



Notes on the Dirichlet problem of a class of second order elliptic partial differential equations on a Riemannian manifold

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Abstract. In these notes we study the Dirichlet problem for critical points of a convex functional of the form

$$F(u) = \int_{\Omega} \phi(|\nabla u|),$$

where Ω is a bounded domain of a complete Riemannian manifold \mathcal{M} . We also study the asymptotic Dirichlet problem when $\Omega = \mathcal{M}$ is a Cartan-Hadamard manifold. Our aim is to present a unified approach to this problem which comprises the classical examples of the p -Laplacian ($\phi(s) = s^p$, $p > 1$) and the minimal surface equation ($\phi(s) = \sqrt{1 + s^2}$). Our approach does not use the direct method of the Calculus of Variations which seems to be common in the case of the p -Laplacian. Instead, we use the classical method of a-priori C^1 estimates of smooth solutions of the Euler-Lagrange equation. These estimates are obtained by a coordinate free calculus. Degenerate elliptic equations like the p -Laplacian are dealt with by an approximation argument.

These notes address mainly researchers and graduate students interested in elliptic partial differential equations on Riemannian manifolds and may serve as a material for corresponding courses and seminars.

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Chapter 1

Introduction

In these notes we study the Dirichlet problem for critical points of a convex functional of the form

$$F(u) = \int_{\Omega} \phi(|\nabla u|), \quad (1.1)$$

where Ω is a bounded domain of a complete Riemannian manifold \mathcal{M} . We also study the asymptotic Dirichlet problem when $\Omega = \mathcal{M}$ is a Cartan-Hadamard manifold.

As minimal conditions on ϕ we require that

$$\begin{cases} \phi \in C^1([0, \infty)) \cap C^2((0, \infty)) \\ \phi'(s) > 0 \text{ and } \phi''(s) > 0 \text{ for } s > 0. \end{cases} \quad (1.2)$$

These conditions imply the strict convexity of F and ensure the ellipticity of the associated Euler-Lagrange equation.

There is a vast literature on this class of problems, mainly on the Euclidean space, which we do not discuss here. Our aim is to present a unified approach to this problem, in the Riemannian setting, which comprises the classical examples of the p -Laplacian ($\phi(s) = s^p$, $p > 1$) and the minimal surface equation ($\phi(s) = \sqrt{1 + s^2}$). Our approach does not use the minimization technique of the Calculus of Variations which seems to be common in the case of the p -Laplacian. Instead we use the classical method of a-priori C^1 estimates which are obtained from the Euler-Lagrange equation using a coordinate free calculus. Degenerate elliptic equations like the p -Laplacian are dealt with by an approximation argument.

The p -energy and the area are typical representatives for two classes of functionals which we shall distinguish in what follows. With the abbreviation $a = \phi'$ the Euler-Lagrange equation of F is

$$Q[u] := \operatorname{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0, \quad (1.3)$$

which may be written in the equivalent form

$$|\nabla u|^2 \Delta u + b(|\nabla u|) \nabla^2 u (\nabla u, \nabla u) = 0, \quad (1.4)$$

where

$$b(s) = \frac{sa'(s)}{a(s)} - 1 \quad (1.5)$$

and ∇^2 denotes the Hessian. It follows from (1.2) that $1 + b(s) > 0$ for $s > 0$.

As it is well known from the theory of elliptic equations, the behavior of the eigenvalues of the quadratic form associated with (1.4)

$$q(\xi, \xi) = |\nabla u|^2 |\xi|^2 + b(|\nabla u|) \langle \xi, \nabla u \rangle^2 \quad (1.6)$$

is crucial. Precisely, it is the quotient of the eigenvalue λ in direction ∇u given by

$$\lambda = |\nabla u|^2 (1 + b(|\nabla u|))$$

and the maximal eigenvalue given by

$$\Lambda = |\nabla u|^2 \max \{1, 1 + b(|\nabla u|)\}$$

which is decisive. We may easily see that

$$\frac{\lambda}{\Lambda} = 1 + b^-,$$

where $b^- = \min\{b, 0\}$. The construction of barriers at the boundary depends on the behavior of the function $1 + b^-$. We consider the two following possibilities:

Condition I. *Mild decay of the eigenvalue ratio:*

$$(1 + b^-(s)) s^2 \geq \varphi(s), \quad s \geq s_0 > 0, \quad (1.7)$$

where φ is non decreasing and

$$\int_{s_0}^{\infty} \frac{\varphi(s)}{s^2} ds = +\infty. \quad (1.8)$$

Condition II. *Strong decay of the eigenvalue ratio:*

$$(1 + b^-(s)) s^2 \geq \varphi(s), \quad s \geq s_0 > 0,$$

where φ is non increasing and

$$\int_{s_0}^{\infty} \frac{\varphi(s)}{s} ds = +\infty. \quad (1.9)$$

As we will see, with mild decay of the eigenvalue ratio one is able to construct barriers on arbitrary bounded smooth domains. However, for partial differential equations with strong decay of the eigenvalue ratio, it is necessary to require the mean convexity of the domain (see Section 3.2).

Let us mention that the p -Laplace equation falls into class I and the minimal surface equation into class II. Further conditions besides I and II will have to be imposed for global and local gradient estimates (see Section 3.3). Let us also mention that the behavior of type I was introduced by Serrin as *regularly elliptic* (see [20]).

These notes address mainly researchers and graduate students interested in elliptic partial differential equations on Riemannian manifolds and may serve as a material for corresponding courses and seminars. Indeed, the first author gave a course at Federal University of Santa Maria, Rio Grande do Sul, Brazil, on the second semester of 2017, based on these notes.

Our goal in this text was to carve out structural conditions on the integrand ϕ which lead to global and local C^1 -estimates for solutions of the corresponding Euler-Lagrange equations. The text gives a complete, self-contained presentation of this part of the theory; no prerequisites besides elementary Riemannian geometry are required. Once the crucial C^1 -estimates are established a general machinery may be applied to obtain higher order estimates. For this machinery we refer to the literature [10], [15], it is not a subject of this text.

We believe that the techniques of these notes can be extended to more general partial differential equations, such as equations with a nonzero right hand side $Q = f$ with f depending on the point of the manifold, the function and its first derivatives.

We would like to express our thanks to Roberto Nuñez for checking part of this manuscript and for contributing the useful estimate in Remark 3.2.

Chapter 2

Overview of the technique

We resume in this section the main ideas used in these notes to investigate the Dirichlet problem in bounded smooth domains of a Riemannian manifold.

2.1 The method of a priori bounds

The case in which the partial differential equation (1.3) is singular or degenerate, as the p -Laplacian and a similar family of partial differential equations, is reduced to the regular case by a perturbation technique (see Section 4). The main and largest part of our notes concerns the existence of solutions of regular partial differential equations. Regular means that (1.3) is elliptic and has at least Hölder-continuous coefficients. To be more precise we write (1.3) in the equivalent form

$$A(|\nabla u|) \Delta u + \frac{A'(|\nabla u|)}{|\nabla u|} \nabla^2 u(\nabla u, \nabla u) = 0, \quad a(s) = sA(s).$$

In terms of a local orthonormal frame E_1, \dots, E_n this equation may be written as

$$\sum_{i,j} a_{ij}(|\nabla u|) (\nabla^2 u)_{ij} = 0$$

with

$$a_{ij} = A(|\nabla u|) \delta_{ij} + \frac{A'(|\nabla u|)}{|\nabla u|} u_i u_j, \quad \nabla u = u_i E_i.$$

By an elementary but careful computation one sees that for $u \in C^2(\bar{\Omega})$ the coefficients a_{ij} are α -Hölder continuous provided that $A \in C^{1,\alpha}([0, \infty))$. Moreover, the eigenvalues of the matrix (a_{ij}) are $A(s)$ and $A(s) + sA'(s)$ so that ellipticity amounts to the inequalities

$$A(s) > 0, \quad A(s) + sA'(s) > 0$$

for all $s \geq 0$.

To investigate the Dirichlet problem for our partial differential equations which are regular, for *smooth boundary data*, we use the classical method of a priori bounds. In the abstract setting of Functional Analysis, this method is conveniently exposed in form of the following fixed point theorem of Leray-Schauder.

Let \mathcal{B} be a Banach space and denote by $\| \cdot \|$ the norm of \mathcal{B} . A continuous mapping $T : \mathcal{B} \rightarrow \mathcal{B}$ is called compact if the image by T of bounded subsets of \mathcal{B} are precompact that is, their closures are compact subsets of \mathcal{B} .

Theorem 2.1. *Let $T : \mathcal{B} \rightarrow \mathcal{B}$ be a compact mapping and set*

$$V := \{v \in \mathcal{B} \mid v = \sigma T(v) \text{ for some } \sigma \in [0, 1]\}. \quad (2.1)$$

Assume that there is a constant C such that

$$\|v\|_{\mathcal{B}} \leq C$$

for all $v \in V$. Then T has a fixed point that is, there is $u \in \mathcal{B}$ such that $T(u) = u$.

This theorem may lastly be derived from Brouwer fixed point theorem (which asserts that a continuous mapping from a ball of \mathbb{R}^n into itself has a fixed point, see Chapter 11 of [10]). We now show how one can use Theorem 2.1 for investigating the existence of solutions to the Dirichlet problem

$$\begin{cases} Q[u] = A(|\nabla u|) \Delta u + \frac{A'(|\nabla u|)}{|\nabla u|} \nabla^2 u (\nabla u, \nabla u) = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = g \end{cases} \quad (2.2)$$

for a given $g \in C^{2,\alpha}(\overline{\Omega})$, where Ω is a $C^{2,\alpha}$ fixed bounded domain of \mathcal{M} , $A(|\nabla u|) = a(|\nabla u|)/|\nabla u|$. We assume that Q is regular.

In order to apply Theorem 2.1 we take $\mathcal{B} = C^2(\overline{\Omega})$ and define the operator $T : C^2(\overline{\Omega}) \rightarrow C^2(\overline{\Omega})$ as follows. Any $u \in C^2(\overline{\Omega})$ determines a linear operator

$$\mathcal{L}_u[v] = A(|\nabla u|) \Delta v + \frac{A'(|\nabla u|)}{|\nabla u|} \nabla^2 v (\nabla u, \nabla u)$$

which satisfies $\mathcal{L}_u[u] = 0$ if and only if $Q[u] = 0$. From the regularity assumption of Q , \mathcal{L}_u is elliptic and has Hölder coefficients. It then follows from Theorem 6.14 of [10] that the Dirichlet problem

$$\begin{cases} \mathcal{L}_u[w] = 0 \text{ in } \Omega \\ w|_{\partial\Omega} = g \end{cases}$$

has a unique solution $w \in C^{2,\alpha}(\overline{\Omega})$. We may then define $T[u] = w \Leftrightarrow \mathcal{L}_u[w] = 0$ and $w|_{\partial\Omega} = g$. It is clear that $u \in C^2(\overline{\Omega})$ is a solution of (2.2) if and only if u is a fixed point of T .

To prove that T is a compact operator we first note that \mathcal{L}_u satisfies the maximum principle (Theorem 3.5 of [10]) which implies, in combination with Theorem 6.6 of [10], that if $w \in C^{2,\alpha}(\overline{\Omega})$ satisfies $\mathcal{L}_u[w] = 0$ then there is a constant C depending only on a $C^2(\overline{\Omega})$ bound for u such that

$$|w|_{C^{2,\alpha}} \leq C |g|_{C^{2,\alpha}}.$$

Since the embedding of $C^{2,\alpha}(\overline{\Omega})$ into $C^2(\overline{\Omega})$ is compact it follows that T maps bounded subsets into precompact subsets. The continuity of T follows by a similar argument.

Now, let $v \in V$ where V is given by (2.1). We may assume that v is nonzero. Then there is $\sigma \in (0, 1]$ such that $v = \sigma T[v]$. Obviously $v/\sigma \in C^{2,\alpha}(\overline{\Omega})$ and $\mathcal{L}_v[v/\sigma] = \mathcal{L}_v[T[v]] = 0$. Hence, by the linearity of \mathcal{L}_v , $\mathcal{L}_v[v] = \sigma \mathcal{L}_v[v/\sigma] = 0$ that is, v satisfies $Q[v] = 0$ with $v|_{\partial\Omega} = \sigma g$. Hence, the applicability of Theorem 2.1 depends on obtaining a uniform estimate of the C^2 norm of v that is, an upper bound of $|v|_{C^2}$ depending only on $|g|_{C^{2,\alpha}(\overline{\Omega})}$ (besides Ω and a , but not on σ). This is what one calls in partial differential equations theory as “a priori” estimates since they can be obtained independently of the existence of a solution of (2.2). How one obtains these estimates in our case is a matter of a preliminary discussion in the next section.

2.2 A priori estimates

We assume here that Q is regular and that Ω is a $C^{2,\alpha}$ bounded domain. A fundamental tool to obtain a priori estimates is the *comparison principle*. It says that if $v, w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ are sub and supersolutions respectively of Q (that is $Q(v) \geq 0$ and $Q(w) \leq 0$, in the classical or weak sense, see Section 3.1 for details), and if $v|_{\partial\Omega} \leq w|_{\partial\Omega}$ then $v \leq w$ in Ω . In our case, once we prove a comparison principle for our partial differential equations, since the constant functions are solutions of $Q[u] = 0$ we immediately have

$$\inf_{\Omega} g \leq u \leq \sup_{\Omega} g$$

if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of (2.2). We then have an a priori estimate for the C^0 norm

$$|u|_{C^0} \leq \sup_{\Omega} |g| \tag{2.3}$$

for any possible solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of (2.2).

The case of smooth boundary data

We begin by obtaining a priori gradient estimates, first by using barriers to control the gradient at the boundary of the domain. Barriers at a given point $p \in \partial\Omega$ are sub and supersolutions $v, w \in C^1(\bar{\Omega})$ of Q , respectively, such that

$$v(p) = w(p) = g(p)$$

and

$$v(x) \leq g(x) \leq w(x)$$

for all $x \in \partial\Omega$. From the comparison principle it then follows that if $u \in C^1(\bar{\Omega})$ is a solution of (2.2) then

$$v \leq u \leq w$$

and, from elementary analysis, it follows that

$$|\nabla u(p)| \leq \max\{|\nabla w(p)|, |\nabla v(p)|\}.$$

In many cases, when \mathcal{M} has nonnegative Ricci curvature a maximum principle holds for the norm of the gradient, that is

$$\sup_{\Omega} |\nabla u| = \sup_{\partial\Omega} |\nabla u|$$

from which, with the help of barriers, one obtains an a priori C^1 estimates of a solution of (2.2). In a general manifold, where the Ricci curvature can be negative or change sign, and for the general class of partial differential equations considered here, the maximum of the gradient can, in principle, occur at an interior point of the domain. If this happens we prove that the gradient at such a point is controlled by the C^0 norm of the solution. Together with the boundary gradient estimate we then obtain an a priori C^1 estimate of a solution of (2.2).

Next we need Hölder estimates for the gradient. For this step we refer to well established theories in the literature (see [10], Theorem 13.2). We may then assert that there is $\gamma > 0$ such that u has an a priori $C^{1,\gamma}$ norm bound in $\bar{\Omega}$ (with γ depending only on $|u|_{C^1}$ and hence only on $|g|_{C^{2,\alpha}}$) and therefore the coefficients of the operator \mathcal{L}_u have uniformly bounded norm in $C^{\alpha\gamma}(\bar{\Omega})$. We may then again apply the linear theory (see Theorem 6.6 of [10]) to the equation

$$\mathcal{L}_u[u] = 0, \quad u|_{\partial\Omega} = \sigma g$$

to obtain an a priori $C^2(\bar{\Omega})$ bound for u .

We have seen that the solvability of the Dirichlet problem (2.2) reduces to proving a comparison principle for Q and obtaining global a priori estimates for the gradient. This will be done in the next sections. It remains to consider the Dirichlet problem (2.2) for continuous boundary data. The main points of this are discussed in the next section.

The case of continuous boundary data

We assume that Q is regular and that (2.2) is solvable for $C^{2,\alpha}$ boundary data. To study the problem (2.2) in the case that Ω is bounded and of $C^{2,\alpha}$ class but g is only continuous we need to obtain a priori *local* gradient estimates. We begin by obtaining a maximum principle for the difference of two solutions (Proposition 3.2) that is, if $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $Q[u] = Q[v] = 0$ in Ω then

$$\sup_{\Omega} |u - v| = \sup_{\partial\Omega} |u - v|. \quad (2.4)$$

We then consider a sequence $g_k \in C^{2,\alpha}(\overline{\Omega})$ converging in the C^0 norm to g . From the maximum principle the corresponding solutions $u_k \in C^2(\overline{\Omega})$ to the Dirichlet problem with boundary data g_k satisfy

$$\sup_{\Omega} |u_k - u_j| = \sup_{\Omega} |g_k - g_j|, \quad j, k \in \mathbb{N},$$

being hence a Cauchy sequence on $C^0(\overline{\Omega})$. The sequence (u_k) then converges in the C^0 norm to some $u \in C^0(\overline{\Omega})$. To prove that in fact $u \in C^2(\Omega)$ we obtain local gradient estimates. This is done by fixing an arbitrary $x \in \Omega$ and choosing $r > 0$ such that the closed geodesic ball $B_r(x)$ is contained in Ω and is normal. Then we prove that there is a constant C depending only on a, r, g such that

$$|\nabla u_k(x)| \leq C$$

for all $k \in \mathbb{N}$. Then, as before, we can make use of Theorem 13.2 of [10] to get an uniform $C^{1,\gamma}$ -norm bound of (u_k) for some $\gamma > 0$ and the linear theory (Theorem 6.6 of [10]) to prove that the sequence (u_k) has equibounded $C^{2,\gamma\alpha}$ norm on any relatively compact subdomain of Ω . Therefore, it contains a subsequence converging in the C^2 norm to u on any such subdomain and thus $u \in C^2(\Omega)$. By the continuity of $Q : C^2(\Omega) \rightarrow C^0(\Omega)$ it follows that $Q[u] = 0$.

