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Infinitesimal variations of submanifolds

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Abstract. The main purpose of these lecture notes is to present recent results in the theory of infinitesimal variations of submanifolds. The smooth variations under consideration are infinitesimally isometric or, in greater generality, infinitesimally conformal. The concept of infinitesimal variation is the infinitesimal analogue of an isometric variation and refers to smooth variations that preserve lengths "up to the first order". In the more general case of conformal infinitesimal variations, lengths are preserved similarly but now up to a conformal factor. The study of such variations is realized by means of the corresponding variational vector fields, called infinitesimal bendings and conformal infinitesimal bendings respectively. Hence, these lecture notes contain results about nontrivial infinitesimal bendings and the geometry of the submanifolds that carry them.

Keywords. Infinitesimal variations; Infinitesimal bendings; Conformal infinitesimal bendings.

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Contents

Preface

The main purpose of these lecture notes is to present recent results in the theory of infinitesimal variations of submanifolds. The smooth variations under consideration are infinitesimally isometric or, in greater generality, infinitesimally conformal. For the most part, we are devoted to Euclidean submanifolds but there is one chapter where the round sphere and the hyperbolic space are the ambient spaces. The results contained in this chapter have not been published elsewhere, at least, in the present form.

The concept of infinitesimal variation is the infinitesimal analogue of an isometric variation and refers to smooth variations that preserve lengths "up to the first order". In the more general case of conformal infinitesimal variations, lengths are preserved similarly but now up to a conformal factor. It is already known from classical differential geometry that the convenient approach to study infinitesimal variations is to focus on the variational vector field of the variation. We call these vector fields in the isometric case an infinitesimal bending, and a conformal infinitesimal bending in the more general conformal case. Consequently, the arguments in these lecture notes are mostly about nontrivial infinitesimal bendings and the geometry of the submanifolds that carry them.

The study of smooth variations of Euclidean surfaces was already a hot topic in differential geometry in the 19^{th} century. In fact, initially there was no distinction made between isometric variations and the ones that are only infinitesimally isometric, but that changed mostly due to the work of G. Darboux. The subject of isometric variations of surfaces was proposed for competition by the French Academy of Sciences in 1859, the main problem being to establish the differential equations that determine all the surfaces isometric to a given one. The prize was obtained by a young engineer called E. Bour, while competing with O. Bonnet and the Italian geometer D. Codazzi. Even though Bonnet had already solved the problem, it was Bour who won because for surfaces of revolution he managed to integrate the differential equations. Part of Bour's work was published in [4]. For a modern account of some aspects of the theory of variations of surfaces we refer to the book of Spivak [36].

The study of isometric variations of hypersurfaces M^n, $n \geq 3$, in

Euclidean space $\mathbb{R}^{n+1}$ is also a classical subject going back to the first part of the last century. In fact, the local classification of isometrically deformable hypersurfaces, by means of two alternative parametrizations, is due to U. Sbrana [34] in 1909 and E. Cartan [6] in 1916. Sbrana seems to have been a student of L. Bianchi [2], who in 1905 already considered the three-dimensional case. A modern presentation of the local parametric classifications by Sbrana and Cartan, as well as several further results, have been given by Dajczer-Florit-Tojeiro [12], and can also be seen in [21].

Although produced later but published earlier than his other work, the case of infinitesimal variations of hypersurfaces was taken up by Sbrana [33] in 1908. It turns out that the class of hypersurfaces that admit an infinitesimal variation is much larger than the class that allow an isometric variation, a fact that may be seen as a surprise. A complete classification was given by Dajczer-Vlachos [23]. Finally, as for complete hypersurfaces see Dajczer-Gromoll for isometric variations [13] and Jimenez [28] in the infinitesimal case. The latter work is also part of these lecture notes.

In a rather long and very difficult paper, Cartan [6] in 1917 gave a parametric classification of the conformally deformable Euclidean hypersurfaces M^n of dimension $n \geq 5$ with the use of the method of moving frames. These are smooth variations of a hypersurface by conformal ones. A modern version of his result, as well as an alternative classification, has been given by Dajczer-Tojeiro [20]. This result is also contained in [21]. The case $n = 4$ was subsequently treated by Cartan in [8] but only in a special case, thus the full classification remains an open problem. Finally, the parametric classification of the Euclidean hypersurfaces that admit conformal infinitesimal bendings is due to Dajczer-Jimenez-Vlachos [18], and is one of the topics considered here.

In these lecture notes we discuss the case of submanifolds other than surfaces. A so-called Fundamental Theorem for infinitesimal bendings, extending to any codimension the result for hypersurfaces in [23], was obtained by Dajczer-Jimenez [15]. As in the theory of isometric immersions, a system formed by three equations, called the fundamental equations, is given. These equations are obtained in terms of a pair of tensors associated to the bending and are shown to be the integrability conditions for the equations that determine an infinitesimal bending. A Fundamental Theorem in the more general case of conformal infinitesimal bendings is due to Dajczer-Jimenez [17], and is also part of these lecture notes.

Dajczer and Rodríguez [19] showed that submanifolds in low codimension are generically infinitesimally rigid, that is, generically only trivial infinitesimal variations are possible. In fact, they proved that certain algebraic conditions on the second fundamental form of an immersion, known to give isometric rigidity, yield infinitesimal rigidity as well. For instance, a necessary condition (but far from being sufficient!)

for a hypersurface M^n in $\mathbb{R}^{n+1}$ to admit an infinitesimal variation is to have at any point at most two nonzero principal curvatures. In fact, this result is already contained in the book of Cesàro [9] published in 1886. For higher codimension, algebraic conditions that yield rigidity are rather strong requirements. They are given in terms of either the type number or the s-nullities of the immersion. A rigidity result for conformal infinitesimal variations is due to Dajczer-Jimenez in [17]. The proof turns out to be much more elaborate than in the case of infinitesimal variations and is also part of these lecture notes.

A brief outline of each chapter is given next.

Chapter 1 establishes several basic facts of the theory of submanifolds that are intensively used throughout the rest of these lecture notes. First, the second fundamental form and normal connection of an isometric immersion are recalled by means of the Gauss and Weingarten formulas, and their compatibility equations are given, namely, the Gauss, Codazzi and Ricci equations. Then the so-called Fundamental Theorem for isometric immersions is stated, according to which these data are sufficient to determine uniquely any Euclidean submanifold. One topic covered in this chapter is the most basic but fundamental result in the theory of flat bilinear forms. Another topic is the differential equations satisfied by the splitting tensor of a submanifold carrying a foliation of relative nullity, and some of their consequences.

The remaining of these lecture notes can be seen as formed by two parts. Constituted of five chapters, the first part is devoted to infinitesimal variations of submanifolds, whereas the second part of two chapters deals with the more general class of conformal infinitesimal variations.

Chapter 2 first introduces the notion of infinitesimal variation of an Euclidean submanifold. It is then discussed why, in order to study infinitesimal variations, one has to understand their variational vectors fields, called infinitesimal bendings. The first result is a Fundamental Theorem for infinitesimal variations. It is shown that a certain system of three equations for two tensors are the integrability conditions for the equations that determine an infinitesimal bending. Moreover, it turns out that this infinitesimal bending is unique in a precise sense. The second part of the chapter deals with the rigidity problem of infinitesimal variations for submanifolds in low codimension. It is shown that certain conditions on the second fundamental form of the submanifold imply rigidity in the sense that any infinitesimal variation has to be trivial.

Chapter 3 first observes that, if an Euclidean submanifold admits an infinitesimal variation, then any embedded submanifold of that manifold inherits, by composition of immersions, an infinitesimal variation. Thus, in order to study the geometry of the submanifolds that admit a nontrivial infinitesimal variation, this situation should somehow be excluded, and this leads to the concept of genuine infinitesimal variation. The main purpose of this chapter is to characterize the Euclidean submanifolds in low codimension that admit a genuine infinitesimal variations. Two local and one global results are given, the latter for compact submanifolds.

Chapter 4 considers the case when the ambient space is either the round sphere or the hyperbolic space. Some results of the previous chapters are extended by means of similar techniques to submanifolds of these space forms.

Chapter 5 is devoted to analyzing the structure of the infinitesimal variation of an Euclidean submanifold that is intrinsically a Riemannian product of manifolds. Conditions are given, both local and global, that imply that any infinitesimal variation of the submanifold has to be the sum of infinitesimal variations of isometric immersions of each of the factors.

Chapter 6 is about the classification of the complete Euclidean hypersurfaces that admit nontrivial infinitesimal variations. It is shown that the variations can only occur along a ruled strip. A ruled strip is a ruled hypersurface with complete rulings, and possible boundary, such that the rulings are tangent to the boundary. In other words, it is an affine vector bundle over a curve with or without end points.

Chapter 7 deals with conformal infinitesimal variations. This concept belongs to the realm of conformal geometry since, by composing the submanifold with a conformal transformation of the ambient Euclidean space, we obtain a new conformal infinitesimal variation. This class of variations had received limited attention until recently; see Yano [37] for an exception. The main contents of the chapter are a Fundamental Theorem for conformal infinitesimal variations and a rigidity theorem for these objects. Both results are in a similar spirit than in the case of infinitesimal variations discussed above.

Chapter 8 gives a parametric classification of the Euclidean hypersurfaces of dimension at least five that admit nontrivial conformal infinitesimal variations. The key ingredients in the classification are the so-called conformal Gauss parametrization and a class of surfaces in either the Euclidean or Lorentzian sphere. Finally, the classification in the conformal case is used to give a parametric classification of the hypersurfaces that admit nontrivial infinitesimal variations.

Finally, we point out that results already contained in [21] that strongly relate to the subjects in these lecture notes will be described or referred to but not proved again, at least, with a similar proof.

Chapter 1

Preliminaries

The purpose of this chapter is to recall several concepts and basic results concerning isometric immersions between Riemannian manifolds.

Let M^n be an n-dimensional connected differentiable manifold endowed with a Riemannian metric $\langle\, ,\, \rangle$ and denote by ∇ the associated Levi-Civita connection. The latter is the only torsion-free connection on the tangent bundle TM of the manifold compatible with the metric, that is, it satisfies the conditions

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

and

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{1.1}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Here and elsewhere $\mathfrak{X}(M)$ stands for the set of smooth local vector fields of M^n. The set of smooth local sections of a more general vector bundle E over M^n is denoted by $\Gamma(E)$. The *curvature tensor* of M^n is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $X, Y, Z \in \mathfrak{X}(M)$. The *Ricci tensor* is defined by

$$\mathrm{Ric}(X, Y) = \mathrm{tr}\,(Z \to R(Z, X)Y),$$

where $X, Y \in \mathfrak{X}(M)$ and tr denotes taking the trace. The *Ricci curvature* in the direction of a unit vector field $X \in \mathfrak{X}(M)$ is given by

$$\mathrm{Ric}(X) = \frac{1}{n-1}\mathrm{Ric}(X, X).$$

1.1 Isometric immersions

A smooth map $f\colon M^n \to \tilde{M}^m$ between two differentiable manifolds is called an *immersion* if the differential $f_*\colon T_xM \to T_{f(x)}\tilde{M}$ is injective for all $x \in M^n$. Usually $f(M)$, or just f for simplicity, is referred to as a submanifold of $\tilde{M}^m$. The manifold $\tilde{M}^m$ is called the *ambient space* and $m-n$ the *codimension* of f.

An immersion $f\colon M^n \to \tilde{M}^m$ between Riemannian manifolds is said to be an *isometric immersion* if the metric induced by f coincides with that of M^n, that is, if

$$\langle f_*X, f_*Y\rangle_{\tilde{M}} = \langle X, Y\rangle_M \tag{1.2}$$

holds for any $X, Y \in \mathfrak{X}(M)$. For simplicity of notation, in the sequel we drop the subindices of the inner products.

Let $f\colon M^n \to \tilde{M}^m$ be an isometric immersion. The orthogonal complement of f_*T_xM in $T_{f(x)}\tilde{M}$ at $x \in M^n$ is denoted by $N_fM(x)$ and called the normal vector space of f at x. Hence, according to this decomposition the pull-back vector bundle $f^*T\tilde{M}$ decomposes orthogonally as

$$f^*T\tilde{M} = f_*TM \oplus N_fM,$$

where N_fM is called the *normal bundle*. The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}^m$ induces a connection on $f^*T\tilde{M}$ which, for simplicity, is also denoted by $\tilde{\nabla}$. Given $X, Y \in \mathfrak{X}(M)$ and taking the tangent and normal components of $\tilde{\nabla}_X f_*Y$, we obtain the relation

$$\tilde{\nabla}_X f_*Y = f_*\nabla_X Y + \alpha(X, Y)$$

known as the *Gauss formula*.

The map $\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(N_fM)$ above is called the *second fundamental form* of f. Since $[f_*X, f_*Y] = f_*[X, Y]$ and both $\tilde{\nabla}$ and ∇ satisfy (1.1), then we have that α is symmetric. It is easily seen that α is C^∞-bilinear, hence it can be regarded as a symmetric tensor, namely, that $\alpha \in \mathrm{Hom}^2(TM, TM; N_fM)$.

Fix $x \in M^n$ and let $\xi \in N_fM(x)$, then the *shape operator* A_ξ of f at x in the direction of ξ is defined by

$$\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle$$

for any $X, Y \in T_xM$. Notice that A_ξ is symmetric. Hence, any $\xi \in \Gamma(N_fM)$ determines a symmetric endomorphism A_ξ of TM. If $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_fM)$, then taking the derivative of $\langle f_*Y, \xi\rangle = 0$ in the direction of $X \in \mathfrak{X}(M)$ gives

$$\langle A_\xi X, Y\rangle = -\langle f_*Y, \tilde{\nabla}_X \xi\rangle.$$

Thus $-A_\xi X$ is the tangent component of $\tilde{\nabla}_X \xi$. The normal component of $\tilde{\nabla}_X \xi$ for $\xi \in \Gamma(N_f M)$ determines a connection on $N_f M$ compatible with the induced metric from $\tilde{M}^m$. Denoted by $\nabla^\perp$ this Riemannian connection is called the *normal connection* of f. Then, we have the *Weingarten formula* given by

$$\tilde{\nabla}_X \xi = -f_* A_\xi X + \nabla^\perp_X \xi$$

for any $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_f M)$.

Finally, the *normal curvature tensor* is the curvature tensor of the normal connection and thus given by

$$R^\perp(X,Y)\xi = \nabla^\perp_X \nabla^\perp_Y \xi - \nabla^\perp_Y \nabla^\perp_X \xi - \nabla^\perp_{[X,Y]}\xi,$$

where $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_f M)$.

1.2 The fundamental equations

Given an isometric immersion $f\colon M^n \to \tilde{M}^m$, then comparing the curvature tensors of both manifolds yields a set of three equations called the fundamental equations of the immersion. In fact, if the ambient space possesses constant sectional curvature then these are the compatibility equations of an isometric immersion, namely, of equation (1.2).

Using the Gauss and Weingarten formulas gives

$$\begin{aligned}\tilde{\nabla}_X \tilde{\nabla}_Y f_* Z &= \tilde{\nabla}_X f_* \nabla_Y Z + \tilde{\nabla}_X \alpha(Y,Z)\\ &= f_*(\nabla_X \nabla_Y Z - A_{\alpha(Y,Z)} X) + \alpha(X, \nabla_Y Z) + \nabla^\perp_X \alpha(Y,Z)\end{aligned} \tag{1.3}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Let $\tilde{R}$ and R denote the curvature tensors of $\tilde{M}^m$ and M^n respectively. Taking the tangent component of $\tilde{R}(X,Y)f_*Z$ and using (1.3) yields

$$\begin{aligned}(\tilde{R}&(X,Y)f_*Z)_{f_*TM}\\ &= (\tilde{\nabla}_X \tilde{\nabla}_Y f_* Z - \tilde{\nabla}_Y \tilde{\nabla}_X f_* Z - \tilde{\nabla}_{[X,Y]} f_* Z)_{f_*TM}\\ &= f_*(\nabla_X \nabla_Y Z - A_{\alpha(Y,Z)} X - \nabla_Y \nabla_X Z + A_{\alpha(X,Z)} Y - \nabla_{[X,Y]} Z)\\ &= f_*(R(X,Y)Z - A_{\alpha(Y,Z)} X + A_{\alpha(X,Z)} Y)\end{aligned}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. The *Gauss equation* is obtained taking the inner product of both sides of the proceeding equation with $W \in \mathfrak{X}(M)$, that is,

$$\langle R(X,Y)Z, W\rangle = \langle \tilde{R}(X,Y)Z, W\rangle + \langle \alpha(X,W), \alpha(Y,Z)\rangle - \langle \alpha(X,Z), \alpha(Y,W)\rangle,$$

where f_*W and W have been identified for simplicity.

Computing the normal component of $\tilde{R}(X,Y)Z$ yields the *Codazzi equation*

$$(\tilde{R}(X,Y)Z)^\perp = (\nabla^\perp_X \alpha)(Y,Z) - (\nabla^\perp_Y \alpha)(X,Z),$$

where

$$(\nabla^\perp_X\alpha)(Y,Z)=\nabla^\perp_X\alpha(Y,Z)-\alpha(\nabla_XY,Z)-\alpha(Y,\nabla_XZ)$$

is the covariant derivative of the second fundamental form.

Taking the normal component of $\tilde{R}(X,Y)\xi$ for $\xi\in\Gamma(N_fM)$ and using the Gauss and Weingarten formulas gives

$$(\tilde{R}(X,Y)\xi)^\perp=R^\perp(X,Y)\xi-\alpha(X,A_\xi Y)+\alpha(A_\xi X,Y).$$

This equation is known as the *Ricci equation*. After taking the inner product with $\eta\in N_fM$ it takes the form

$$\langle R^\perp(X,Y)\xi,\eta\rangle=\langle\tilde{R}(X,Y)\xi,\eta\rangle+\langle[A_\xi,A_\eta]X,Y\rangle,$$

where $[\ ,\ ]$ stands for the commutator of operators.

Next we focus on immersions $f\colon M^n\to\mathbb{Q}_c^m$, where $\mathbb{Q}_c^m$ denotes a simply connected complete *space form* with sectional curvature c, that is, the Euclidean space $\mathbb{R}^m$, the sphere $\mathbb{S}_c^m$ or the hyperbolic space $\mathbb{H}_c^m$, according to whether $c=0$, $c>0$ or $c<0$, respectively. The ambient space is endowed with the usual metric given by the inner product denoted by $\langle\ ,\ \rangle$. In fact, we also use $\langle\ ,\ \rangle$ for the metric induced by f on M^n. In this case, the fundamental equations take the following forms:

The Gauss equation

$$\langle R(X,Y)Z,W\rangle=c\langle(X\wedge Y)Z,W\rangle+\langle\alpha(X,W),\alpha(Y,Z)\rangle-\langle\alpha(X,Z),\alpha(Y,W)\rangle,\tag{1.4}$$

or equivalently

$$R(X,Y)Z=c(X\wedge Y)Z+A_{\alpha(Y,Z)}X-A_{\alpha(X,Z)}Y.$$

The Codazzi equation

$$(\nabla^\perp_X\alpha)(Y,Z)=(\nabla^\perp_Y\alpha)(X,Z).\tag{1.5}$$

The Ricci equation

$$\langle R^\perp(X,Y)\xi,\eta\rangle=\langle[A_\xi,A_\eta]X,Y\rangle.\tag{1.6}$$

The fundamental equations for a hypersurface $f\colon M^n\to\mathbb{Q}_c^{n+1}$ have a simpler form. Let $N\in\Gamma(N_fM)$ be a (local) unit normal vector field. If $c=0$, we can also regard N as the smooth map $N\colon M^n\to\mathbb{S}_1^n$ called the *Gauss map* of f. Associated to N we have the shape operator A_N, which we just denote by A. In this case, we also call A the *second fundamental form* of f. Since the Ricci equation in this case is clearly trivial, then the fundamental equations written in terms of A are as follows:

The Gauss equation

$$R(X,Y)Z = c(X \wedge Y)Z + (AX \wedge AY)Z.$$

The Codazzi equation

$$(\nabla_X A)Y = (\nabla_Y A)X.$$

The Fundamental Theorem of submanifolds is only stated while a proof can be seen in [21].

