists $\rho_3 < \rho_2$ with the property that for all $r < \rho_3$, there exists $\sigma_3 := \sigma_3(r) < \sigma_2(r)$ such that if $s < \sigma_3$, if $x \in X_1$ and if $d'_s(x) = r$, then $(d'_s)^{-1}(\{r\})$ is smooth near x and has gaussian curvature at least $k - 1/m$ at x. It follows that if $K_m := (d'_s)^{-1}(]0, r])$, then K_m is a strong barrier of gaussian curvature at least $k - 1/m$ in W_m. Furthermore, by Lemma 6.29, $\|d' - d'_s\|_0 < \sigma_3 < r$ so that

$$K' = (d')^{-1}(]-\infty, 0]) \subseteq (d'_s)^{-1}(]-\infty, r]) = K_m,$$

and

$$K_m = (d'_s)^{-1}(]-\infty, r]) \subseteq d^{-1}(]-\infty, 2r]) = \overline{B}_{2r}(K').$$

Since $r < 1/2m$, it follows that $d_H(K', K_m) \leq 1/m$, and since m is arbitrary, we conclude that $(K_m)_{m\in\mathbb{N}}$ converges to K' in the Hausdorff sense, so that K' is a weak barrier of gaussian curvature at least k over U, as desired. □

6.7 The Plateau problem

The machinery developed in the preceeding sections allows us to solve a general version of the Plateau problem. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ with smooth boundary and non-trivial interior. Let X be a closed subset of the boundary of K such that $\mathrm{Conv}(X)$ also has non-trivial interior. Choose $k > 0$, and suppose that ∂K has gaussian curvature at least k at every point of $(\partial K) \setminus X$. Observe that, using the terminology of the preceeding section, this means that K is a strong barrier of gaussian curvature at least k in $\mathbb{R}^{n+1} \setminus X$. We define the family $\mathcal{B}(k, K, X)$ to be the set of all compact, convex subsets K' of $\mathbb{R}^{n+1}$ such that $X \subseteq K' \subseteq K$ and K' is a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1} \setminus X$. Since strong barriers are also weak barriers, we see that K itself is an element of $\mathcal{B}(k, K, X)$ so that this family is non-empty.

Lemma 6.39. *If L is an element of $\mathcal{B}(k, K, X)$, then L has non-trivial interior.*

Proof. By definition, L is compact and convex. Since $X \subseteq L$, using Lemma 4.16, we have $\mathrm{Conv}(X) \subseteq \mathrm{Conv}(L) = L$. Since $\mathrm{Conv}(X)$ has non-trivial interior, it follows that L too has non-trivial interior, as desired. □

Lemma 6.40. *Let L be an element of $\mathcal{B}(k, K, X)$. If Σ is a smooth embedded hypersurface (without boundary) such that $\Sigma \subseteq L$, then Σ has gaussian curvature at least k at every point of $(\Sigma \cap \partial L) \setminus X$.*

Remark. *In other words, every element of $\mathcal{B}(k, K, X)$ is a viscosity supersolution of the Gauss curvature equation (c.f. [8]).*

Proof. Consider $x \in (\Sigma \cap \partial L) \setminus X$. Without loss of generality, we may suppose that $x = 0$. Since every supporting normal to L at 0 is also normal to Σ, L has only one supporting normal at this point, which we may take to be $-e_{n+1}$. By Theorem 4.12, there exist $C, \rho > 0$ and a convex, C-Lipschitz function $\omega : B'_\rho(0) \to]-C\rho, C\rho[$ such that $\partial L \cap (B'_\rho(0) \times]-2C\rho, 2C\rho[)$ coincides with the graph of ω. Upon reducing ρ if necessary, we may suppose furthermore that there exists a smooth function $f : B'_\rho(0) \to]-C\rho, C\rho[$ such that $\Sigma \cap (B'_\rho(0) \times]-2C\rho, 2C\rho[)$ coincides with the graph of f. In particular, since $\Sigma \subseteq L$, $f \geq \omega$.

Fix $r < \rho$. For $0 < t < C/2r$ and for $|s| < t$, denote $f_{s,t}(x') := f(x') + t\|x'\|^2 + sr^2$ and let $\Sigma_{s,t}$ be the graph of $f_{s,t}$ over $\overline{B}'_r(0)$. Observe that for all $|s| < t$, $\partial\Sigma_{s,t}$ lies in the interior of L, and for all $s > 0$, the whole of $\Sigma_{s,t}$ lies in the interior of L.