The boundary gradient estimates, the gradient estimates at interior points for smooth boundary data, and the local gradient estimates are obtained in the next sections.

Chapter 3

Gradient estimates

In this section we derive global and local estimates for solutions for regular partial differential equations under the assumption that the solutions are of C^3 class.

3.1 Comparison and maximum principles

Let Ω be a domain in \mathcal{M} , $\bar{\Omega}$ compact. We assume that

$$Q[u] = \operatorname{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right)$$

is such that $a : [0, \infty) \rightarrow \mathbb{R}$ is strictly increasing and $a(0) = 0$.

Denote by $C_0^{0,1}(\Omega)$ the space of Lipschitz functions which compact support on Ω . We say that $u \in C^{0,1}(\Omega)$ is a weak solution of Q if

$$\int_{\Omega} \left\langle \frac{a(|\nabla u|)}{|\nabla u|} \nabla u, \nabla \xi \right\rangle dx = 0 \quad (3.1)$$

for all $\xi \in C_0^{0,1}(\Omega)$. We say that $v \in C^{0,1}(\Omega)$ is a weak supersolution (subsolution) of Q if (3.1) holds with “ \geq ” (“ \leq ”) instead of “ $=$ ” for all $\xi \in C_0^{0,1}(\Omega)$ with $\xi \geq 0$ on Ω .

Proposition 3.1 (Comparison Principle). *Let Ω be open and bounded, $u \in C^{0,1}(\Omega)$ a weak subsolution of Q and $v \in C^{0,1}(\Omega)$ a weak supersolution of Q such that*

$$\limsup_k (u(x_k) - v(x_k)) \leq 0 \quad (3.2)$$

for any sequence x_k in Ω which leaves any compact subset of Ω . Then it follows that $u \leq v$ in Ω .

Proof. Let $\varepsilon > 0$ and let us choose

$$\zeta := (u - v - \varepsilon)^+ = \max \{u - v - \varepsilon, 0\}.$$

From (3.2) it follows that u and v have bounded first derivatives on the support of ζ , a compact subset of Ω , and

$$\nabla \zeta = \begin{cases} \nabla(u - v) & \text{if } u - v > \varepsilon \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore $|\nabla u|$ and $|\nabla v|$ are integrable on the set

$$\Lambda_\varepsilon := \{x \in \Omega \mid u(x) - v(x) > \varepsilon\}$$

and we have

$$\int_{\Lambda_\varepsilon} \left\langle \frac{a(|\nabla u|)}{|\nabla u|} \nabla u - \frac{a(|\nabla v|)}{|\nabla v|} \nabla v, \nabla u - \nabla v \right\rangle dx \leq 0. \quad (3.3)$$

On the other hand,

$$\begin{aligned} & \left\langle \frac{a(|\nabla u|)}{|\nabla u|} \nabla u - \frac{a(|\nabla v|)}{|\nabla v|} \nabla v, \nabla u - \nabla v \right\rangle \\ &= a(|\nabla u|)|\nabla u|^2 - \frac{a(|\nabla u|)}{|\nabla u|} \langle \nabla u, \nabla v \rangle - \frac{a(|\nabla v|)}{|\nabla v|} \langle \nabla u, \nabla v \rangle + a(|\nabla v|)|\nabla v| \\ &\geq a(|\nabla u|)|\nabla u| - a(|\nabla u|)|\nabla v| - a(|\nabla v|)|\nabla u| + a(|\nabla v|)|\nabla v| \\ &= (a(|\nabla u|) - a(|\nabla v|)) (|\nabla u| - |\nabla v|), \end{aligned}$$

where the inequality is implied by Cauchy-Schwarz inequality. Since a is increasing it follows from (3.3) that $|\nabla u| = |\nabla v|$ a.e. on Λ_ε . From this, in connection with (3.3), we conclude that $\nabla \zeta = 0$ a.e. on Ω . It follows then, again from (3.3), that $\zeta = 0$ since $\zeta \in C_0^{0,1}(\Omega)$. We conclude that $u - v \leq 0$ in Ω since $\varepsilon > 0$ is arbitrary, concluding with the proof of the proposition. \square

Proposition 3.2 (Maximum Principle). *Let $\Omega \subset \mathcal{M}$ be an open bounded and $u, v \in C^{0,1}(\Omega) \cap C^0(\bar{\Omega})$ be weak solutions of Q . Then*

$$\max_{\bar{\Omega}} |u - v| = \max_{\partial\Omega} |u - v|. \quad (3.4)$$

In particular, since $v = 0$ is a solution we have the maximum principle

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Proof. Set

$$M := \max_{\partial\Omega} |u - v|.$$

Then, from the Comparison Principle

$$u = u - v + v \leq v + M.$$

Reversing the roles of u and v we get $v \leq u + M$ from which we get (3.4). \square

3.2 Boundary gradient estimates. Barriers.

In this section we assume that Ω is a bounded domain of C^2 class in \mathcal{M} . We follow Serrin's treatment with some simplifications [20]. The cases of mild and strong decay of the eigenvalue ratio have to be treated separately. Nevertheless the type of barrier that we use, defined in what follows, will be the same in both cases.

We fix a number $\delta_0 > 0$ such that the function

$$d(x) = \text{distance}(x, \partial\Omega)$$

is of C^2 class on the boundary strip

$$\bar{\Omega}_{\delta_0} = \{x \in \bar{\Omega} \mid 0 \leq d(x) \leq \delta_0\}$$

and we seek barriers of the form

$$w = g + f(d),$$

where the function f is defined in some interval $[0, \delta]$, $0 < \delta \leq \delta_0$, $f(0) = 0$, and w is a supersolution with $f(\delta) = M$ or a subsolution with $f(\delta) = -M$, $M > 0$ a preassigned number.

To simplify later calculations we introduce the linear operator

$$\mathcal{L}_w v = \Delta v + b(|\nabla w|) \nabla^2 v \left(\frac{\nabla w}{|\nabla w|}, \frac{\nabla w}{|\nabla w|} \right)$$

which satisfies

$$|\mathcal{L}_w v| \leq (n - 1 + 1 + b) |\nabla^2 v| \leq nB |\nabla^2 v| \quad (3.5)$$

with $B = \max\{1, 1 + b\}$. Using (1.4) and (1.5) we may write the operator Q as

$$Q[w] = \mathcal{L}_w g + f' \mathcal{L}_w d + f'' \left(1 + b \left\langle \nabla d, \frac{\nabla w}{|\nabla w|} \right\rangle^2 \right). \quad (3.6)$$

We shall also need the following obvious estimates of $|\nabla w|$ in terms of f' :

$$f' - c_1 \leq |\nabla w| \leq f' + c_1, \quad (3.7)$$

where $c_1 = \max_{\Omega_{\delta_0}} |\nabla g|$ and hence

$$\frac{2}{3}f' \leq |\nabla w| \leq \frac{4}{3}f' \quad (3.8)$$

provided that

$$f' \geq \alpha \geq \max\{1, 3c_1\}, \quad (3.9)$$

where the number α will be appropriately chosen later on. We construct only a supersolution and assume then that $f'' \leq 0$.

The case of mild decay of the eigenvalue ratio

As mentioned at the introduction, this class falls into the Serrin's category of "regularly elliptic" equations (see [20]). We have

$$\begin{aligned} 1 + b \left\langle \nabla d, \frac{\nabla w}{|\nabla w|} \right\rangle^2 &\geq (1 + b) \left\langle \nabla d, \frac{\nabla w}{|\nabla w|} \right\rangle^2 \\ &= \frac{1 + b}{f'^2} \left\langle \nabla w - \nabla g, \frac{\nabla w}{|\nabla w|} \right\rangle^2 \geq \frac{1 + b}{f'^2} (|\nabla w| - c_1)^2 \\ &\geq \frac{1}{4} \frac{1 + b}{f'^2} |\nabla w|^2 \end{aligned} \quad (3.10)$$

on account of (3.8) and (3.9). Inserting into (3.6), recalling that $f'' \leq 0$ and observing (3.5) we obtain

$$\frac{4}{f'B} Q[w] \leq 4n (|\nabla^2 g| + |\nabla^2 d|) + \frac{f''}{f'^3} \frac{b+1}{B} |\nabla w|^2.$$

Setting

$$C = 4n \max_{\Omega_\delta} (|\nabla^2 g| + |\nabla^2 d|)$$

and observing that

$$\frac{b+1}{B} = 1 + b^-$$

we obtain from (1.7) and (1.8)

$$\frac{4}{f'B} Q[w] \leq C + \frac{f''}{f'^3} \varphi \left(\frac{2}{3}f' \right). \quad (3.11)$$

In the last step having used (3.8) and the fact that φ is non-decreasing. Our task will be complete if we can find a solution to the ordinary differential equation

$$f'' + C \frac{f'^3}{\varphi\left(\frac{2}{3}f'\right)} = 0 \quad (3.12)$$

which is defined in some interval $[0, \delta,]$ to be explicitly determined later, $0 \leq \delta \leq \delta_0$, and satisfies $f(0) = 0$, $f(\delta) = M$ and $f' \geq \alpha$ in $[0, \delta]$, where M is a given positive number and now α is chosen as

$$\alpha = \max \left\{ \frac{M}{\delta_0}, 1, 3c_1 \right\}. \quad (3.13)$$

We rewrite (3.12) as an equation for the inverse function of f' , denoted by h , that is

$$h'(s) = -\frac{\varphi\left(\frac{2}{3}s\right)}{Cs^3},$$

leading to

$$h(s) = \int_s^\beta \frac{\varphi\left(\frac{2}{3}t\right)}{Ct^3} dt, \quad \alpha \leq s \leq \beta,$$

where β is still to be determined.

The domain of the definition of f is the interval $[h(\beta), h(\alpha)] = [0, \delta]$ with

$$\delta = \int_\alpha^\beta \frac{\varphi\left(\frac{2}{3}t\right)}{Ct^3} dt$$

and

$$f(\delta) = \int_0^\delta f'(s) ds = - \int_\beta^\alpha t \frac{\varphi\left(\frac{2}{3}t\right)}{Ct^3} dt = \int_\alpha^\beta \frac{\varphi\left(\frac{2}{3}t\right)}{Ct^2} dt.$$

Due to (1.8) we may now choose β and hence δ so that $f(\delta) = M$. Moreover,

$$\delta = \int_\alpha^\beta \frac{\varphi\left(\frac{2}{3}t\right)}{Ct^3} dt \leq \frac{1}{\alpha} \int_\alpha^\beta \frac{\varphi\left(\frac{2}{3}t\right)}{Ct^2} dt = \frac{M}{\alpha} \leq \delta_0,$$

where we used that $\alpha \geq 1$ for the first inequality and that $\alpha \geq M/\delta_0$ for the second one. Replacing f by $-f$ we obtain a subsolution. This completes the construction of barriers for the class of partial differential equations with mild decay of the eigenvalue ratio.

Remark 3.1. We shall now illustrate the significance of the divergence of the integral in (1.8) by showing that the existence of a supersolution w with the above properties fails in general in the case that the boundary is concave and condition (1.8) is violated. More precisely, let us assume that

$$\int_{s_0}^\infty (1 + b(s)) ds < +\infty \quad (3.14)$$

and that $\Delta d \geq c$ in the boundary strip Ω_{δ_0} for some constant $c > 0$. Then we may construct a subsolution of the form

$$v = f(d), \quad 0 \leq d \leq \delta \leq \delta_0,$$

such that

$$f(0) = 0, f' > 0, f'(0) = +\infty, f(\delta) = M$$

for some positive number M which is determined by the integral (3.14).

If there were a supersolution $w \in C^1(\overline{\Omega_\delta})$ such that $w(x) \geq M$ for $d(x) = \delta$ and $w|_{\partial\Omega} = 0$ then it would follow from the comparison principle that $w \geq v$ in Ω_δ . This is however impossible since the interior normal derivative of $w - v$ equals to $-\infty$ along $\partial\Omega$.

In order to construct v we compute

$$Q[v] = f' \Delta d + f'' (1 + b(f')) \geq c f' + f'' (1 + b(f'))$$

so that it will be enough to find a solution to the equation

$$f'' + c \frac{f'}{1 + b(f')} = 0$$

with the required properties.

As above, we switch to the inverse function h of f' and get

$$h(t) = \frac{1}{c} \int_t^\infty \frac{1 + b(s)}{s} ds, \quad \alpha \leq t < +\infty,$$

where α is chosen in such a way that $\delta = h(\alpha) \leq \delta_0$. It follows that

$$f(\delta) = \int_\alpha^\infty (1 + b(s)) ds =: M > 0.$$

The case of strong decay of the eigenvalue ratio

In this case it becomes necessary to restrict the geometry of $\partial\Omega$; we require that the mean curvature of $\partial\Omega$ as well as of the level hypersurfaces of d , $0 \leq d \leq \delta_0$, is nonnegative with respect to the normal vector ∇d . This is equivalent to the condition

$$\Delta d \leq 0 \text{ in } \overline{\Omega_{\delta_0}}. \quad (3.15)$$

Since $\nabla^2 d(\nabla d, \nabla d) = 0$ we then obtain

$$\begin{aligned} f' \mathcal{L}_w d &\leq f' b \nabla^2 d \left(\frac{\nabla w}{|\nabla w|}, \frac{\nabla w}{|\nabla w|} \right) \\ &\leq B \frac{f'^2}{|\nabla w|^2} \left| 2 \nabla^2 d(\nabla d, \nabla g) + \frac{1}{f'} \nabla^2 d(\nabla g, \nabla g) \right| \\ &\leq B c_0, \end{aligned}$$

where the constant c_0 only depends on

$$\max_{\overline{\Omega}_{\delta_0}} (|\nabla g| + |\nabla^2 d|)$$

and (3.9) is assumed to hold further on. Inserting this last estimate and (3.10) in (3.6) we arrive at

$$\begin{aligned} \frac{4}{B} Q[w] &\leq C + \frac{f''}{f'^2} \frac{1+b}{B} |\nabla w|^2 \\ &\leq C + \frac{f''}{f'^2} \varphi(|\nabla w|) \\ &\leq C + \frac{f''}{f'^2} \varphi\left(\frac{4}{3} f'\right), \end{aligned}$$

where the constant C depends only on

$$\max_{\overline{\Omega}_{\delta_0}} (|\nabla g| + |\nabla^2 g| + |\nabla^2 d|)$$

and we used (3.8), (1.9) and the fact that φ is non-increasing. We again choose α according to (3.13) and consider h , the inverse function of f' , which is given by

$$h(s) = C \int_s^\beta \frac{\varphi\left(\frac{4t}{3}\right)}{t^2} dt, \quad \alpha \leq s \leq \beta.$$

We obtain

$$\delta = C \int_\alpha^\beta \frac{\varphi\left(\frac{4t}{3}\right)}{t^2} dt$$

and

$$f(\delta) = C \int_\alpha^\beta \frac{\varphi\left(\frac{4t}{3}\right)}{t} dt.$$

Condition 1.9 allows to choose β such that $f(\delta) = M$ and, as before,

$$\delta \leq \frac{M}{\alpha} \leq \delta_0$$

and the barrier construction for the class of minimal surface equation is complete.

From the previous calculations and also (2.3), we obtain:

Theorem 3.3. *Let Ω be a bounded domain of class C^2 in \mathcal{M} and $u \in C^1(\overline{\Omega})$ be a weak solution of (1.3) such that $u = g$ on $\partial\Omega$ with $g \in C^2(\overline{\Omega})$. We assume that either Condition I or II of Section 1 are satisfied and in case that Condition II holds we require furthermore that the mean curvature of $\partial\Omega$ with respect to the interior normal of $\partial\Omega$ as well as of the inner parallel hypersurfaces of $\partial\Omega$ in some neighborhood of $\partial\Omega$ is non negative. Then the normal derivative of u on $\partial\Omega$ can be estimated by a constant depending only on $|g|_{C_2(\Omega)}$.*

3.3 Global and local gradient estimates

In this section we prove global and local estimates of solutions of the partial differential equation (1.3) on bounded domains. We assume that u is a solution of class C^3 and use the equivalent form of (1.3), namely:

$$|\nabla u|^2 \Delta u + b \nabla^2 u (\nabla u, \nabla u) = 0 \quad (3.16)$$

recalling that

$$b(s) = \frac{sa'(s)}{a(s)} - 1.$$

In order to derive gradient bounds for the solutions of (3.16) we consider a point of Ω , say x_0 , where a certain auxiliary function attains a local maximum. We need slightly different such auxiliary functions, all of them of the form

$$G(x) = g(x)f(u)F(|\nabla u|).$$

The gradient estimates (local and global) are obtained by writing

$$\nabla^2 G(x_0) (\nabla u, \nabla u)$$

as a polynomial in $|\nabla u|$. Analyzing its leading coefficient, after an appropriate choice of g , f and F , the constraint

$$\nabla^2 G(x_0) (\nabla u, \nabla u) \leq 0$$

will impose an upper bound for $|\nabla u|$.