Theorem 1.1. Existence*: Let M^n be a simply connected Riemannian manifold, let $\mathcal{E}$ be a Riemannian vector bundle of rank p over M^n with compatible connection $\nabla^{\mathcal{E}}$ and curvature tensor $R^{\mathcal{E}}$, and let $\alpha^{\mathcal{E}} \in \Gamma(Hom^2(TM,TM;\mathcal{E}))$ be a symmetric tensor. For each $\xi \in \Gamma(\mathcal{E})$, define $A^{\mathcal{E}}_\xi \in \Gamma(End(TM))$ by*

$$\langle A^{\mathcal{E}}_\xi X, Y\rangle = \langle \alpha^{\mathcal{E}}(X,Y), \xi\rangle.$$

If $(\nabla^{\mathcal{E}}, \alpha^{\mathcal{E}}, A^{\mathcal{E}}_\xi)$ satisfies the Gauss, Codazzi and Ricci equations, then there exist an isometric immersion $f\colon M^n \to \mathbb{Q}_c^{n+p}$ and a vector bundle isometry $\phi\colon \mathcal{E} \to N_f M$ such that $\nabla^\perp \phi = \phi \nabla^{\mathcal{E}}$ and $\alpha^f = \phi \circ \alpha^{\mathcal{E}}$.

Uniqueness: *Let $f, g\colon M^n \to \mathbb{Q}_c^{m+p}$ be isometric immersions of a Riemannian manifold M^n. Assume that there is a vector bundle isometry $\phi\colon N_f M \to N_g M$ such that $\phi^f \nabla^\perp =^g \nabla^\perp \phi$ and $\phi \circ \alpha^f = \alpha^g$. Then there is an isometry $\tau\colon \mathbb{Q}_c^{n+p} \to \mathbb{Q}_c^{n+p}$ such that $\tau \circ f = g$ and $\tau_*|_{N_f M} = \phi$.*

In the case of hypersurfaces the above result is as follows.

Theorem 1.2. Existence: *Let M^n be a simply connected Riemannian manifold, and let $A \in \Gamma(End(TM))$ be a symmetric tensor satisfying the Gauss and Codazzi equations. Then there exist an isometric immersion $f\colon M^n \to \mathbb{Q}_c^{n+1}$and a unit normal vector field N such that A coincides with the shape operator A_N of f with respect to N.*

Uniqueness: *Let $f, g\colon M^n \to \mathbb{Q}_c^{n+1}$ be isometric immersions of an orientable Riemannian manifold M^n with, respectively, unit normal vector fields N^f and N^g. If the corresponding shape operators satisfy $A^f = \pm A^g$, then there exists an isometry $\tau\colon \mathbb{Q}_c^{n+1} \to \mathbb{Q}_c^{n+1}$ such that $\tau \circ f = g$ and $\tau_* N^f = \pm N^g$.*

Remark 1.3. The compatibility conditions (1.4), (1.5) and (1.6) also hold when we consider an isometric immersion $f\colon M^n \to \mathbb{L}^m$ of a Riemannian manifold M^n into the standard flat Lorentzian space form $\mathbb{L}^m$. Moreover, Theorem 1.1 also holds if we let $\mathcal{E}$ be a semi-Riemannian vector bundle over M^n. More precisely, if the metric on $\mathcal{E}$ is Lorentzian then Theorem 1.1 holds with $\mathbb{L}^{n+p}$ as ambient space.

1.3 The relative nullity

An isometric immersion $f\colon M^n \to \tilde{M}^m$ is said to be totally geodesic at $x \in M^n$ if its second fundamental form α at x vanishes. The submanifold f is called *totally geodesic* if it is totally geodesic at every point, that is, if α is identically zero. An isometric immersion $f\colon M^n \to \tilde{M}^m$ is said to be umbilical at $x \in M^n$ if there is $\eta \in N_f M(x)$ such that the second fundamental form satisfies

$$\alpha(X,Y)(x) = \langle X,Y\rangle\eta$$

for any $X, Y \in T_xM$. Then a submanifold is said to be *umbilical* if it is umbilical at every point. Of course, totally geodesic submanifolds are also umbilical.

The totally geodesic (respectively, umbilical) submanifolds of $\mathbb{R}^m$ are open subsets of affine subspaces (respectively, round spheres). Regarding the sphere $\mathbb{S}_c^m$ as a hypersurface of $\mathbb{R}^{m+1}$, its totally geodesic (respectively, umbilical) submanifolds are open subsets of the intersections of $\mathbb{S}_c^m$ with linear (respectively, affine) subspaces of $\mathbb{R}^{m+1}$, and similarly for the hyperbolic space $\mathbb{H}_c^m$ seen as a hypersurface of the Lorentzian space $\mathbb{L}^{m+1}$.

Given a symmetric bilinear form $\gamma\colon V \times V \to W$, where V and W are finite dimensional real vector spaces, the nullity subspace $\mathcal{N}(\gamma) \subset V$ of γ is

$$\mathcal{N}(\gamma) = \{X \in V : \gamma(X,Y) = 0 \ \text{ for all } \ Y \in V\}.$$

The *relative nullity* subspace $\Delta(x) \subset T_xM$ at $x \in M^n$ of an isometric immersion $f\colon M^n \to \tilde{M}^m$ is $\Delta(x) = \mathcal{N}(\alpha)(x)$. The dimension $\nu(x)$ of $\Delta(x)$ is called the *index of relative nullity* of f at x.

A smooth distribution $E \subset TM$ on a Riemannian manifold M^n is said to be *totally geodesic* if $\nabla_T S \in \Gamma(E)$ for any $S, T \in \Gamma(E)$.

Proposition 1.4. *Let $f\colon M^n \to \mathbb{Q}_c^m$ be an isometric immersion. Then the index of relative nullity is ν upper semi-continuous. In particular, the subset*

$$M_0 = \{x \in M^n \colon \nu(x) = \nu_0\},$$

where ν attains its minimum value ν_0 is open. Moreover, on any open subset $U \subset M^n$ where ν is constant $\Delta(x)$ determines a smooth totally geodesic distribution. Thus Δ is integrable on U and the restriction of f to each leaf is a totally geodesic submanifold.

Proof. This is Exercise 1.1. □

The leaves of the relative nullity distribution on an open subset $U \subset M^n$ where the index of relative nullity $\nu > 0$ is constant form the *relative nullity foliation* of U.

For an isometric immersion $f\colon M^n \to \mathbb{R}^m$, a vector $\eta \in N_fM(x)$ is called a principal normal of f at $x \in M^n$ if the subspace

$$E_\eta(x) = \{T \in T_xM\colon \alpha(T,X) = \langle T,X\rangle\eta \ \text{ for all } \ X \in T_xM\}$$

is nontrivial. A normal vector field $\eta \in \Gamma(N_fM)$ is called a *principal normal vector field* of f with multiplicity $q > 0$ if $E_\eta(x)$ has dimension q at any point $x \in M^n$. In particular, if $f\colon M^n \to \mathbb{R}^{n+1}$ is a hypersurface with Gauss map N, then a normal vector field $\eta(x) = \lambda(x)N(x)$ is a principal normal at x if and only if $\lambda(x)$ is a *principal curvature* of f at x, that is, if and only if $\lambda(x)$ is an eigenvalue of $A(x)$.

A smooth distribution $E \subset TM$ of a Riemannian manifold is called *umbilical* if there exists a smooth section $\delta \in \Gamma(E^\perp)$ such that

$$\langle \nabla_T S, X\rangle = \langle T,S\rangle\langle \delta, X\rangle$$

for all $T,S \in \Gamma(E)$ and $X \in \Gamma(E^\perp)$. It follows that an umbilical distribution is integrable and its leaves are umbilical submanifolds of M^n. For an umbilical distribution E of M^n, if the vector field $\delta \in \Gamma(E^\perp)$ as above satisfies that

$$(\nabla_T \delta)_{E^\perp} = 0$$

for all $T \in \Gamma(E)$ then we say that E is a *spherical distribution.*

Proposition 1.5. *Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion with a principal normal vector field η of multiplicity q. Then E_η is a smooth distribution. Moreover, if $q \geq 2$ then E_η is a spherical distribution and the restriction of f to each leaf is an umbilical submanifold of $\mathbb{R}^m$.*

Proof. See Exercise 1.2. □

1.4 The splitting tensor

Let $D \subset TM$ denote a tangent smooth distribution of a Riemannian manifold M^n. Then the tangent bundle splits orthogonally as $TM = D \oplus D^\perp$ and any tangent vector field $X \in \mathfrak{X}(M)$ decomposes accordingly as

$$X = X_D + X_{D^\perp}.$$

The *splitting tensor* $C\colon \Gamma(D) \times \Gamma(D^\perp) \to \Gamma(D^\perp)$ of D is defined by

$$C(T,X) = C_T X = -(\nabla_X T)_{D^\perp}.$$

Clearly the tensor is $C^\infty(M)$-linear with respect to the second variable. This is also the case for the first variable since

$$(\nabla_X \varphi T)_{D^\perp} = \varphi(\nabla_X T)_{D^\perp}$$

for any $\varphi \in \mathbb{C}^\infty(M)$. Thus the value of $C_T X(x)$ only depends on the values of T and X at $x \in M^n$. Any $T \in D(x)$ determines an endomorphism $C_T \colon D^\perp(x) \to D^\perp(x)$ called the splitting tensor of D at $x \in M^n$ with respect to T.

The splitting tensor encodes information of the distribution $D^\perp$. For instance $D^\perp$ is integrable if and only if C_T is self-adjoint for any $T \in \Gamma(D)$, in which case C_T is the shape operator with respect to T of the inclusion of the leaves of $D^\perp$. Then the distribution $\mathcal{D}^\perp$ is totally geodesic if and only if C vanishes. More generally, $D^\perp$ is umbilical if and only if there is $S \in \Gamma(D)$ such that

$$C_T = \langle T, S\rangle I$$

for any $T \in \Gamma(D)$. The proofs of these facts are left as Exercise 1.3.

We now focus on the properties of the splitting tensor of the relative nullity distribution Δ of an isometric immersion $f \colon M^n \to \mathbb{R}^m$. In fact, we consider the slightly more general case of a totally geodesic distribution $D \subset \Delta$ of M^n with splitting tensor C. In the sequel ∇^h stands for the induced connection on $D^\perp$ and $\nabla_T C_S$ denotes the tensor

$$(\nabla_T C_S)X = (\nabla_T C_S X)_{D^\perp} - C_S(\nabla_T X)_{D^\perp},$$

where $S, T \in \Gamma(D)$ and $X \in \Gamma(D^\perp)$.

Proposition 1.6. *The splitting tensor of $D \subset \Delta$ satisfies the equations:*

$$\nabla_T C_S = C_S C_T + C_{\nabla_T S}, \tag{1.7}$$

$$(\nabla^h_X C_T)Y - (\nabla^h_Y C_T)X = C_{(\nabla_X T)_D} Y - C_{(\nabla_Y T)_D} X \tag{1.8}$$

and

$$\nabla_T A_\xi = A_\xi C_T + A_{\nabla^\perp_T \xi} \tag{1.9}$$

for any $S, T \in \Gamma(D)$, $X, Y \in \Gamma(D^\perp)$ and $\xi \in \Gamma(N_f M)$. In particular, we have that

$$\frac{D}{dt} C_{\gamma'} = C^2_{\gamma'}, \tag{1.10}$$

where $\gamma = \gamma(t)$ is a unit speed geodesic contained in a leaf of D.

Proof. See Exercise 1.4. □

Proposition 1.7. *Given an isometric immersion $f \colon M^n \to \mathbb{R}^m$ and a smooth symmetric bilinear form $\beta \colon TM \times TM \to N_f M$ assume that $\Delta^*(x) = \Delta \cap \mathcal{N}(\beta)(x)$ has constant dimension $\nu^* > 0$ on an open subset $U \subset M^n$. Suppose further that on U the smooth distribution Δ^* is totally geodesic with splitting tensor C and that*

$$(\nabla^\perp_X \beta)(Y, Z) = (\nabla^\perp_Y \beta)(X, Z) \tag{1.11}$$

holds for any $X \in \Gamma(\Delta^*)$ *and* $Y, Z \in \mathfrak{X}(M)$. *If* $\gamma\colon [0,b] \to M^n$ *is a unit speed geodesic such that* $\gamma([0,b))$ *is contained in a leaf of* Δ^* *in* U, *then* $\Delta^*(\gamma(b)) = \mathcal{P}_0^b(\Delta^*(\gamma(0)))$ *where* $\mathcal{P}_0^t$ *is the parallel transport along* γ *from* $\gamma(0)$ *to* $\gamma(t)$. *In particular, we have that* $\nu^*(\gamma(b)) = \nu^*(\gamma(0))$ *and the tensor* $C_{\gamma'}$ *extends smoothly to* $[0,b]$.

Proof. Let E be given by the orthogonal decomposition $TM = \Delta^* \oplus E$. Define the tensor $J\colon E \to E$ as the solution in $[0,b)$ of

$$\frac{D}{dt}J + C_{\gamma'} \circ J = 0$$

with initial condition $J(0) = I$. In fact, if we take the parallel transport along γ of an orthonormal basis of $E(\gamma(0))$, the previous equation can be seen as an ordinary differential matrix equation. We have from (1.10) that $D^2J/dt^2 = 0$, and hence J extends smoothly to $\mathcal{P}_0^b(E(0))$ in $\gamma(b)$. Let Y and Z be parallel vector fields along γ such that $Y(t) \in E(t)$ for each $t \in [0,b)$. Since $\gamma' \in \Delta^*$, it follows from (1.11) and the definition of J that

$$\begin{aligned}
\nabla^\perp_{\gamma'}\beta(JY,Z) &= (\nabla^\perp_{\gamma'}\beta)(JY,Z) + \beta(DJY/dt, Z)\\
&= (\nabla^\perp_{JY}\beta)(\gamma',Z) + \beta(DJY/dt, Z)\\
&= \beta(C_{\gamma'}JY + DJY/dt, Z)\\
&= 0.
\end{aligned}$$

Thus $\beta(JY,Z)$ and $\alpha(JY,Z)$ are parallel along γ. In particular J is invertible in $[0,b]$. By continuity $\mathcal{P}_0^b(\Delta^*(\gamma(0))) \subset \Delta^*(\gamma(b))$, and then $\mathcal{P}_0^b(\Delta^*(\gamma(0))) = \Delta^*(\gamma(b))$. Finally the tensor $C_{\gamma'}$ extends to $[0,b]$ as $C_{\gamma'} = -DJ/dt \circ J^{-1}$. $\square$

If the leaves of the relative nullity foliation are complete manifolds we have the following result.

Proposition 1.8. *Let* $f\colon M^n \to \mathbb{R}^{n+p}$ *be an isometric immersion. Assume that* $U \subset M^n$ *is an open subset where the index of relative nullity* $\nu(x) = \nu_0$ *is constant and the relative nullity leaves are complete. Then, for any* $x_0 \in U$ *and* $T_0 \in \Delta(x_0)$ *the only possible real eigenvalue of* C_{T_0} *is zero. Moreover, if* $\gamma(t)$ *is a geodesic through* x_0 *tangent to* T_0 *then*

$$C_{\gamma'(t)} = \mathcal{P}_0^t C_{T_0}(I - tC_{T_0})^{-1}(\mathcal{P}_0^t)^{-1},$$

where $\mathcal{P}_0^t$ *is the parallel transport along* γ *from* x_0. *In particular,* $\ker C_{\gamma'}$ *is parallel along* γ.

Proof. This is Exercise 1.5. $\square$

An isometric immersion $g\colon M^n \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^k$ of a Riemannian product manifold $M^n \times \mathbb{R}^k$ is called a *k-cylinder* (or just a *cylinder*) over the isometric immersion $f\colon M^n \to \mathbb{R}^m$ if it factors as

$$g = f \times I\colon M^n \times \mathbb{R}^k \to \mathbb{R}^{m+k},$$

where $I\colon \mathbb{R}^k \to \mathbb{R}^k$ is the identity map.

Cylinders are the simplest examples of Euclidean submanifolds carrying a totally geodesic distribution contained in the relative nullity subspaces. In fact, we have that $\{x\} \times \mathbb{R}^k$ is contained in the relative nullity subspace of g at $(x, y) \in M^n \times \mathbb{R}^k$.

Proposition 1.9. *Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion and let D be a totally geodesic tangent distribution of rank k such that $D \subset \Delta$. If the splitting tensor of D vanishes then f is locally a k-cylinder over an isometric immersion $g\colon L^{n-k} \to \mathbb{R}^{m-k}$.*

Proof. We have from Exercise 1.3 that the distribution $D^\perp$ is totally geodesic. In particular, it is integrable. Since $D \subset \Delta$, we have that f_*D is a constant subspace along M^n in $\mathbb{R}^m$. Let $i\colon L^{n-k} \to M^n$ be the inclusion of a leaf of $D^\perp$. Since f_*D is constant, then $g = f \circ i$ satisfies $g(L) \subset \mathbb{R}^{m-k}$ where $\mathbb{R}^{m-k} = (f_*D)^\perp$, and hence f coincides locally with the k-cylinder over g. ☐

1.5 Flat bilinear forms

Flat bilinear forms were introduced by J. D. Moore [31] after the pioneering work of E. Cartan as a tool to deal with rigidity questions on isometric immersions in space forms. In fact, they are also very helpful in the study of similar questions for infinitesimal variations of submanifolds.

Let V^n and U^m denote finite dimensional real vector spaces and let $W^{p,q}$ be a real vector space of dimension $p+q$ endowed with an indefinite inner product $\langle\, ,\, \rangle$ of *signature* (p, q). This means that p (respectively, q) is the maximal dimension of a subspace of $W^{p,q}$ restricted to which the inner product is positive definite (respectively, negative definite).

Let $\gamma\colon V^n \times U^m \to W^{p,q}$ be a bilinear form. An element $X \in V^n$ is called a (left) *regular element* of γ if

$$\dim \gamma_X(U) = \max\{\dim \gamma_Y(U)\colon Y \in V^n\},$$

where $\gamma_X(Y) = \gamma(X, Y)$ for any $Y \in U^m$. It is easy to see that the set $RE(\gamma)$ of regular elements of γ is open and dense in V^n.

A bilinear form $\gamma\colon V^n \times U^m \to W^{p,q}$ is said to be *flat* if

$$\langle\gamma(X, Z), \gamma(Y, W)\rangle - \langle\gamma(X, W), \gamma(Y, Z)\rangle = 0$$

for all $X,Y \in V^n$ and $W,Z \in U^m$. The bilinear form γ is called *null* if

$$\langle \gamma(X,Z), \gamma(Y,W) \rangle = 0$$

for all $X,Y \in V^n$ and $W,Z \in U^m$. Thus null bilinear forms are trivially flat.

The following basic fact was observed by Moore [31].

Proposition 1.10. *Let $\gamma\colon V^n \times U^m \to W^{p,q}$ be a flat bilinear form. If $X \in RE(\gamma)$ then*

$$\gamma(Y, \ker \gamma_X) \subset \gamma_X(U) \cap \gamma_X(U)^\perp$$

for any $Y \in V^n$.

If $\gamma\colon V^n \times V^n \to W^{p,q}$ is a bilinear form its image is the subspace $\mathcal{S}(\gamma) \subset W^{p,q}$ given by

$$\mathcal{S}(\gamma) = \operatorname{span}\{\gamma(X,Y) : X, Y \in V^n\}.$$

We conclude this chapter with a fundamental result in the theory of symmetric flat bilinear forms. It turns out to be false for $p \geq 6$, as shown in [11] by means of a counterexample.

Theorem 1.11. *Let $\gamma\colon V^n \times V^n \to W^{p,q}$, $1 \leq p \leq 5$ and $p+q<n$, be a symmetric flat bilinear form. If $\dim \mathcal{N}(\gamma) \leq n-p-q-1$ there is an orthogonal decomposition*

$$W^{p,q} = W_1^{\ell,\ell} \oplus W_2^{p-\ell,q-\ell},\ 1 \leq \ell \leq p,$$

such that the W_j-components γ_j of γ satisfy:

(i) γ_1 is nonzero but is null since $\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma) \cap \mathcal{S}(\gamma)^\perp$.

(ii) γ_2 is flat and $\dim \mathcal{N}(\gamma_2) \geq n-p-q+2\ell$.

Proof. See Theorem 3 in [10] or Lemma 4.22 in [21]. □

1.6 Exercises

Exercise 1.1. Prove Proposition 1.4.

Hint: Given $x \in M^n$ observe that

$$\Delta^\perp(x) = \operatorname{span}\{A_\xi X : X \in T_xM, \xi \in N_fM(x)\}.$$

Let $\{X_i\}_{1 \leq i \leq n-\nu} \in T_xM$ and $\{\xi_i\}_{1 \leq i \leq n-\nu} \in N_fM(x)$ be such that the set of vectors $\{A_{\xi_i} X_i\}_{1 \leq i \leq n-\nu}$ span $\Delta^\perp(x)$. Then the smooth extensions of these vectors on a small neighborhood of x are linearly independent vector fields. Use this to prove the first assertions whereas for last one use the Codazzi equation.