Fix $0 < t < C/2r$. Let $(\epsilon_m)_{m\in\mathbb{N}}$ be a sequence of positive numbers converging to 0, let $(V_m)_{m\in\mathbb{N}}$ be an increasing sequence of open subsets of $\mathbb{R}^{n+1} \setminus X$ and let $(L_m)_{m\in\mathbb{N}}$ be a sequence of convex sets converging to L in the Hausdorff sense such that $\mathbb{R}^{n+1} \setminus X = \cup_{m\in\mathbb{N}} V_m$ and, for all m, L_m is a strong barrier of gaussian curvature at least $k - \epsilon_m$ inside V_m. Fix $s_0 < 0 < s_1$ such that $|s_0|, |s_1| < t$. For sufficiently large m, the whole of $\Sigma_{s_1,t}$ is contained in the interior of L_m, $\partial\Sigma_{s,t}$ is contained in the interior of L_m for all $s \in [-s_0, s_1]$, but $\Sigma_{-s_0,t}$ intersects the complement of L_m non-trivially. There therefore exists $s \in]-s_0, s_1[$ such that $\Sigma_{s,t}$ is an interior tangent to ∂L_m at some point, $(x'_m, f_{s,t}(x'_m))$, say. By the maximum principal, $\Sigma_{s,t}$ has gaussian curvature at least $k - \epsilon_m$ at this point. By compactness, letting t tend to 0, we conclude that there exists $x' \in \overline{B}'_r(0)$ such that Σ has gaussian curvature at least k at x', and since $r > 0$ is arbitrary, we conclude that Σ has gaussian curvature at least k at 0, as desired. □

For any Borel measurable subset X of $\mathbb{R}^{n+1}$, we define the **volume** of X to be its $(n+1)$-dimensional Lebesgue measure, and we denote it by $\mathrm{Vol}(X)$. We denote

$$V_0 := \operatorname*{Inf}_{L\in\mathcal{B}(k,K,X)} \mathrm{Vol}(L).$$

Lemma 6.41. *$V_0 > 0$.*

Proof. Choose $L \in \mathcal{B}(k,K,X)$. By definition, L is compact and convex. Since $X \subseteq L$, and bearing in mind Lemma 4.16, $\mathrm{Conv}(X) \subseteq \mathrm{Conv}(L) = L$. Thus, by monotonicity of Lebesgue measure, $\mathrm{Vol}(L) \geq \mathrm{Vol}(\mathrm{Conv}(X))$. However, since $\mathrm{Conv}(X)$ has non-trivial interior, $\mathrm{Vol}(\mathrm{Conv}(X)) > 0$, and so

$$V_0 = \operatorname*{Inf}_{L\in\mathcal{B}(k,K,X)} \mathrm{Vol}(L) \geq \mathrm{Vol}(\mathrm{Conv}(X)) > 0,$$

as desired. □

Lemma 6.42. *Let $K_0 \subseteq K_1$ be compact, convex subsets of $\mathbb{R}^{n+1}$ with non-trivial interiors. If $K_0 \neq K_1$, then $\mathrm{Vol}(K_0) < \mathrm{Vol}(K_1)$.*

Proof. Choose $x \in K_1 \setminus K_0$. Since K_0 is compact, there exists $\delta_1 > 0$ such that $B_{\delta_1}(x) \cap K_0 = \emptyset$. Let y be an interior point of K_0. There exists $\delta_2 > 0$ such that $B_{\delta_2}(y) \subseteq K_0$. By convexity, for all $t \in]0,1]$, and for all $z \in B_{t\delta_2}(0)$,

$$(1-t)x + ty + z = (1-t)x + t(y + z/t) \in K_1.$$

That is, for all $t \in]0,1]$, $B_{t\delta_2}((1-t)x + ty) \subseteq K_1$. Choose $t > 0$ such that $t(\|y-x\| + \delta_2) < \delta_1$. In particular $B_{t\delta_2}((1-t)x+ty) \cap K_0 = \emptyset$, and so, by additivity and monotonicity of Lebesgue measure,

$$\mathrm{Vol}(K_1) \geq \mathrm{Vol}(K_0) + \mathrm{Vol}(B_{t\delta_2}((1-t)x+ty)) > \mathrm{Vol}(K_0),$$

as desired. □

Theorem 6.43. *There exists a unique element $K_0 \in \mathcal{B}(k,K,X)$ such that*

$$\mathrm{Vol}(K_0) = V_0.$$

Proof. We first show uniqueness. Indeed, suppose that there exists $K_0 \neq K_0' \in \mathcal{B}(k,K,X)$ such that, $\mathrm{Vol}(K_0) = \mathrm{Vol}(K_0') = V_0$. Since $K_0 \neq K_0'$, without loss of generality, we may assume that $K_0 \cap K_0' \neq K_0$. Since X is contained in each of K_0 and K_0', it is also contained in $K_0 \cap K_0'$. Moreover, since both K_0 and K_0' are contained in K, so too is $K_0 \cap K_0'$. Finally, by Lemma 6.36, $K_0 \cap K_0'$ is a weak barrier of gaussian curvature at least k over $\mathbb{R}^{n+1} \setminus X$ and we conclude that $K_0 \cap K_0'$ is an element of $\mathcal{B}(k,K,X)$. However, by Lemma 6.42, $\mathrm{Vol}(K_0 \cap K_0') < \mathrm{Vol}(K_0)$. This contradicts minimality of K_0, and uniqueness follows.