We shall make use of the well known Bochner formula:

Proposition 3.4 (Bochner formula). *If \mathcal{M}^n is a Riemannian manifold and $u \in C^3(\mathcal{M})$ then*

$$\langle \nabla \Delta u, \nabla u \rangle = \frac{1}{2} \Delta |\nabla u|^2 - |\nabla^2 u|^2 - \text{Ric}(\nabla u, \nabla u). \quad (3.17)$$

Proof. Let $p \in M$ and E_1, \dots, E_n a local orthonormal frame field in a neighborhood V of p such that

$$\nabla_{E_i} E_j(p) = 0, \quad i, j = 1, \dots, n. \quad (3.18)$$

Hence, we have at p

$$\begin{aligned} \Delta |\nabla u|^2 &= \sum_j \nabla^2 |\nabla u|^2 (E_j, E_j) \\ &= \sum_j \left\langle \nabla_{E_j} \nabla |\nabla u|^2, E_j \right\rangle \\ &= \sum_j E_j \langle \nabla |\nabla u|^2, E_j \rangle. \end{aligned} \quad (3.19)$$

Since

$$\begin{aligned}
\langle \nabla |\nabla u|^2, E_j \rangle &= E_j |\nabla u|^2 = E_j \langle \nabla u, \nabla u \rangle \\
&= 2 \langle \nabla_{E_j} \nabla u, \nabla u \rangle = 2 \nabla^2 u (E_j, \nabla u) \\
&= 2 \nabla^2 u \left(E_j, \sum_i \langle \nabla u, E_i \rangle E_i \right) \\
&= 2 \sum_i \langle \nabla u, E_i \rangle \nabla^2 u (E_j, E_i) \tag{3.20}
\end{aligned}$$

hold at every point of M we obtain

$$\begin{aligned}
\Delta |\nabla u|^2 &= 2 \sum_{i,j} E_j (\langle \nabla u, E_i \rangle \nabla^2 u (E_j, E_i)) \\
&= 2 \sum_{i,j} [E_j (\langle \nabla u, E_i \rangle) \nabla^2 u (E_j, E_i) + \langle \nabla u, E_i \rangle E_j (\nabla^2 u (E_j, E_i))]
\end{aligned}$$

and then

$$\Delta |\nabla u|^2 = 2 \sum_{i,j} \left[\nabla^2 u (E_j, E_i)^2 + \langle \nabla u, E_i \rangle E_j (\nabla^2 u (E_j, E_i)) \right]. \tag{3.21}$$

By the symmetry of $\nabla^2 u$ we have at p

$$\begin{aligned}
E_j (\nabla^2 u (E_j, E_i)) &= E_j (\nabla^2 u (E_i, E_j)) \\
&= E_j \langle \nabla_{E_i} \nabla u, E_j \rangle \\
&= \langle \nabla_{E_j} \nabla_{E_i} \nabla u, E_j \rangle \\
&= \langle R (E_j, E_i) \nabla u + \nabla_{E_i} \nabla_{E_j} \nabla u, E_j \rangle \\
&= \langle R (E_j, E_i) \nabla u, E_j \rangle + E_i \langle \nabla_{E_j} \nabla u, E_j \rangle \\
&= \langle R (E_j, E_i) \nabla u, E_j \rangle + E_i (\nabla^2 u (E_j, E_j)), \tag{3.22}
\end{aligned}$$

where R denotes the curvature tensor of \mathcal{M} . Inserting (3.22) in (3.21) we finally arrive at

$$\begin{aligned}
\frac{\Delta |\nabla u|^2}{2} &= \sum_{i,j} \left\{ \nabla^2 u (E_j, E_i)^2 + \langle \nabla u, E_i \rangle [\langle R (E_j, E_i) \nabla u, E_j \rangle + E_i (\nabla^2 u (E_j, E_j))] \right\} \\
&= |\nabla^2 u|^2 + \sum_j \langle R (E_j, \nabla u) \nabla u, E_j \rangle + \sum_i \langle \nabla u, E_i \rangle E_i (\Delta u) \\
&= |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u \rangle.
\end{aligned}$$

□

We now obtain an equation for $|\nabla u|$ by differentiating (3.16) in direction ∇u .

Lemma 3.5. *If u solves (3.16) then, in an orthonormal frame E_1, \dots, E_n with $E_1 = |\nabla u|^{-1} \nabla u$ on a neighborhood of Ω where ∇u is non zero, the following equality holds*

$$\begin{aligned} & (b+1) |\nabla u| \nabla^2 |\nabla u| (E_1, E_1) + |\nabla u| \sum_{i=2}^n \nabla^2 |\nabla u| (E_i, E_i) \\ & + b' |\nabla u| \nabla^2 u (E_1, E_1)^2 + b \sum_{i=2}^n \nabla^2 u (E_1, E_i)^2 \\ & - \sum_{i=1, j=2}^n \nabla^2 u (E_i, E_j)^2 - \text{Ric}(\nabla u, \nabla u) = 0, \end{aligned}$$

where Ric denotes the Ricci tensor of \mathcal{M} .

Proof. Differentiating (3.16) in direction ∇u gives

$$\begin{aligned} & |\nabla u|^2 \langle \nabla \Delta u, \nabla u \rangle + \langle \nabla |\nabla u|^2, \nabla u \rangle \Delta u \\ & + b' \frac{1}{2} |\nabla u|^{-1} \langle \nabla |\nabla u|^2, \nabla u \rangle \nabla^2 u (\nabla u, \nabla u) \\ & + b \nabla u (\nabla^2 u (\nabla u, \nabla u)) = 0. \end{aligned}$$

From the relation

$$\nabla^2 u (\nabla u, \nabla u) = \frac{1}{2} \langle \nabla |\nabla u|^2, \nabla u \rangle$$

and Bochner formula (3.17) we obtain

$$\begin{aligned} & \frac{1}{2} \Delta |\nabla u|^2 - |\nabla^2 u|^2 - \text{Ric}(\nabla u, \nabla u) \\ & + (b' |\nabla u| - 2b) |\nabla u|^{-4} \nabla^2 u (\nabla u, \nabla u)^2 \\ & + \frac{1}{2} b |\nabla u|^{-2} \nabla u \langle \nabla |\nabla u|^2, \nabla u \rangle = 0. \end{aligned}$$

For the last term we have

$$\begin{aligned} \nabla u \langle \nabla |\nabla u|^2, \nabla u \rangle &= |\nabla u| \left[\langle \nabla_{E_1} \nabla |\nabla u|^2, \nabla u \rangle + \langle \nabla |\nabla u|^2, \nabla_{E_1} \nabla u \rangle \right] \\ &= |\nabla u| \left[\nabla^2 |\nabla u|^2 (E_1, \nabla u) + \sum_{i=1}^n \langle E_i (|\nabla u|^2) E_i, \nabla_{E_1} \nabla u \rangle \right] \\ &= |\nabla u|^2 \left[\nabla^2 |\nabla u|^2 (E_1, E_1) + 2 \sum_{i=1}^n \nabla^2 u (E_1, E_i)^2 \right]. \end{aligned}$$

This leads to

$$\begin{aligned} & \frac{1}{2}(b+1)\nabla^2|\nabla u|^2(E_1, E_1) + \frac{1}{2}\sum_{i=2}^n\nabla^2|\nabla u|^2(E_i, E_i) \\ & + (b'|\nabla u| - b)\nabla^2 u(E_1, E_1)^2 + b\sum_{i=2}^n\nabla^2 u(E_1, E_i)^2 \\ & - \sum_{i,j=1}^n\nabla^2 u(E_i, E_j)^2 - \text{Ric}(\nabla u, \nabla u) = 0. \end{aligned}$$

Using finally the relation

$$\frac{1}{2}\nabla^2|\nabla u|^2(E_i, E_i) = |\nabla u|\nabla^2|\nabla u|(E_i, E_i) + \nabla^2 u(E_1, E_i)^2, \quad 1 \leq i \leq n,$$

to convert the last equation into one for $|\nabla u|$ instead of $|\nabla u|^2$ we arrive at the equation in the lemma. \square

We resume some computations used in the estimates in the following lemma:

Lemma 3.6. *If u solves (3.16) and the function $G(x) = g(x)f(u)F(|\nabla u|)$ attains a local maximum in an interior point y_0 of Ω with $\nabla u(y_0) \neq 0$ then, in terms of a local orthonormal basis $E_1 := |\nabla u|^{-1}\nabla u, E_2, \dots, E_n$ of $T_{y_0}\mathcal{M}$ we obtain, at y_0 , the relations*

$$\frac{F'}{F}\nabla^2 u(E_1, E_i) = -\frac{1}{g}\langle\nabla g, E_i\rangle - \frac{f'}{f}\langle\nabla u, E_i\rangle$$

and

$$\begin{aligned} 0 \geq & \left[-\frac{F'b'}{F} + (b+1)\left(\frac{F''}{F} - \frac{F'^2}{F^2}\right) \right] \nabla^2 u(E_1, E_1)^2 + \frac{F'}{F|\nabla u|} \sum_{\substack{i,j \\ i \geq 2}} \nabla^2 u(E_i, E_j)^2 \\ & + \left[-\frac{F'b}{F|\nabla u|} + \frac{F''}{F} - \frac{F'^2}{F^2} \right] \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 + (b+1)\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)|\nabla u|^2 \\ & + \frac{|\nabla u|F'}{F} \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b+1)\nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\ & - \frac{1}{g^2} \left[(b+1)\langle\nabla g, E_1\rangle^2 + \sum_{i \geq 2} \langle\nabla g, E_i\rangle^2 \right]. \end{aligned}$$

Proof. We compute

$$\nabla \ln G = \frac{1}{g}\nabla g + \frac{f'}{f}\nabla u + \frac{F'}{F}\nabla|\nabla u| \quad (3.23)$$

and, for $\xi, \eta \in T_{y_0}M$,

$$\begin{aligned}
\nabla^2 \ln G(\xi, \eta) &= \langle \nabla_\xi \nabla \ln G, \eta \rangle = \langle \nabla_\xi \left(\frac{1}{g} \nabla g + \frac{f'}{f} \nabla u + \frac{F'}{F} \nabla |\nabla u| \right), \eta \rangle \\
&= \langle -\frac{1}{g^2} \xi(g) \nabla g + \frac{1}{g} \nabla_\xi \nabla g + [\frac{f''}{f} \xi(u) - \frac{f'^2}{f^2} \xi(u)] \nabla u + \frac{f'}{f} \nabla_\xi \nabla u \\
&\quad + [\frac{F''}{F} \xi(|\nabla u|) - \frac{F'^2}{F^2} \xi(|\nabla u|)] \nabla |\nabla u| + \frac{F'}{F} \nabla_\xi \nabla |\nabla u|, \eta \rangle \\
&= -\frac{1}{g^2} \langle \nabla g, \xi \rangle \langle \nabla g, \eta \rangle + \frac{1}{g} \nabla^2 g(\xi, \eta) + [\frac{f''}{f} - \frac{f'^2}{f^2}] \langle \nabla u, \xi \rangle \langle \nabla u, \eta \rangle \\
&\quad + \frac{f'}{f} \nabla^2 u(\xi, \eta) + [\frac{F''}{F} - \frac{F'^2}{F^2}] \frac{1}{|\nabla u|^2} \nabla^2 u(\nabla u, \xi) \nabla^2 u(\nabla u, \eta) \\
&\quad + \frac{F'}{F} \nabla^2 |\nabla u|(\xi, \eta),
\end{aligned}$$

where we used the relation

$$\begin{aligned}
\nabla^2 u(\nabla u, \eta) &= \langle \nabla_\eta \nabla u, \nabla u \rangle = \frac{1}{2} \eta(|\nabla u|^2) \\
&= \frac{1}{2} \langle \nabla |\nabla u|^2, \eta \rangle = |\nabla u| \langle \nabla |\nabla u|, \eta \rangle.
\end{aligned} \tag{3.24}$$

By (3.23) and (3.24) we have at y_0

$$\begin{aligned}
\frac{F'}{F} \nabla^2 u(E_1, E_i) &= \frac{F'}{F |\nabla u|} \nabla^2 u(\nabla u, E_i) \\
&= \frac{F'}{F} \langle \nabla |\nabla u|, E_i \rangle = -\frac{1}{g} \langle \nabla g, E_i \rangle - \frac{f'}{f} \langle \nabla u, E_i \rangle.
\end{aligned}$$

Since (3.16) is elliptic and the matrix $(\nabla^2 \ln G_{y_0}(E_i, E_j))$ is nonpositive, we obtain at y_0

$$\begin{aligned}
0 \geq \theta &= (b+1) \nabla^2 \ln G(E_1, E_1) + \sum_{i \geq 2} \nabla^2 \ln G(E_i, E_i) \\
&= -\frac{b+1}{g^2} \langle \nabla g, E_1 \rangle^2 + \frac{b+1}{g} \nabla^2 g(E_1, E_1) + (b+1) \left(\frac{f''}{f} - \frac{f'^2}{f^2} \right) \langle \nabla u, E_1 \rangle^2 \\
&\quad + (b+1) \frac{f'}{f} \nabla^2 u(E_1, E_1) + (b+1) \left(\frac{F''}{F} - \frac{F'^2}{F^2} \right) \frac{1}{|\nabla u|^2} \nabla^2 u(\nabla u, E_1)^2 \\
&\quad + (b+1) \frac{F'}{F} \nabla^2 |\nabla u|(E_1, E_1) - \frac{1}{g^2} \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 + \frac{1}{g} \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \\
&\quad + \left(\frac{f''}{f} - \frac{f'^2}{f^2} \right) \sum_{i \geq 2} \langle \nabla u, E_i \rangle^2 + \frac{f'}{f} \sum_{i \geq 2} \nabla^2 u(E_i, E_i) \\
&\quad + \left(\frac{F''}{F} - \frac{F'^2}{F^2} \right) \frac{1}{|\nabla u|^2} \sum_{i \geq 2} \nabla^2 u(\nabla u, E_i)^2 + \frac{F'}{F} \sum_{i \geq 2} \nabla^2 |\nabla u|(E_i, E_i).
\end{aligned}$$

Hence, by (3.16),

$$\begin{aligned}
\theta &= \frac{F'}{F|\nabla u|} \left\{ |\nabla u|(b+1)\nabla^2|\nabla u|(E_1, E_1) + |\nabla u| \sum_{i \geq 2} \nabla^2|\nabla u|(E_i, E_i) \right\} \\
&+ (b+1) \left(\frac{F''}{F} - \frac{F'^2}{F^2} \right) \nabla^2 u(E_1, E_1)^2 + \left(\frac{F''}{F} - \frac{F'^2}{F^2} \right) \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 \\
&+ (b+1) \left(\frac{f''}{f} - \frac{f'^2}{f^2} \right) |\nabla u|^2 + \frac{1}{g} \left[(b+1)\nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&- \frac{1}{g^2} \left[(b+1)\langle \nabla g, E_1 \rangle^2 + \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \right] + \frac{f'}{f} \underbrace{[b\nabla^2 u(E_1, E_1) + \Delta u]}_{=0}.
\end{aligned} \tag{3.25}$$

By Lemma 3.5 and (3.25), we obtain

$$\begin{aligned}
\theta &= \frac{F'}{F|\nabla u|} \left\{ -b' |\nabla u| \nabla^2 u(E_1, E_1)^2 - b \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 \right. \\
&\quad \left. + \sum_{\substack{i,j \\ i \geq 2}} \nabla^2 u(E_i, E_j)^2 + \text{Ric}(\nabla u, \nabla u) \right\} \\
&+ (b+1) \left(\frac{F''}{F} - \frac{F'^2}{F^2} \right) \nabla^2 u(E_1, E_1)^2 + \left(\frac{F''}{F} - \frac{F'^2}{F^2} \right) \sum_{i \geq 2} \nabla^2 u(E_1, E_i) \\
&+ (b+1) \left(\frac{f''}{f} - \frac{f'^2}{f^2} \right) |\nabla u|^2 + \frac{1}{g} \left[(b+1)\nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&- \frac{1}{g^2} \left[(b+1)\langle \nabla g, E_1 \rangle^2 + \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \right]
\end{aligned}$$

and so

$$\begin{aligned}
\theta &= \left[-\frac{F'b'}{F} + (b+1)\left(\frac{F''}{F} - \frac{F'^2}{F^2}\right) \right] \nabla^2 u(E_1, E_1)^2 + \frac{F'}{F|\nabla u|} \sum_{\substack{i,j \\ i \geq 2}} \nabla^2 u(E_i, E_j)^2 \\
&+ \left[-\frac{F'b}{F|\nabla u|} + \frac{F''}{F} - \frac{F'^2}{F^2} \right] \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 + (b+1)\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right) |\nabla u|^2 \\
&+ \frac{|\nabla u|F'}{F} \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b+1)\nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&- \frac{1}{g^2} \left[(b+1)\langle \nabla g, E_1 \rangle^2 + \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \right].
\end{aligned}$$

□

The class of mild decay of the eigenvalue ratio

Taking the function $F(s) = s$ in Lemma 3.6, we obtain

$$\frac{\nabla^2 u(E_1, E_1)^2}{|\nabla u|^2} = \frac{f'^2}{f^2} |\nabla u|^2 + \frac{1}{g^2} \langle \nabla g, E_1 \rangle^2 + \frac{2f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u|, \quad (3.26)$$

$$\frac{1}{|\nabla u|^2} \nabla^2 u(E_1, E_i)^2 = \frac{1}{g^2} \langle \nabla g, E_i \rangle^2, \quad \forall i = 2, \dots, n \quad (3.27)$$

and

$$\begin{aligned}
0 &\geq -\frac{b'|\nabla u| + b + 1}{|\nabla u|^2} \nabla^2 u(E_1, E_1)^2 + \frac{1}{|\nabla u|^2} \sum_{\substack{i,j \\ i \geq 2}} \nabla^2 u(E_i, E_j)^2 \\
&- \frac{b+1}{|\nabla u|^2} \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 + (b+1)\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right) |\nabla u|^2 \\
&+ \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b+1)\nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&- \frac{1}{g^2} \left[(b+1)\langle \nabla g, E_1 \rangle^2 + \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \right]. \quad (3.28)
\end{aligned}$$