Exercise 1.2. Prove Proposition 1.5. If the hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ has a principal curvature λ of multiplicity $q \geq 2$ conclude that λ is constant along the spherical leaves.

Hint: That E_η is smooth follows from similar arguments as in Exercise 1.1. Set $\eta = \lambda\zeta$ where $\zeta \in \Gamma(N_fM)$ has unit length. Use the assumption that $q \geq 2$ and the Codazzi equation to prove that $T(\lambda) = 0$ and $\nabla^\perp_T\zeta = 0$ for $T \in \Gamma(E_\eta)$. Now use the Codazzi to show that

$$(A_\zeta - \lambda I)\nabla_T S = -\langle T, S\rangle \operatorname{grad}\lambda \;\text{ and }\; \langle A_\xi \nabla_T S, X\rangle = \lambda\langle T, S\rangle\langle \nabla^\perp_X \xi, \zeta\rangle$$

for any $S, T \in \Gamma(E_\eta)$, $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_fM)$ with $\langle \xi, \eta\rangle = 0$. From this and the definition of E_η conclude that E_η is an umbilical distribution. In fact, show that

$$\langle \nabla_T S, X\rangle = \langle T, S\rangle\langle \delta, X\rangle \tag{1.12}$$

for all $T, S \in \Gamma(E_\eta)$ and $X \in E^\perp_\eta$ where δ satisfies

$$(A_\zeta - \lambda I)\delta = -\operatorname{grad}\lambda \;\text{ and }\; \langle A_\xi\delta, X\rangle = \lambda\langle \nabla^\perp_X \xi, \zeta\rangle.$$

Now use the above equations together with the Codazzi and Ricci equations to prove that $\nabla_T\delta \in \Gamma(E_\eta)$ for any $\mathcal{T} \in \Gamma(\eta)$, and thus that E_η is spherical. Finally, from the definition of E_η and (1.12) see that the restriction of f to a leaf of E_η is an umbilical submanifold of $\mathbb{R}^m$.

Exercise 1.3. Let $D \subset TM$ be a tangent distribution of a Riemannian manifold M^n and let C be its splitting tensor.

(i) Prove that $D^\perp$ is integrable if and only if C_T is self-adjoint for any $T \in \Gamma(D)$.

(ii) Show that $D^\perp$ is umbilical if and only if there is $S \in \Gamma(D)$ such that $C_T = \langle T, S\rangle I$ for any $T \in \Gamma(D)$. Conclude that $D^\perp$ is totally geodesic if and only if C vanishes.

Exercise 1.4. Prove Proposition 1.6.

Hint: Equations (1.7) and (1.8) follow from the facts that D is totally geodesic, $D \subset \Delta$ and the Gauss equation. As for (1.9) use the Codazzi equation. See Propositions 7.1, 7.2 and 7.3 in [21].

Exercise 1.5. Prove Proposition 1.8.

Hint: Assume on the contrary that $C_0 = C_{T_0}$ has nonzero real eigenvalues λ_i, $1 \leq i \leq k$, and let $\tau^{-1} = \max_i |\lambda_i|$. Then $I - tC_0$ is invertible for $-\tau < t < \tau$. Show that

$$C_t = \mathcal{P}^t_0 C_{T_0}(I - tC_{T_0})^{-1}(\mathcal{P}^t_0)^{-1}$$

solves the equation

$$\frac{D}{dt}C_t = C_t^2$$

with initial condition C_0 at $t = 0$. Use (1.10) to show that $C_{\gamma'(t)}$ coincides with C_t. Observe that $(\tau - t)^{-1}$ or $-(\tau + t)^{-1}$ is an eigenvalue of $C_{\gamma'(t)}$ which diverges as t tends to τ or $-\tau$ respectively. Reach to a contradiction with the fact that $C_{\gamma'(t)}$ is well defined for all $t \in \mathbb{R}$ from the completeness assumption. See also Proposition 13.8 in [21].

Chapter 2

Infinitesimal variations

The first part of this chapter is devoted to introduce the notion of an infinitesimal variation of an Euclidean submanifold and to establish a Fundamental Theorem for that class of variations. In the theory of isometric immersions, the so-called Fundamental Theorem, discussed in Chapter 1, shows that the Gauss-Codazzi-Ricci equations are the integrability conditions for the system of differential equations that gave the existence of an isometric immersion of a given Riemannian manifold into Euclidean space. A similar result is given here for the class of infinitesimal variations. In fact, it is shown that a system of three equations involving two tensors are the integrability conditions for the equations that determine the infinitesimal variations, and that in a certain sense there is uniqueness.

The second part of the chapter deals with the rigidity problem for submanifolds in low codimension. It is shown that certain conditions on the second fundamental form of the submanifold, that are well-known to yield isometric rigidity in the usual sense, also give rigidity for infinitesimal variations.

2.1 Infinitesimal variations

In this section, the notions of infinitesimal variation and infinitesimal bending of an Euclidean submanifold are introduced. Then, it is explained why the study of the infinitesimal variations of a submanifold is done by analyzing the possible infinitesimal bendings.

Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion of a Riemannian manifold into Euclidean space. A *smooth variation* of f is a smooth map $\mathcal{F}\colon I \times M^n \to \mathbb{R}^m$, where $0 \in I \subset \mathbb{R}$ is an open interval, such that $f_t = \mathcal{F}(t,\cdot)\colon M^n \to \mathbb{R}^m$ is an immersion for any $t \in I$ and $f_0 = f$. The *variational vector field* of a variation $\mathcal{F}$ of f is the section $\mathcal{T} \in \Gamma(f^*T\mathbb{R}^m)$ defined as

$$\mathcal{T} = \mathcal{F}_* \partial/\partial t|_{t=0} = \tilde{\nabla}_{\partial/\partial t} f_t|_{t=0}.$$

Let X, Y still denote the extensions in a trivial way of vector fields $X, Y \in \mathfrak{X}(M)$ to vector fields in $\mathfrak{X}(I \times M)$. Since

$$[X, \partial/\partial t] = 0 = [Y, \partial/\partial t]$$

holds and the ambient space is flat, we have

$$\begin{aligned} \frac{\partial}{\partial t}\langle f_{t*}X, f_{t*}Y\rangle &= \langle \tilde{\nabla}_{\partial/\partial t}\mathcal{F}_*X, \mathcal{F}_*Y\rangle + \langle \mathcal{F}_*X, \tilde{\nabla}_{\partial/\partial t}\mathcal{F}_*Y\rangle \\ &= \langle \tilde{\nabla}_X \mathcal{F}_*\partial/\partial t, \mathcal{F}_*Y\rangle + \langle \mathcal{F}_*X, \tilde{\nabla}_Y \mathcal{F}_*\partial/\partial t\rangle. \end{aligned} \tag{2.1}$$

An *isometric variation* of $f\colon M^n \to \mathbb{R}^m$ is a smooth variation $\mathcal{F}\colon I \times M^n \to \mathbb{R}^m$ such that $f_t\colon M^n \to \mathbb{R}^m$ is an isometric immersion for any $t \in I$.

Given an isometric variation $\mathcal{F}$ of $f\colon M^n \to \mathbb{R}^m$, we have that

$$\frac{\partial}{\partial t}\langle f_{t*}X, f_{t*}Y\rangle = 0 \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(M)$ and $t \in I$. In particular, it follows from (2.1) that the variational vector field $\mathcal{T}$ of $\mathcal{F}$ satisfies the condition

$$\langle \tilde{\nabla}_X \mathcal{T}, f_*Y\rangle + \langle f_*X, \tilde{\nabla}_Y \mathcal{T}\rangle = 0$$

for any $X, Y \in \mathfrak{X}(M)$.

An isometric variation can be produced by composing an isometric immersion $f\colon M^n \to \mathbb{R}^m$ with a family of isometries of $\mathbb{R}^m$ as follows: Let $C\colon I \to O(m)$ be a smooth family of orthogonal transformations of $\mathbb{R}^m$ and let $v\colon I \to \mathbb{R}^m$ be a smooth map such that $(C(0), v(0)) = (I, 0)$. Then, we define an isometric variation $\mathcal{F}$ of f by

$$\mathcal{F}(t, x) = C(t)f(x) + v(t)$$

for all $(t, x) \in I \times M^n$. Such an $\mathcal{F}$ is called a *trivial isometric variation.*

An infinitesimal variation is the infinitesimal analogue of an isometric variation. In fact, as seen next these are the variations that preserve lengths but just "up to the first order".

A smooth variation $\mathcal{F}\colon I \times M^n \to \mathbb{R}^m$ of an isometric immersion $f\colon M^n \to \mathbb{R}^m$ is called an *infinitesimal variation* if it satisfies the condition

$$\frac{\partial}{\partial t}|_{t=0}\langle f_{t*}X, f_{t*}Y\rangle = 0 \tag{2.3}$$

for any $X, Y \in \mathfrak{X}(M)$.

It is known from classical differential geometry that the convenient approach to study variations is to look at the variational vector fields. That this is the way to proceed in the case of infinitesimal variations is justified in the sequel.

A section $\mathcal{T}$ of $f^*T\mathbb{R}^m$ is called an *infinitesimal bending* of an isometric immersion $f\colon M^n \to \mathbb{R}^m$ if the condition

$$\langle \tilde{\nabla}_X \mathcal{T}, f_* Y \rangle + \langle f_* X, \tilde{\nabla}_Y \mathcal{T} \rangle = 0 \tag{2.4}$$

holds for any tangent vector fields $X, Y \in \mathfrak{X}(M)$.

Since the condition (2.3) gives (2.4), then there is an infinitesimal bending associated to any infinitesimal variation. On the other hand, associated to an infinitesimal bending $\mathcal{T}$ of $f\colon M^n \to \mathbb{R}^m$ we have that the infinitesimal variation $\mathcal{F}\colon \mathbb{R} \times M^n \to \mathbb{R}^m$ given by

$$\mathcal{F}(t, x) = f(x) + t\mathcal{T}(x) \tag{2.5}$$

has variational vector field $\mathcal{T}$. But by no means (2.5) is unique with this property, although it may be seen as the simplest one. In fact, new infinitesimal variations with variational vector field $\mathcal{T}$ are obtained by adding to (2.5) terms of the type $t^k\delta$, $k > 1$, where $\delta \in \Gamma(f^*T\mathbb{R}^m)$ and, maybe, for restricted values of the parameter t.

We say that an infinitesimal bending is trivial if it is induced by a trivial isometric variation. More precisely, a *trivial infinitesimal bending* is the restriction to the submanifold of a Killing vector field of the ambient space. That is, there is a skew-symmetric linear endomorphism $\mathcal{D}$ of $\mathbb{R}^m$ and a vector $w \in \mathbb{R}^m$ such that $\mathcal{T} = \mathcal{D}f + w$. Conversely, given a trivial infinitesimal bending we have that

$$\mathcal{F}(t, x) = e^{t\mathcal{D}} f(x) + tw$$

is a trivial isometric variation of f.

Multiplying an infinitesimal bending by a constant and adding a trivial infinitesimal bending yields a new infinitesimal bending. Since it is not convenient to distinguish between these two bendings, from now on we identify two infinitesimal bendings $\mathcal{T}_1$ and $\mathcal{T}_2$ if there exists $0 \neq c \in \mathbb{R}$ and a trivial infinitesimal bending $\mathcal{T}_0$ such that

$$\mathcal{T}_2 = \mathcal{T}_0 + c\mathcal{T}_1. \tag{2.6}$$

Then (2.5) will be seen as the representative of the class of infinitesimal variations that share a common infinitesimal bending.

To conclude this section we observe that, beside the trivial infinitesimal bendings, there are the following examples of infinitesimal bendings of a rather simple geometric nature.

Examples 2.1. (i) Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion. If $Z \in \mathfrak{X}(M)$ is a Killing vector field of M^n and $\xi \in \Gamma(N_fM)$ satisfies $A_\xi = 0$, then $\mathcal{T} = f_*Z + \xi$ is an infinitesimal bending of f. In particular, if f is contained in an affine subspace, say $f(M) \subset \mathbb{R}^\ell \subset \mathbb{R}^m$, then any vector field η normal to $\mathbb{R}^\ell$ determines an infinitesimal bending of f.

(ii) Given two isometric immersions $f, g\colon M^n \to \mathbb{R}^m$ suppose that the map $h = f + g$ is an immersion. Then the map $\mathcal{T} = f - g$ is an infinitesimal bending of h.

2.2 The associated pair

We show next that an infinitesimal bending $\mathcal{T} \in \Gamma(f^*T\mathbb{R}^m)$ of an isometric immersion $f\colon M^n \to \mathbb{R}^m$ together with its second fundamental form $\alpha\colon TM \times TM \to N_fM$ determine an *associate pair* of tensors $(\beta, \mathcal{E})$ to $\mathcal{T}$, where $\beta\colon TM \times TM \to N_fM$ is symmetric and $\mathcal{E}\colon TM \times N_fM \to N_fM$ satisfies the compatibility condition

$$\langle \mathcal{E}(X,\eta), \xi\rangle + \langle \mathcal{E}(X,\xi), \eta\rangle = 0 \tag{2.7}$$

for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_fM)$.

Let $L \in \Gamma(\mathrm{Hom}(TM, f^*T\mathbb{R}^m))$ be the tensor defined by

$$LX = \tilde\nabla_X \mathcal{T} = \mathcal{T}_*X$$

for any $X \in \mathfrak{X}(M)$. Notice that in terms of this tensor (2.4) acquires the form

$$\langle LX, f_*Y\rangle + \langle f_*X, LY\rangle = 0 \tag{2.8}$$

for any $X, Y \in \mathfrak{X}(M)$. Let $B\colon TM \times TM \to f^*T\mathbb{R}^m$ be the tensor given by

$$B(X,Y) = (\tilde\nabla_X L)Y = \tilde\nabla_X LY - L\nabla_X Y \tag{2.9}$$

for any $X, Y \in \mathfrak{X}(M)$. The flatness of the ambient space and

$$B(X,Y) = \tilde\nabla_X \tilde\nabla_Y \mathcal{T} - \tilde\nabla_{\nabla_X Y}\mathcal{T}$$

yield that B is symmetric. Hence, the tensor $\beta\colon TM \times TM \to N_fM$ defined by

$$\beta(X,Y) = (B(X,Y))_{N_fM}$$

is also symmetric. For later use, associated to a given $\xi \in \Gamma(N_fM)$ we define the symmetric tensor $B_\xi \in \Gamma(\mathrm{End}(TM))$ by

$$\langle B_\xi X, Y\rangle = \langle \beta(X,Y), \xi\rangle$$

for any $X, Y \in \mathfrak{X}(M)$.

Let $\mathcal{Y} \in \Gamma(\mathrm{Hom}(N_fM, TM))$ be given by

$$\langle \mathcal{Y}\eta, X\rangle + \langle \eta, LX\rangle = 0. \tag{2.10}$$

Then, we define the tensor $\mathcal{E}\colon TM \times N_fM \to N_fM$ by

$$\mathcal{E}(X,\eta) = \alpha(X, \mathcal{Y}\eta) + (LA_\eta X)_{N_fM}.$$

Hence, we have

$$\begin{aligned}
\langle \mathcal{E}(X,\eta), \xi\rangle &= \langle \alpha(X,\mathcal{Y}\eta) + LA_\eta X, \xi\rangle \\
&= \langle A_\xi X, \mathcal{Y}\eta\rangle - \langle \mathcal{Y}\xi, A_\eta X\rangle \\
&= -\langle LA_\xi X, \eta\rangle - \langle \alpha(X, \mathcal{Y}\xi), \eta\rangle \\
&= -\langle \mathcal{E}(X,\xi), \eta\rangle,
\end{aligned}$$

and thus the compatibility condition (2.7) is satisfied.

Proposition 2.2. *We have that*

$$B(X,Y) = f_*\mathcal{Y}\alpha(X,Y) + \beta(X,Y) \tag{2.11}$$

for any $X, Y \in \mathfrak{X}(M)$*.*

Proof. We need to show that

$$C(X,Y,Z) = \langle (B - f_*\mathcal{Y}\alpha)(X,Y), f_*Z\rangle$$

vanishes for any $X, Y, Z \in \mathfrak{X}(M)$. Equation (2.8) and its derivative give

$$\begin{aligned}
0 &= \langle \tilde\nabla_Z LX, f_*Y\rangle + \langle LX, \tilde\nabla_Z f_*Y\rangle + \langle \tilde\nabla_Z LY, f_*X\rangle + \langle LY, \tilde\nabla_Z f_*X\rangle \\
&= \langle B(Z,X), f_*Y\rangle + \langle L\nabla_Z X, f_*Y\rangle + \langle LX, f_*\nabla_Z Y + \alpha(Z,Y)\rangle \\
&\quad + \langle B(Z,Y), f_*X\rangle + \langle L\nabla_Z Y, f_*X\rangle + \langle LY, f_*\nabla_Z X + \alpha(Z,X)\rangle \\
&= \langle B(Z,X), f_*Y\rangle + \langle LX, \alpha(Z,Y)\rangle + \langle B(Z,Y), f_*X\rangle + \langle LY, \alpha(Z,X)\rangle \\
&= \langle (B - f_*\mathcal{Y}\alpha)(Z,X), f_*Y\rangle + \langle (B - f_*\mathcal{Y}\alpha)(Z,Y), f_*X\rangle.
\end{aligned}$$

From the symmetry of B and the above, we obtain

$$C(X,Y,Z) = C(Y,X,Z) \quad \text{and} \quad C(Z,X,Y) = -C(Z,Y,X)$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
C(X,Y,Z) &= -C(X,Z,Y) = -C(Z,X,Y) = C(Z,Y,X) \\
&= C(Y,Z,X) = -C(Y,X,Z) = -C(X,Y,Z) \\
&= 0,
\end{aligned}$$

as we wished. $\square$

Remark 2.3. The last manipulation in the above proof is known as the Braid Lemma.

2.3 The fundamental equations

In this section, it is shown that the pair of tensors associated to an infinitesimal bending satisfy a set of three equations that form the *Fundamental system of equations* of an infinitesimal variation. The term fundamental means that they are the integrability condition of the system of differential equations whose solutions yield an infinitesimal bendings, a fact that is proved in the following section.

Proposition 2.4. *The pair* $(\beta, \mathcal{E})$ *associated to an infinitesimal bending* $\mathcal{T}$ *satisfies the following system of three equations:*

$$A_{\beta(Y,Z)}X + B_{\alpha(Y,Z)}X = A_{\beta(X,Z)}Y + B_{\alpha(X,Z)}Y, \tag{2.12}$$

$$(\nabla^\perp_X\beta)(Y,Z) - (\nabla^\perp_Y\beta)(X,Z) = \mathcal{E}(Y,\alpha(X,Z)) - \mathcal{E}(X,\alpha(Y,Z)) \tag{2.13}$$

and

$$\begin{aligned}(\nabla^\perp_X\mathcal{E})(Y,\eta) &- (\nabla^\perp_Y\mathcal{E})(X,\eta)\\ &= \beta(X,A_\eta Y) - \beta(A_\eta X,Y) + \alpha(X,B_\eta Y) - \alpha(B_\eta X,Y)\end{aligned} \tag{2.14}$$

for all $X,Y,Z \in \mathfrak{X}(M)$ *and* $\eta \in \Gamma(N_fM)$*. Moreover, equation* (2.13) *is equivalent to*

$$(\nabla_X B_\eta)Y - (\nabla_Y B_\eta)X - B_{\nabla^\perp_X\eta}Y + B_{\nabla^\perp_Y\eta}X = A_{\mathcal{E}(X,\eta)}Y - A_{\mathcal{E}(Y,\eta)}X \tag{2.15}$$

for all $X,Y,Z \in \mathfrak{X}(M)$ *and* $\eta \in \Gamma(N_fM)$*.*

Proof. We first show that

$$(\tilde\nabla_X\mathcal{Y})\eta = -f_*B_\eta X - LA_\eta X + \mathcal{E}(X,\eta) \tag{2.16}$$

for any $X \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$, where we used the notation

$$(\tilde\nabla_X\mathcal{Y})\eta = \tilde\nabla_X f_*\mathcal{Y}\eta - f_*\mathcal{Y}\nabla^\perp_X\eta.$$

Taking the derivative of (2.10), we have from (2.8) and (2.10) that

$$\begin{aligned}0 &= \langle\tilde\nabla_X f_*\mathcal{Y}\eta, f_*Y\rangle + \langle\mathcal{Y}\eta, \nabla_X Y\rangle + \langle\tilde\nabla_X LY,\eta\rangle + \langle LY,\tilde\nabla_X\eta\rangle\\ &= \langle(\tilde\nabla_X\mathcal{Y})\eta, f_*Y\rangle + \langle B_\eta X, Y\rangle + \langle LA_\eta X, f_*Y\rangle.\end{aligned}$$

Since $\langle f_*\mathcal{Y}\eta,\xi\rangle = 0$, we obtain

$$\begin{aligned}0 &= \langle\tilde\nabla_X f_*\mathcal{Y}\eta,\xi\rangle + \langle f_*\mathcal{Y}\eta,\tilde\nabla_X\xi\rangle = \langle(\tilde\nabla_X\mathcal{Y})\eta,\xi\rangle - \langle\alpha(X,\mathcal{Y}\eta),\xi\rangle\\ &= \langle(\tilde\nabla_X\mathcal{Y})\eta,\xi\rangle + \langle LA_\eta X - \mathcal{E}(X,\eta),\xi\rangle\end{aligned}$$

for any $X \in \mathfrak{X}(M)$ and $\eta,\xi \in \Gamma(N_fM)$, and hence (2.16) follows.