Let $(L_m)_{m\in\mathbb{N}} \in \mathcal{B}(k,K,X)$ be a sequence such that $(\mathrm{Vol}(L_m))_{m\in\mathbb{N}}$ converges to V_0. For all m, define $K_m := L_1\cap\ldots\cap L_m$. For all m, $X\subseteq K_m\subseteq K$, and, by Lemma 6.36, K_m is also a weak barrier of gaussian curvature at least k over $\mathbb{R}^{n+1}\setminus X$, so that $K_m\in\mathcal{B}(k,K,X)$. Moreover, by monotonicity of the Lebesgue measure, for all m, $V_0\leq\mathrm{Vol}(K_m)\leq\mathrm{Vol}(L_m)$. In particular, $(\mathrm{Vol}(K_m))_{m\in\mathbb{N}}$ also converges to V_0. Define

$$K_\infty := \bigcap_{m\in\mathbb{N}} K_m.$$

Since $\mathrm{Vol}(K_\infty)\leq\mathrm{Vol}(K_m)$ for all m, we have $\mathrm{Vol}(K_\infty)\leq V_0$. However $X\subseteq K_\infty\subseteq K$, and, by Lemma 4.1, $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense. It follows by Lemma 6.35 that K_∞ is a weak barrier of gaussian curvature at least k over $\mathbb{R}^{n+1}\setminus X$. That is, $K_\infty\in\mathcal{B}(k,K,X)$ and so $V_0\leq\mathrm{Vol}(K_\infty)$. We conclude that $\mathrm{Vol}(K_\infty)=V_0$, and this completes the proof. □

6.8 Singularities and smoothness

Continuing to use the notation of the preceeding section, we now show that the volume minimiser solves the Plateau problem modulo singularities of a type that are now well understood.

Theorem 6.44. *Let $K_0\in\mathcal{B}(k,K,X)$ be the volume minimiser. Then $(\partial K_0)\cap K^o$ has gaussian curvature equal to k in the viscosity sense. Furthermore, if $x\in(\partial K_0)\setminus X$, then either*

(1) ∂K_0 is smooth near x and has gaussian curvature equal to k at x; or

(2) K_0 satisfies the local geodesic property at x.

Proof. By Lemma 6.40, $(\partial K_0)\cap K^o$ has gaussian curvature at least k in the viscosity sense. Now choose $x\in(\partial K_0)\setminus X$. Suppose that K_0 satisfies the local geodesic property at x. Then, if Σ is a smooth, embedded surface (without boundary) contained in $\overline{K_o^c}$ and if $x\in\Sigma$, then, provided Σ is oriented such that its normal points outwards from K_o, this surface has non-positive curvature at x. In particular, $(\partial K_0)\cap K^o$ has curvature at most $0\leq k$ in the viscosity sense at x.

Now suppose that K_0 does not satisfy the local geodesic property at x. Let U be a relatively compact neighbourhood of x whose closure is contained in $\mathbb{R}^{n+1}\setminus X$. Let $(K_m)_{m\in\mathbb{N}}$ be a sequence of compact, convex subsets of $\mathbb{R}^{n+1}$ with the properties that $(K_m)_{m\in\mathbb{N}}$ converges to K_0 in the Hausdorff sense and, for all m, K_m is a strong barrier of gaussian curvature at least $k-1/m$ in U. Denote $K_\infty:=K_0$ and $x_\infty=x_0$.

Let $(x_m)_{m\in\mathbb{N}}$ be a sequence converging to x_∞ such that $x_m\in\partial K_m$ for all m. Upon applying a convergent sequence of affine isometries, we may suppose that $x_m = 0$ for all m. Since K_∞ has non-trivial interior, by Lemma 4.26, $\mathcal{N}(0;K_\infty)$ is strictly contained in a hemisphere. By Lemma 4.27, there exists $\mathsf{N} \in \mathcal{N}(0;K_\infty)$ such that $\langle \mathsf{N}, \mathsf{M}\rangle > 0$ for all $\mathsf{M} \in \mathcal{N}(0;K_\infty)$. By compactness of $\mathcal{N}(0;K_\infty)$, there exists $\theta \in [0,\pi/2[$ such that $\langle \mathsf{N},\mathsf{M}\rangle > 3\cos(\theta)$ for all $\mathsf{M} \in \mathcal{N}(0;K_\infty)$. Denote $C := \tan(\theta)$.