Inserting (3.26) and (3.27) in (3.28), we arrive at

$$\begin{aligned}
0 &\geq -(b'|\nabla u| + b + 1) \left[\frac{f'^2}{f^2} |\nabla u|^2 + \frac{1}{g^2} \langle \nabla g, E_1 \rangle^2 + \frac{2f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u| \right] \\
&\quad - (b + 1) \sum_{i \geq 2} \frac{1}{g^2} \langle \nabla g, E_i \rangle^2 + (b + 1) \left(\frac{f''}{f} - \frac{f'^2}{f^2} \right) |\nabla u|^2 \\
&\quad + \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b + 1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&\quad - \frac{1}{g^2} \left[(b + 1) \langle \nabla g, E_1 \rangle^2 + \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \right] \\
&= -b' |\nabla u| \frac{f'^2}{f^2} |\nabla u|^2 + (b + 1) \left(\frac{f''}{f} - 2 \frac{f'^2}{f^2} \right) |\nabla u|^2 + \text{Ric}(E_1, E_1) \\
&\quad - (b' |\nabla u| + b + 1) \left[\frac{\langle \nabla g, E_1 \rangle^2}{g^2} + \frac{2f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u| \right] - \frac{b + 1}{g^2} \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \\
&\quad + \frac{1}{g} \left[(b + 1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&\quad - \frac{1}{g^2} \left[(b + 1) \langle \nabla g, E_1 \rangle^2 + \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \right]. \tag{3.29}
\end{aligned}$$

We first consider the global estimate where we set $g \equiv 1$ and choose $f(u) = (\ln(K + u))^{-1}$ with a constant $K > 0$. For convenience we also assume that $u \geq 0$. For this f we have

$$f' = -(K + u)^{-1} (\ln(K + u))^{-2}, \quad f'' = (K + u)^{-2} (\ln(K + u))^{-3} (\ln(K + u) + 2)$$

and thus

$$\frac{f'}{f} = -\frac{1}{(K + u) \ln(K + u)}, \quad \frac{f''}{f} = \frac{1}{(K + u)^2 \ln(K + u)} + \frac{2f'^2}{f^2}.$$

Then (3.29) becomes

$$0 \geq -b' |\nabla u| \frac{f'^2}{f^2} |\nabla u|^2 + (b + 1) \left[\frac{f''}{f} - 2 \frac{f'^2}{f^2} \right] |\nabla u|^2 + \text{Ric}(E_1, E_1)$$

and so

$$(K + u)^{-2} (\ln(K + u))^{-1} \left[b + 1 - \frac{b' |\nabla u|}{\ln(K + u)} \right] |\nabla u|^2 \leq |\text{Ric}^-|, \tag{3.30}$$

where $b'^+ = \max\{b', 0\}$ and

$$\text{Ric}^- := \min_{|\eta|=1} \min \{\text{Ric}(\eta, \eta), 0\}.$$

We now require the condition:

Condition 3.7. *There are numbers $s_0, \beta > 0$ and a function $\varphi \in C^0([0, \infty))$ with $\lim_{s \rightarrow +\infty} \varphi(s) = +\infty$ such that*

$$(b(s) + 1 - \beta b'^+(s)s)s^2 \geq \varphi(s)$$

for $s \geq s_0$.

Then, since (3.30) holds at a point where the function $G = |\nabla u|/\ln(K + u)$ attains a maximum, choosing $K = \exp(1/\beta)$, we obtain

Theorem 3.8. *Under Condition 3.7 there is a constant C depending only on φ, β and $\sup_{\Omega} (|u| + \text{Ric}^-)$ such that if the function $|\nabla u|/\ln(K + u)$ attains a local maximum at an interior point y_0 of Ω , then*

$$|\nabla u(y_0)| \leq C.$$

We now turn our attention to the local estimates. Here we consider the solution u in a closed geodesic ball $B_r(x_0)$ with center x_0 and radius r smaller than the distance of x_0 to its cut locus, if the latter is nonempty, and we choose

$$g(x) = 1 - \frac{\rho^2}{r^2}, \quad \rho(x) = \text{dist}(x, x_0).$$

Unless $\nabla u \equiv 0$ in $B_r(x_0)$, what would make any further estimate superfluous the function $\ln G$ attains a local maximum in some point y_0 in the interior of $B_r(x_0)$. In case that

$$g(y_0)|\nabla u(y_0)| \leq \frac{4}{r} \frac{f(u(y_0))}{|f'(u(y_0))|} \quad (3.31)$$

would hold, then the estimate of $|\nabla u(y_0)|$ will turn out to be trivial. Hence we shall assume that

$$\frac{1}{g} \leq \frac{r|f'|}{4f} |\nabla u| \quad \text{at } y_0$$

what implies

$$\begin{aligned}
2 \left| \frac{f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u| \right| &\leq 2 \frac{|f'|}{fg} |\nabla g| |\nabla u| \\
&\leq 2 \frac{|f'| |\nabla u|}{f} \frac{r |f'| |\nabla u|}{4f} \frac{|\nabla \rho^2|}{r^2} \\
&= \frac{f'^2 |\nabla \rho^2| |\nabla u|^2}{2f^2 r} \\
&= \frac{f'^2 \rho |\nabla \rho| |\nabla u|^2}{f^2 r} \\
&\leq \frac{f'^2 |\nabla u|^2}{f^2}.
\end{aligned} \tag{3.32}$$

Now we find it necessary to require a much stronger condition than Condition 3.7, namely:

Condition 3.9. *There exist positive numbers α , β and s_0 such that*

$$B(s)^{-1}(b(s) + 1 - \beta |b'(s)|s) \geq \alpha, \quad \forall s \geq s_0,$$

where, as before, $B(s) = \max\{1, 1 + b(s)\}$.

It is immediate to see that Condition 3.9 implies Conditions I and 3.7.

From (3.29) we have

$$\begin{aligned}
0 &\geq -|b'| |\nabla u| \frac{f'^2}{f^2} |\nabla u|^2 + (b+1) \left(\frac{f''}{f} - 2 \frac{f'^2}{f^2} \right) |\nabla u|^2 + \text{Ric}(E_1, E_1) \\
&\quad - (|b'| |\nabla u| + b+1) \left[\frac{|\nabla g|^2}{g^2} + \left| \frac{2f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u| \right| \right] - \frac{b+1}{g^2} |\nabla g|^2 \\
&\quad - \frac{1}{g} [(b+1)|\nabla^2 g| + \sqrt{n-1}|\nabla^2 g|] - \frac{1}{g^2} [(b+1)|\nabla g|^2 + |\nabla g|^2].
\end{aligned}$$

By means of (3.32) one has

$$\begin{aligned}
0 &\geq -|b'|\|\nabla u\|\frac{f'^2}{f^2}|\nabla u|^2 + (b+1)\left(\frac{f''}{f} - 2\frac{f'^2}{f^2}\right)|\nabla u|^2 + \text{Ric}(E_1, E_1) \\
&\quad - (|b'|\|\nabla u\| + b+1)\left[\frac{4}{g^2r^2} + \frac{f'^2|\nabla u|^2}{f^2}\right] - \frac{4(b+1)}{g^2r^2} \\
&\quad - \frac{1}{g}\left[(b+1)\frac{|\nabla^2\rho^2|}{r^2} + \frac{\sqrt{n-1}|\nabla^2\rho^2|}{r^2}\right] - \frac{1}{g^2}\left[(b+1)\frac{4}{r^2} + \frac{4}{r^2}\right] \\
&= -|b'|\|\nabla u\|\frac{f'^2}{f^2}|\nabla u|^2 + (b+1)\left(\frac{f''}{f} - 2\frac{f'^2}{f^2}\right)|\nabla u|^2 + \text{Ric}(E_1, E_1) \\
&\quad - (|b'|\|\nabla u\| + b+1)\frac{4}{g^2r^2} - \frac{|b'|\|\nabla u\|f'^2|\nabla u|^2}{f^2} - (b+1)\frac{f'^2|\nabla u|^2}{f^2} \\
&\quad - \frac{4(b+1)}{g^2r^2} - \frac{1}{g}\left[(b+1)\frac{|\nabla^2\rho^2|}{r^2} + \frac{\sqrt{n-1}|\nabla^2\rho^2|}{r^2}\right] - \frac{1}{g^2}\left[(b+1)\frac{4}{r^2} + \frac{4}{r^2}\right]
\end{aligned}$$

and so

$$\begin{aligned}
0 &\geq -2|b'|\|\nabla u\|\frac{f'^2}{f^2}|\nabla u|^2 + (b+1)\left(\frac{f''}{f} - 3\frac{f'^2}{f^2}\right)|\nabla u|^2 + \text{Ric}(E_1, E_1) \\
&\quad - (|b'|\|\nabla u\| + b+1)\frac{4}{g^2r^2} - \frac{b+1 + \sqrt{n-1}}{gr^2}|\nabla^2\rho^2| - \frac{1}{g^2}\left[(b+1)\frac{8}{r^2} + \frac{4}{r^2}\right] \\
&\geq -2|b'|\|\nabla u\|\frac{f'^2}{f^2}|\nabla u|^2 + (b+1)\left(\frac{f''}{f} - 3\frac{f'^2}{f^2}\right)|\nabla u|^2 + \text{Ric}(E_1, E_1) \\
&\quad - (|b'|\|\nabla u\| + b+1)\frac{4}{g^2r^2} - \frac{(1 + \sqrt{n-1})B}{gr^2}|\nabla^2\rho^2| - \frac{12B}{g^2r^2}.
\end{aligned}$$

Our Condition 3.9 implies

$$|b's| \leq \frac{1}{\beta}(b+1)$$

and hence

$$b+1 + |b's| \leq \left(1 + \frac{1}{\beta}\right)(b+1) \leq \left(1 + \frac{1}{\beta}\right)B.$$

Thus

$$\begin{aligned}
&(b+1)\left(\frac{f''}{f} - 3\frac{f'^2}{f^2}\right)|\nabla u|^2 - 2|b'|\|\nabla u\|\frac{f'^2}{f^2}|\nabla u|^2 \\
&\leq \frac{B}{g^2r^2}\left(14 + \frac{4}{\beta} + (2 + \sqrt{n-1})|\nabla^2\rho^2|\right) + |\text{Ric}^-|
\end{aligned}$$

Hence there is a constant $C_0 = C_0(n, \beta)$ such that

$$\begin{aligned} & B \left[(b+1) \left(\frac{f''}{f} - 3 \frac{f'^2}{f^2} \right) |\nabla u|^2 - 2|b'| |\nabla u| \frac{f'^2}{f^2} |\nabla u|^2 \right] \\ & \leq \frac{C_0}{r^2 g^2} (1 + |\nabla^2 \rho^2|) + |\text{Ric}^-|. \end{aligned}$$

We again choose $f(u) = 1/\ln(K+u)$. From the assumption that $u \geq 0$, one has

$$\begin{aligned} & (b+1) \left(\frac{f''}{f} - 3 \frac{f'^2}{f^2} \right) - \frac{2|b'| s f'^2}{f^2} \\ & = \frac{1}{(K+u)^2 \ln(K+u)} \left[\left(1 - \frac{1}{\ln(K+u)}\right) (b+1) - \frac{2|b'| s}{\ln(K+u)} \right] \\ & = \frac{\left(1 - \frac{1}{\ln(K+u)}\right)}{(K+u)^2 \ln(K+u)} \left[b+1 - \frac{2|b'| s}{\ln(K+u) \left(1 - \frac{1}{\ln(K+u)}\right)} \right] \\ & \geq \frac{\left(1 - \frac{1}{\ln K}\right)}{(K+u)^2 \ln(K+u)} \left[b+1 - \frac{2|b'| s}{\ln K - 1} \right] \end{aligned}$$

We choose K with $2/(\ln K - 1) = \beta$ so that

$$(b+1) \left(\frac{f''}{f} - 3 \frac{f'^2}{f^2} \right) - \frac{2|b'| s f'^2}{f^2} \geq \frac{\alpha B (1 - (\ln K)^{-1})}{(K+u)^2 \ln(K+u)}, \quad \forall s \geq s_0$$

Therefore we get at y_0 ,

$$g^2 |\nabla u|^2 \leq \frac{1}{\alpha} \left(\frac{C_0}{r^2} (1 + |\nabla^2 \rho^2|) + |\text{Ric}^-| \right) \frac{(K+u)^2 \ln(K+u)}{(1 - (\ln K)^{-1})},$$

provided that $|\nabla u(y_0)| \geq s_0$ and (3.31) does not hold. Since by construction $G(x_0) \leq G(y_0)$, setting

$$M = \max_{B_r(x_0)} u,$$

we therefore either have

$$\begin{aligned} & \frac{|\nabla u(x_0)|}{\ln(K+u(x_0))} \leq \frac{g(y_0) |\nabla u(y_0)|}{\ln(K+u(y_0))} \\ & \leq \left\{ \frac{1}{\alpha} \left[\frac{C_0}{r^2} (1 + |\nabla^2 \rho^2|) + |\text{Ric}^-| \right] \right\}^{1/2} \frac{(K+u(y_0)) (\ln(K+u(y_0)))^{1/2}}{\ln(K+u(y_0)) (1 - (\ln K)^{-1})^{1/2}} \\ & \leq \left\{ \frac{1}{\alpha} \left[\frac{C_0}{r^2} (1 + |\nabla^2 \rho^2|) + |\text{Ric}^-| \right] \right\}^{1/2} \frac{(K+M)}{(\ln K)^{1/2} (1 - (\ln K)^{-1})^{1/2}} \end{aligned}$$

and so

$$|\nabla u(x_0)| \leq \left\{ \frac{1}{\alpha} \left[\frac{C_0}{r^2} (1 + \max |\nabla^2 \rho^2|) + |\text{Ric}^-| \right] \right\}^{1/2} \frac{(K+M)(\ln(K+M))^{\frac{3}{2}}}{(1 - (\ln K)^{-1})^{1/2}}$$

or $|\nabla u(y_0)| \leq s_0$, leading to

$$\frac{|\nabla u(x_0)|}{\ln(K+u(x_0))} \leq \frac{g(y_0)|\nabla u(y_0)|}{\ln(K+u(y_0))} \leq \frac{|\nabla u(y_0)|}{\ln K},$$

and so

$$|\nabla u(x_0)| \leq s_0 \frac{\ln(K+M)}{\ln K},$$

or, finally (3.31) holds, leading to

$$\begin{aligned} \frac{|\nabla u(x_0)|}{\ln(K+u(x_0))} &\leq \frac{g(y_0)|\nabla u(y_0)|}{\ln(K+u(y_0))} \\ &\leq \frac{4f}{r|f'|} \frac{1}{\ln(K+u(y_0))} \\ &= \frac{4}{r}(K+u(y_0)) \end{aligned}$$

and so

$$|\nabla u(x_0)| \leq \frac{4(K+M)\ln(K+M)}{r}.$$

We thus proved

Theorem 3.10. *Let u be a nonnegative solution of (3.16) in $B_r(x_0)$, where r is smaller than the distance of x_0 to its cut locus, and let Condition 3.9 be satisfied. Then there are constants C and K depending only on n and the numbers α , β , and s_0 in Condition 3.9 such that*

$$|\nabla u(x_0)| \leq C \left\{ 1 + \frac{1}{r} + \frac{1}{r^2} \left(1 + \max_{B_r(x_0)} |\nabla^2 \rho^2| \right) + \max_{B_r(x_0)} |\text{Ric}^-| \right\}^{1/2} (K+M)(\ln(K+M))^2$$

with $M = \max_{B_r(x_0)} u$.

We remark that Conditions 3.7 and 3.9 are for example fulfilled if $b(s) + 1 = cs^m$ for some $c > 0$, $m \geq 0$. Thus, Theorems 3.8 and 3.10 are valid for the p -Laplacian where $b = p - 2$, $p > 1$.

The case of strong decay of the eigenvalue ratio

For the global estimates we may use the same type of auxiliary function as before, though with a different f , namely, $f(u) = \exp(Ku)$ with a positive constant K . With $g \equiv 1$ we obtain from (3.29)

$$\begin{aligned} 0 &\geq -b'|\nabla u| \frac{f'^2}{f^2} |\nabla u|^2 + (b+1) \left(\frac{f''}{f} - 2 \frac{f'^2}{f^2} \right) |\nabla u|^2 + \text{Ric}(E_1, E_1) \\ &\geq K^2 (-b'|\nabla u| - (b+1)) |\nabla u|^2 - |\text{Ric}^-|. \end{aligned}$$

If we introduce the condition:

Condition 3.11. *There are positive numbers α and s_0 such that*

$$(-b'(s)s - (b(s) + 1)) s^2 \geq \alpha, \quad s \geq s_0$$

and choose

$$K > \left(\frac{1}{\alpha} \max_{\Omega} |\text{Ric}^-| \right)^{1/2},$$

then we immediately get the result:

Theorem 3.12. *If Condition 3.11 holds and the function $\exp(Ku)|\nabla u|$ attains a local maximum in an interior point y_0 of Ω then it follows that $|\nabla u(y_0)| \leq s_0$, where K must be chosen as above.*

In order to derive an analogous local gradient estimate it seems to be necessary to slightly modify the auxiliary function G and choose

$$G(x) = g(x)f(u) \ln |\nabla u|, \quad g = 1 - \frac{\rho^2}{r^2}.$$

This choice and the subsequent calculations are inspired by a paper of Wang [21] which deals with the Euclidean mean curvature equation. It is sufficient to consider this function in a range where $|\nabla u| > 1$ such that also $\ln G$ is well defined. Again we consider G in the neighborhood of a point $y_0 \in \Omega$ where G attains a local maximum.