Using

$$(\tilde{\nabla}_X B)(Y,Z) = \tilde{\nabla}_X(\tilde{\nabla}_Y L)Z - (\tilde{\nabla}_{\nabla_X Y} L)Z - (\tilde{\nabla}_Y L)\nabla_X Z \tag{2.17}$$

it is easy to see that

$$(\tilde{\nabla}_X B)(Y,Z) - (\tilde{\nabla}_Y B)(X,Z) = -LR(X,Y)Z \tag{2.18}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. It follows using (2.11) that

$$\langle(\tilde{\nabla}_X B)(Y,Z), f_*W\rangle = \langle(\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) + f_*\mathcal{Y}(\nabla^\perp_X\alpha)(Y,Z) - f_*A_{\beta(Y,Z)}X, f_*W\rangle$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$. Then (2.18) and the Codazzi equation give

$$\langle(\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) - (\tilde{\nabla}_Y \mathcal{Y})\alpha(X,Z), f_*W\rangle = \langle LR(Y,X)Z + A_{\beta(Y,Z)}X - A_{\beta(X,Z)}Y, W\rangle.$$

Now using the Gauss equation, we obtain

$$\begin{aligned}&\langle(\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) - (\tilde{\nabla}_Y \mathcal{Y})\alpha(X,Z), f_*W\rangle\\ &\qquad = \langle LA_{\alpha(X,Z)}Y - LA_{\alpha(Y,Z)}X + A_{\beta(Y,Z)}X - A_{\beta(X,Z)}Y, f_*W\rangle.\end{aligned}$$

On the other hand, it follows from (2.16) that

$$\begin{aligned}&\langle(\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) - (\tilde{\nabla}_Y \mathcal{Y})\alpha(X,Z), f_*W\rangle\\ &\qquad = \langle B_{\alpha(X,Z)}Y + LA_{\alpha(X,Z)}Y - B_{\alpha(Y,Z)}X - LA_{\alpha(Y,Z)}X, f_*W\rangle.\end{aligned}$$

From the last two equations, we obtain

$$\langle B_{\alpha(X,Z)}Y - B_{\alpha(Y,Z)}X, f_*W\rangle = \langle A_{\beta(Y,Z)}X - A_{\beta(X,Z)}Y, W\rangle,$$

and this is (2.12).

From (2.11) and (2.17) we obtain

$$((\tilde{\nabla}_X B)(Y,Z))_{N_fM} = \alpha(X, \mathcal{Y}\alpha(Y,Z)) + (\nabla^\perp_X\beta)(Y,Z).$$

Then, we have from (2.18) and the Gauss equation that

$$\begin{aligned}&(\nabla^\perp_X\beta)(Y,Z) - (\nabla^\perp_Y\beta)(X,Z)\\ &\qquad = (LR(Y,X)Z)_{N_fM} - \alpha(X, \mathcal{Y}\alpha(Y,Z) + \alpha(Y, \mathcal{Y}\alpha(X,Z)\\ &\qquad = (LA_{\alpha(X,Z)}Y - LA_{\alpha(Y,Z)}X)_{N_fM} - \alpha(X, \mathcal{Y}\alpha(Y,Z) + \alpha(Y, \mathcal{Y}\alpha(X,Z),\end{aligned}$$

and this is (2.13). Since $\mathcal{E}$ satisfies the compatibility condition (2.7), then

$$\langle\mathcal{E}(X, \alpha(Y,Z)), \eta\rangle = -\langle A_{\mathcal{E}(X,\eta)}Y, Z\rangle,$$

and this gives (2.15).

We have

$$
\begin{aligned}
(\nabla^\perp_X \mathcal{E})(Y,\eta) &= \nabla^\perp_X \mathcal{E}(Y,\eta) - \mathcal{E}(\nabla_X Y,\eta) - \mathcal{E}(Y,\nabla^\perp_X \eta)\\
&= (\nabla^\perp_X \alpha)(Y,\mathcal{Y}\eta) + (L(\nabla_X A)(Y,\eta))_{N_fM} + \alpha(Y,\nabla_X \mathcal{Y}\eta)\\
&\quad - \alpha(Y,\mathcal{Y}\nabla^\perp_X\eta) - (L\nabla_X A_\eta Y)_{N_fM} + \nabla^\perp_X (LA_\eta Y)_{N_fM}.
\end{aligned}
$$

Then (2.16) gives

$$
\begin{aligned}
(\nabla^\perp_X \mathcal{E})(Y,\eta) = (\nabla^\perp_X \alpha)(Y,\mathcal{Y}\eta) + (L(\nabla_X A)(Y,\eta))_{N_fM} - \alpha(Y,B_\eta X)\\
- \alpha(Y,(LA_\eta X)_{TM}) - (L\nabla_X A_\eta Y)_{N_fM} + \nabla^\perp_X (LA_\eta Y)_{N_fM}.
\end{aligned}
$$

Using the Codazzi equation, we obtain

$$
\begin{aligned}
&(\nabla^\perp_X \mathcal{E})(Y,\eta) - (\nabla^\perp_Y \mathcal{E})(X,\eta)\\
&\qquad = \alpha(X,B_\eta Y) - \alpha(Y,B_\eta X) + \alpha(X,(LA_\eta Y)_{TM})\\
&\qquad\quad - \alpha(Y,(LA_\eta X)_{TM}) - (L\nabla_X A_\eta Y)_{N_fM} + \nabla^\perp_X (LA_\eta Y)_{N_fM}\\
&\qquad\quad + (L\nabla_Y A_\eta X)_{N_fM} - \nabla^\perp_Y (LA_\eta X)_{N_fM}.
\end{aligned}
$$

Since

$$
\beta(X,A_\eta Y) = \alpha(X,(LA_\eta Y)_{TM}) - (L\nabla_X A_\eta Y)_{N_fM} + \nabla^\perp_X (LA_\eta Y)_{N_fM},
$$

then (2.14) follows. $\square$

Remark 2.5. An alternative way to obtain the equations in Proposition 2.4 would be to follow the "classical procedure", which goes as follows. Since the metrics g_t induced by the infinitesimal variation $f_t = f + t\mathcal{T}$ satisfy $\partial/\partial t|_{t=0} g_t = 0$, hence the Levi-Civita connections and curvature tensors of g_t satisfy

$$
\partial/\partial t|_{t=0} \nabla^t_X Y = 0
$$

and

$$
\partial/\partial t|_{t=0} g_t(R^t(X,Y)Z,W) = 0
$$

for any $X,Y,Z,W \in \mathfrak{X}(M)$. Then this is used to compute the derivatives with respect to t at $t=0$ of the Gauss, Codazzi and Ricci equations for f_t. In fact, this works quite nicely to obtain (2.12) since we have that

$$
B(X,Y) = \partial/\partial t|_{t=0} \alpha^t(X,Y),
$$

where α^t is the second fundamental form of f_t. On the other hand, the computation for the other two equations becomes really cumbersome outside the hypersurface case. For hypersurfaces this was done in [21] and [23]. See also Exercise 2.1. A result in coordinates for general codimension was stated in [29].

The *first normal space* $N_1(x) \subset N_fM(x)$ at $x \in M^n$ of an isometric immersion $f\colon M^n \to \mathbb{R}^m$ is the vector subspace given by

$$N_1(x) = \operatorname{span}\{\alpha(X,Y) : X, Y \in T_xM\}.$$

We say that f has *full first normal spaces* if $N_1(x) = N_fM(x)$ at any $x \in M^n$.

The following result shows that for a submanifold with full first normal spaces the tensor β determines $\mathcal{E}$.

Proposition 2.6. *Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion with full first normal spaces. If $(\beta, \mathcal{E})$ is the associated pair to an infinitesimal bending $\mathcal{T}$ of f then $\mathcal{E}$ is the unique tensor that satisfies* (2.7) *and* (2.13).

Proof. If $\mathcal{E}_0\colon TM \times N_fM \to N_fM$ is a tensor that satisfies (2.7) and (2.13), it follows from (2.13) that

$$(\mathcal{E} - \mathcal{E}_0)(X, \alpha(Y,Z)) = (\mathcal{E} - \mathcal{E}_0)(Y, \alpha(X,Z))$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Since both $\mathcal{E}$ and $\mathcal{E}_0$ satisfy (2.7), we have

$$\langle(\mathcal{E} - \mathcal{E}_0)(X_1, \alpha(X_2,X_3)), \alpha(X_4,X_5)\rangle = -\langle(\mathcal{E} - \mathcal{E}_0)(X_1, \alpha(X_4,X_5)), \alpha(X_2,X_3)\rangle,$$

where $X_i \in \mathfrak{X}(M)$, $1 \leq i \leq 5$. We denote

$$\langle(\mathcal{E} - \mathcal{E}_0)(X_1, \alpha(X_2,X_3)), \alpha(X_4,X_5)\rangle = (X_1, X_2, X_3, X_4, X_5).$$

It follows from the relations above and the symmetry of α that

$$\begin{aligned}
&(X_1,X_2,X_3,X_4,X_5) = -(X_1,X_4,X_5,X_2,X_3) = -(X_5,X_4,X_1,X_3,X_3)\\
&= (X_5,X_2,X_3,X_4,X_1) = (X_3,X_2,X_5,X_4,X_1) = -(X_3,X_4,X_1,X_2,X_5)\\
&= -(X_4,X_3,X_1,X_2,X_5) = (X_4,X_2,X_5,X_3,X_1) = (X_2,X_4,X_5,X_3,X_1)\\
&= -(X_2,X_3,X_1,X_4,X_5) = -(X_2,X_1,X_3,X_4,X_5) = -(X_1,X_2,X_3,X_4,X_5)\\
&= 0,
\end{aligned}$$

and thus $\mathcal{E} - \mathcal{E}_0 = 0$. ∎

Finally, we characterize trivial infinitesimal bending in terms of the associated pair of tensors.

Let $\mathcal{T}$ be a trivial infinitesimal bending $f\colon M^n \to \mathbb{R}^m$, that is, we have that

$$\mathcal{T} = \mathcal{D}f + w,$$

where $\mathcal{D} \in \operatorname{End}(\mathbb{R}^m)$ is skew-symmetric and $w \in \mathbb{R}^m$. Then, we obtain that

$$L = \mathcal{D}|_{f_*TM} \;\text{ and }\; B(X,Y) = \mathcal{D}\alpha(X,Y).$$

Let $\mathcal{D}^N \in \Gamma(\mathrm{End}(N_f M))$ be skew-symmetric and given by

$$\mathcal{D}^N \eta = (\mathcal{D}\eta)_{N_f M}$$

for any $\eta \in \Gamma(N_f M)$. Then, we have

$$\beta(X,Y) = \mathcal{D}^N \alpha(X,Y) \ \text{ and } \ \mathcal{E}(X,\eta) = -(\nabla^\perp_X \mathcal{D}^N)\eta,$$

where the second equation follows computing $(\tilde{\nabla}_X D)\eta = 0$.

Proposition 2.7. *An infinitesimal bending $\mathcal{T}$ of f is trivial if and only if there is a skew-symmetric $C \in \Gamma(End(N_f M))$ such that*

$$\beta(X,Y) = C\alpha(X,Y) \ \text{ and } \ \mathcal{E}(X,\eta) = -(\nabla^\perp_X C)\eta. \tag{2.19}$$

Proof. Define $\mathcal{D} \in \Gamma(\mathrm{End}(f^*T\mathbb{R}^m))$ by

$$\mathcal{D}(x)X = L(x)X \ \text{ and } \ \mathcal{D}(x)\eta = \mathcal{Y}(x)\eta + C(x)\eta$$

for any $X \in T_x M$ and $\eta \in N_{f(x)}M$. Using the assumption on β, we obtain that

$$\begin{aligned}\tilde{\nabla}_X \mathcal{D}Y &= (\tilde{\nabla}_X L)Y + L\nabla_X Y\\ &= f_*\mathcal{Y}\alpha(X,Y) + C\alpha(X,Y) + L\nabla_X Y\\ &= \mathcal{D}\tilde{\nabla}_X Y\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. The assumptions on $\mathcal{E}$ and (2.16) give

$$\begin{aligned}\tilde{\nabla}_X \mathcal{D}\eta &= \tilde{\nabla}_X f_*\mathcal{Y}\eta + \tilde{\nabla}_X C\eta\\ &= (\tilde{\nabla}_X \mathcal{Y})\eta + f_*\mathcal{Y}\nabla^\perp_X\eta + (\nabla^\perp_X C)\eta + C\nabla^\perp_X\eta - f_*A_{C\eta}X\\ &= -f_*B_\eta X - LA_\eta X + f_*\mathcal{Y}\nabla^\perp_X\eta + C\nabla^\perp_X\eta - f_*A_{C\eta}X\end{aligned}$$

for any $X \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_f M)$. But $B_\eta = -A_{C\eta}$ from $\beta = C\alpha$, hence

$$\tilde{\nabla}_X \mathcal{D}\eta = -LA_\eta X + f_*\mathcal{Y}\nabla^\perp_X\eta + C\nabla^\perp_X\eta = \mathcal{D}\tilde{\nabla}_X\eta.$$

Therefore, we have shown that $\mathcal{D}(x) = \mathcal{D}$ is constant along M^n, and thus the map $\mathcal{T} - \mathcal{D}f$ is constant. $\square$

In accordance with the identification (2.6), from now on we also identify two pairs $(\beta_1, \mathcal{E}_1)$ and $(\beta_2, \mathcal{E}_2)$ if there is $0 \neq c \in \mathbb{R}$ such that the pair $(\beta_1 - c\beta_2, \mathcal{E}_1 - c\mathcal{E}_2)$ has the form (2.19).

2.4 The Fundamental Theorem

This section gives the Fundamental Theorem of infinitesimal variations.

Theorem 2.8. *Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion of a simply connected Riemannian manifold. Let $\beta\colon TM \times TM \to N_fM$ be a symmetric tensor and let the tensor $\mathcal{E}\colon TM \times N_fM \to N_fM$ satisfy the compatibility condition* (2.7)*. If the pair $(\beta, \mathcal{E}) \neq 0$ satisfies* (2.12)*,* (2.13) *and* (2.14)*, then there is a unique infinitesimal bending $\mathcal{T}$ of f having $(\beta, \mathcal{E})$ as associated pair.*

Proof. Given $(\beta, \mathcal{E})$ as in the statement, we argue that there is $\mathcal{D} \in \Gamma(\mathrm{End}(f^*T\mathbb{R}^m))$ satisfying

$$(\tilde{\nabla}_X \mathcal{D})(Y + \eta) = -f_* B_\eta X + \beta(X, Y) + \mathcal{E}(X, \eta) \tag{2.20}$$

for any $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$. To prove this, henceforth we check the integrability condition of (2.20), namely, that

$$(\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta) = 0$$

holds for any $X, Y, Z \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$. For simplicity, in the following we write X instead of f_*X. We have

$$\begin{aligned}
&(\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)\\
&= \tilde{\nabla}_X (\tilde{\nabla}_Y \mathcal{D})(Z + \eta) - (\tilde{\nabla}_Y \mathcal{D})\tilde{\nabla}_X (Z + \eta) - \tilde{\nabla}_Y (\tilde{\nabla}_X \mathcal{D})(Z + \eta)\\
&\quad + (\tilde{\nabla}_X \mathcal{D})\tilde{\nabla}_Y (Z + \eta) - (\tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)\\
&= \tilde{\nabla}_X [-B_\eta Y + \beta(Y, Z) + \mathcal{E}(Y, \eta)] + B_{\alpha(X,Z) + \nabla^\perp_X \eta} Y - \beta(Y, \nabla_X Z - A_\eta X)\\
&\quad - \mathcal{E}(Y, \alpha(X, Z) + \nabla^\perp_X \eta) + \tilde{\nabla}_Y [B_\eta X - \beta(X, Z) - \mathcal{E}(X, \eta)]\\
&\quad - B_{\alpha(Y,Z) + \nabla^\perp_Y \eta} X + \beta(X, \nabla_Y Z - A_\eta Y) + \mathcal{E}(X, \alpha(Y, Z) + \nabla^\perp_Y \eta)\\
&\quad + B_\eta [X, Y] - \beta([X, Y], Z) - \mathcal{E}([X, Y], \eta).
\end{aligned}$$

Hence

$$\begin{aligned}
&(\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)\\
&= -A_{\beta(Y,Z)} X + B_{\alpha(X,Z)} Y + A_{\beta(X,Z)} Y - B_{\alpha(Y,Z)} X\\
&\quad + (\nabla^\perp_X \beta)(Y, Z) - (\nabla^\perp_Y \beta)(X, Z) + \mathcal{E}(X, \alpha(Y, Z)) - \mathcal{E}(Y, \alpha(X, Z))\\
&\quad - (\nabla_X B_\eta) Y + (\nabla_Y B_\eta) X + B_{\nabla^\perp_X \eta} Y - B_{\nabla^\perp_Y \eta} X - A_{\mathcal{E}(Y,\eta)} X + A_{\mathcal{E}(X,\eta)} Y\\
&\quad + (\nabla^\perp_X \mathcal{E})(Y, \eta) - (\nabla^\perp_Y \mathcal{E})(X, \eta) - \alpha(X, B_\eta Y) + \alpha(Y, B_\eta X)\\
&\quad + \beta(Y, A_\eta X) - \beta(X, A_\eta Y)\\
&= 0,
\end{aligned}$$

where to obtain the final conclusion we made use of (2.12) to (2.15).

Fix $x_0 \in M^n$ and a solution $\mathcal{D}^* \in \Gamma(\mathrm{End}(f^*T\mathbb{R}^m))$ of (2.20). Set $\mathcal{D}_0 = \mathcal{D}^*(x_0)$ and let $\phi\colon f^*T\mathbb{R}^m \times f^*T\mathbb{R}^m \to \mathbb{R}$ be the tensor defined by

$$\phi(\rho, \sigma) = \langle (\mathcal{D}^* - \mathcal{D}_0)\rho, \sigma \rangle + \langle (\mathcal{D}^* - \mathcal{D}_0)\sigma, \rho \rangle.$$

Using (2.7) and (2.20) we obtain

$$\begin{aligned}(\tilde{\nabla}_X\phi)(\rho,\sigma) &= X(\phi(\rho,\sigma)) - \phi(\tilde{\nabla}_X\rho,\sigma) - \phi(\rho,\tilde{\nabla}_X\sigma)\\ &= \langle(\tilde{\nabla}_X\mathcal{D}^*)\rho,\sigma\rangle + \langle(\tilde{\nabla}_X\mathcal{D}^*)\sigma,\rho\rangle\\ &= 0.\end{aligned}$$

Hence we have $\phi = 0$, and therefore the map $\mathcal{D}(x) = \mathcal{D}^*(x) - \mathcal{D}_0$ is a skew-symmetric endomorphism of $\mathbb{R}^m$.