By Lemma 4.4, upon extracting a subsequence, there exists $r > 0$ such that $\overline{B}_r(0) \subseteq U$ and, for all m, for all $x \in (\partial K_m)\cap B_r(0)$ and for all $\mathsf{M} \in \mathcal{N}(x;K_m)$, $\langle \mathsf{N},\mathsf{M}\rangle > 2\cos(\theta)$. We denote $\rho = r/\sqrt{1+4C^2}$. By Lemma 4.22, there exists N', which we may choose as close to N as we wish such that for all $x \in K_\infty \setminus B_{\rho/2}(0)$, $\langle x, \mathsf{N}'\rangle < 0$. Moreover, we may assume that for all m, for all $x \in (\partial K_m)\cap B_r(0)$ and for all $\mathsf{M} \in \mathcal{N}(x;K_m)$, $\langle \mathsf{N}',\mathsf{M}\rangle > \cos(\theta)$.

Upon applying a rotation, we may suppose that $\mathsf{N}' = -e_{n+1}$. By Theorem 4.12, for all m, there exists a convex, C-Lipschitz function $\hat{f}_m : B'_\rho(0) \to]-C\rho, C\rho[$ such that $\hat{f}_m(0) = 0$ and $(\partial K_m)\cap(B'_\rho(0)\times]-2C\rho,2C\rho[)$ coincides with the graph of $\hat{f}_m$ over $B'_\rho(0)$. By the Arzelà-Ascoli theorem, every subsequence of $(\hat{f}_m)_{m\in\mathbb{N}}$ has a subsubsequence converging in the local uniform sense over $B'_\rho(0)$ to some limit $\hat{f}'_\infty$ say. Furthermore, since $(K_m)_{m\in\mathbb{N}}$ converges to K_∞ in the Hausdorff sense, we conclude that $\hat{f}'_\infty = \hat{f}_\infty$. It follows that $(\hat{f}_m)_{m\in\mathbb{N}}$ converges in the local uniform sense over $B'_\rho(0)$ to $\hat{f}_\infty$.

By construction, $\hat{f}_\infty(x') > 2\delta > 0$ for all $x' \in \partial B'_{\rho/2}(0)$ and for some $\delta > 0$. Since $(\hat{f}_m)_{m\in\mathbb{N}}$ converges locally uniformly to $\hat{f}_\infty$ over $B'_\rho(0)$, we may suppose that $\hat{f}_m(x') > \delta$ for all m and for all $x' \in \partial B'_{\rho/2}(0)$.

Choose $m < \infty$. Observe that $\hat{f}_m$ is smooth and strictly convex. Denote $\overline{\Omega}_m := \hat{f}_m^{-1}(]-\infty,\delta])$ and observe that $\overline{\Omega}_m$ is a compact, convex subset of $B'_{\rho/2}(0)$. By strict convexity, $D\hat{f}_m$ only vanishes at the unique absolute minimum of $\hat{f}_m$ over $B'_{\rho/2}(0)$. However, since $\hat{f}_m(0) = 0$, this absolute minimum is contained in the interior of $\overline{\Omega}_m$. In particular, $D\hat{f}_m$ does not vanish at any boundary point of $\overline{\Omega}_m$, so that $\overline{\Omega}_m$ has smooth boundary. Thus, by Theorem 3.16, there exists a unique, smooth, strictly convex function $f_m : \overline{\Omega}_m \to]-\infty,\delta]$ such that $f_m(x') = \delta$ for all $x' \in \partial\Omega_m$ and the graph of f_m has constant gaussian curvature equal to k. By convexity, $f_m \leq \delta$, and by Lemma 2.9, $f_m \geq \hat{f}_m$.

Define $V_m := \Omega_m\times]-2C\rho,2C\rho[$. Observe that V_m is open and convex. Moreover, $\overline{V}_m \subseteq \overline{B}_r(0) \subseteq U$. We define the subset L_m of $\overline{V}_m$ by

$$L_m := \left\{(x',t) \mid x' \in \overline{\Omega}_m,\ f_m(x') \leq t \leq 2C\rho\right\}.$$

Observe that L_m is compact and convex, $K_m\cap\partial V_m \subseteq L_m$ and $L_m \subseteq K_m\cap\overline{V}_m$. Define $K'_m := (K_m\setminus\overline{V})\cup(K_m\cap L_m) = (K_m\setminus\overline{V})\cup L_m$. We claim that K'_m is a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1}\setminus X$.