Taking the function $F(s) = \ln(s)$ in Lemma 3.6, we obtain at y_0

$$\frac{\nabla^2 u(E_1, E_i)}{|\nabla u| \ln |\nabla u|} = -\frac{f'}{f} \langle \nabla u, E_i \rangle - \frac{\langle \nabla g, E_i \rangle}{g} \quad (3.33)$$

and

$$\begin{aligned}
0 &\geq -\frac{1}{|\nabla u|^2 \ln |\nabla u|} \left[b' |\nabla u| + (b+1) \left(1 + \frac{1}{\ln |\nabla u|} \right) \right] \nabla^2 u(E_1, E_1)^2 \\
&+ \sum_{\substack{i,j \\ i \geq 2}} \frac{\nabla^2 u(E_i, E_j)^2}{|\nabla u|^2 \ln |\nabla u|} - \frac{1}{|\nabla u|^2 \ln |\nabla u|} \left[b+1 + \frac{1}{\ln |\nabla u|} \right] \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 \\
&+ (b+1) \frac{f''}{f} |\nabla u|^2 + \frac{1}{\ln |\nabla u|} \text{Ric}(E_1, E_1) \\
&+ \frac{1}{g} \left[(b+1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&- (b+1) \left[\frac{f'^2}{f^2} |\nabla u|^2 + \frac{1}{g^2} \langle \nabla g, E_1 \rangle^2 \right] - \frac{1}{g^2} \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2. \tag{3.34}
\end{aligned}$$

From (3.33) we have

$$\frac{\nabla^2 u(E_1, E_1)^2}{|\nabla u|^2 \ln^2 |\nabla u|} = \frac{f'^2}{f^2} |\nabla u|^2 + \frac{\langle \nabla g, E_1 \rangle^2}{g^2} + \frac{2f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u| \tag{3.35}$$

and

$$\frac{\nabla^2 u(E_1, E_i)^2}{|\nabla u|^2 \ln^2 |\nabla u|} = \frac{\langle \nabla g, E_i \rangle^2}{g^2}, \quad \forall i = 2, \dots, n. \tag{3.36}$$

Since $b+1 \geq 0$, it follows that

$$\begin{aligned}
\sum_{\substack{i,j \\ i \geq 2}} \nabla^2 u(E_i, E_j)^2 &= \sum_{i \geq 2} \nabla^2 u(E_i, E_1)^2 + \sum_{\substack{i \geq 2 \\ j \geq 2}} \nabla^2 u(E_i, E_j)^2 \\
&= -b \sum_{i \geq 2} \nabla^2 u(E_i, E_1)^2 + (b+1) \sum_{i \geq 2} \nabla^2 u(E_i, E_1)^2 \\
&+ \sum_{\substack{i \geq 2 \\ j \geq 2}} \nabla^2 u(E_i, E_j)^2 \\
&\geq -b \sum_{i \geq 2} \nabla^2 u(E_i, E_1)^2. \tag{3.37}
\end{aligned}$$

By (3.34), (3.35), (3.36) and (3.37), we have

$$\begin{aligned}
0 &\geq -\frac{1}{|\nabla u|^2 \ln |\nabla u|} \left[b' |\nabla u| + (b+1) \left(1 + \frac{1}{\ln |\nabla u|} \right) \right] \nabla^2 u(E_1, E_1)^2 \\
&\quad - \frac{1}{|\nabla u|^2 \ln |\nabla u|} \left[2b+1 + \frac{1}{\ln |\nabla u|} \right] \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 + (b+1) \frac{f''}{f} |\nabla u|^2 \\
&\quad + \frac{1}{\ln |\nabla u|} \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b+1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&\quad - (b+1) \left[\frac{\nabla^2 u(E_1, E_1)^2}{|\nabla u|^2 \ln^2 |\nabla u|} - \frac{2f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u| \right] - \sum_{i \geq 2} \frac{\nabla^2 u(E_1, E_i)^2}{|\nabla u|^2 \ln^2 |\nabla u|} \\
&= -\frac{1}{|\nabla u|^2 \ln |\nabla u|} \left[b' |\nabla u| + (b+1) \left(1 + \frac{2}{\ln |\nabla u|} \right) \right] \nabla^2 u(E_1, E_1)^2 \\
&\quad - \frac{1}{|\nabla u|^2 \ln |\nabla u|} \left[2b+1 + \frac{2}{\ln |\nabla u|} \right] \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 + (b+1) \frac{f''}{f} |\nabla u|^2 \\
&\quad + \frac{1}{\ln |\nabla u|} \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b+1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&\quad + \frac{2(b+1)f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u|. \tag{3.38}
\end{aligned}$$

For a given $\varepsilon > 0$ only depending on a structural condition for (3.16) we may assume that

$$\ln |\nabla u(y_0)| \geq \frac{2}{\varepsilon} \tag{3.39}$$

because if (3.39) does not hold the estimate for $|\nabla u|$ will turn out to be trivial.

From (3.38) and (3.39) we obtain

$$\begin{aligned}
0 &\geq -\frac{1}{|\nabla u|^2 \ln |\nabla u|} [b' |\nabla u| + (b+1)(1+\varepsilon)] \nabla^2 u(E_1, E_1)^2 \\
&\quad - \frac{1}{|\nabla u|^2 \ln |\nabla u|} (2b+1+\varepsilon) \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 + (b+1) \frac{f''}{f} |\nabla u|^2 \\
&\quad + \frac{1}{\ln |\nabla u|} \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b+1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right] \\
&\quad + \frac{2(b+1)f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u|.
\end{aligned}$$

Let us assume that

$$2b(|\nabla u|) + 1 + \varepsilon \leq 0. \tag{3.40}$$

Hence,

$$\begin{aligned}
0 &\geq -\frac{1}{|\nabla u|^2 \ln |\nabla u|} [b'|\nabla u| + (b+1)(1+\varepsilon)] \nabla^2 u(E_1, E_1)^2 \\
&\quad + (b+1) \frac{f''}{f} |\nabla u|^2 \frac{1}{\ln |\nabla u|} \text{Ric}(E_1, E_1) + \frac{2(b+1)f'}{fg} \langle \nabla g, E_1 \rangle |\nabla u| \\
&\quad + \frac{1}{g} \left[(b+1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right]. \tag{3.41}
\end{aligned}$$

We shall choose f in such a way that

$$\frac{f'}{f} \geq 1$$

and assume that

$$rg(y_0) |\nabla u(y_0)| \geq 4, \tag{3.42}$$

because otherwise

$$g(y_0) \ln |\nabla u(y_0)| \leq \frac{1}{4}$$

and the estimative of $|\nabla u(x_0)|$ becomes trivial. By (3.42) we have

$$\left| \frac{\langle \nabla g, E_i \rangle}{g} \right| \leq \frac{2\rho |\nabla \rho|}{gr^2} \leq \frac{2}{gr} \leq \frac{|\nabla u|}{2}. \tag{3.43}$$

It follows then from (3.33) and (3.43) that

$$\begin{aligned}
\left| \frac{\nabla^2 u(E_1, E_1)}{|\nabla u| \ln |\nabla u|} \right| &\geq \left| \frac{f' \langle \nabla u, E_1 \rangle}{f} \right| - \left| \frac{\langle \nabla g, E_1 \rangle}{g} \right| \\
&\geq \frac{f' |\nabla u|}{f} - \frac{|\nabla u|}{2} \\
&\geq \frac{f' |\nabla u|}{f} - \frac{f' |\nabla u|}{2f} = \frac{f' |\nabla u|}{2f} \tag{3.44}
\end{aligned}$$

and

$$\begin{aligned}
\frac{2f'}{fg} (b+1) \langle \nabla g, E_1 \rangle |\nabla u| &\geq -\frac{2f'}{f} (b+1) \left| \frac{\langle \nabla g, E_1 \rangle}{g} \right| |\nabla u| \\
&\geq -\frac{f'}{f} (b+1) |\nabla u|^2. \tag{3.45}
\end{aligned}$$

From (3.41), (3.44) and (3.45) one obtains

$$\begin{aligned}
0 &\geq -\frac{f'^2}{4f^2} [b'|\nabla u| + (b+1)(1+\varepsilon)] |\nabla u|^2 \ln |\nabla u| + (b+1) \left(\frac{f''}{f} - \frac{f'}{f} \right) |\nabla u|^2 \\
&\quad + \frac{1}{\ln |\nabla u|} \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b+1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right],
\end{aligned}$$

where we assumed that

$$b'|\nabla u| + (1 + \varepsilon)(b + 1) \leq 0. \quad (3.46)$$

Finally we therefore arrive at

$$\begin{aligned} 0 \geq & -\frac{f'^2}{4f^2} [b'|\nabla u| + (b + 1)(1 + \varepsilon)] |\nabla u|^2 \ln |\nabla u| + (b + 1) \left(\frac{f''}{f} - \frac{f'}{f} \right) |\nabla u|^2 \\ & + \frac{1}{\ln |\nabla u|} \text{Ric}(E_1, E_1) + \frac{1}{g} \left[(b + 1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right]. \end{aligned} \quad (3.47)$$

We now introduce the following hypothesis:

Condition 3.13. *There are positive numbers ε , α , and s_0 such that*

$$(-b'(s)s - (1 + \varepsilon)(b(s) + 1))s^2 \geq \alpha \text{ for } s \geq s_0. \quad (3.48)$$

Remark 3.2. *From (3.48) we have*

$$-\frac{b'(s)}{b(s) + 1} > \frac{1 + \varepsilon}{s}.$$

Integrating between s_0 and s we arrive at

$$-\ln \left(\frac{b(s) + 1}{b(s_0) + 1} \right) > (1 + \varepsilon) \ln \left(\frac{s}{s_0} \right)$$

and so

$$\frac{b(s_0) + 1}{b(s) + 1} > \left(\frac{s}{s_0} \right)^{1 + \varepsilon}.$$

Then

$$2b(s) + 1 + \varepsilon < -1 + \varepsilon + 2 \left(\frac{s_0}{s} \right)^{1 + \varepsilon} (b(s_0) + 1) \rightarrow -1 + \varepsilon < 0$$

when $s \rightarrow +\infty$. Taking $s_0 > 0$ large enough we arrive at

$$2b(s) + 1 + \varepsilon < 0 \text{ for } s \geq s_0.$$

It follows from the Condition 3.13 that (3.40) is valid.

Remark 3.3. *In the case $b(s) + 1 = \frac{1}{s^2 + 1}$ and $\varepsilon < 1$, we have*

$$(-b'(s)s - (1 + \varepsilon)(b(s) + 1))s^2 = \frac{2s^4}{(s^2 + 1)^2} - \frac{(1 + \varepsilon)s^2}{s^2 + 1} \rightarrow 1 - \varepsilon > 0$$

and

$$2b(s) + 1 + \varepsilon = \frac{-2s^2}{s^2 + 1} + 1 + \varepsilon \rightarrow -1 + \varepsilon < 0$$

when $s \rightarrow +\infty$.

It follows from Condition 3.13 that (3.46) is also valid and that $b'(s) \leq 0$ for $s \geq s_0$ and hence there is a number $\beta \geq 1$ such that $b(s) + 1 \leq \beta$ for $s \geq s_0$. We now choose $f(u) = \exp(Ku)$ with $K > 1$. From (3.39), (3.47) and the condition (3.13) we have

$$\begin{aligned} 0 &\geq -\frac{K^2}{4} [b'|\nabla u| + (1 + \varepsilon)(b + 1)] |\nabla u|^2 \ln |\nabla u| \\ &\quad + (b + 1)(K^2 - K) |\nabla u|^2 - \frac{1}{\ln |\nabla u|} |\text{Ric}^-| \\ &\quad - \frac{1}{g} \left[(b + 1) \frac{|\nabla^2 \rho|^2}{r^2} + \sqrt{n-1} \frac{|\nabla^2 \rho|^2}{r^2} \right] \\ &\geq \frac{K^2 \alpha}{4} \ln |\nabla u| - \frac{\varepsilon}{2} |\text{Ric}^-| - \frac{\beta + \sqrt{n-1}}{g} \frac{|\nabla^2 \rho|^2}{r^2}. \end{aligned}$$

We can now conclude that

$$g(y_0) \ln |\nabla u(y_0)| \leq \frac{4}{K^2 \alpha} \left(\frac{\varepsilon}{2} |\text{Ric}^-| + \frac{\beta + \sqrt{n-1}}{r^2} |\nabla^2 \rho|^2 \right),$$

unless

$$\ln |\nabla u(y_0)| \leq \max \left\{ \frac{4}{g(y_0)r}, \ln s_0 \right\}.$$

Since by construction $G(x_0) \leq G(y_0)$ we therefore either have

$$\begin{aligned} \exp(Ku(x_0)) \ln |\nabla u(x_0)| &\leq g(y_0) \exp(Ku(y_0)) \ln |\nabla u(y_0)| \\ &\leq \frac{4 \exp(Ku(y_0))}{K^2 \alpha} \left(\frac{\varepsilon}{2} |\text{Ric}^-| + \frac{\beta + \sqrt{n-1}}{r^2} |\nabla^2 \rho|^2 \right). \end{aligned}$$

Hence there is a constant $C = C(n, \beta, \alpha, \varepsilon)$ such that

$$\begin{aligned} \ln |\nabla u(x_0)| &\leq C \frac{\exp(KM)}{K^2} \max \left\{ |\text{Ric}^-| + \frac{1}{r^2} |\nabla^2 \rho|^2 \right\} \\ &\leq C \exp(KM) \max \left\{ |\text{Ric}^-| + \frac{1}{r^2} |\nabla^2 \rho|^2 \right\}, \end{aligned}$$

where

$$M = \max_{x \in B_r(x_0)} |u(x) - u(x_0)|$$

or

$$g(y_0) \ln |\nabla u(y_0)| \leq \frac{4}{r},$$

leading to

$$\exp(Ku(x_0)) \ln |\nabla u(x_0)| \leq g(y_0) \exp(Ku(y_0)) \ln |\nabla u(x_0)| \leq \frac{4}{r} \exp(Ku(y_0))$$

and thus

$$\ln |\nabla u(x_0)| \leq \frac{4}{r} \exp(KM)$$

or, finally $|\nabla u(y_0)| \leq s_0$ holds, leading to

$$\exp(Ku(x_0)) \ln |\nabla u(x_0)| \leq g(y_0) \exp(Ku(y_0)) \ln s_0 \leq \exp(Ku(y_0)) \ln s_0$$

and so

$$\ln |\nabla u(x_0)| \leq \exp(KM) \ln s_0.$$

We thus proved

Theorem 3.14. *Let u be a solution of (3.16) in Ω such that the Condition 3.13 is satisfied and let $x_0 \in \Omega$ and $r > 0$ be such that the geodesic ball $B_r(x_0)$, is contained in Ω and is disjoint of the cut locus of x_0 . Then there is a constant C depending only n and the numbers ε , α , and s_0 in the Condition 3.13 such that*

$$\ln |\nabla u(x_0)| \leq C \exp(KM) \max \left\{ 1 + \frac{1}{r} + \max \left(|\text{Ric}^-| + \frac{1}{r^2} |\nabla^2 \rho^2| \right) \right\}$$

for all $K > 1$, where

$$M = \max_{x \in B_r(x_0)} |u(x) - u(x_0)|.$$

We remark that Condition 3.11 and Condition 3.13, the latter with $\varepsilon = 1/4$, are satisfied if

$$b(s) + 1 = \frac{1}{1 + s^2}.$$

Hence, Theorems 3.14 and 3.12 are valid for the minimal surface equation.

Chapter 4

Existence theorems for regular equations on bounded domains

In the course of the proof of the existence theorems we shall make references to results of [10] by using local coordinate systems.

Let \mathcal{M} be a complete Riemannian manifold. We note that in local coordinates $\{\partial/\partial x_i, 1 \leq i \leq n\}$ of $T\Omega \subset T\mathcal{M}$ defined on an open set U of \mathbb{R}^n , the functional

$$F(u) = \int_{\Omega} \phi(|\nabla u|) \omega$$

can be written as

$$F(u) = \int_U \phi(|\nabla u|) \sqrt{g} dx,$$

where $g = \det(g_{ij})$, $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$, $(g^{ij}) = (g_{ij})^{-1}$. We may see that given $l \in \{1, \dots, n\}$ the l -local component of $(\nabla u)^l$ is

$$(\nabla u)^l = g^{kl} \frac{\partial u}{\partial x_k},$$

where the summation is understood and so

$$\begin{aligned} |\nabla u|^2 &= g_{ij} (\nabla u)^i (\nabla u)^j = g_{ij} g^{ik} \frac{\partial u}{\partial x_k} g^{jl} \frac{\partial u}{\partial x_l} \\ &= \delta_{jk} \frac{\partial u}{\partial x_k} g^{jl} \frac{\partial u}{\partial x_l} = g^{jl} \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_l}. \end{aligned} \tag{4.1}$$

To obtain the Euler-Lagrange equations we note that, writing

$$A(|\nabla u|) = \frac{a(|\nabla u|)}{|\nabla u|},$$

we have for $\varphi \in C_0^\infty(U)$

$$\begin{aligned} \int_U \frac{\phi'(|\nabla u|)}{|\nabla u|} \langle \nabla u, \nabla \varphi \rangle \sqrt{g} dx &= \int_U A(|\nabla u|) g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \sqrt{g} dx \\ &= - \int_U \frac{\partial}{\partial x_j} \left(\sqrt{g} g^{ij} A(|\nabla u|) \frac{\partial u}{\partial x_i} \right) \varphi dx \end{aligned}$$

so that

$$\sum_j \frac{\partial}{\partial x_j} A^j(x, Du) = 0, \quad (4.2)$$

where

$$A^j = \sum_i \left(\sqrt{g} g^{ij} A(|\nabla u|) \frac{\partial u}{\partial x_i} \right).$$

4.1 The case of mild decay of the eigenvalue ratio

We begin with the case of smooth boundary data.