Define $L \in \Gamma(\mathrm{Hom}(TM, f^*T\mathbb{R}^m))$ by $L(x) = \mathcal{D}(x)|_{T_xM}$. Using (2.20) we obtain

$$\begin{aligned}(\tilde{\nabla}_X L)Y &= \tilde{\nabla}_X\mathcal{D}Y - \mathcal{D}\nabla_X Y\\ &= \beta(X,Y) + \mathcal{D}\alpha(X,Y).\end{aligned}$$

Thus

$$(\tilde{\nabla}_X L)Y = (\tilde{\nabla}_Y L)X.$$

Hence, there is $\mathcal{T} \in \Gamma(f^*T\mathbb{R}^m)$ such that

$$\tilde{\nabla}_X\mathcal{T} = LX$$

for any $X \in \mathfrak{X}(M)$. Since $\mathcal{D}$ is skew-symmetric then L satisfies

$$\langle LX, Y\rangle + \langle LY, X\rangle = 0,$$

and thus $\mathcal{T}$ is an infinitesimal bending of f. Moreover, its associate pair $(\tilde{\beta},\tilde{\mathcal{E}})$ is

$$\tilde{\beta}(X,Y) = \beta(X,Y) + \mathcal{D}^N\alpha(X,Y) \;\; \textit{and} \;\; \tilde{\mathcal{E}}(X,\eta) = \mathcal{E}(X,\eta) - (\nabla^\perp_X\mathcal{D}^N)\eta.$$

In fact, in this case $\mathcal{Y}\eta = (\mathcal{D}\eta)_{TM}$. Using (2.20), we have

$$\begin{aligned}\tilde{\mathcal{E}}(X,\eta) &= \alpha(X,(\mathcal{D}\eta)_{TM}) + (LA_\eta X)_{N_fM}\\ &= (\tilde{\nabla}_X(\mathcal{D}\eta)_{TM})_{N_fM} + (LA_\eta X)_{N_fM}\\ &= (\tilde{\nabla}_X\mathcal{D}\eta)_{N_fM} - \nabla^\perp_X\mathcal{D}^N\eta + (LA_\eta X)_{N_fM}\\ &= \mathcal{E}(X,\eta) + (\mathcal{D}\tilde{\nabla}_X\eta)_{N_fM} - \nabla^\perp_X\mathcal{D}^N\eta + (LA_\eta X)_{N_fM}\\ &= \mathcal{E}(X,\eta) - (LA_\eta X)_{N_fM} - (\nabla^\perp_X\mathcal{D}^N)\eta + (LA_\eta X)_{N_fM}\\ &= \mathcal{E}(X,\eta) - (\nabla^\perp_X\mathcal{D}^N)\eta.\end{aligned}$$

Another solution $\mathcal{D}_1^*$ of (2.20) gives rise to an infinitesimal bending $\mathcal{T}_1$ of f. It now follows from Proposition 2.7 that $\mathcal{T} - \mathcal{T}_1$ is a trivial infinitesimal bending. $\square$

Remark 2.9. Let $f\colon M^n \to \mathbb{L}^m$ be an isometric immersion of a Riemannian manifold M^n into the Lorentzian space form $\mathbb{L}^m$. Recall that $\mathbb{L}^m$ has $\mathbb{R}^m$ as underlying space and that its Levi-Civita connection coincides with the Euclidean one. Hence, the arguments in the proof of Theorem 2.8 also hold in this case. In fact, they hold for an immersion of a Riemannian manifold M^n into $\mathbb{R}^m$ where the latter is endowed with any possible indefinite metric.

2.5 The hypersurfaces case

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface with shape operator A corresponding to the Gauss map $N \in \Gamma(N_fM)$. If $\mathcal{T}$ is an infinitesimal bending of f, then associated to $\mathcal{T}$ we have the symmetric tensor $\mathcal{B} \in \Gamma(\mathrm{End}(TM))$ given by

$$\beta(X,Y) = \langle \mathcal{B}X, Y\rangle N.$$

In codimension one, any tensor $\mathcal{E}\colon TM \times N_fM \to N_fM$ satisfying (2.7) vanishes. Therefore, the fundamental equations of an infinitesimal bending take the form

$$\mathcal{B}X \wedge AY - \mathcal{B}Y \wedge AX = 0 \tag{2.21}$$

and

$$(\nabla_X \mathcal{B})Y = (\nabla_Y \mathcal{B})X$$

for any $X, Y \in \mathfrak{X}(M)$. Notice that the second equation says that $\mathcal{B}$ satisfies the condition of being a *Codazzi tensor*.

Proposition 2.7 gives the following characterization of trivial infinitesimal bendings of hypersurfaces.

Proposition 2.10. *An infinitesimal bending $\mathcal{T}$ of an hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is trivial if and only if its associated tensor $\mathcal{B}$ vanishes.*

The Fundamental Theorem for infinitesimal variations of hypersurfaces goes as follows.

Theorem 2.11. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold. Let $0 \neq \mathcal{B} \in \Gamma(End(TM))$ be a symmetric Codazzi tensor that satisfies* (2.21)*. Then there exists a unique infinitesimal bending $\mathcal{T}$ of f having $\mathcal{B}$ as associated tensor.*

Proof. Let $\beta\colon TM \times TM \to N_fM$ be the symmetric tensor given by

$$\beta(X,Y) = \langle \mathcal{B}X, Y\rangle N.$$

Then (2.14) trivially holds for β and $\mathcal{E} = 0$. Moreover, by the assumptions on $\mathcal{B}$ we have that $(\beta, 0)$ satisfies (2.12) and (2.13). Thus, by Theorem 2.8

there is a unique infinitesimal bending $\mathcal{T}$ of f having $(\beta, 0)$ as associated pair. $\square$

2.6 Infinitesimal rigidity

That a submanifold $f\colon M^n \to \mathbb{R}^m$ is *infinitesimally rigid* means that any infinitesimal bending of f is trivial. The goal of this section is to provide conditions on the submanifold that yield infinitesimal rigidity.

The proof of the rigidity theorems in this section will make use of an elementary but very useful result already contained in the classical literature of infinitesimal variations of surfaces, for instance, see Bianchi [3].

Proposition 2.12. *Let $\mathcal{T}$ be an infinitesimal bending of an isometric immersion $f\colon M^n \to \mathbb{R}^m$ and let $G_t\colon M^n \to \mathbb{R}^m$, $t \in \mathbb{R}$, be the map defined by*

$$G_t(x) = f(x) + t\mathcal{T}(x). \tag{2.22}$$

The following assertions hold:

(i) The maps G_t and G_{-t} are immersions that induce the same metric.

(ii) If f is substantial and there exists $0 \neq t_0 \in \mathbb{R}$ such that G_{t_0} and G_{-t_0} are congruent then $\mathcal{T}$ is trivial.

Proof. The assertion in part (i) follows from

$$\|G_{t*}X\|^2 = \|f_*X\|^2 + t^2\|\mathcal{T}_*X\|^2.$$

By the assumption of part (ii) there exist an orthogonal transformation S of $\mathbb{R}^m$ and a vector $w \in \mathbb{R}^m$ such that

$$f + t_0\mathcal{T} = S(f - t_0\mathcal{T}) + w.$$

Thus

$$f_*X + t_0\tilde{\nabla}_X\mathcal{T} = S(f_*X - t_0\tilde{\nabla}_X\mathcal{T}),$$

and hence

$$t_0(S+I)\tilde{\nabla}_X\mathcal{T} = (S-I)f_*X \tag{2.23}$$

for all $X \in \mathfrak{X}(M)$.

Suppose that $S + I$ is not invertible, that is, that there exists

$$0 \neq \delta \in \ker(S+I) = \ker(S+I)^t,$$

where $(\ )^t$ denotes taking the transpose. Then $(S-I)^t\delta = -2\delta$. Taking the inner product of (2.23) with δ gives

$$\langle f_*X, \delta\rangle = 0$$

for all $X \in \mathfrak{X}(M)$, contradicting the fact that f is substantial.

Thus $S+I$ is invertible, and hence (2.23) yields

$$\tilde{\nabla}_X \mathcal{T} = \mathcal{D} f_* X, \tag{2.24}$$

where

$$\mathcal{D} = \frac{1}{t_0}(S+I)^{-1}(S-I).$$

Since f is substantial, it follows from

$$\langle \tilde{\nabla}_X \mathcal{T}, f_* Y\rangle + \langle f_* X, \tilde{\nabla}_Y \mathcal{T}\rangle = 0$$

and (2.24) that $\mathcal{D}$ is skew-symmetric. Moreover, since $\mathcal{D} f_* X = \tilde{\nabla}_X \mathcal{D} f$ then (2.24) also yields

$$\tilde{\nabla}_X(\mathcal{T} - \mathcal{D} f) = 0$$

for all $X \in \mathfrak{X}(M)$, thus showing that $\mathcal{T}$ is trivial. $\square$

Proposition 2.12 was used to prove the following global result due to Dajczer-Rodríguez [19]. The proof is also contained in [21].

Theorem 2.13. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an isometric immersion of a compact Riemannian manifold such that there are no open subsets of M^n where f is totally geodesic. Then f is infinitesimally rigid.*

Next we state two well-known rigidity results for submanifolds. Recall that an isometric immersion $f\colon M^n \to \mathbb{R}^m$ is said to be rigid if any other isometric immersion $g\colon M^n \to \mathbb{R}^m$ is congruent to f by an isometry of $\mathbb{R}^m$. That is, there is an isometry (rigid motion) $\tau\colon \mathbb{R}^m \to \mathbb{R}^m$ such that $g = \tau \circ f$. The first result is the classical Allendoerfer's theorem and the second is due to do Carmo-Dajczer [5]. The proofs of both results can be seen in [21].

It is said that an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ has *type number* $\tau \geq 3$ if at any point $x \in M^n$ there are three vectors $X_1, X_2, X_3 \in T_xM$ and a basis $\xi_1, \ldots, \xi_p$ of $N_fM(x)$ such that the $3p$ vectors $A_{\xi_j} X_i$ $1 \leq i \leq 3$, $1 \leq j \leq p$, are linearly independent. This condition is independent of the normal basis.

Proposition 2.14. *An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ with type number $\tau \geq 3$ is rigid.*

The *s-nullity* $\nu_s(x)$, $1 \leq s \leq p$, of an immersion $f\colon M^n \to \mathbb{R}^{n+p}$ at $x \in M^n$ is defined as

$$\nu_s(x) = \max_{U^s \subset N_fM(x)} \{\dim \mathcal{N}(\alpha_{U^s})\},$$

where $\alpha_{U^s} = \pi_{U^s} \circ \alpha$ and $\pi_{U^s}\colon N_fM \to U^s$ is the orthogonal projection onto the normal subspace U^s.

Proposition 2.15. *An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \leq 5$, whose s-nullities satisfy $\nu_s \leq n-2s-1$ for all $1 \leq s \leq p$ at any point of M^n is rigid.*

Remark 2.16. It is easy to see that the assumption on the s-nullities is weaker than the one on the type number. In fact, that $\tau \geq 3$ implies $\nu_s \leq n-3s$, $1 \leq s \leq p$. On the other hand, it is not known if the Theorem 2.15 holds for higher codimensions since its

The following is the infinitesimal version of the above two results.

Theorem 2.17. *An isometric immersion $f\colon M^n \to \mathbb{R}^m$ which satisfies the conditions in either Proposition 2.14 or Proposition 2.15 is infinitesimally rigid.*

Proof. Let $\mathcal{T}$ be an infinitesimal bending of f and let $G_t\colon M^n \to \mathbb{R}^m$ be defined by (2.22) for any $t \in \mathbb{R}$. By Proposition 2.12, the immersions G_t and G_{-t} are isometric. Moreover, any point of M^n lies in an open neighborhood U where G_t still satisfies the assumptions if t is small enough. By either Proposition 2.14 or Proposition 2.15, we have that the restrictions $G_t|_U$ and $G_{-t}|_U$ are congruent, and hence $\mathcal{T}$ is trivial on U by Proposition 2.12 since the assumptions include that $f|_U$ has full first normal spaces and thus is substantial.

We have seen that $\mathcal{T}$ is locally trivial, that is, each point of M^n lies in an open subset U such that $\tilde{\nabla}_X \mathcal{T} = \mathcal{D}_U f_* X$ along U. If two such open subsets U and V intersect, then

$$(\mathcal{D}_U - \mathcal{D}_V)|_{f_* T_x M} = 0 \ \text{ for all } \ x \in U \cap V.$$

Since $f|_{U\cap V}$ is substantial,

$$\operatorname{span}\{f_* T_x M : x \in U \cap V\} = \mathbb{R}^m.$$

Hence $\mathcal{D}_U = \mathcal{D}_V$, and thus $\mathcal{T}$ is globally trivial. □

2.7 Exercises

Exercise 2.1. Given an infinitesimal bending $\mathcal{T}$ of a submanifold $f\colon M^n \to \mathbb{R}^m$, let $\mathcal{F}\colon I \times M^n \to \mathbb{R}^m$ be a variation of $f = f_0$ by immersions $f_t\colon M^n \to \mathbb{R}^m$ with variational vector field $\mathcal{T}$.

(i) Show that the Levi-Civita connections and curvature tensors of the metrics g_t induced by f_t for $t \in I$ satisfy

$$\partial/\partial t|_{t=0} \nabla^t_X Y = 0$$

and
$$\partial/\partial t|_{t=0} g_t(R^t(X,Y)Z,W) = 0$$
for any $X,Y,Z,W \in \mathfrak{X}(M)$.

(ii) If α^t denotes the second fundamental form of f_t for $t \in I$ prove that
$$B(X,Y) = \partial/\partial t|_{t=0}\alpha^t(X,Y).$$

(iii) Extend $\eta \in \Gamma(N_f M)$ to a map $\eta(t) \in \Gamma(N_{f_t} M)$, $t \in I$. Show that
$$f_* \mathcal{Y}\eta = (\partial/\partial t|_{t=0}\eta(t))_{f_* TM}.$$

(iv) Give an alternative proof of (2.11) using the above items.

(v) Using the above give an alternative proof of (2.12).

Exercise 2.2. Let $\mathcal{F}\colon I \times M \to \mathbb{R}^m$ be an infinitesimal variation of an isometric immersion $f\colon M^n \to \mathbb{R}^m$. Let $\{X_1^t, \dots, X_n^t\}$ be a one-parameter family of tangent vectors fields such that for each t fixed $\{X_1^t, \dots, X_n^t\}$ is an orthonormal frame for the metric induced by f_t. Let $X_i' \in \mathfrak{X}(M)$, $1 \leq i \leq n$, be given at each point $x \in M^n$ by
$$X_i'(x) = \frac{\partial}{\partial t}|_{t=0} X_i^t(x).$$

(i) Prove that
$$\langle X_i', X_j\rangle + \langle X_j', X_i\rangle = 0$$
for any $1 \leq i,j \leq n$.

(ii) Assume further that f is minimal and let $\mathcal{H}^t = \frac{1}{n}\sum_{i=1}^n \alpha^t(X_i^t, X_i^t)$ be the mean curvature vector field of f_t, $t \in I$. Show that
$$\frac{\partial}{\partial t}|_{t=0}\mathcal{H}^t = 0$$
if and only if the tensor β associated to the corresponding infinitesimal bending satisfies
$$\sum_{i=1}^n \beta(X_i, X_i) = 0.$$

Hint: For *(ii)* use the previous item, part *(ii)* of Exercise 2.1 and (2.11).

Exercise 2.3. Prove the statements in Examples 2.1.

Chapter 3

Genuine infinitesimal variations

If an isometric immersion of a Riemannian manifold into Euclidean space admits an isometric deformation, then any submanifold of that manifold inherits an isometric deformation obtained via the composition of immersions. Therefore, to study the geometry of the isometrically deformable submanifolds that lie in codimension larger than one it is clear that deformations produced via compositions should somehow be excluded. Consequently, it is convenient to restrict the study to the class of isometric deformations called genuine. For results in this direction we refer to Chapter 12 of [21]. The goal of this chapter is to deal with local and global infinitesimal variations of submanifolds by means of a similar approach.

Let $\tilde{\mathcal{T}}$ be an infinitesimal bending of an isometric immersion $F\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$, $0 < \ell < p$, and let $j\colon M^n \to \tilde{M}^{n+\ell}$ be an embedding. Then $\mathcal{T} = \tilde{\mathcal{T}}|_{j(M)}$ is an infinitesimal bending of $f = F \circ j\colon M^n \to \mathbb{R}^{n+p}$. This observation motivates the following definitions where a more general situation is considered since certain singularities are allowed. In fact, the necessity to admit the existence of singularities of F along $j(M)$ for isometric deformations, in the local as well as in the global situation, was already well established in [14] and [26].

A smooth map $F\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$, $0 < \ell < p$, from a differentiable manifold into Euclidean space is said to be a *singular extension* of a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ if there is an embedding $j\colon M^n \to \tilde{M}^{n+\ell}$, $0 < \ell < p$, such that F is an immersion along $\tilde{M}^{n+\ell} \backslash j(M)$ and $f = F \circ j$. Hence, the map F may fail (but not necessarily) to be an immersion along points of $j(M)$.

It is said that an infinitesimal bending $\mathcal{T}$ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ *extends in the singular sense* if there is a singular extension $F\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$ of f and a smooth map $\tilde{\mathcal{T}}\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$ such that $\tilde{\mathcal{T}}$ is an infinitesimal bending of $F|_{\tilde{M}\setminus j(M)}$ and $\mathcal{T} = \tilde{\mathcal{T}}|_{j(M)}$.

An infinitesimal bending $\mathcal{T}$ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, is called a *genuine infinitesimal bending* if $\mathcal{T}$ does not extend in the singular sense when restricted to any open subset of M^n. If f admits such a bending we say that it is *genuinely infinitesimally bendable*. By a *genuine infinitesimal variation* we mean an infinitesimal variation whose associated infinitesimal bending is genuine. Finally, we say that f is *genuinely infinitesimally rigid* if given any infinitesimal bending $\mathcal{T}$ of f there is an open dense subset of M^n such that $\mathcal{T}$ restricted to any connected component extends in the singular sense.

As one may expect trivial infinitesimal bendings are never genuine. Moreover, if we have that $f(M) \subset \mathbb{R}^{n+\ell} \subset \mathbb{R}^{n+p}$, $\ell < p$, and that $e \in \mathbb{R}^{n+p}$ is orthogonal to $\mathbb{R}^{n+\ell}$, then $\mathcal{T} = \phi e$ for $\phi \in C^\infty(M)$ is another example of an infinitesimal bending that it is not genuine.

3.1 The local results

What can be said about the geometry of an Euclidean submanifold in low codimension that admits an genuine infinitesimal variation? In this section, we give two answers to the local version of this question. The case when the submanifold is compact is treated in the subsequent section.

The following is the first main local result of this section. An isometric immersion $f\colon M^n \to \mathbb{R}^m$ is *r-ruled* if M^n carries a smooth r-dimensional totally geodesic tangent distribution whose leaves (called rulings) are mapped diffeomorphically by f to open subsets of affine subspaces of $\mathbb{R}^m$.

Theorem 3.1. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p \geq 4$, be an isometric immersion and let $\mathcal{T}$ be an infinitesimal bending of f. Then along each connected component of an open dense subset either $\mathcal{T}$ extends in the singular sense or f is r-ruled with $r \geq n - 2p$.*

An immediate consequence is the following result.

Corollary 3.2. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p \geq 4$, be a genuinely infinitesimally bendable isometric immersion. Then f is r-ruled with $r \geq n - 2p$ along connected components of an open dense subset of M^n.*

Theorem 3.1 also has the next two immediate consequences since the possibility of the submanifold being ruled is excluded by the assumptions.

Corollary 3.3. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p \geq 4$, be an isometric immersion. If M^n has positive Ricci curvature then f is genuinely infinitesimally rigid.*

Corollary 3.4. *Let $g\colon M^n \to \mathbb{S}^{n+p-1}$, $n > 2p \geq 4$, be an isometric immersion and let $f = i \circ g$ where $i\colon \mathbb{S}^{n+p-1} \to \mathbb{R}^{n+p}$ denotes the umbilical inclusion. Then f is genuinely infinitesimally rigid.*

A key ingredient in the proofs of the theorems in this section is the next result due to Florit-Guimarães [26]; see also [21].

Proposition 3.5. *Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion and let D be a smooth tangent distribution of dimension $d > 0$. Assume that there does not exist an open subset $U \subset M^n$ and $Z \in \Gamma(D|_U)$ such that the map $F\colon U \times \mathbb{R} \to \mathbb{R}^m$ given by*

$$F(x,t) = f(x) + tf_*Z(x)$$

is a singular extension of f on an open neighborhood of $U \times \{0\}$. Then, for any $x \in M^n$ there is an open neighborhood V of the origin in $D(x)$ such that $f_*(x)V \subset f(M)$. Hence f is d-ruled along each connected component of an open dense subset of M^n.

Next we associate to an infinitesimal bending a flat bilinear form.