Indeed, for all $s \in [0, C\rho[$, define the subset $L_{m,s}$ of $\overline{V}_m$ by

$$L_{m,s} := \left\{(x', t) \mid x' \in \overline{\Omega}_m,\ f_m(x') - s \leq t \leq 2C\rho\right\},$$

and define $K'_{m,s} := (K_m \setminus \overline{V}) \cup (K_m \cap L_{m,s})$. For all $s > 0$, $K_m \cap \partial V_m$ is contained in the relative interior of L_m in V_m. Thus, by Lemma 6.38, $K'_{m,s}$ is a weak barrier of gaussian curvature at least k over $\mathbb{R}^{n+1} \setminus X$. Thus, since $(K'_{m,s})_{s\in[0,(1/2)C\rho[}$ converges to $K'_{m,0} = K'_m$ in the Hausdorff sense as s tends to 0, it follows by Lemma 6.35 that K'_m is also a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1}$, as asserted.

By Lemmas 4.2 and 4.3, we may suppose that $(K'_m)_{m\in\mathbb{N}}$ converges towards a compact, convex subset, K'_∞, say, of $\mathbb{R}^{n+1}$. We claim that $K'_\infty = K_\infty$. Indeed, by Lemma 6.35, K'_∞ is a weak barrier of gaussian curvature at least k in $\mathbb{R}^{n+1} \setminus X$. Moreover, for all m, $X \subseteq K_m \setminus B_r(0) \subseteq K_m \setminus V_m \subseteq K'_m$ and so $X \subseteq K'_\infty$. Finally, for all m, $K'_m \subseteq K_m \subseteq K$, so that $K'_\infty \subseteq K_\infty \subseteq K$. We conclude that K'_∞ is an element of $\mathcal{B}(k, K, X)$. In particular, $V_0 \leq \mathrm{Vol}(K'_\infty)$. However, since $K'_\infty \subseteq K_\infty$, $\mathrm{Vol}(K'_\infty) \leq \mathrm{Vol}(K_\infty) \leq V_0$, so that $\mathrm{Vol}(K'_\infty) = V_0$. It follows by uniqueness that $K'_\infty = K_\infty$, as asserted.

By continuity, there exists $\rho' < \rho$ such that for all $x' \in \overline{B}'_{\rho'}(0)$, $\hat{f}_\infty(x') < \delta/2$. Since $(\hat{f}_m)_{m\in\mathbb{N}}$ converges to $\hat{f}_\infty$ uniformly over $\overline{B}'_{\rho'}(0)$, we may suppose that for all m and for all $x' \in \overline{B}'_{\rho'}(0)$, $\hat{f}_m(x') \leq \delta$. In particular, for all m, $\overline{B}'_{\rho'}(0) \subseteq \overline{\Omega}_m$. We therefore define $W := B'_{\rho'}(0) \times]-(3/2)C\rho, (3/2)C\rho[$, and, for all $m < \infty$, $(\partial K_m) \cap W = (\partial L_m) \cap W$ is also smooth with constant gaussian curvature equal to k. However, since K_∞ does not satisfy the local geodesic property at x, it follows by Theorem 4.28, that $(\partial K_\infty) \cap W$ is smooth with constant gaussian curvature equal to k, and this completes the proof. □

The boundary of the volume minimiser, K_0, therefore solves the Plateau problem in the very general setting where X is any closed subset of ∂K. We say that a point $x \in \partial K_0 \setminus X$ is **regular** if ∂K_0 is smooth near that point. We define the **singular set**, $\mathrm{Sing}(K_0)$, to be the set of all points of ∂K_0 that are not regular. In particular, this includes every point of X. We obtain the following characterisation.

Theorem 6.45. *There exists a family $(X_\alpha)_{\alpha\in A}$ of subsets of X such that*

$$\mathrm{Sing}(K_0) = \bigcup_{\alpha\in A} \mathrm{Conv}(X_\alpha).$$

Proof. By definition, $\mathrm{Sing}(K_0)$ is closed. Furthermore, by Theorem 6.44, $\mathrm{Sing}(K_0) \setminus X$ consists of all points of $\partial K_0 \setminus X$ satisfying the local geodesic property, so that, by Theorem 4.18, $\mathrm{Sing}(K_0) \subseteq \mathrm{Conv}(X)$.