Theorem 4.1. *Let $\Omega \subset \mathcal{M}$ be a domain of class $C^{2,\alpha}$ for some $\alpha > 0$, $\bar{\Omega}$ compact and let $g \in C^{2,\alpha}(\bar{\Omega})$. We write a as $a(s) = sA(s)$ and assume*

$$(i) \ A \in C^{1,\alpha}([0, \infty)) \cap C^{2,\alpha}((0, \infty)),$$

$$\min_{0 \leq s \leq s_0} \left\{ A, 1 + \frac{sA'(s)}{A(s)} \right\} > 0 \quad (4.3)$$

for any $s_0 > 0$

(ii) *there is a non-decreasing function φ defined on some interval $[s_0, \infty)$ such that*

$$\int_{s_0}^{\infty} \frac{\varphi(s)}{s^2} ds = \infty$$

and

$$(1 + b^-(s)) s^2 \geq \varphi(s),$$

where

$$b(s) = \frac{sa'(s)}{a(s)} - 1 = \frac{sA'(s)}{A(s)}$$

(iii) *there are a C^1 function ψ on some interval $[s_0, \infty)$ with $\psi(s) \rightarrow \infty$ as $s \rightarrow \infty$ and $\beta > 0$ such that*

$$(1 + b(s) - \beta(b')^+(s)s) s^2 \geq \psi(s).$$

Then the Dirichlet problem

$$\begin{cases} Q[u] = \operatorname{div}(A(|\nabla u|)\nabla u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = g \end{cases} \quad (4.4)$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Proof. As explained in Subsection 2.2, it is enough to obtain a priori gradient bounds for a one parameter family of solutions of (4.4). Precisely, it is enough to prove that there is a constant C depending only on Ω , A and g such that

$$|\nabla v| \leq N \quad (4.5)$$

if $v \in C^2(\overline{\Omega})$ satisfies $Q[v] = 0$ and $v|_{\partial\Omega} = \sigma g$ for some $\sigma \in [0, 1]$. But this is immediate from (ii) and (iii) and Theorems 3.3 and 3.8. This proves Theorem 4.1. \square

Remark 4.1. *The assumption of $A \in C^{2,\alpha}((0, \infty))$ is necessary to guarantee that u is of class C^3 (in the set $|\nabla u| > s_0 > 0$ for large s_0) and to apply the results of Section 3.3.*

Theorem 4.2. *Let Ω be as in Theorem 4.1 and $g \in C^0(\partial\Omega)$. We assume Condition (i) as in Theorem 4.1 but instead of (ii) and (iii) as in this theorem we require the stronger condition*

$$B(s)^{-1}(1 + b(s) - \beta|b'(s)|s) \geq \alpha \quad (4.6)$$

with some positive constant α and

$$B(s) := \max\{1, 1 + b(s)\}, \quad s > 0.$$

Then the above Dirichlet problem has a unique solution $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$. Moreover, for any relatively compact subdomain Λ of Ω there is a $C^{2,\alpha}(\Lambda)$ bound depending only on Λ and $\sup_{\Omega}|u|$.

Proof. Let us first remark that condition (4.6) implies conditions (ii) and (iii) in Theorem 4.1. Indeed, since $B(s) \geq 1$, the condition (iii) is clearly satisfied with $\varphi(s) = \alpha s^2$. Recalling that

$$\frac{1+b}{B} = 1 + b^-,$$

it follows from (4.6) that $1 + b^- \geq \alpha$ and hence (ii) is fulfilled with $\varphi(s) = \alpha s^2$. We therefore have Theorem 4.1 at our disposal and then, with Theorem 3.10, it is enough to apply the technique explained at Section 2.2 to conclude the proof of the theorem. \square

4.2 The case of strong decay of the eigenvalue ratio

Theorem 4.3. *Let Ω be a bounded domain of class $C^{2,\alpha}$ in M such that the mean curvature of $\partial\Omega$ with respect to the interior normal vector of $\partial\Omega$ as well as of the inner parallel hypersurfaces of $\partial\Omega$ in some neighborhood of $\partial\Omega$ is non negative. We write a as $a(s) = sA(s)$ and assume:*

$$(i) \ A \in C^{1,\alpha}([0, \infty)) \cap C^{2,\alpha}((0, \infty)),$$

$$\min_{0 \leq s \leq s_0} \left\{ A, 1 + \frac{sA'(s)}{A(s)} \right\} > 0 \quad (4.7)$$

for any $s_0 > 0$;

(ii) there are positive numbers α and s_0 such that

$$(-b'(s)s - (b(s) + 1))s^2 \geq \alpha, \quad s \geq s_0. \quad (4.8)$$

Then the Dirichlet problem

$$\begin{cases} Q[u] = \operatorname{div}(A(|\nabla u|)\nabla u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = g, \end{cases}$$

where $g \in C^{2,\alpha}(\overline{\Omega})$ has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Proof. Using our Theorems 3.3 and 3.12 the proof of the above theorem is completely analogous to that of Theorem 4.1 once we prove that condition (ii) implies Condition II in the introduction.

Clearly (4.8) implies that $b' \leq 0$ for $s \geq s_0$ and hence b is non increasing for $s \geq s_0$. If there were numbers $C > 0$ and $s_1 > 0$ such that $-b'(s)s \geq C$ for $s \geq s_1$ then $b(s) \rightarrow -\infty$ ($s \rightarrow \infty$), a contradiction since $b \geq -1$. Therefore, there is a sequence $s_k \rightarrow +\infty$ such that $-b(s_k)s_k \rightarrow 0$ for $k \rightarrow \infty$. Then condition (4.8) implies

$$1 + b(s_k) \leq -b'(s_k)s_k \rightarrow 0$$

and so

$$1 + b(s) \rightarrow 0 \quad (4.9)$$

as $s \rightarrow \infty$ since b is non increasing. Now condition (4.8) also implies

$$b'(s) \leq -\frac{\alpha}{s^3}$$

for $s \geq s_0$. Therefore we have for $s_0 < s < \sigma$

$$b(\sigma) - b(s) \leq -\alpha \int_s^\sigma \frac{dt}{t^3} = \frac{\alpha}{2} \left(\frac{1}{\sigma^2} - \frac{1}{s^2} \right).$$

Since $b(\sigma) \rightarrow -1$ ($\sigma \rightarrow \infty$) by (4.9) we obtain

$$(1 + b(s)) s^2 \geq \frac{\alpha}{2} > 0 \quad (s > s_0).$$

Since $b(s) < 0$ for all sufficient large s we have

$$(1 + b^-(s)) s^2 \geq \frac{\alpha}{2} > 0$$

and so Condition II is satisfied with $\varphi(s) = \alpha/2$. □

Since the condition (4.10) below obviously implies condition (4.8), the proof of the next theorem is completely analogous to the proof of Theorem 4.2 using the local gradient estimates of Theorem 3.14:

Theorem 4.4. *Let Ω be as in the previous theorem. We assume Condition (i) as in Theorem 4.3 but instead of (ii) as in this theorem we require the stronger condition*

$$(-b'(s)s - (1 + \varepsilon)(b(s) + 1)) s^2 \geq \alpha \text{ for } s \geq s_0 \quad (4.10)$$

for some positive numbers ε , α and s_0 . Then the Dirichlet problem

$$\begin{cases} Q[u] = \operatorname{div}(A(|\nabla u|)\nabla u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = g, \end{cases}$$

where $g \in C^0(\overline{\Omega})$ has a unique solution $u \in C^0(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.

Chapter 5

Existence theorems for some degenerate or singular equations on bounded domains

In order to prove similar existence theorem for certain singular or degenerate equations like the p -Laplace equations we show how such equations may be approximated by regular ones satisfying the conditions of Theorems 4.1 and 4.2. Precisely, we now assume that the coefficient a is of the form

$$\begin{aligned} a(s) &= s^{p-1}A(s), \quad s \geq 0, \\ \text{where } p > 1, \quad A &\in C^{1,\alpha}([0, \infty)) \cap C^{2,\alpha}(0, \infty), \quad A(s) > 0 \text{ for } s \geq 0. \end{aligned} \tag{5.1}$$

We observe that the equation (1.3) becomes singular for $p < 2$ and degenerate elliptic for $p > 2$. We now regularize (5.1) in the form

$$a_\kappa(s) = (\kappa + s^2)^{\frac{p}{2}-1} A(s)s, \tag{5.2}$$

where $\kappa > 0$. We compute

$$\begin{aligned} a'_\kappa &= (\kappa + s^2)^{\frac{p}{2}-1} A + (\kappa + s^2)^{\frac{p}{2}-1} A' s + \left(\frac{p}{2} - 1\right) (\kappa + s^2)^{\frac{p}{2}-2} 2s^2 A, \\ 1 + b_\kappa &= \frac{s a'_\kappa}{a_\kappa} = (\kappa + s^2)^{1-\frac{p}{2}} A^{-1} a'_\kappa = 1 + \frac{s A'}{A} + (p-2) \frac{s^2}{\kappa + s^2}. \end{aligned}$$

Since

$$a' = (p-1) s^{p-2} A + s^{p-1} A'$$

we get

$$\frac{sa'}{a} = \frac{sa'}{s^{p-1}A} = (p-1) + \frac{sA'}{A} \quad (5.3)$$

and hence

$$\begin{aligned} 1 + b_\kappa &= 1 + \frac{sa'}{a} - (p-1) + (p-2) \frac{s^2}{\kappa + s^2} \\ &= 1 + b - (p-2) \left(1 - \frac{s^2}{\kappa + s^2} \right) \end{aligned}$$

that is

$$1 + b_\kappa = 1 + b - (p-2) \frac{\kappa}{\kappa + s^2}. \quad (5.4)$$

It follows that

$$b'_\kappa = b' + (p-2) \frac{2\kappa s}{(\kappa + s^2)^2}. \quad (5.5)$$

Now we would like to check that the Conditions I, 3.7, 3.9 hold for the family a_κ uniformly for $\kappa \in (0, 1]$ provided that they are true for the original a .

We have from Condition I and (5.4)

$$\begin{aligned} (1 + b_\kappa^-) s^2 &\geq (1 + b) s^2 + (p-2)^- \frac{\kappa s^2}{\kappa + s^2} \\ &\geq \varphi(s) + (p-2)^- =: \tilde{\varphi}(s), \quad 0 \leq \kappa \leq 1, \end{aligned}$$

where $\tilde{\varphi}$ is non decreasing as φ is and

$$\int_1^\infty \frac{\tilde{\varphi}(s)}{s^2} ds = \infty$$

by Condition I. As next we turn to the analogue of Condition 3.7. We have

$$sb'_\kappa = sb' + (p-2) \frac{2\kappa s^2}{(\kappa + s^2)^2}$$

so that

$$s(b'_\kappa)^+ \leq s(b')^+ + (p-2)^+ \frac{2\kappa s^2}{(\kappa + s^2)^2}.$$

It follows then from Condition 3.7 that

$$\begin{aligned} &\left(1 + b_\kappa(s) - \beta (b'_\kappa(s))^+ s \right) s^2 \\ &\geq \left(1 + b(s) - \beta (b'(s))^+ s \right) s^2 - (p-2)^+ \left(\frac{\kappa s^2}{\kappa + s^2} + \beta \frac{2\kappa s^4}{(\kappa + s^2)^2} \right) \\ &\geq \varphi(s) - (p-2)^+ (\kappa + 2\beta\kappa) \rightarrow \infty \quad (s \rightarrow \infty) \end{aligned}$$

uniformly for $\kappa \in (0, 1]$. Finally we come to Condition 3.9. There holds

$$\begin{aligned} 1 \leq B_\kappa = \max \{1, 1 + b_\kappa\} &\leq \max \{1, 1 + b\} + (2 - p)^+ \frac{\kappa}{\kappa + s^2} \\ &\leq B \left(1 + (2 - p)^+ \frac{\kappa}{\kappa + s^2} \right) \leq 2B \end{aligned}$$

since $p > 1$. It follows from Condition 3.9 and (5.5) above that

$$\begin{aligned} &B_\kappa^{-1} (1 + b_\kappa(s) - \beta |b'_\kappa(s)| s) \\ &\geq \frac{1}{2} B^{-1} (1 + b(s) - \beta |b'(s)| s) \\ &- (p - 2)^+ \frac{\kappa}{\kappa + s^2} + \beta |p - 2| \frac{2\kappa s^2}{(\kappa + s^2)^2} \geq \frac{1}{4} \alpha \end{aligned}$$

if $s \geq s_0(p, \beta)$ and $\kappa \in (0, 1]$. We thus showed that the Conditions I, 3.7 and 3.9 imply the corresponding conditions for the regularized equation, uniformly for $\kappa \in (0, 1]$.

Finally we come to Condition 3.13 which clearly includes Condition 3.11, the latter also implying Condition II in the introduction as was shown in the proof of Theorem 4.3. We have

$$\begin{aligned} &(-sb'_\kappa - (1 + \varepsilon)(1 + b_\kappa)) s^2 = (-sb' - (1 + \varepsilon)(1 + b)) s^2 \\ &+ (2 - p) \left[\frac{2\kappa s^4}{(\kappa + s^2)^2} - (1 + \varepsilon) \frac{\kappa s^2}{\kappa + s^2} \right] \\ &\geq (-sb' - (1 + \varepsilon)(1 + b)) s^2 - |2 - p| \kappa (2 + 1 + \varepsilon) \geq \frac{\alpha}{2} \end{aligned}$$

for $s \geq s_0$, uniformly for $\kappa \in (0, \kappa_0(p))$, $0 < \varepsilon \leq 1$, provided that Condition 3.13 holds.

Let us finally check for the ellipticity of the regularized equation. We compute

$$\begin{aligned} &\operatorname{div} \left(\frac{a_\kappa(|\nabla u|)}{|\nabla u|} \nabla u \right) = \\ &(\kappa + |\nabla u|^2)^{\frac{p}{2}-1} A \Delta u + \left(\frac{p}{2} - 1 \right) (\kappa + |\nabla u|^2)^{\frac{p}{2}-2} A \langle \nabla |\nabla u|^2, \nabla u \rangle \\ &+ (\kappa + |\nabla u|^2)^{\frac{p}{2}-1} A' \langle \nabla |\nabla u|, \nabla u \rangle = \\ &(\kappa + |\nabla u|^2)^{\frac{p}{2}-1} A \Delta u + \left[(p - 2) (\kappa + |\nabla u|^2)^{\frac{p}{2}-2} A \right. \\ &\left. + (\kappa + |\nabla u|^2)^{\frac{p}{2}-1} \frac{A'}{|\nabla u|} \right] \nabla^2 u (\nabla u, \nabla u), \end{aligned}$$

that is

$$\operatorname{div} \left(\frac{a_\kappa(|\nabla u|)}{|\nabla u|} \nabla u \right) = \left(\kappa + |\nabla u|^2 \right)^{\frac{p}{2}-1} A \left\{ \Delta u + \left[(p-2) \frac{|\nabla u|^2}{\kappa + |\nabla u|^2} + \frac{A'|\nabla u|}{A} \right] \nabla^2 u \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right) \right\}.$$

In order to test the ellipticity we replace $\nabla^2 u$ by $\langle \xi, \cdot \rangle \otimes \langle \xi, \cdot \rangle$ with a unit length vector ξ and with

$$\eta := |\nabla u|^{-1} \nabla u, \quad \xi^\perp := \xi - \langle \xi, \eta \rangle \eta$$

we must consider the quadratic form

$$q(s, \xi) = (\kappa + s^2)^{\frac{p}{2}-1} A(s) \left\{ |\xi^\perp|^2 + \left[1 + (p-2) \frac{s^2}{\kappa + s^2} + \frac{sA'(s)}{A(s)} \right] \langle \xi, \eta \rangle^2 \right\}.$$

Since

$$1 + (p-2) \frac{s^2}{\kappa + s^2} \geq \min \{1, p-1\}$$

we obtain

Lemma 5.1. *We assume that for some interval $[0, s_0]$ there are positive numbers c, C such that*

$$\begin{aligned} c &\leq A(s) \leq C, \\ c &\leq \min \{1, p-1\} + \frac{sA'(s)}{A(s)} \leq 1 + (p-2)^+ + \frac{sA'(s)}{A(s)} \leq C, \end{aligned}$$

for $s \in [0, s_0]$. Then the estimate

$$(\kappa + s^2)^{\frac{p}{2}-1} c^2 \leq q(s, \xi) \leq (\kappa + s^2)^{\frac{p}{2}-1} C^2$$

holds for $s \in [0, s_0]$ and $|\xi| = 1$.

Lemma 5.1 gives the ellipticity condition for the regularized equations and we may therefore get the following immediate consequences of Theorems 4.1 and 4.2. Since we showed above that the Condition I, 3.7 and 3.9 hold uniformly for $\kappa \in (0, 1]$ and Condition 3.13 uniformly for $\kappa \in (0, \kappa_0(p)]$ we get C^1 bounds for the solutions of the regularized equations independent of $\kappa \in (0, 1]$ and $\kappa \in (0, \kappa_0(p)]$, respectively.

Corollary 5.2 (of Theorem 4.1). *We assume all the conditions of Theorem 4.1 but replacing (4.3) by*

$$\min_{0 \leq s \leq s_0} \left\{ A(s), \min \{1, p-1\} + \frac{sA'(s)}{A(s)} \right\} > 0 \quad (5.6)$$

for any $s_0 > 0$. Then the regularized Dirichlet problem

$$\begin{cases} \operatorname{div} \left(\frac{a_\kappa(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0, \\ u|_{\partial\Omega} = g, \end{cases}$$

where a_κ is given by (5.2), has a unique solution $u_\kappa \in C^{2,\alpha}(\overline{\Omega})$ for all $\kappa > 0$. Moreover, the family u_κ , $0 < \kappa \leq 1$, is uniformly bounded in $C^1(\overline{\Omega})$.