Proposition 3.6. *Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion and let $\mathcal{T}$ be an infinitesimal bending with associated pair $(\beta, \mathcal{E})$. Then, at any point of M^n the bilinear form $\theta\colon TM \times TM \to N_fM \oplus N_fM$ defined by*

$$\theta(X,Y) = (\alpha(X,Y) + \beta(X,Y), \alpha(X,Y) - \beta(X,Y)) \tag{3.1}$$

is flat with respect to the inner product in $N_fM \oplus N_fM$ given by

$$\langle\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle\rangle_{N_fM \oplus N_fM} = \langle \xi_1, \xi_2 \rangle_{N_fM} - \langle \eta_1, \eta_2 \rangle_{N_fM}.$$

Proof. A straightforward computation shows that

$$\begin{aligned}\frac{1}{2}\left(\langle\langle \theta(X,Z), \theta(Y,W) \rangle\rangle - \langle\langle \theta(X,W), \theta(Y,Z) \rangle\rangle\right) &= \langle \beta(X,Z), \alpha(Y,W) \rangle \\ + \langle \alpha(X,Z), \beta(Y,W) \rangle - \langle \beta(X,W), \alpha(Y,Z) \rangle &- \langle \alpha(X,W), \beta(Y,Z) \rangle,\end{aligned}$$

and the proof follows from (2.12). □

An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is called 1-*regular* if the first normal spaces $N_1(x)$, $x \in M^n$, have constant dimension $k \leq p$ on M^n and thus form a subbundle N_1 of rank k of the normal bundle. Under the 1-regularity assumption we have the following statement that is equivalent to the above one.

Proposition 3.7. *Let $f\colon M^n \to \mathbb{R}^m$ be 1-regular and let $\beta_1\colon TM\times TM \to N_1$ be the N_1-component of β. Then the bilinear form $\hat{\theta}\colon TM \times TM \to N_1 \oplus N_1$ defined at any point by*

$$\hat{\theta}(X,Y) = (\alpha(X,Y) + \beta_1(X,Y), \alpha(X,Y) - \beta_1(X,Y)) \tag{3.2}$$

is flat with respect to the inner product induced on $N_1 \oplus N_1$.

Proof of Theorem 3.1: Let $\mathcal{T}$ be an infinitesimal bending of f. From (2.10) we have

$$\langle \alpha(X,Y), LZ\rangle + \langle \mathcal{Y}\alpha(X,Y), Z\rangle = 0. \tag{3.3}$$

Then, we easily obtain using (2.8) that

$$\langle f_*X + \tilde{\nabla}_X Y, LX + \tilde{\nabla}_X LY\rangle = \langle \alpha(X,Y), \beta(X,Y)\rangle \tag{3.4}$$

for any $X,Y \in \mathfrak{X}(M)$.

By Proposition 3.6 we have that the symmetric tensor θ is flat at any point of M^n. Given $Y \in RE(\theta(x)) \subset T_xM$ at $x \in M^n$, denote $D = \ker\theta_Y$ where $\theta_Y(X) = \theta(Y,X)$. Notice that $Z \in D$ means that $\alpha(Y,Z) = 0 = \beta(Y,Z)$.

Let $U \subset M^n$ be an open subset where $Y \in \mathfrak{X}(U)$ satisfies $Y \in RE(\theta)$ and D has dimension d at any point. Proposition 1.10 gives

$$\langle\!\langle \theta(X,Z), \theta(X,Z)\rangle\!\rangle = 0$$

for any $X \in \mathfrak{X}(U)$ and $Z \in \Gamma(D)$. Equivalently, the right hand side of (3.4) vanishes and hence

$$\langle f_*X + \tilde{\nabla}_X Z, LX + \tilde{\nabla}_X LZ\rangle = 0 \tag{3.5}$$

for any $X \in \mathfrak{X}(U)$ and $Z \in \Gamma(D)$.

Assume that there exists a nowhere zero $Z \in \Gamma(D)$ defined on an open subset V of U such that $F\colon V \times (-\epsilon,\epsilon) \to \mathbb{R}^{n+p}$ given by

$$F(x,t) = f(x) + tf_*Z(x)$$

is a singular extension of $f|_V$. The map $\tilde{\mathcal{T}}\colon V \times (-\epsilon,\epsilon) \to \mathbb{R}^{n+p}$ given by

$$\tilde{\mathcal{T}}(x,t) = \mathcal{T}(x) + tLZ(x)$$

extends $\mathcal{T}|_V$ and is an infinitesimal bending of F on the open subset where F is an immersion. In fact,

$$\langle F_*\partial/\partial t, \tilde{\nabla}_{\partial/\partial t}\tilde{\mathcal{T}}\rangle = \langle f_*Z, LZ\rangle = 0,$$

$$\langle \tilde{\nabla}_{\partial/\partial t}\tilde{\mathcal{T}}, F_*X\rangle + \langle \tilde{\nabla}_X\tilde{\mathcal{T}}, F_*\partial/\partial t\rangle = \langle LZ, f_*X + t\tilde{\nabla}_X Z\rangle + \langle LX + t\tilde{\nabla}_X LZ, f_*Z\rangle = 0$$

and

$$\langle F_*X, \tilde{\nabla}_X \tilde{\mathcal{T}} \rangle = \langle f_*X + t\tilde{\nabla}_X Z, LX + t\tilde{\nabla}_X LZ \rangle = 0,$$

where the last equality follows from (3.5).

Let $W \subset U$ be an open subset such that $Z \in \Gamma(D)$ as above does not exist along any open subset of W. By Proposition 3.5 the immersion is d-ruled along any connected component of an open dense subset of W. Moreover, we have that $d = \dim D = n - \dim \mathrm{Im}(\theta_Y) \geq n - 2p$. □

Remark 3.8. In Theorem 3.1 assume further that f is 1-regular with $\dim N_1 = q < p$. Then we obtain the better lower bound $r \geq n - 2q$ since the proof still works making use of Proposition 3.7 instead of Proposition 3.6.

In the case of low codimension the following result, obtained with a substantial additional effort, gives a better lower bound for the dimension of the rulings. The proof is given at the end of this section after several considerations.

Theorem 3.9. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p$, be a genuinely infinitesimally bendable isometric immersion. If $2 \leq p \leq 5$, then one of the following facts holds along any connected component, say U, of an open dense subset of M^n:*

(i) $f|_U$ is ν-ruled by leaves of relative nullity with $\nu \geq n - 2p$.

(ii) $f|_U$ has index of relative nullity $\nu < n - 2p$ at any point of U and is r-ruled with $r \geq n - 2p + 3$.

Remark 3.10. For $p = 2$ we are always in case (i) since a $(n-1)$-ruled submanifold in that codimension has index of relative nullity $\nu \geq n - 3$ at any point.

Let $F\colon \tilde{M}^{n+1} \to \mathbb{R}^{n+p}$ be an isometric immersion and let $\tilde{\mathcal{T}}$ be an infinitesimal bending of F. Given an isometric embedding $j\colon M^n \to \tilde{M}^{n+1}$ consider the composition of isometric immersions $f = F \circ j\colon M^n \to \mathbb{R}^{n+p}$. Then $\mathcal{T} = \tilde{\mathcal{T}}|_{j(M)}$ is an infinitesimal bending of f. It is easy to see that the corresponding tensors B and $\tilde{B}$ given by (2.9) have the relation

$$B(X,Y) = \tilde{B}(X,Y) + \langle \tilde{\nabla}_X Y, F_*\eta \rangle \tilde{L}\eta,$$

where $\eta \in \Gamma(N_j M)$ is of unit length and $X, Y \in \mathfrak{X}(M)$. Then (3.3) gives

$$\langle \beta(X,Y), F_*\eta \rangle + \langle \alpha^f(X,Y), \tilde{L}\eta \rangle = 0$$

for any $X, Y \in \mathfrak{X}(M)$. We will see that satisfying a condition of this type may guarantee that an infinitesimal bending is not genuine. In fact, this was already proved by Florit [25] in a special case.

We say that an infinitesimal bending of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, satisfies the *condition* $(*)$ if there is $\eta \in \Gamma(N_fM)$ of unit length and $\xi \in \Gamma(R)$, where R is determined by the orthogonal splitting $N_fM = P \oplus R$ and $P = \operatorname{span}\{\eta\}$, such that

$$B_\eta + A_\xi = 0, \tag{3.6}$$

where $B_\eta = \langle \beta, \eta \rangle$. Thus, that (3.6) holds means that

$$\langle \beta(X,Y), \eta \rangle + \langle \alpha(X,Y), \xi \rangle = 0 \tag{3.7}$$

for any $X, Y \in \mathfrak{X}(M)$.

The following result proved below is of independent interest since it does not require the codimension to satisfy $p \leq 5$ as is the case in Theorem 3.9.

Theorem 3.11. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, be an isometric immersion and let $\mathcal{T}$ be an infinitesimal bending of f that satisfies the condition $(*)$. Then along each connected component of an open dense subset of M^n either $\mathcal{T}$ extends in the singular sense or f is r-ruled with $r \geq n - 2p + 3$.*

Similarly as above, there is the following immediate consequence.

Corollary 3.12. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, be an isometric immersion and let $\mathcal{T}$ be a genuine infinitesimal bending of f that satisfies the condition $(*)$. Then f is r-ruled with $r \geq n - 2p + 3$ on connected components of an open dense subset of M^n.*

In the case that $\mathcal{T}$ satisfies the condition $(*)$ we may extend the tensor L to a tensor $\bar{L} \in \Gamma(\operatorname{End}(TM \oplus P, f^*T\mathbb{R}^{n+p})$ by defining

$$\bar{L}\eta = f_*\mathcal{Y}\eta + \xi.$$

Then $\bar{L}$ satisfies

$$\langle \bar{L}X, \eta \rangle + \langle f_*X, \bar{L}\eta \rangle = 0 \tag{3.8}$$

for any $X \in \mathfrak{X}(M)$.

Given $\lambda \in \Gamma(f_*TU \oplus P)$ nowhere vanishing along an open subset U of M^n, let the map $F\colon U \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+p}$ be given by

$$F(x,t) = f(x) + t\lambda(x). \tag{3.9}$$

Notice that at least for $t = 0$ the map F is not an immersion at points where λ is tangent to U. Then let $\tilde{\mathcal{T}}\colon U \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+p}$ be the map given by

$$\tilde{\mathcal{T}}(x,t) = \mathcal{T}(x) + t\bar{L}\lambda(x). \tag{3.10}$$

We have

$$\langle F_*\partial/\partial t, \tilde{\nabla}_{\partial/\partial t}\tilde{\mathcal{T}} \rangle = 0.$$

Moreover, since $\langle \bar{L}\lambda, \lambda\rangle = 0$ we obtain

$$\langle \tilde{\nabla}_{\partial/\partial t}\tilde{\mathfrak{T}}, F_*X\rangle + \langle \tilde{\nabla}_X\tilde{\mathfrak{T}}, F_*\partial/\partial t\rangle = \langle \bar{L}\lambda, f_*X\rangle + \langle LX, \lambda\rangle + tX\langle \bar{L}\lambda, \lambda\rangle = 0$$

for any $X \in \mathfrak{X}(M)$ and $t \in (-\epsilon, \epsilon)$. Thus $\tilde{\mathfrak{T}}$ is an infinitesimal bending of F on the open subset $\tilde{U}$ of $U \times (-\epsilon, \epsilon)$ where F is an immersion if and only if

$$\langle F_*X, \tilde{\nabla}_X\tilde{\mathfrak{T}}\rangle = 0,$$

or equivalently, if and only if

$$\langle f_*X + t\tilde{\nabla}_X\lambda, LX + t\tilde{\nabla}_X\bar{L}\lambda\rangle = 0$$

for any $X \in \mathfrak{X}(M)$.

In the sequel, we take F restricted to $\tilde{U}$. By the above, in order to have that $\tilde{\mathfrak{T}}$ is an infinitesimal bending of F the strategy is to make use of the condition $(*)$ to construct a subbundle $D \subset f_*TM \oplus P$ such that

$$\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda\rangle = 0$$

for any $X \in \mathfrak{X}(M)$ and any $\lambda \in \Gamma(D)$.

Lemma 3.13. *Assume that $\mathfrak{T}$ satisfies the condition $(*)$. Then*

$$\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda\rangle = \langle (\tilde{\nabla}_X\lambda)_R, (\tilde{\nabla}_X\bar{L})\lambda\rangle, \tag{3.11}$$

where $X \in \mathfrak{X}(M)$, $\lambda \in \Gamma(f_*TM \oplus P)$ and

$$(\tilde{\nabla}_X\bar{L})\lambda = \tilde{\nabla}_X\bar{L}\lambda - \bar{L}\nabla'_X\lambda,$$

being ∇' the connection induced on $f_*TM \oplus P$.

Proof. We have that

$$\langle f_*X+\tilde{\nabla}_X\lambda, LX+\tilde{\nabla}_X\bar{L}\lambda\rangle = \langle \tilde{\nabla}_X\lambda, \tilde{\nabla}_X\bar{L}\lambda\rangle + \langle \tilde{\nabla}_X\lambda, LX\rangle + \langle f_*X, \tilde{\nabla}_X\bar{L}\lambda\rangle.$$

Set $\lambda = f_*Z + \phi\eta$ where $Z \in \mathfrak{X}(M)$ and $\phi \in C^\infty(M)$. We have from (2.8), (2.10), (3.7) and (3.8) that

$$\begin{aligned}
\langle \tilde{\nabla}_X\lambda, &LX\rangle + \langle f_*X, \tilde{\nabla}_X\bar{L}\lambda\rangle \\
&= \langle \tilde{\nabla}_X\lambda, LX\rangle + X\langle f_*X, \bar{L}\lambda\rangle - \langle \tilde{\nabla}_Xf_*X, \bar{L}\lambda\rangle \\
&= -\langle \tilde{\nabla}_Xf_*X, \bar{L}\lambda\rangle - \langle \lambda, \tilde{\nabla}_XLX\rangle \\
&= -\langle f_*\nabla_XX, \bar{L}\lambda\rangle - \langle \alpha(X,X), \bar{L}\lambda\rangle - \langle \lambda, L\nabla_XX\rangle - \langle \lambda, (\tilde{\nabla}_XL)X\rangle \\
&= -\langle \alpha(X,X), LZ + \phi\xi\rangle - \langle f_*Z + \phi\eta, f_*\mathcal{Y}\alpha(X,X) + \beta(X,X)\rangle \\
&= 0.
\end{aligned}$$

Thus
$$\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda\rangle = \langle\tilde{\nabla}_X\lambda, \tilde{\nabla}_X\bar{L}\lambda\rangle.$$
For simplicity, from now on we write X for both $X \in \mathfrak{X}(M)$ and its image under f_*. Calling $Y = (\tilde{\nabla}_X\lambda)_{f_*TM} = \nabla_X Z - \phi A_\eta X$, we have

$$\begin{aligned}
&\langle\tilde{\nabla}_X\lambda, \tilde{\nabla}_X\bar{L}\lambda\rangle\\
&= \langle Y + (\tilde{\nabla}_X\lambda)_P + (\tilde{\nabla}_X\lambda)_R, \tilde{\nabla}_X\bar{L}\lambda\rangle\\
&= \langle Y, (\tilde{\nabla}_X L)Z + L\nabla_X Z + X(\phi)\bar{L}\eta + \phi\tilde{\nabla}_X\bar{L}\eta\rangle + \langle(\tilde{\nabla}_X\lambda)_R, \tilde{\nabla}_X\bar{L}\lambda\rangle\\
&\quad + (\langle A_\eta X, Z\rangle + X(\phi))\langle\eta, (\tilde{\nabla}_X L)Z + L\nabla_X Z + X(\phi)\bar{L}\eta + \phi\tilde{\nabla}_X\bar{L}\eta\rangle
\end{aligned} \tag{3.12}$$

for any $X \in \mathfrak{X}(M)$. Using (2.8), (2.11) and (3.3) we obtain

$$\langle Y, (\tilde{\nabla}_X L)Z + L\nabla_X Z\rangle = -\langle LY, \alpha(X,Z)\rangle - \phi\langle A_\eta X, L\nabla_X Z\rangle \tag{3.13}$$

and

$$\langle Y, X(\phi)\bar{L}\eta + \phi\tilde{\nabla}_X\bar{L}\eta\rangle = \phi\langle Y, \nabla_X \mathfrak{Y}\eta\rangle - X(\phi)\langle LY, \eta\rangle - \phi\langle\alpha(X,Y), \xi\rangle, \tag{3.14}$$

where using (3.7) for the first term in the right hand side of (3.14) gives

$$\begin{aligned}
\langle Y, \nabla_X \mathfrak{Y}\eta\rangle &= X\langle Y, \mathfrak{Y}\eta\rangle - \langle\nabla_X Y, \mathfrak{Y}\eta\rangle\\
&= -X\langle LY, \eta\rangle + \langle L\nabla_X Y, \eta\rangle\\
&= -\langle(\tilde{\nabla}_X L)Y, \eta\rangle - \langle LY, \tilde{\nabla}_X\eta\rangle\\
&= \langle\alpha(X,Y), \xi\rangle - \langle LY, \tilde{\nabla}_X\eta\rangle.
\end{aligned} \tag{3.15}$$

Moreover,

$$\langle\eta, (\tilde{\nabla}_X L)Z + L\nabla_X Z\rangle = -\langle\alpha(X,Z), \xi\rangle + \langle\eta, L\nabla_X Z\rangle \tag{3.16}$$

and

$$\begin{aligned}
\langle\eta, X(\phi)\bar{L}\eta + \phi\tilde{\nabla}_X\bar{L}\eta\rangle &= -\phi\langle\tilde{\nabla}_X\eta, \bar{L}\eta\rangle\\
&= -\phi\langle LA_\eta X, \eta\rangle - \phi\langle\nabla^\perp_X\eta, \xi\rangle.
\end{aligned} \tag{3.17}$$

Now, a straightforward computation replacing (3.13) through (3.17) in (3.12) and using (2.8) yields

$$\begin{aligned}
&\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda\rangle\\
&\qquad = \langle(\tilde{\nabla}_X\lambda)_R, \tilde{\nabla}_X\bar{L}\lambda\rangle - \langle LY, \alpha(X,Z)_R\rangle - \phi\langle LY, \nabla^\perp_X\eta\rangle\\
&\qquad\quad - \langle\alpha(X,Z), \bar{L}(\tilde{\nabla}_X\lambda)_P\rangle - \phi\langle\nabla^\perp_X\eta, \bar{L}(\tilde{\nabla}_X\lambda)_P\rangle\\
&\qquad = \langle(\tilde{\nabla}_X\lambda)_R, (\tilde{\nabla}_X\bar{L})\lambda\rangle
\end{aligned}$$

as we wished. □

In view of (3.11) the next step is to construct a subbundle $D \subset f_*TM \oplus P$ that satisfies that

$$\langle(\tilde{\nabla}_X\lambda)_R, (\tilde{\nabla}_X\bar{L})\lambda\rangle = 0 \tag{3.18}$$

for any $X \in \mathfrak{X}(M)$ and $\lambda \in \Gamma(D)$.