Now choose $x \in \mathrm{Sing}(K_0)$. Let H be a supporting tangent hyperplane to K_0 at x. Since $\mathrm{Conv}(X) \subseteq K_0$, H is also a supporting tangent hyperplane to $\mathrm{Conv}(X)$ at x. Denote $X_x := X \cap H$. Since $H \cap \mathrm{Conv}(X) = \mathrm{Conv}(X_x)$, it follows that $x \in \mathrm{Conv}(X_x)$. Furthermore, since $\mathrm{Conv}(X_x) \subseteq K_0 \cap H$, every point of $\mathrm{Conv}(X_x)$ is a boundary point of K_0. However, by Theorem 4.19, the set $\mathrm{Conv}(X_x)$ satisfies the local geodesic property at every point of $\mathrm{Conv}(X_x) \setminus X_x$. In particular, K_0 also satisfies the local geodesic property at every point of this subset, so that, by Theorem 6.44, $\mathrm{Conv}(X)$ is contained in $\mathrm{Sing}(K_0)$. Since $x \in \mathrm{Sing}(K_0)$ is arbitrary, we conclude that

$$\mathrm{Sing}(K_0) = \bigcup_{x \in \mathrm{Sing}(K_0)} \mathrm{Conv}(X_x),$$

as desired. □

Various ad-hoc arguments can now be used to eliminate singularities. For example, by Lemma 6.40, if the boundary of $\mathrm{Conv}(X)$ is smooth at some point, then that point must lie in the interior of K_0. In the particular case at hand, however, singularites are removed as follows.

Lemma 6.46. *Suppose that for every point x of ∂X, there exists a C^2 function $f : \partial K \to \mathbb{R}$ such that $f(x) = 0$, $Df(x) \neq 0$ and $f^{-1}(]-\infty, 0]) \subseteq X$. Then,* $\mathrm{Sing}(K_0) = X$.

Proof. Suppose the contrary. Let x be a point of $\mathrm{Sing}(K_0) \setminus X$. By Theorem 6.45, there exists a subset $X' \subseteq X$ such that $x \in \mathrm{Conv}(X') \subseteq \partial K_0$. Furthermore, since $x \notin X$, X' contains at least two distinct points, y_1 and y_2, say, and, without loss of generality, x lies along the straight line, Γ, passing through these two points. Let N be a supporting normal to K_0 at x. In particular, N is normal to Γ. For $\epsilon > 0$, denote

$$C_\epsilon := \bigcup_{z \in \Gamma} \overline{B}_\epsilon(x - \epsilon \mathsf{N}),$$

so that, for all ϵ, C_ϵ is the closed cylinder of radius ϵ about the straight line, Γ_ϵ, obtained by displacing Γ a distance ϵ in the $-\mathsf{N}$ direction. We claim that for all sufficiently small ϵ, $C_\epsilon \cap K \subseteq \mathrm{Conv}(X)$.

It suffices to show that for sufficiently small ϵ, $C_\epsilon \cap \partial K \subseteq X$ near y_1 and y_2. Without loss of generality, we may suppose that $y_1 = 0$, that $\mathsf{N} = e_n$ and that x lies on the positive x_{n+1} axis. Since x is an interior point of K, for all $\mathsf{N} \in \mathcal{N}(y_1; K)$, $\langle \mathsf{N}, e_{n+1} \rangle = \|x - y\|^{-1} \langle x - y_1, \mathsf{N} \rangle < 0$. By compactness of $\mathcal{N}(y_1; K)$, we may suppose that there exists $\theta \in]0, \pi/2[$ such that $\langle \mathsf{N}, e_{n+1} \rangle > 2\cos(\theta)$ for all $\mathsf{N} \in \mathcal{N}(y_1)$. By Lemma 4.8, there exists $r > 0$ such that for all $y \in \partial K \cap B_r(y_1)$ and for all $\mathsf{N} \in \mathcal{N}(y; K)$, $\langle \mathsf{N}, e_{n+1} \rangle > \cos(\theta)$. Denote $C := \tan(\theta)$ and $\rho = r/\sqrt{1 + 4C^2}$. By Theorem 4.12, there exists a convex, C-Lipschitz function $\omega : B'_\rho(0) \to]-C\rho, C\rho[$ such that $\partial K \cap (B'_\rho(0) \times]-2C\rho, 2C\rho[)$ coincides with the graph of ω. Furthermore, since ∂K is smooth, so too is ω.

Let $f : \partial K \to \mathbb{R}$ be a C^2 function such that $f(y_1) = 0$, $Df(y_1) \neq 0$ and $f^{-1}(]-\infty, 0]) \subseteq X$. Define $g : B'_\rho(0) \to \mathbb{R}$ by $g(x') := f(x', \omega(x'))$.