Corollary 5.3 (of Theorem 4.2). *We assume all the conditions of Theorem 4.2 but replacing (4.3) by (5.6). Then the regularized Dirichlet problem*

$$\begin{cases} \operatorname{div} \left(\frac{a_\kappa(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0, \\ u|_{\partial\Omega} = g, \end{cases}$$

where a_κ is given by (5.2), has a unique solution $u_\kappa \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$, for all $\kappa > 0$. Moreover, the family u_κ , $0 < \kappa \leq 1$, has a uniform C^1 bound on each compact subset of Ω .

Corollary 5.4 (of Theorem 4.3). *We assume all the conditions of Theorem 4.3 but with $a(s) = s^{p-1}A(s)$ and replacing (4.7) by (5.6). Then the regularized Dirichlet problem*

$$\begin{cases} \operatorname{div} \frac{a_\kappa(|\nabla u|)}{|\nabla u|} \nabla u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ for all $\kappa \in (0, \kappa_0(p))$ and for all $g \in C^{2,\alpha}(\overline{\Omega})$. Moreover, there is a $C^1(\overline{\Omega})$ bound for u independent of κ .

Corollary 5.5 (of Theorem 4.4). *We assume all the conditions of Theorem 4.4 but with $a(s) = s^{p-1}A(s)$ and replacing (4.7) by (5.6). Then the regularized Dirichlet problem*

$$\begin{cases} \operatorname{div} \frac{a_\kappa(|\nabla u|)}{|\nabla u|} \nabla u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

has a unique solution $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ for all $\kappa \in (0, \kappa_0(p))$ and for all $g \in C^0(\overline{\Omega})$. Moreover, the family u_κ , $\kappa \in (0, \kappa_0(p))$, admits a uniform C^1 bound on each relatively compact open subset Λ of Ω .

We are now able to obtain existence theorems when the coefficient a behaves as $a(s) = s^{p-1}A(s)$ (for details see (5.1)). As an intermediate step we first demonstrate how from the results obtained so far one may easily derive the existence of Lipschitz continuous weak solutions in the

degenerate or singular case. A much deeper analysis is required if one wants to show the optimal $C^{1,\beta}$ regularity of solutions. A complete proof of the latter would go far beyond the intentions of these notes and we shall therefore confine ourselves with a reduction to known results in the literature.

Theorem 5.6. *We assume either the conditions of Theorem 4.1 or Theorem 4.2, but replacing (4.3) and (4.7) by (5.6). Then the Dirichlet problem*

$$\begin{cases} \operatorname{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0, \\ u|_{\partial\Omega} = g, \end{cases}$$

where $a(s) = s^{p-1}A(s)$, $p > 1$, has a unique weak solution in $C^{0,1}(\bar{\Omega})$.

Proof. The solution is obtained as a limit of a sequence (u_{κ_i}) , $\kappa_i > 0$, $\kappa_i \rightarrow 0$ ($i \rightarrow \infty$), where u_{κ_i} are solutions of the regularized Dirichlet problems according to Corollary 5.2 or Corollary 5.4, respectively. By these corollaries, the families (u_{κ}) are uniformly bounded in $C^1(\bar{\Omega})$ and hence, in particular, there is a uniform $L^2(\bar{\Omega})$ bound for ∇u_{κ_i} . We may therefore find a function $u \in C^{0,1}(\bar{\Omega})$ such that u_{κ_i} converges (up to a subsequence) uniformly to u in $\bar{\Omega}$ and ∇u_{κ_i} converges weakly to ∇u in $L^2(\Omega)$. The statement of the theorem follows immediately if we can show that u minimizes the functional

$$F(v) = \int_{\Omega} \phi(|\nabla v|) dV$$

subject to the boundary condition $v|_{\partial\Omega} = g$, i.e. that $F(u) \leq F(v)$ for any $v \in C^1(\bar{\Omega})$ with $v|_{\partial\Omega} = g$.

We clearly have

$$F_{\kappa}(u_{\kappa}) \leq F_{\kappa}(v)$$

for all such v where

$$F_{\kappa}(v) = \int_{\Omega} \phi_{\kappa}(|\nabla v|) dw, \quad \phi_{\kappa}(s) = \int_0^s a_{\kappa}(t) dt \geq \phi(s).$$

By a well known result in the Calculus of Variations, using the convexity of ϕ [16] we get for arbitrary $v \in C^1(\bar{\Omega})$, $v|_{\partial\Omega} = g$:

$$F(u) \leq \liminf F(u_{\kappa_i}) \leq \liminf F_{\kappa_i}(u_{\kappa_i}) \leq \liminf F_{\kappa_i}(v) = F(v).$$

□

As announced above we shall now show using a result of Lieberman how one may deduce the $C^{1,\beta}$ regularity in the singular or degenerate case. As a consequence of this, the set where the gradient of the solution does not vanish is open and hence the solution is of class C^2 on this set.

Theorem 5.7. *We assume all the conditions of Theorem 4.1 but with $a(s) = s^{p-1}A(s)$, $p > 1$, and replacing (4.3) by (5.6). Then there is $\beta > 0$ such that the Dirichlet problem*

$$\begin{cases} \operatorname{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0, \\ u|_{\partial\Omega} = g \end{cases} \quad (5.7)$$

has a unique weak solution $u \in C^{1,\beta}(\overline{\Omega})$.

Proof. We shall obtain the solution of (5.7) as limit of solutions u_{κ_i} of the corresponding regularized equations when $\kappa_i \rightarrow 0$ as $i \rightarrow \infty$. The existence of u_{κ_i} and the uniform $C^1(\overline{\Omega})$ bound is guaranteed by Corollary 5.2. We shall show that Theorem 1 of [15] may be applied to obtain a uniform $C^{1,\beta}(\overline{\Omega})$ bound for the sequence u_{κ_i} . From this the statement of the theorem is a straightforward consequence. In order to meet the assumptions of Lieberman we must modify our equation as follows. We replace the coefficient A by

$$\tilde{A}(s) = \begin{cases} A(s) & \text{if } s \leq s_0, \\ A(s_0) & \text{if } s > s_0, \end{cases}$$

where s_0 is an upper bound for the $C^1(\overline{\Omega})$ norm of our solution u_κ which is guaranteed by Corollary 5.2. It is then obvious that u_κ will be a solution of the modified equation

$$\operatorname{div} \left(\left(\kappa + |\nabla u_\kappa|^2 \right)^{\frac{p}{2}-1} \tilde{A}(|\nabla u_\kappa|) \nabla u_\kappa \right) = 0,$$

in local coordinates (see (4.2))

$$\sum \frac{\partial}{\partial x_j} \tilde{A}^j(x, Du) = 0$$

with

$$\tilde{A}^j(x, Du) = \sum \sqrt{g} g^{ij} \left(\kappa + |\nabla u|^2 \right)^{\frac{p}{2}-1} \tilde{A}(|\nabla u|) \frac{\partial u}{\partial x_i}.$$

Let us remark that for a function u on the manifold \mathcal{M} the norm of its gradient ∇u on \mathcal{M} is equivalent to the Euclidean norm of its Euclidean gradient Du . Moreover the quotient

$$\frac{\kappa + s^2}{(\sqrt{\kappa} + s)^2}$$

being bounded by positive constants from above and from below independent of $0 < \kappa \leq 1$ and $s \geq 0$ the term $\left(\kappa + |\nabla u|^2 \right)^{\frac{p}{2}-1}$ may

be replaced in all estimates by $(\sqrt{\kappa} + |Du|)^{p-2}$. Therefore, by Lemma 5.1, the ellipticity required in Lieberman's theorem is satisfied. It remains to check the Holder estimates of the coefficients $\tilde{A}^j(x, Du)$ with respect to the coordinates $x = (x_1, \dots, x_n)$. We even show a Lipschitz condition by computing a bound for the partial derivatives $\partial \tilde{A}^j / \partial x_k$.

Denoting by C a generic constant which only depends on the metric tensor g_{ij} and its first derivatives we get

$$\frac{\partial |\nabla u|^2}{\partial x_k} \leq C |Du|^2, \quad \frac{\partial |\nabla u|}{\partial x_k} \leq C |Du|$$

and hence

$$\begin{aligned} \left| \frac{\partial \tilde{A}^j}{\partial x_k}(x, Du) \right| &\leq C \left\{ (\kappa + |\nabla u|^2)^{\frac{p}{2}-1} \tilde{A}(|\nabla u|) |Du| \right. \\ &\quad + (\kappa + |\nabla u|^2)^{\frac{p}{2}-2} |Du|^2 \tilde{A}'(|\nabla u|) |Du| \\ &\quad \left. + (\kappa + |\nabla u|^2)^{\frac{p}{2}-1} \tilde{A}'(|\nabla u|) |Du|^2 \right\}. \end{aligned}$$

Setting

$$K := \max_{0 \leq s \leq s_0} \{A(s), A'(s)\} = \max_{0 \leq s \leq s_0} \{\tilde{A}(s), \tilde{A}'(s)\}$$

and observing that $\tilde{A}'(s) = 0$ for $s > s_0$ we obtain the estimate

$$\begin{aligned} \left| \frac{\partial \tilde{A}^j}{\partial x_k}(x, Du) \right| &\leq CK \left\{ (1 + |Du|)^{p-1} + (1 + s_0^2)^{\frac{p}{2}-1} s_0^2 \right\} \\ &\leq \tilde{C} (1 + |Du|)^{p-1} \end{aligned}$$

with a constant \tilde{C} also depending on s_0 . This completes the proof. \square

On the basis of Corollary 5.3 the proof of the following theorem proceeds along the same lines as that of the previous theorem by replacing Ω by a relatively compact subdomain Λ of Ω and using a local version of the $C^{1,\beta}$ estimates in Lieberman's theorem. We note that though such a local version is not explicitly stated in Lieberman's result, it is obviously a necessary ingredient to get global estimates. Indeed the interior estimates can be obtained by the same methods as the estimates at the boundary.

Theorem 5.8. *We assume all the conditions of Theorem 4.2 but replacing (4.3) by (5.6). Then there is a unique weak solution of the Dirichlet problem*

$$\begin{cases} \operatorname{div} \left(\frac{\alpha(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0, \\ u|_{\partial\Omega} = g \end{cases}$$

which, on each subdomain Λ of Ω with $\bar{\Lambda} \subset \Omega$, belongs to $C^{1,\beta}(\bar{\Lambda})$ for some β possibly depending on Λ . Moreover, for each such relatively compact subdomain Λ there is a $C^{1,\beta}(\bar{\Lambda})$ bound for u depending only on Λ and $\sup_{\Omega} |u|$.

Theorem 5.9. *We assume all the conditions of Theorem 4.2 but with $a(s) = s^{p-1}A(s)$, $p > 1$, and replacing (4.7) by (5.6). Then there is a unique weak solution of the Dirichlet problem*

$$\begin{cases} \operatorname{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0, \\ u|_{\partial\Omega} = g \end{cases}$$

which belongs to $C^{1,\beta}(\bar{\Omega})$.

Proof. On the basis of Corollary 5.4 the proof is completely analogous to that of Theorem 5.7. \square

Theorem 5.10. *We assume all the conditions of Theorem 4.3 but with $a(s) = s^{p-1}A(s)$, $p > 1$, and replacing (4.7) by (5.6). Then the Dirichlet problem*

$$\begin{cases} \operatorname{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0, \\ u|_{\partial\Omega} = g \end{cases}$$

has a unique weak solution $u \in C^0(\bar{\Omega}) \cap C^1(\Omega)$ for any $g \in C^0(\bar{\Omega})$. Moreover, for each relatively compact subdomain Λ of Ω the solution u belongs to $C^{1,\beta}(\bar{\Lambda})$ for some $\beta > 0$ and there is an a-priori bound for the $C^{1,\beta}$ norm of u depending only on $\sup_{\Omega} |u|$, Λ and Ω .

We close this section by presenting an example where Theorems 5.9 and 5.10 apply. This example can be seen as a p -area version of the minimal surface equation, analogous to the p -energy of the Laplace partial differential equation, $1 < p \leq 2$.

Example 5.1. We consider the integrand

$$\phi(s) = (1 + s^p)^{\frac{1}{p}}$$

for which we get

$$\begin{aligned} a(s) = \phi'(s) &= \frac{s^{p-1}}{(s^p + 1)^{\frac{p-1}{p}}}, \quad 1 + b(s) = \frac{sa'(s)}{a(s)} = \frac{p-1}{s^p + 1} \\ b'(s) &= -\frac{p(p-1)s^{p-1}}{(s^p + 1)^2}. \end{aligned}$$

Hence we obtain

$$(-sb'(s) - (1 + \varepsilon)(1 + b(s)))s^2 = \frac{p-1}{(s^p + 1)^2} ((p-1-\varepsilon)s^{p+2} - (1+\varepsilon)s^2) \geq \alpha > 0$$

for $s \geq s_0(p, \varepsilon)$ provided that $\varepsilon < p-1$ and $p \leq 2$. The ellipticity condition (5.6) is verified by a similar direct calculation.

Chapter 6

The asymptotic Dirichlet problem

The existence or nonexistence of non constant entire bounded harmonic functions on a Cartan-Hadamard manifold \mathcal{M} is a topic of study in Differential Geometry that dates back to the 70's (see [9], [19]). In the last years this problem has been studied with other partial differential equations such as the p -Laplacian ([11]) and the minimal surface equation ([8], [4], [5], [18]). The class of partial differential equations considered here has been studied in [17] with the purpose of proving Liouville type theorems that is, *non existence* of non constant bounded solutions. In this last part of our notes we investigate the *existence* of bounded non constant solutions of this class of partial differential equations by studying the associated asymptotic Dirichlet problem.

6.1 Existence theorems

A natural way of finding bounded entire solutions to a partial differential equation on a Cartan-Hadamard manifold is by solving the asymptotic Dirichlet problem with a prescribed non constant boundary data given at infinity.

Recall that a Cartan-Hadamard manifold is a complete, connected and simply connected Riemannian n -manifold \mathcal{M} , $n \geq 2$, of non-positive sectional curvature. By the Cartan-Hadamard theorem, the exponential map $\exp_o: T_o\mathcal{M} \rightarrow \mathcal{M}$ is a diffeomorphism for every point $o \in \mathcal{M}$. Consequently, \mathcal{M} is diffeomorphic to \mathbb{R}^n .

A Cartan-Hadamard manifold \mathcal{M} can be compactified by adding a *sphere at infinity* which is also called the asymptotic boundary of \mathcal{M} . The sphere at infinity of \mathcal{M} , denoted by $\partial_\infty\mathcal{M}$, is defined as the set of all

equivalence classes $[\gamma]$ of unit speed geodesic rays γ of \mathcal{M} ; two such rays γ_1 and γ_2 are equivalent if $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$. The compactification $\overline{\mathcal{M}}$ of \mathcal{M} , also known as the geometric compactification of \mathcal{M} , is then $\overline{\mathcal{M}} := \mathcal{M} \cup \partial_\infty \mathcal{M}$, with the following topology.

Given $p \in \mathcal{M}$ let $B \subset \mathbb{S}^{n-1}$ be an open geodesic ball of the unit sphere \mathbb{S}^{n-1} of $T_p \mathcal{M}$. Given $v \in \mathbb{S}^{n-1}$, denote by $\gamma_v : [0, \infty) \rightarrow \mathcal{M}$ the geodesic ray such that $\gamma_v(0) = p$ and $\gamma'_v(0) = v$. Then, given $t > 0$ the sets

$$T = \{[\gamma_v] \mid v \in B\} \cup \{\gamma_v((t, \infty)) \mid v \in B\}$$

with p varying in \mathcal{M} , B varying on the unit sphere of $T_p \mathcal{M}$ and t varying in the positive real numbers form a basis for a topology on $\overline{\mathcal{M}}$, called the cone topology. The space $\overline{\mathcal{M}}$, equipped with the cone topology, is homeomorphic to a closed Euclidean ball. For more details see [7].

The asymptotic Dirichlet problem on \mathcal{M} for a differential operator Q on \mathcal{M} consists in finding a (unique) function $u \in C^0(\overline{\mathcal{M}})$ such that $Q[u] = 0$ on \mathcal{M} and $u|_{\partial_\infty \mathcal{M}} = g$, for a given function $g \in C^0(\partial_\infty \mathcal{M})$.

We consider on \mathcal{M} the same family of operators already considered here so far, namely, operators of the form

$$Q[u] = \operatorname{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right), \quad (6.1)$$

where $a \in C^2((0, \infty)) \cap C^0([0, \infty))$ satisfies conditions to be discussed in the course of the text.

For further references in the text, it is convenient to state the asymptotic Dirichlet problem for Q in the following short form

$$\begin{cases} Q[u] = \operatorname{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0 \text{ on } \mathcal{M} \\ u|_{\partial_\infty \mathcal{M}} = g, \quad u \in C^1(\mathcal{M}) \cap C^0(\overline{\mathcal{M}}). \end{cases} \quad (6.2)$$

We follow now closely the paper [18] of J. Ripoll and M. Telichevesky. In the case of bounded domains, the continuous extension to the boundary of the domain of a prospective solution to the Dirichlet problem of an elliptic partial differential equation (for example, the one obtained by Perron's method is typical, [10]) depends on the *regularity* of the domain with respect to the partial differential equation, that is, on the existence of barriers at each point of the boundary of the domain (see Subsection 2.2 and also [10]). To deal with the asymptotic Dirichlet problem in \mathcal{M} we extend this notion of regularity to the asymptotic boundary $\partial_\infty \mathcal{M}$ of \mathcal{M} .