Lemma 3.14. *Assume that $\mathcal{T}$ satisfies the condition $(*)$. Then, the bilinear form $\varphi\colon TM \times f_*TM \oplus P \to R \oplus R$ defined by*

$$\varphi(X,\lambda) = ((\tilde{\nabla}_X\lambda)_R + ((\tilde{\nabla}_X\bar{L})\lambda)_R, (\tilde{\nabla}_X\lambda)_R - ((\tilde{\nabla}_X\bar{L})\lambda)_R)$$

is flat with respect to the indefinite inner product given by

$$\langle\!\langle(\xi_1,\mu_1),(\xi_2,\mu_2)\rangle\!\rangle_{R\oplus R} = \langle\xi_1,\xi_2\rangle_R - \langle\mu_1,\mu_2\rangle_R.$$

Proof. We need to show that

$$\Theta = \langle\!\langle\varphi(X,\lambda),\varphi(Y,\delta)\rangle\!\rangle - \langle\!\langle\varphi(X,\delta),\varphi(Y,\lambda)\rangle\!\rangle = 0$$

for any $X, Y \in \mathfrak{X}(M)$ and $\lambda, \delta \in f_*TM \oplus P$. We have

$$\begin{aligned}\frac{1}{2}\Theta = {} & \langle(\tilde{\nabla}_X\lambda)_R, ((\tilde{\nabla}_Y\bar{L})\delta)_R\rangle + \langle(\tilde{\nabla}_Y\delta)_R, ((\tilde{\nabla}_X\bar{L})\lambda)_R\rangle \\ & - \langle(\tilde{\nabla}_X\delta)_R, ((\tilde{\nabla}_Y\bar{L})\lambda)_R\rangle - \langle(\tilde{\nabla}_Y\lambda)_R, ((\tilde{\nabla}_X\bar{L})\delta)_R\rangle.\end{aligned}$$

Clearly $\Theta = 0$ if $\lambda, \delta \in \Gamma(P)$. If $\lambda, \delta \in \mathfrak{X}(M)$, then

$$\begin{aligned}\frac{1}{2}\Theta = {} & \langle\alpha(X,\lambda)_R, ((\tilde{\nabla}_Y\bar{L})\delta)_R\rangle + \langle\alpha(Y,\delta)_R, ((\tilde{\nabla}_X\bar{L})\lambda)_R\rangle \\ & - \langle\alpha(X,\delta)_R, ((\tilde{\nabla}_Y\bar{L})\lambda)_R\rangle - \langle\alpha(Y,\lambda)_R, ((\tilde{\nabla}_X\bar{L})\delta)_R\rangle \\ = {} & \langle\alpha(X,\lambda)_R, ((\tilde{\nabla}_Y L)\delta)_R\rangle - \langle A_\eta Y,\delta\rangle\langle\alpha(X,\lambda)_R, \bar{L}\eta\rangle \\ & + \langle\alpha(Y,\delta)_R, ((\tilde{\nabla}_X L)\lambda)_R\rangle - \langle A_\eta X,\lambda\rangle\langle\alpha(Y,\delta)_R, \bar{L}\eta\rangle \\ & - \langle\alpha(X,\delta)_R, ((\tilde{\nabla}_Y L)\lambda)_R\rangle + \langle A_\eta Y,\lambda\rangle\langle\alpha(X,\delta)_R, \bar{L}\eta\rangle \\ & - \langle\alpha(Y,\lambda)_R, ((\tilde{\nabla}_X L)\delta)_R\rangle + \langle A_\eta X,\delta\rangle\langle\alpha(Y,\lambda)_R, \bar{L}\eta\rangle.\end{aligned}$$

Using first (3.7) and then (2.12) we obtain that

$$\begin{aligned}\frac{1}{2}\Theta = {} & \langle\alpha(X,\lambda),\beta(Y,\delta)\rangle + \langle\alpha(Y,\delta),\beta(X,\lambda)\rangle \\ & - \langle\alpha(X,\delta),\beta(Y,\lambda)\rangle - \langle\alpha(Y,\lambda),\beta(X,\delta)\rangle \\ = {} & 0.\end{aligned}$$

Finally, we consider the case $\lambda = \eta$ and $\delta = Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned}\frac{1}{2}\Theta = {} & \langle\nabla^\perp_X\eta, ((\tilde{\nabla}_Y L)Z)_R\rangle - \langle A_\eta Y, Z\rangle\langle\nabla^\perp_X\eta, \bar{L}\eta\rangle + \langle\alpha(Y,Z)_R, ((\tilde{\nabla}_X\bar{L})\eta)_R\rangle \\ & - \langle\nabla^\perp_Y\eta, ((\tilde{\nabla}_X L)Z)_R\rangle + \langle A_\eta X, Z\rangle\langle\nabla^\perp_Y\eta, \bar{L}\eta\rangle - \langle\alpha(X,Z)_R, ((\tilde{\nabla}_Y\bar{L})\eta)_R\rangle \\ = {} & \langle\nabla^\perp_X\eta, \beta(Y,Z)\rangle - \langle A_\eta Y, Z\rangle\langle\nabla^\perp_X\eta, \xi\rangle + \langle\alpha(Y,Z)_R, (\tilde{\nabla}_X\bar{L}\eta + LA_\eta X)_R\rangle \\ & - \langle\alpha(X,Z)_R, (\tilde{\nabla}_Y\bar{L}\eta + LA_\eta Y)_R\rangle - \langle\nabla^\perp_Y\eta, \beta(X,Z)\rangle + \langle A_\eta X, Z\rangle\langle\nabla^\perp_Y\eta, \xi\rangle.\end{aligned}$$

Notice that

$$\begin{aligned}(\tilde\nabla_X\bar L\eta+LA_\eta X)_R&=(\alpha(X,\mathcal{Y}\eta)+LA_\eta X)_R+(\nabla^\perp_X\xi)_R\\&=\mathcal{E}(X,\eta)_R+(\nabla^\perp_X\xi)_R\\&=\mathcal{E}(X,\eta)+(\nabla^\perp_X\xi)_R,\end{aligned}$$

where the last step follows from (2.7). Then

$$\begin{aligned}\frac{1}{2}\Theta&=\langle\nabla^\perp_X\eta,\beta(Y,Z)\rangle-\langle A_\eta Y,Z\rangle\langle\nabla^\perp_X\eta,\xi\rangle+\langle\alpha(Y,Z)_R,\mathcal{E}(X,\eta)+(\nabla^\perp_X\xi)_R\rangle\\&\quad-\langle\alpha(X,Z)_R,\mathcal{E}(Y,\eta)+(\nabla^\perp_Y\xi)_R\rangle-\langle\nabla^\perp_Y\eta,\beta(X,Z)\rangle+\langle A_\eta X,Z\rangle\langle\nabla^\perp_Y\eta,\xi\rangle\\&=X\langle\eta,\beta(Y,Z)\rangle-\langle\eta,\nabla^\perp_X\beta(Y,Z)\rangle+\langle\alpha(Y,Z),\mathcal{E}(X,\eta)\rangle+\langle\alpha(Y,Z),\nabla^\perp_X\xi\rangle\\&\quad-\langle\alpha(X,Z),\mathcal{E}(Y,\eta)\rangle-\langle\alpha(X,Z),\nabla^\perp_Y\xi\rangle-Y\langle\eta,\beta(X,Z)\rangle+\langle\eta,\nabla^\perp_Y\beta(X,Z)\rangle.\end{aligned}$$

Now using (3.7) we obtain

$$\begin{aligned}\frac{1}{2}\Theta&=-X\langle\xi,\alpha(Y,Z)\rangle-\langle\eta,\nabla^\perp_X\beta(Y,Z)\rangle+\langle\alpha(Y,Z),\mathcal{E}(X,\eta)\rangle+\langle\alpha(Y,Z),\nabla^\perp_X\xi\rangle\\&\quad-\langle\alpha(X,Z),\mathcal{E}(Y,\eta)\rangle-\langle\alpha(X,Z),\nabla^\perp_Y\xi\rangle+Y\langle\xi,\alpha(X,Z)\rangle+\langle\eta,\nabla^\perp_Y\beta(X,Z)\rangle\\&=-\langle\xi,(\nabla^\perp_X\alpha)(Y,Z))\rangle-\langle\eta,(\nabla^\perp_X\beta)(Y,Z)\rangle+\langle\alpha(Y,Z),\mathcal{E}(X,\eta)\rangle\\&\quad-\langle\alpha(X,Z),\mathcal{E}(Y,\eta)\rangle+\langle\xi,(\nabla^\perp_Y\alpha)(X,Z)\rangle+\langle\eta,(\nabla^\perp_Y\beta)(X,Z)\rangle\\&=0,\end{aligned}$$

where the last equality follows from (2.7), (2.13) and the Codazzi equation. □

Proof of Theorem 3.11: By Lemma 3.14 there is the flat bilinear form φ. Let U be an open subset of M^n where there is $Y\in\mathfrak{X}(U)$ such that $Y\in RE(\varphi)$ and $D=\ker\varphi_Y$ has dimension d at any point. Then Proposition 1.10 gives

$$\langle\!\langle\varphi(X,\lambda),\varphi(X,\lambda)\rangle\!\rangle=0$$

for any $X\in\mathfrak{X}(U)$ and $\lambda\in\Gamma(D)$. Notice that this implies that (3.18) holds for any $\lambda\in\Gamma(D)$. Whenever there is a nonvanishing $\lambda\in\Gamma(D)$ on an open subset $V\subset U$ such that (3.9) defines a singular extension of $f|_V$, then $\mathcal{T}|_V$ extends in the singular sense by means of (3.10).

Let $W\subset U$ be an open subset where $\lambda\in\Gamma(D)$ as above does not exist along any open subset of W. Hence D must be a tangent distribution on W, and Proposition 3.5 gives that $f|_W$ is d-ruled on connected components of an open dense subset of W. Moreover, the dimension of the rulings is bounded from below by $n+1-\dim\operatorname{Im}(\varphi_Y)\geq n-2p+3$. □

Proof of Theorem 3.9: We work on the open dense subset of M^n where f is 1-regular on any connected component. Consider an open subset of a connected component where the index of relative nullity is $\nu \leq n-2p-1$ at any point. Theorem 1.11 applies and thus the flat bilinear form $\hat{\theta}$ in (3.2) decomposes at any point as $\hat{\theta} = \theta_1 + \theta_2$ where θ_1 is as in part (i) of that result. Hence, on any open subset where the dimension of $\mathcal{S}(\theta_1) = \mathcal{S}(\hat{\theta}) \cap \mathcal{S}(\hat{\theta})^{\perp}$ is constant, there are smooth local unit vector fields $\zeta_1, \zeta_2 \in N_1$ such that $(\zeta_1, \zeta_2) \in \mathcal{S}(\theta_1)$. Equivalently, we have

$$\langle \beta(X,Y), \zeta_1 + \zeta_2 \rangle + \langle \alpha(X,Y), \zeta_1 - \zeta_2 \rangle = 0 \tag{3.19}$$

for any $X, Y \in \mathfrak{X}(M)$. Then $\zeta_1 + \zeta_2 \neq 0$ since otherwise $\zeta_1 - \zeta_2 \in N_1^{\perp}$. Hence $\mathcal{T}$ satisfies the condition $(*)$ and the proof follows from Corollary 3.12. $\square$

3.2 The global result

Dajczer-Gromoll [14] proved that along connected components of an open dense subset an isometrically deformable compact Euclidean submanifold in codimension two and of dimension at least five is either isometrically rigid or it is contained in a deformable hypersurface (with possible singularities) and that any isometric deformation of the former is given by an isometric deformation of the latter. This result was extended by Florit-Guimarães [26] to other low codimensions. The next result that concerns infinitesimal bendings of submanifolds in codimension two is of a similar nature.

Theorem 3.15. *Let $f\colon M^n \to \mathbb{R}^{n+2}$, $n \geq 5$, be an isometric immersion of a compact Riemannian manifold that does not contain an open flat subset. For any infinitesimal bending $\mathcal{T}$ of f at least one of the following facts holds along any connected component, say U, of an open dense subset of M^n:*

(i) The infinitesimal bending $\mathcal{T}|_U$ extends in the singular sense.

(ii) There is an orthogonal splitting $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus span\{e\}$ so that $f(U) \subset \mathbb{R}^{n+1}$ and $\mathcal{T}|_U = \mathcal{T}_1 + \mathcal{T}_2$ is a sum of infinitesimal bendings that extend in the singular sense where $\mathcal{T}_1 \in \mathbb{R}^{n+1}$ and $\mathcal{T}_2 = \phi e$ for $\phi \in C^{\infty}(U)$.

It will follow from the proof that the assumption on the open flat subset can be replaced by the weaker hypothesis that there is no open subset of M^n where the index of relative nullity satisfies $\nu \geq n-1$. Moreover, we will see that cases (i) and (ii) are not disjoint.

For the proof the following two results of independent interest are used.

Proposition 3.16. *Let $\mathcal{T}$ be an infinitesimal bending of $f\colon M^n \to \mathbb{R}^{n+p}$ and let θ be the flat bilinear form defined by* (3.1). *At $x \in M^n$ denote $\nu^*(x) = \dim \Delta^*(x)$ where*

$$\Delta^*(x) = \mathcal{N}(\theta)(x) = \Delta \cap \mathcal{N}(\beta)(x).$$

Then, on any open subset of M^n where ν^* is constant the distribution Δ^* is totally geodesic and its leaves are mapped by f onto open subsets of affine subspaces of $\mathbb{R}^{n+p}$.

Proof. We have from (2.13) and the definition of Δ^* that

$$(\nabla^\perp_X \beta)(Z, Y) = (\nabla^\perp_Z \beta)(X, Y) = 0$$

for any $X, Y \in \Gamma(\Delta^*)$ and $Z \in \mathfrak{X}(M)$. Let $\nabla^* = (\nabla^\perp, \nabla^\perp)$ be the compatible connection in $N_fM \oplus N_fM$. Hence

$$0 = (\nabla^*_X \theta)(Z, Y) = \theta(Z, \nabla_X Y)$$

for any $X, Y \in \Gamma(\Delta^*)$ and $Z \in \mathfrak{X}(M)$. Thus $\Delta^* \subset \Delta$ is totally geodesic. $\square$

On an open subset $U \subset M^n$ where $\nu^* > 0$ is constant, consider the orthogonal splitting $TM = \Delta^* \oplus E$. Then let $C\colon \Gamma(\Delta^*) \times \Gamma(E) \to \Gamma(E)$ be the splitting tensor of Δ^*. Notice that $\mathcal{E}(T, \eta) = 0$ for any $T \in \Gamma(\Delta)$. Then, we have from (2.13) that β verifies the conditions on Proposition 1.7, in fact, we have the following result.

Lemma 3.17. *If $\gamma\colon [0, b] \to M^n$ is a unit speed geodesic such that $\gamma([0, b))$ is contained in a leaf of Δ^* in U, then $\Delta^*(\gamma(b)) = \mathcal{P}_0^b(\Delta^*(\gamma(0)))$ where $\mathcal{P}_0^t$ is the parallel transport along γ from $\gamma(0)$ to $\gamma(t)$. In particular, we have $\nu^*(\gamma(b)) = \nu^*(\gamma(0))$ and that the tensor $C_{\gamma'}$ extends smoothly to $[0, b]$.*

We also need the following result.

Lemma 3.18. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $p \le 5$ and $n > 2p$, be an isometric immersion of a compact Riemannian manifold and let $\mathcal{T}$ be an infinitesimal bending of f. Then, at any $x \in M^n$ there is a pair of vectors $\zeta_1, \zeta_2 \in N_fM(x)$ of unit length such that $(\zeta_1, \zeta_2) \in (\mathcal{S}(\theta))^\perp(x)$ where*

$$\mathcal{S}(\theta)(x) = \mathit{span}\{\theta(X, Y) : X, Y \in T_xM\}.$$

Moreover, on any connected component of an open dense subset of M^n the pair ζ_1, ζ_2 at $x \in M^n$ extend to smooth vector fields ζ_1 and ζ_2 parallel along Δ^* that satisfy the same conditions.

Proof. We claim that the subset $U \subset M^n$ of points where there is not a pair ζ_1, ζ_2 as in the statement, that is, where the metric induced on $(\mathcal{S}(\theta))^\perp$ is positive or negative definite, is empty. We first argue that U is open. If otherwise, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ of points in $M^n \setminus U$ that converges to $x \in U$. Hence, at each x_i there is a pair of unit vectors $\zeta_1^i, \zeta_2^i \in N_fM(x_i)$ such that $\langle\!\langle \theta(X,Y)(x_i), (\zeta_1^i, \zeta_2^i)\rangle\!\rangle = 0$ for all $X, Y \in T_{x_i}M$. This determines a sequence $\{(x_i, \zeta_1^i, \zeta_2^i)\}_{i\in\mathbb{N}} \subset V \times \mathbb{S}^n \times \mathbb{S}^n$ where V is a small neighborhood of x. Take a subsequence $\{(x_j, \zeta_1^j, \zeta_2^j)\}_{j\in\mathbb{N}}$ such that $\{(\zeta_1^j, \zeta_2^j)\}_{j\in\mathbb{N}}$ converges to some (ζ_1, ζ_2) in $\mathbb{S}^n \times \mathbb{S}^n$. Then, we have that $\langle\!\langle \theta(X,Y)(x), (\zeta_1, \zeta_2)\rangle\!\rangle = 0$ for any $X, Y \in T_xM$, and this is a contradiction.

From Theorem 1.11 we have $\nu^* > 0$ in U. Let $V \subset U$ be the open subset where $\nu^* = \nu_0^*$ is minimal. Take $x_0 \in V$ and a unit speed geodesic γ in M^n contained in a maximal leaf of Δ^* with $\gamma(0) = x_0$. Since M^n is compact, there is $b > 0$ such that $\gamma([0,b)) \subset V$ and $\gamma(b) \notin V$. Lemma 3.17 gives $\nu^*(\gamma(b)) = \nu_0^*$ which implies $\gamma(b) \notin U$. Hence, there are unit vectors $\zeta_1, \zeta_2 \in N_fM(\gamma(b))$ such that $(\zeta_1, \zeta_2) \in (\mathcal{S}(\theta))^\perp(\gamma(b))$.

Let $\zeta_i(t)$ be the parallel transport along $\gamma = \gamma(t)$ of ζ_i, $i = 1, 2$. Then

$$\langle\!\langle \theta(X,Y), (\zeta_1, \zeta_2)\rangle\!\rangle = \langle (A_{\zeta_1 - \zeta_2} + B_{\zeta_1+\zeta_2})X, Y\rangle.$$

It follows from (2.13) and the Codazzi equation that

$$(\nabla_T^*\theta)(X,Y) = (\nabla_X^*\theta)(T,Y), \tag{3.20}$$

where $T \in \Gamma(\Delta^*)$ extends γ' and $X, Y \in \mathfrak{X}(M)$. Along γ this gives

$$\frac{D}{dt}\mathcal{C}_{\zeta_1,\zeta_2} = \mathcal{C}_{\zeta_1,\zeta_2} C_{\gamma'} = C_{\gamma'}^t \mathcal{C}_{\zeta_1,\zeta_2},$$

where $\mathcal{C}_{\zeta_1,\zeta_2} = A_{\zeta_1-\zeta_2} + B_{\zeta_1+\zeta_2}$ and $C_{\gamma'}^t$ is the transpose of $C_{\gamma'}$. Moreover, by Lemma 3.17 this ODE holds on $[0,b]$. Given that $\mathcal{C}_{\zeta_1,\zeta_2}(\gamma(b)) = 0$, then $\mathcal{C}_{\zeta_1,\zeta_2}$ vanishes along γ. This is a contradiction and proves the claim.

We have from (3.20) that

$$(\nabla_T^*\theta)(X,Y) = -\theta(\nabla_X T, Y) \in \Gamma(\mathcal{S}(\theta))$$

for any $T \in \Gamma(\Delta^*)$ and $X, Y \in \mathcal{X}(M)$. Thus $\mathcal{S}(\theta)$ is parallel along the leafs of Δ^*. Let U_0 be a connected component of the open dense subset of M^n where the dimensions of Δ^*, $\mathcal{S}(\theta), \mathcal{S}(\theta) \cap \mathcal{S}(\theta)^\perp$ as well as the index of the metric induced on $\mathcal{S}(\theta)^\perp \times \mathcal{S}(\theta)^\perp$ are all constant. Hence, on U_0 the vector fields ζ_1, ζ_2 can be taken parallel along the leafs of Δ^*. □

Finally, for the proof of Theorem 3.15 we also need a result about extensions of submanifolds in codimension two.

Let $f\colon M^n \to \mathbb{R}^{n+2}$ be an isometric immersion. Assume that there is a smooth line subbundle $R \subset N_fM$, such that the tangent subspace

$D(x) = \mathcal{N}(\alpha_R)(x)$ has dimension $n-k$ at every point $x \in M^n$. Then D is a smooth tangent distribution. Assume further that R is parallel along D with respect to the normal connection. Then, we have that R is constant along D. Decompose the tangent and normal bundles orthogonally as

$$TM = D \oplus E, \;\; N_f M = P \oplus R.$$

At each point $x \in M^n$, define

$$\Gamma(x) = \operatorname{span}\{(\tilde{\nabla}_X \eta)_{f_* E \oplus P} \colon X \in E(x), \eta \in R(x)\}.$$

Then it follows from our assumptions that Γ is a smooth rank-k subbundle of $f_* E \oplus P$. Let Λ be given by the orthogonal decomposition $f_* E \oplus P = \Gamma \oplus \Lambda$ and let $\lambda \in \Gamma(\Lambda)$ be a nowhere vanishing section of Λ. The following holds.

Proposition 3.19. *The map $F\colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+2}$ given, for some $\epsilon > 0$, by*

$$F(x,t) = f(x) + t\lambda(x)$$

parametrizes a hypersurface whose second fundamental form has constant rank k. Moreover, its relative nullity subspaces are $\Delta^F = D \oplus \mathrm{span}\{\partial/\partial t\}$.