Observe that g is C^2, $g(0) = 0$ and $Dg(0) \neq 0$. Furthermore, if $g(z') \leq 0$, then $(z', \omega(z')) \in X$, and so, recalling that N is a supporting normal to $\mathrm{Conv}(X)$ at y_1, $\langle z', e_n \rangle = \langle (z', \omega(z')), \mathsf{N} \rangle \leq 0$. It follows that $Dg(0) = \lambda e_n$ for some $\lambda > 0$. Thus, since g is C^2, for sufficiently small $\epsilon > 0$, $\overline{B}_\epsilon(-\epsilon e_n) \subseteq g^{-1}(]-\infty, 0])$, so that

$$C_\epsilon \cap \partial K \cap (B'_\rho(0) \times]-2C\rho, 2C\rho[) = \left\{ (z', \omega(z')) \mid z' \in \overline{B}_\epsilon(-\epsilon e_n) \right\} \subseteq X.$$

That is, $C_\epsilon \cap \partial K \subseteq X$ near y_1. In like manner, we show that $C_\epsilon \cap \partial K \subseteq X$ also near y_2 so that, for sufficiently small ϵ, $C_\epsilon \subseteq \mathrm{Conv}(X)$, as desired. However, ∂C_ϵ has zero curvature at every point. This is absurd, by Lemma 6.40, and we conclude that $\mathrm{Sing}(K_0)$ is empty, as desired. $\square$

In particular, the classical existence result follows as an immediate corollary.

Theorem (1.2). *Choose $k > 0$. Let K be a compact, convex subset of $\mathbb{R}^{n+1}$ with smooth boundary. Let X be a closed subset of ∂K with C^2 boundary $C = \partial X$. If ∂K has gaussian curvature bounded below by k at every point of $(\partial K) \setminus X$, then there exists a compact, strictly convex, $C^{0,1}$ embedded hypersurface $S \subseteq \mathbb{R}^{n+1}$ with the properties that*

(1) $S \subseteq K$;

(2) $\partial S = C$; and

(3) $S \setminus \partial S$ is smooth and has constant gaussian curvature equal to k.

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Appendix

Terminology

Derivatives: For any vector spaces E, F, let $\text{Symm}(n, E) \otimes F$ denote the space of symmetric multilinear forms from E into F. When $F = \mathbb{R}$, we denote simply $\text{Symm}(n, E) = \text{Symm}(n, E) \otimes \mathbb{R}$. For any open subset $U \subseteq E$ and for any k-times differentiable function $f : U \to F$, we denote the k'th total derivative by $D^k f : U \to \text{Symm}(k, E) \otimes F$. For any point $p \in U$ and for k vectors $V_1, ..., V_k \in E$, we denote $D^k f(p)(V_1, ..., V_k) \in F$ the image of the k-tuplet $(V_1, ..., V_k)$ under the action of $D^k f$ at the point P.
For any vector spaces E and F, for any open subset U of E, and for all $k \in \mathbb{N}$, we denote by $C^k(U, F)$ the space of k-times continuously differentiable functions from U into F. We denote by $C^\infty(U, F)$ the space of functions from U into F which have continuous derivatives of arbitrarily high order. When $F = \mathbb{R}$, we denote simply $C^k(U) = C^k(U, \mathbb{R})$ and $C^\infty(U) = C^\infty(U, \mathbb{R})$. We denote by $\| \cdot \|_k$ the C^k norm over $C^k(U, F)$, and, for $\lambda = k + \alpha$ where $k \in \mathbb{N}$ and $\alpha \in]0, 1]$, we denote by $\| \cdot \|_\lambda$ the C^λ-Hölder norm over $C^k(U, F)$.
For any $k \in \mathbb{N}$ and for any $f \in C^k(U)$, we define $J^k(f) \in C^0(U, \oplus_{k=0}^m \text{Symm}(n, E))$ by:

$$J^k(f)(x) = (f(x), Df(x), ..., D^k f(x)).$$

We refer to $J^k(f)$ as the k-jet of f.
Canonical Basis of Euclidean Space: For all n, we denote by $\mathbb{R}^n$, n-dimensional, real space and by $e_1, ..., e_n$ its canonical basis. We denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product and by $\| \cdot \|$ the Euclidean norm. For any open subset $U \subseteq \mathbb{R}^n$, for any k-times differentiable function $f : U \to \mathbb{R}$, and for any k-tuple of indices $1 \leq i_1, ..., i_k \leq n$, we define the function $(\partial_{i_1} ... \partial_{i_k} f)$ such that for all $x \in U$:

$$(\partial_{i_1} ... \partial_{i_k} f)(x) = D^k f(x)(e_{i_1}, ..., e_{i_k}).$$

We will also use the more concise notation:

$$f_{i_1\ldots i_k} := (\partial_{i_1}\ldots\partial_{i_k}f).$$

Distributional Derivatives: Let E be a vector space furnished with a volume form dVol. Let U be an open subset of E and let $f:U\rightarrow\mathbb{R}$ be a real valued function which is locally L^1. Let $g=(g_0,g_1,\ldots,g_k):U\rightarrow\oplus_{k=0}^m\text{Symm}(n,E)$ be locally L^1. We say that g is the k'th order distributional derivative of f whenever it has the property that for any smooth function ϕ with compact support, for all $1\leq k\leq n$, and for all vectors $V_1,\ldots,V_k$:

$$\int_E f(x)(D^k\phi)(x)(V_1,\ldots V_k)\text{dVol}=(-1)^k\int_E g_k(x)(V_1,\ldots,V_k)\phi(x)\text{dVol}.$$