We consider here weak C^1 solutions to the equation $Q[u] = 0$ in \mathcal{M} that is, we require that $u \in C^1(\mathcal{M})$ and satisfies

$$\int_{\mathcal{M}} \left\langle \frac{a(|\nabla u|)}{|\nabla u|} \nabla u, \nabla \zeta \right\rangle dx = 0 \quad (6.3)$$

for every $\zeta \in C_0^\infty(\mathcal{M})$. A function $v \in C^{0,1}(\Omega)$ is a subsolution of Q in a domain Ω of \mathcal{M} if $Q[v] \geq 0$ weakly in Ω , that is

$$\int_{\mathcal{M}} \left\langle \frac{a(|\nabla v|)}{|\nabla v|} \nabla v, \nabla \zeta \right\rangle dx \leq 0 \quad (6.4)$$

for every non-negative $\zeta \in C_0^\infty(\Omega)$. A function $w \in C^{0,1}(\Omega)$ is called a supersolution of Q in Ω if $-w$ is a subsolution for Q in Ω .

Given $x \in \partial_\infty \mathcal{M}$ and an open subset $\Omega \subset \mathcal{M}$ such that $x \in \partial_\infty \Omega$, an upper barrier for Q relative to x and Ω with height C is a function $w \in C^{0,1}(\mathcal{M})$ such that

- (i) w is a supersolution for Q
- (ii) $w \geq 0$ and $\lim_{p \in \mathcal{M}, p \rightarrow x} w(p) = 0$
- (iii) $w_{\mathcal{M} \setminus \Omega} \geq C$.

Lower barriers are defined similarly.

We say that \mathcal{M} is regular at infinity with respect to Q if, given $C > 0$, $x \in \partial_\infty \mathcal{M}$ and an open subset $W \subset \partial_\infty \mathcal{M}$ with $x \in W$, there exist an open set $\Omega \subset \mathcal{M}$ such that $x \in \text{Int } \partial_\infty \Omega \subset W$ and upper and lower barriers $w, v \in C^1(\mathcal{M})$ relative to x and Ω , with height C .

The regularity at infinity has already been considered by other authors for the p -Laplacian. The reader should compare the above definition with Definition 2.6 and Theorem 2.7 in [3] for the case of the Laplace operator and also with Theorem 3.3 and Definition 3.4 in [14] for the case of the p -Laplacian.

Theorem 6.1. *Let \mathcal{M} be a Hadamard manifold which is regular at infinity with respect to Q . Assume moreover that*

- (a) *given $\phi \in C^0(\mathcal{M})$, there is a sequence of bounded $C^{2,\alpha}$ domains $\Omega_k \subset \mathcal{M}$, $k \in \mathbb{N}$, satisfying $\Omega_k \subset \Omega_{k+1}$, $\cup \Omega_k = \mathcal{M}$ such that, given k , there is a weak solution $u_k \in C^0(\overline{\Omega_k}) \cap C^1(\Omega_k)$ of the Dirichlet problem for $Q[u] = 0$ in Ω_k such that $u_k|_{\partial\Omega_k} = \phi|_{\partial\Omega_k}$*
- (b) *sequences of solutions with uniformly bounded C^0 norm are compact in the C^1 norm in precompact subsets of \mathcal{M} .*

Then the asymptotic Dirichlet (6.2) is solvable for any continuous asymptotic boundary data $g \in C^0(\partial\mathcal{M}_\infty)$.

Proof. Let $\tilde{g} \in C^0(\overline{M})$ be a continuous extension to \overline{M} of the asymptotic boundary data g of problem (6.2). From condition (a) there is a solution $u_k \in C^0(\overline{\Omega_k}) \cap C^1(\Omega_k)$ of the Dirichlet problem

$$\begin{cases} Q[u] = 0 & \text{in } \Omega_k, \\ u|_{\partial\Omega_k} = \tilde{g}|_{\partial\Omega_k}. \end{cases}$$

Condition **(b)** together with the diagonal method show that there exists a subsequence of (u_k) converging uniformly on compact subsets of \mathcal{M} in the C^1 norm to a global solution $u \in C^1(\mathcal{M})$ of $Q[u] = 0$. From the comparison principle it follows that

$$\sup_{\mathcal{M}} |u| \leq \sup_{\mathcal{M}} |\tilde{g}|.$$

One needs to show that u extends continuously to $\partial_\infty \mathcal{M}$ and satisfies $u|_{\partial_\infty \mathcal{M}} = g$. Let $x \in \partial_\infty \mathcal{M}$ and $\varepsilon > 0$ be given.

Since g is continuous, there exists an open neighborhood $W \subset \partial_\infty \mathcal{M}$ of x such that $g(y) < g(x) + \varepsilon/2$ for all $y \in W$. Furthermore, regularity of $\partial_\infty \mathcal{M}$ implies that exists an open subset $\Omega \subset \mathcal{M}$ such that $x \in \text{Int}(\partial_\infty \Omega) \subset W$ and $w : \mathcal{M} \rightarrow \mathbb{R}$ upper barrier with respect to x and Ω with height $C := 2 \max_{\overline{M}} |g|$.

Defining

$$v(p) := w(p) + g(x) + \varepsilon, \quad p \in \Omega,$$

we claim that $u \leq v$ in Ω .

From the continuity of \tilde{g} we may find k_0 such that $\tilde{g}(p) < g(x) + \varepsilon/2$ for all $p \in \partial\Omega_k \cap \Omega$, $k \geq k_0$. Moreover, we may choose k_0 such that $\Omega_{k_0} \cap \Omega \neq \emptyset$. Set $V_k := \Omega \cap \Omega_k$, $k \geq k_0$. We note that $u_k \leq v$ in V_k . Indeed, this inequality holds on

$$\partial V_k = \overline{(\partial\Omega_k \cap \Omega)} \cup \overline{(\partial\Omega \cap \Omega_k)}.$$

On $\partial\Omega_k \cap \Omega$ is due to the choice of k_0 ; on $\partial\Omega \cap \Omega_k$ it holds because $w \geq \max |g|$ on $\partial\Omega$, which implies that $w \geq u_k$, by the comparison principle. Also the comparison principle implies that $u_k \leq v$ in V_k . Since it holds for all $k \geq k_0$, we have $u \leq v$ on Ω .

It is also possible to define $v_- : \mathcal{M} \rightarrow \mathbb{R}$ by $v_-(p) := g(x) - \varepsilon - w(p)$ in order to obtain $u \geq v_-$ in Ω . We then have

$$|u(p) - g(x)| < \varepsilon + w(p),$$

for all $p \in \Omega$ and hence

$$\limsup_{p \rightarrow x} |u(p) - g(x)| \leq \varepsilon.$$

The proof is complete, since $\varepsilon > 0$ is arbitrary. \square

Theorem 6.1 brings up the problem of when the Hadamard manifold \mathcal{M} is regular at infinity. The following definition, introduced in [18], turned out to be a key concept.

Definition 6.1. Let \mathcal{M} be a Hadamard manifold. We say that \mathcal{M} satisfies the *strict convexity condition (SC condition)* if, given $x \in \partial_\infty \mathcal{M}$ and a relatively open subset $W \subset \partial_\infty \mathcal{M}$ containing x , there exists a C^2 open subset $\Omega \subset \overline{\mathcal{M}}$ such that $x \in \text{Int}(\partial_\infty \Omega) \subset W$, where $\text{Int}(\partial_\infty \Omega)$ denotes the interior of $\partial_\infty \Omega$ in $\partial_\infty \mathcal{M}$, and $\mathcal{M} \setminus \Omega$ is convex.

Informally speaking one may say that, as it happens with strictly convex bounded domains of the Euclidean space, \mathcal{M} satisfies the SC condition when one can extract from $\overline{\mathcal{M}}$ a neighborhood of any point of $\partial_\infty \mathcal{M}$ such that what remains is still convex.

Lemma 6.2. *Let \mathcal{M} be a Hadamard manifold with sectional curvature $K_M \leq -k^2 < 0$ and satisfying the SC condition. Assume that*

$$Q[u] = \text{div} \left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0 \quad (6.5)$$

with $a \in C^0([0, \infty)) \cap C^1((0, \infty))$ satisfies

$$a(0) = 0, a'(s) > 0 \text{ for all } s > 0; \quad (6.6)$$

$$\text{there exist } q > 0 \text{ and } \delta > 0 \text{ such that } a(s) \geq s^q, s \in [0, \delta]. \quad (6.7)$$

Then \mathcal{M} is regular at infinity with respect to Q .

Proof. Let $C > 0$ and $x \in W \subset \partial_\infty \mathcal{M}$ be given. Since $Q[-u] = -Q[u]$ it is enough to prove the existence of barriers from above at x . Since \mathcal{M} satisfies the SC condition, there exists a C^2 open subset Ω of \mathcal{M} such that $x \in \text{Int}(\partial_\infty \Omega) \subset W$ and such that $\mathcal{M} \setminus \Omega$ is convex. Let $s : \Omega \rightarrow \mathbb{R}$ be the distance function to $\partial\Omega$. Since $\mathcal{M} \setminus \Omega$ is convex and $K_M \leq -k^2$, $k > 0$, we may apply comparison theorems (see Theorems 4.2 and 4.3 of [3]) to obtain the estimate

$$\Delta s \geq (n-1)k \tanh ks. \quad (6.8)$$

On the other hand, since $a' > 0$, a has an inverse function $a^{-1} \in C^1([0, \alpha])$ where $\alpha = \sup a \leq \infty$. Set

$$c = \frac{a(2C)}{\cosh^{n-1} k}.$$

Since $0 \leq c \cosh^{1-n} kt \leq \alpha$ for all $t \geq 0$ we may define a function $g : [0, \infty) \rightarrow \mathbb{R}$, $g \in C^2((0, \infty))$, possibly with $g(0) = \infty$, by

$$g(s) := \int_s^\infty a^{-1}(c \cosh^{1-n} kt) dt. \quad (6.9)$$

Without loss of generality, we may choose δ small such that $a(\delta) < c$. Then, since

$$c \cosh^{1-n} 0 = c > a(\delta)$$

and $\lim_{t \rightarrow +\infty} c \cosh^{1-n} kt = 0$ there is τ satisfying

$$c \cosh^{1-n} k\tau = a(\delta).$$

Since for $t \in [0, a(\delta)]$ we have $a^{-1}(t) \in [0, \delta]$ it follows from (6.7) that

$$t = a(a^{-1}(t)) \geq a^{-1}(t)^q$$

and hence $a^{-1}(t) \leq t^{1/q}$ for $t \in [0, a(\delta)]$. Therefore

$$\begin{aligned} g(s) &\leq \int_0^\tau a^{-1}(c \cosh^{1-n} kt) dt + \int_\tau^{+\infty} a^{-1}(c \cosh^{1-n} kt) dt \\ &\leq a^{-1}(c)\tau + \int_\tau^{+\infty} (c \cosh^{1-n} kt)^{\frac{1}{q}} dt \\ &\leq a^{-1}(c)\tau + (2c)^{\frac{1}{q}} \int_\tau^{+\infty} e^{-\frac{kt}{q}} dt = a^{-1}(c)\tau + \frac{(2c)^{\frac{1}{q}}}{q} e^{-k\tau} < +\infty \end{aligned}$$

for all $s > 0$. Furthermore,

$$g(0) > \int_0^1 a^{-1}(c \cosh^{1-n} kt) dt \geq a^{-1}(c \cosh^{1-n} k) = 2C$$

and $\lim_{s \rightarrow \infty} g(s) = 0$. Therefore we define $v : \Omega \rightarrow \mathbb{R}$ as

$$v(p) := g(s(p)),$$

and will prove that $Q(v) \leq 0$. We have

$$\nabla v(p) = g'(s(p)) \nabla s(p) = -a^{-1}(c \cosh^{1-n} ks(p)) \nabla s$$

and then

$$|\nabla v| = |g'(s)| = a^{-1}(c \cosh^{1-n} ks).$$

Also $\nabla v / |\nabla v| = -\nabla s$. Combining the previous expressions, we obtain

$$\begin{aligned} Q[v] &= \operatorname{div}(-a(|g'(s)|) \nabla s) \\ &= \operatorname{div}(-a(a^{-1}(c \cosh^{1-n} ks)) \nabla s) \\ &= \operatorname{div}(-c \cosh^{1-n} ks \nabla s) \\ &= -(1-n)ck (\cosh^{-n} ks) (\sinh ks) \langle \nabla s, \nabla s \rangle - c (\cosh^{1-n} ks) \Delta s \\ &\leq (n-1)ck (\cosh^{-n} ks) (\sinh ks) \\ &\quad - (n-1)c (\cosh^{1-n} ks) k (\tanh ks) = 0, \end{aligned}$$

and hence, by Lemma 3.1, v is a supersolution on Ω .

To finish with the proof, define the global supersolution $w \in C^0(\overline{\mathcal{M}}) \cap C^{0,1}(\mathcal{M})$ by

$$w(p) = \begin{cases} \min\{v(p), C\} & \text{if } p \in \Omega \\ C & \text{if } p \in \overline{\mathcal{M}} \setminus \Omega, \end{cases}$$

which is of course an upper barrier relative to x and Ω with height C . \square

As a consequence of Theorems 6.1 and 6.2 we obtain

Theorem 6.3. *Let \mathcal{M} be a Hadamard manifold with sectional curvature satisfying $K_{\mathcal{M}} \leq -k^2 < 0$. Assume that \mathcal{M} satisfies the SC condition together with the conditions (6.6) and (6.7) of Lemma 6.2, and that Q satisfies conditions (a) and (b) of Theorem 6.1. Then the asymptotic Dirichlet problem (6.2) is solvable for any $g \in C^0(\partial_{\infty}\mathcal{M})$.*

Theorem 6.4. *We assume that the operator*

$$Q[u] = \operatorname{div} \frac{a(|\nabla u|)}{|\nabla u|} \nabla u$$

satisfies the following conditions (i) and **either** (ii) **or** (ii'):

(i) *There are numbers $p > 1$ and $\alpha \in (0, 1)$ such that $a(s) = s^{p-1}A(s)$ with $A \in C^{1,\alpha}([0, \infty)) \cap C^{2,\alpha}((0, \infty))$ and*

$$\min_{0 \leq s \leq \sigma} \left\{ A(s), \min\{1, p-1\} + \frac{sA'(s)}{A(s)} \right\} > 0$$

for any $\sigma > 0$

(ii) *There are numbers $\beta > 0$, $\gamma > 0$ and $s_0 > 0$ such that with*

$$\begin{aligned} b(s) &:= \frac{sA'(s)}{A(s)} \\ B(s) &:= \max\{1, 1 + b(s)\} \end{aligned}$$

there holds

$$B(s)^{-1} (1 + b(s) - \beta |b'(s)| s) \geq \gamma$$

for $s \geq s_0$.

(ii') *There are numbers $\varepsilon > 0$, $\gamma > 0$ and $s_0 > 0$ such that*

$$(-sb'(s) - (1 + \varepsilon)(1 + b(s))) s^2 \geq \gamma$$

for $s \geq s_0$.

Then, if the Cartan-Hadamard manifold \mathcal{M} has sectional curvature $K_{\mathcal{M}} \leq -k^2$ for some number $k > 0$ and satisfies the SC condition (Definition 6.1), the asymptotic Dirichlet problem

$$\begin{cases} Q[u] = 0 \text{ on } \mathcal{M} \\ u|_{\partial_{\infty}\mathcal{M}} = g \end{cases}$$

has a unique solution $u \in C^0(\overline{\mathcal{M}}) \cap C^1(\mathcal{M})$ which, for each relatively compact subdomain $\Omega \subset \mathcal{M}$ belongs to $C^{1,\lambda}(\Omega)$ with $\lambda > 0$ possibly depending on Ω . In case $p = 2$ in (i) above the solution u is classical, i.e., $u \in C^{2,\alpha}(\mathcal{M})$.

Proof. We observe that the uniqueness follows immediately from Proposition 3.1. For the existence part, choose a fixed point of \mathcal{M} , say o , and, given $k \geq 1$, let Ω_k be the geodesic ball of \mathcal{M} centered at o and with radius k . Since \mathcal{M} is a Cartan Hadamard manifold Ω_k is a $C^{2,\alpha}$ domain for all k and, from the Hessian comparison theorem, Ω_k is a convex and hence a mean convex domain for all $k \geq 1$ (Theorems 4.2 and 4.3 of [3]). Conditions (a) and (b) in Theorem 6.1 are satisfied as follows from the Theorems 5.8 and 5.10. The conditions 6.6 and 6.7 of Lemma 6.2 clearly hold by assumption (i). Thus, Theorem 6.4 is an immediate consequence of Theorem 6.3. \square

6.2 Final comments

Let \mathcal{M} be a Hadamard manifold. If $\dim \mathcal{M} = 2$ and the sectional curvature of \mathcal{M} satisfies $K_{\mathcal{M}} \leq -k^2$, $k > 0$, then, since any two points at infinity of \mathcal{M} can be connected by a geodesic, it trivially follows that \mathcal{M} satisfies the SC condition. In arbitrary dimensions it is proved in [18] that if $K_{\mathcal{M}} \leq -k^2$ and either \mathcal{M} is rotationally symmetric (see [3]) or the sectional curvature of \mathcal{M} decays at most exponentially then \mathcal{M} satisfies the SC condition. The SC condition also holds in Hadamard manifolds where the sectional curvature may go to zero with a certain rate, but assuming stronger assumptions on the decay of the curvature. The asymptotic Dirichlet problem can be solved in some of these manifolds by using barriers at infinity others than those of Lemma 6.2 (see [6]).

We finally should remark that in dimensions greater than or equal to 3 just an upper bound for the sectional curvature of \mathcal{M} is not enough for the solvability of the asymptotic Dirichlet problem in \mathcal{M} . Indeed, Ancona [1] and Borbely [2] construct examples of 3-dimensional Hadamard manifolds with curvature less than or equal to -1 such that if an entire harmonic function extends continuously to the asymptotic boundary then it is constant. Holopainen extended Borbely's example to the p -Laplacian [12] and in [13] the authors constructed similar counter-examples of manifolds for a class of partial differential equations that includes the p -Laplacian and minimal surface equation. Therefore, in these manifolds, because of Theorem 6.3, the SC condition is not satisfied.

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