Smooth Functions on Sets with Boundary: Ω will always represent a bounded, strictly convex, open subset of $\mathbb{R}^n$. Given any vector space E, a function $f:\overline{\Omega}\rightarrow E$ is said to be C^k whenever there exists an extension $\hat{f}$ of f to $\mathbb{R}^n$ which is k-times continuously diferentiable. By Whitney's Extension Theorem (c.f. [22]), the extension $\hat{f}$ can be chosen such that for all $k\leq l$:

$$\|D^k\hat{f}\|_{L^\infty}=\|D^k\hat{f}|_{\overline{\Omega}}\|_{L^\infty}=\|D^kf\|_{L^\infty}.$$

We say that f is smooth whenever it is C^k for all finite k. Given any open subset U of E, we denote by $C^\infty(\overline{\Omega},U)$ the set of all smooth functions from $\overline{\Omega}$ into E taking values in U. In particular, when $U=E=\mathbb{R}$, we denote $C^\infty(\overline{\Omega})=C^\infty(\overline{\Omega},\mathbb{R})$.

Non-linear Operators: Given open subsets $U\subseteq\mathbb{R}^n$ and $V\subseteq\text{Symm}(2,\mathbb{R}^n)$ and a smooth function $F:\mathbb{R}\times\text{Symm}(1,\mathbb{R}^n)\times V\rightarrow\mathbb{R}$, for any function $f:U\rightarrow\mathbb{R}$ with the property that $D^2f(x)\in V$ for all $x\in U$, we define the function $F(f,Df,D^2f)$ such that, for all $x\in U$:

$$F(f,Df,D^2f)(x)=F(f(x),Df(x),D^2f(x)).$$

F thus represents the most general second-order, non-linear partial differential operator acting on functions over U which is homogeneous in the spatial variables.

Decomposition of Euclidean Space: We often decompose $\mathbb{R}^{n+1}$ as $\mathbb{R}^n\times\mathbb{R}$. For all $r>0$ and for all $x\in\mathbb{R}^{n+1}$, we denote by $B_r(x)$ the open ball of radius r about x in $\mathbb{R}^{n+1}$. For all $r>0$ and for all $x'\in\mathbb{R}^n$, we denote by $B_r'(x)$ the open ball of radius r about x in $\mathbb{R}^n$.

Metrics: Let X and Y be two compact subsets of $\mathbb{R}^{n+1}$. We define the Hausdorff distance between X and Y by:

$$d_H(X,Y)=\mathop{\text{Sup}}_{x\in X}\mathop{\text{Inf}}_{y\in Y}\|x-y\|+\mathop{\text{Sup}}_{y\in Y}\mathop{\text{Inf}}_{x\in X}\|x-y\|.$$

We denote by Σ^n the sphere of unit radius in $\mathbb{R}^{n+1}$. We define the spherical distance $d_\Sigma : \Sigma^n \times \Sigma^n \to \mathbb{R}$ by:

$$d_\Sigma(\mathsf{N}, \mathsf{M}) = \cos^{-1}\langle \mathsf{N}, \mathsf{M} \rangle.$$

The spherical distance thus measures the angle between two points in the sphere. Let X and Y be two compact subsets of Σ^n. We define the spherical-Hausdorff distance between X and Y by:

$$d_{H,\Sigma}(X, Y) = \mathop{\mathrm{Sup}}_{x \in X} \mathop{\mathrm{Inf}}_{y \in Y} d_\Sigma(x, y) + \mathop{\mathrm{Sup}}_{y \in Y} \mathop{\mathrm{Inf}}_{x \in X} d_\Sigma(x, y).$$

Miscellaneous: If X is any subset of $\mathbb{R}^n$, we denote its closure by $\overline{X}$, its interior by X^o and its boundary by ∂X. Let E be a vector space furnished with an inner product. For vectors X and Y in E, we denote by $\langle X, Y \rangle$ the inner product of X with Y. Let E be any vector space. For vectors $X_1, ..., X_n$, we denote by $\langle X_1, ..., X_n \rangle$ the linear subspace of E spanned by $X_1, ..., X_n$. This should not be confused with the inner product. It will be clear from the context which is meant.

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