Proof. F is an immersion since λ is nowhere tangent to f. We have

$$F_* Z = f_* Z + t\tilde{\nabla}_Z \lambda$$

for any $Z \in \mathfrak{X}(M)$. From the definition of Λ we obtain $\langle \tilde{\nabla}_Z \lambda, \eta \rangle = 0$ for $\eta \in \Gamma(R)$. Thus, the normal space of F at (x,t) coincides with the parallel transport of R along the segment parameterized by t. In particular, we have that $\Delta^F = D \oplus \operatorname{span}\{\partial/\partial t\}$, and hence the second fundamental form of F has constant rank k. $\square$

Proof of Theorem 3.15: We assume that there is no open subset of M^n where the index of relative nullity satisfies $\nu \geq n-1$. By Lemma 3.18, on connected components of an open dense subset of M^n there are $\zeta_1, \zeta_2 \in \Gamma(N_f M)$ with $\|\zeta_1\| = \|\zeta_2\| = 1$ that are parallel along the leaves of Δ^* and

$$\langle\!\langle \theta(X,Y), (\zeta_1, \zeta_2) \rangle\!\rangle = 0$$

for any $X, Y \in \mathfrak{X}(M)$. It follows from (3.1) that (3.19) holds on connected components of an open dense subset of M^n. Let $U \subset M^n$ be an open subset where ζ_1, ζ_2 are smooth and $\zeta_1 + \zeta_2 \neq 0$. Thus $\mathcal{T}|_U$ satisfies the condition $(*)$. Let $\tilde{V} \subset U$ be an open subset where $\mathcal{T}$ is a genuine infinitesimal bending. By Corollary 3.12 we have that f is $(n-1)$-ruled on each connected component V of an open dense subset of $\tilde{V}$. Since our goal is to show that V is empty we assume otherwise.

Proposition 3.5 and the proof of Theorem 3.11 yield that the rulings on V are determined by the tangent subbundle $D = \ker \varphi_X$ where φ was given in Lemma 3.14 and $X \in RE(\varphi)$. Also from that proof we have $\dim \mathrm{Im}(\varphi_X) = 2$, and therefore $\mathrm{Im}(\varphi_X) = R \oplus R$ where $N_f M = P \oplus R$ as in Lemma 3.14. Proposition 1.10 gives

$$\varphi_Y(D) \subset \mathrm{Im}(\varphi_X) \cap \mathrm{Im}(\varphi_X)^\perp = 0$$

for any $Y \in \mathfrak{X}(M)$, that is, $D = \mathcal{N}(\varphi)$. In particular, from the definition of φ it follows that $D \subset \mathcal{N}(\alpha_R)$. Hence, by dimension reasons either $\mathcal{N}(\alpha_R) = TM$ or $D = \mathcal{N}(\alpha_R)$. Next we contemplate both possibilities.

Let $V_1 \subset V$ be an open subset where $\mathcal{N}(\alpha_R) = TM$ holds, that is, $N_1 = P$. Thus N_1 is parallel relative to the normal connection since, otherwise, the Codazzi equation gives $\nu = n-1$, and that has been ruled out. Hence $f|_{V_1}$ reduces codimension, that is, $f(V_1)$ is contained in an affine hyperplane $\mathbb{R}^{n+1}$. Decompose $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are tangent and normal to $\mathbb{R}^{n+1}$, respectively. It follows that $\mathcal{T}_1$ is an infinitesimal bending of $f|_{V_1}$ in $\mathbb{R}^{n+1}$. Since $\mathcal{T}$ satisfies the condition $(*)$ it follows from Proposition 2.10 that $\mathcal{T}_1$ is trivial, that is, the restriction of a Killing vector field of $\mathbb{R}^{n+1}$ to $f(V_1)$. Extending $\mathcal{T}_2$ as a vector field normal to $\mathbb{R}^{n+1}$ we have that $\mathcal{T}|_{V_1}$ extends in the singular sense, and this is a contradiction.

Let $V_2 \subset V$ be an open subset where $D = \mathcal{N}(\alpha_R)$. By assumption $D \neq \Delta$. Let $\hat{D}$ be the distribution tangent to the rulings in a neighborhood V_2' of $x_0 \in V_2$. From Proposition 3.5 we have $D(x_0) = \hat{D}(x_0)$. Let $W \subset V_2'$ be an open subset where $D \neq \hat{D}$, that is, where D is not totally geodesic. Then there are two transversal $(n-1)$-dimensional rulings passing through any point $y \in W$. It follows easily that $N_1 = P$ on W. As above, we obtain that $\mathcal{T}|_W$ extends in the singular sense leading to a contradiction. Let $V_3 \subset V_2$ be the interior of the subset where D is totally geodesic. On V_3 the Codazzi equation gives

$$\nabla_X^\perp \alpha(Z, Y) \in \Gamma(P)$$

for all $X, Y \in \Gamma(D)$ and $Z \in \mathfrak{X}(M)$. Thus R is parallel along D relative to the normal connection. We have from Proposition 3.19 that f admits a singular extension

$$F(x,t) = f(x) + t\lambda(x)$$

for $\lambda \in \Gamma(f_*TM \oplus P)$ as a flat hypersurface. Moreover, F has R as normal bundle and $\partial/\partial t$ belongs to the relative nullity distribution. Therefore $(\tilde{\nabla}_X \lambda)_R = 0$ for any $X \in \mathfrak{X}(V_3)$. Hence (3.18) is satisfied and thus $\mathcal{T}|_{V_3}$ extends in the singular sense. This is a contradiction which shows that V is empty, and hence also is $\tilde{V}$.

It remains to consider the existence of an open subset $U' \subset M^n$ where ζ_1, ζ_2 are smooth and $\zeta_1 + \zeta_2 = 0$. It follows from (3.19) that $\zeta_1 - \zeta_2 \perp N_1$.

Once more, we obtain that $f(U') \subset \mathbb{R}^{n+1}$. Thus, we have an orthogonal decomposition of $\mathcal{T}|_{U'}$ as in part (ii) of the statement and $\mathcal{T}_1, \mathcal{T}_2$ extend in the singular sense as follows:

(i) $\bar{\mathcal{T}}_1(x,t) = \mathcal{T}_1(x)$ to $F\colon U \times \mathbb{R} \to \mathbb{R}^{n+2}$ where $F(x,t) = f(x) + te$.

(ii) For instance locally as $\bar{\mathcal{T}}_2(x,t) = \mathcal{T}_2(x)$ to $F\colon U \times I \to \mathbb{R}^{n+2}$ where $F(x,t) = f(x)+tN$ being N is a unit normal field to $f|_U$ in $\mathbb{R}^{n+1}$. $\square$

Remarks 3.20. (1) In case (ii) of Theorem 3.15 if $\mathcal{T}_1$ is trivial then $\mathcal{T}_1$ and $\mathcal{T}_2$ extend in the same direction, and thus $\mathcal{T}$ also extends. Therefore we are also in case (i).

(2) Notice that for $p = 2$ we have shown as part of the proof that an infinitesimal bending of a submanifold without flat points as in part (ii) of Theorem 3.9 cannot be genuine.

3.3 Exercises

Exercise 3.1. Prove the Corollary 3.3.

Exercise 3.2. Verify the assertion in Remark 3.8.

Exercise 3.3. Verify the assertion in Remark 3.10.

Chapter 4

Nonflat ambient spaces

In this chapter, we extend several results from the previous chapters to the case of submanifolds of simply connected complete space form $\mathbb{Q}_c^m$ with sectional curvature $c \neq 0$, that is, either the sphere or the hyperbolic space according to whether $c > 0$ or $c < 0$, respectively.

A section $\mathcal{T}$ of $f^*T\mathbb{Q}_c^m$ is called an *infinitesimal bending* of an isometric immersion $f\colon M^n \to \mathbb{Q}_c^m$ if the condition

$$\langle \tilde{\nabla}_X \mathcal{T}, f_*Y\rangle + \langle f_*X, \tilde{\nabla}_Y \mathcal{T}\rangle = 0$$

holds for any $X, Y \in \mathfrak{X}(M)$. Here $\tilde{\nabla}$ denotes the Levi-Civita connection of $\mathbb{Q}_c^m$. Then

$$\langle LX, f_*Y\rangle + \langle f_*X, LY\rangle = 0, \tag{4.1}$$

where $L \in \Gamma(\mathrm{Hom}(TM, f^*T\mathbb{Q}_c^m))$ is defined by

$$LX = \tilde{\nabla}_X \mathcal{T}$$

for any $X \in \mathfrak{X}(M)$.

Let $B\colon TM \times TM \to f^*T\mathbb{Q}_c^m$ be given by

$$B(X, Y) = (\tilde{\nabla}_X L)Y \tag{4.2}$$

for any $X, Y \in \mathfrak{X}(M)$. Now B is not symmetric differently to the case of the Euclidean ambient space. In fact, we have

$$\begin{aligned} B(X, Y) - B(Y, X) &= \tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{T} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{T} - \tilde{\nabla}_{[X,Y]} \mathcal{T} \\ &= \tilde{R}(X, Y)\mathcal{T} \\ &= c(f_*X \wedge f_*Y)\mathcal{T}, \end{aligned} \tag{4.3}$$

where $\tilde{R}$ denotes the curvature tensor of the ambient space.

Let $\beta\colon TM\times TM\to N_fM$ be the tensor defined by

$$\beta(X,Y)=(B(X,Y)+c\langle X,Y\rangle\mathcal{T})_{N_fM}$$

for any $X,Y\in\mathfrak{X}(M)$. It follows from (4.3) that β is symmetric. Hence, the tensor $B_\xi\in\Gamma(\mathrm{End}(TM))$ associated to $\xi\in\Gamma(N_fM)$ given by

$$\langle B_\xi X,Y\rangle=\langle\beta(X,Y),\xi\rangle$$

is also symmetric.

Let $\mathcal{Y}\in\Gamma(\mathrm{Hom}(N_fM,TM))$ be defined by

$$\langle\mathcal{Y}\eta,X\rangle+\langle LX,\eta\rangle=0 \tag{4.4}$$

for any $X\in\mathfrak{X}(M)$ and $\eta\in\Gamma(N_fM)$. Then, the tensor $\mathcal{E}\colon TM\times N_fM\to N_fM$ given by

$$\mathcal{E}(X,\eta)=\alpha(X,\mathcal{Y}\eta)+(LA_\eta X)_{N_fM}$$

satisfies

$$\langle\mathcal{E}(X,\eta),\xi\rangle+\langle\mathcal{E}(X,\xi),\eta\rangle=0 \tag{4.5}$$

for any $X\in\mathfrak{X}(M)$ and $\eta,\xi\in\Gamma(N_fM)$.

Proposition 4.1. *We have that*

$$B(X,Y)=f_*\mathcal{Y}\alpha(X,Y)+\beta(X,Y)+c(f_*X\wedge\mathcal{T})f_*Y \tag{4.6}$$

for any $X,Y\in\mathfrak{X}(M)$.

Proof. We have to show that

$$C(X,Y,Z)=\langle(B-f_*\mathcal{Y}\alpha)(X,Y),f_*Z\rangle+c\langle X,Y\rangle\langle\mathcal{T},f_*Z\rangle-c\langle X,Z\rangle\langle\mathcal{T},f_*Y\rangle$$

vanishes for any $X,Y,Z\in\mathfrak{X}(M)$. The derivative of (4.1) gives

$$\begin{aligned}
0&=\langle\tilde\nabla_Z LX,f_*Y\rangle+\langle LX,\tilde\nabla_Z f_*Y\rangle+\langle\tilde\nabla_Z LY,f_*X\rangle+\langle LY,\tilde\nabla_Z f_*X\rangle\\
&=\langle B(Z,X),f_*Y\rangle+\langle L\nabla_Z X,f_*Y\rangle+\langle LX,f_*\nabla_Z Y+\alpha(Z,Y)\rangle\\
&\quad+\langle B(Z,Y),f_*X\rangle+\langle L\nabla_Z Y,f_*X\rangle+\langle LY,f_*\nabla_Z X+\alpha(Z,X)\rangle\\
&=\langle B(Z,X),f_*Y\rangle+\langle LX,\alpha(Z,Y)\rangle+\langle B(Z,Y),f_*X\rangle+\langle LY,\alpha(Z,X)\rangle\\
&=\langle(B-f_*\mathcal{Y}\alpha)(Z,X),f_*Y\rangle+\langle(B-f_*\mathcal{Y}\alpha)(Z,Y),f_*X\rangle.
\end{aligned}$$

It follows that

$$C(Z,X,Y)=-C(Z,Y,X)$$

for any $X,Y,Z\in\mathfrak{X}(M)$. On the other hand, from (4.3) we obtain that

$$\begin{aligned}
C(X,Y,Z)-C(Y,X,Z)&=\langle B(X,Y)-B(Y,X),Z\rangle-c\langle\mathcal{T},f_*Y\rangle\langle X,Z\rangle\\
&\quad+c\langle\mathcal{T},f_*X\rangle\langle Y,Z\rangle\\
&=0.
\end{aligned}$$

By the above, we have

$$\begin{aligned}C(X,Y,Z) &= -C(X,Z,Y) = -C(Z,X,Y) = C(Z,Y,X)\\ &= C(Y,Z,X) = -C(Y,X,Z) = -C(X,Y,Z)\\ &= 0,\end{aligned}$$

as we wished. $\square$

Proposition 4.2. *The pair $(\beta, \mathcal{E})$ associated to an infinitesimal bending $\mathcal{T}$ of an isometric immersion $f\colon M^n \to \mathbb{Q}_c^m$ satisfies the following system of three equations:*

$$A_{\beta(Y,Z)}X + B_{\alpha(Y,Z)}X = A_{\beta(X,Z)}Y + B_{\alpha(X,Z)}Y, \tag{4.7}$$

$$(\nabla_X^\perp \beta)(Y,Z) - (\nabla_Y^\perp \beta)(X,Z) = \mathcal{E}(Y,\alpha(X,Z)) - \mathcal{E}(X,\alpha(Y,Z)) \tag{4.8}$$

and

$$\begin{aligned}(\nabla_X^\perp \mathcal{E})(Y,\eta) - (\nabla_Y^\perp \mathcal{E})(X,\eta) = \beta(X, A_\eta Y) - \beta(A_\eta X, Y)\\ +\alpha(X, B_\eta Y) - \alpha(B_\eta X, Y)\end{aligned} \tag{4.9}$$

for any $X,Y,Z \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$.

Proof. We first show that

$$(\tilde{\nabla}_X \mathcal{Y})\eta = c\langle \mathcal{T}, \eta\rangle f_*X - f_*B_\eta X - LA_\eta X + \mathcal{E}(X,\eta) \tag{4.10}$$

for any $X \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$, where

$$(\tilde{\nabla}_X \mathcal{Y})\eta = \tilde{\nabla}_X f_*\mathcal{Y}\eta - f_*\mathcal{Y}\nabla_X^\perp \eta.$$

From the derivative of (4.4) we have using (4.1) and (4.6) that

$$\begin{aligned}0 &= \langle \tilde{\nabla}_X f_*\mathcal{Y}\eta, f_*Y\rangle + \langle \mathcal{Y}\eta, \nabla_X Y\rangle + \langle \tilde{\nabla}_X LY, \eta\rangle + \langle LY, \tilde{\nabla}_X \eta\rangle\\ &= \langle (\tilde{\nabla}_X \mathcal{Y})\eta, f_*Y\rangle + \langle B_\eta X, Y\rangle - c\langle \mathcal{T}, \eta\rangle\langle X, Y\rangle + \langle LA_\eta X, f_*Y\rangle.\end{aligned}$$

Since $\langle f_*\mathcal{Y}\eta, \xi\rangle = 0$, we obtain

$$\begin{aligned}0 &= \langle \tilde{\nabla}_X f_*\mathcal{Y}\eta, \xi\rangle + \langle f_*\mathcal{Y}\eta, \tilde{\nabla}_X \xi\rangle = \langle (\tilde{\nabla}_X \mathcal{Y})\eta, \xi\rangle - \langle \alpha(X, \mathcal{Y}\eta), \xi\rangle\\ &= \langle (\tilde{\nabla}_X \mathcal{Y})\eta, \xi\rangle + \langle LA_\eta X - \mathcal{E}(X,\eta), \xi\rangle\end{aligned}$$

for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_fM)$, and (4.10) follows.

Since

$$(\tilde{\nabla}_X B)(Y,Z) = \tilde{\nabla}_X(\tilde{\nabla}_Y L)Z - (\tilde{\nabla}_{\nabla_X Y} L)Z - (\tilde{\nabla}_Y L)\nabla_X Z$$

it is easy to see that

$$
\begin{aligned}
(\tilde{\nabla}_X B)(Y,Z) - (\tilde{\nabla}_Y B)(X,Z) &= \tilde{R}(X,Y)LZ - LR(X,Y)Z \\
&= c(X \wedge Y)LZ - LR(X,Y)Z \qquad (4.11)
\end{aligned}
$$

for any $X,Y,Z \in \mathfrak{X}(M)$. It follows using (4.6) that

$$
\begin{aligned}
(\tilde{\nabla}_X B)(Y,Z) = (\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) &+ f_*\mathcal{Y}(\nabla^\perp_X \alpha)(Y,Z) + (\nabla^\perp_X \beta)(Y,Z) \\
&+ c\langle Z, \mathcal{T}\rangle \alpha(X,Y) - c\langle Y,Z\rangle LX - f_* A_{\beta(Y,Z)}X \\
&+ c(\langle \alpha(X,Z), \mathcal{T}\rangle + \langle f_*Z, LX\rangle) f_*Y
\end{aligned}
$$

for any $X,Y,Z \in \mathfrak{X}(M)$. Then, the Codazzi equation gives

$$
\begin{aligned}
&\langle (\tilde{\nabla}_X B)(Y,Z) - (\tilde{\nabla}_Y B)(X,Z), f_*W\rangle \\
&\qquad = \langle (\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) - (\tilde{\nabla}_Y \mathcal{Y})\alpha(X,Z), f_*W\rangle + c(\langle \alpha(X,Z), \mathcal{T}\rangle \\
&\qquad\quad + \langle f_*Z, LX\rangle)\langle Y, W\rangle - c(\langle \alpha(Y,Z), \mathcal{T}\rangle + \langle f_*Z, LY\rangle)\langle X, W\rangle \\
&\qquad\quad + \langle A_{\beta(X,Z)}Y - A_{\beta(Y,Z)}X, W\rangle - \langle cL(X\wedge Y)Z, f_*W\rangle
\end{aligned}
$$

for any $X,Y,Z,W \in \mathfrak{X}(M)$. Hence (4.1), (4.11) and the Gauss equation (1.4) yield

$$
\begin{aligned}
&\langle (\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) - (\tilde{\nabla}_Y \mathcal{Y})\alpha(X,Z), f_*W\rangle \\
&= c\langle \alpha(Y,Z), \mathcal{T}\rangle\langle X,W\rangle - c\langle \alpha(X,Z), \mathcal{T}\rangle\langle Y,W\rangle \\
&\quad + \langle LA_{\alpha(X,Z)}Y - LA_{\alpha(Y,Z)}X, f_*W\rangle + \langle A_{\beta(Y,Z)}X - A_{\beta(X,Z)}Y, W\rangle.
\end{aligned}
$$

On the other hand, it follows from (4.10) that

$$
\begin{aligned}
&\langle (\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) - (\tilde{\nabla}_Y \mathcal{Y})\alpha(X,Z), f_*W\rangle \\
&= c\langle \alpha(Y,Z), \mathcal{T}\rangle\langle X,W\rangle - c\langle \alpha(X,Z), \mathcal{T}\rangle\langle Y,W\rangle \\
&\quad + \langle LA_{\alpha(X,Z)}Y - LA_{\alpha(Y,Z)}X, f_*W\rangle + \langle B_{\alpha(X,Z)}Y - B_{\alpha(Y,Z)}X, W\rangle.
\end{aligned}
$$

The last two equations give

$$
\langle A_{\beta(Y,Z)}X - A_{\beta(X,Z)}Y, W\rangle = \langle B_{\alpha(X,Z)}Y - B_{\alpha(Y,Z)}X, W\rangle,
$$

and this is (4.7).

Using (4.6) we obtain

$$
\begin{aligned}
((\tilde{\nabla}_X B)(Y,Z))_{N_fM} = \alpha(X, \mathcal{Y}\alpha(Y,Z)) &+ (\nabla^\perp_X \beta)(Y,Z) \\
&+ c\langle f_*Z, \mathcal{T}\rangle \alpha(X,Y) - c\langle Y,Z\rangle (LX)_{N_fM}.
\end{aligned}
$$

Then, we have from (4.11) and the Gauss equation that

$$
\begin{aligned}
&(\nabla^\perp_X \beta)(Y,Z) - (\nabla^\perp_Y \beta)(X,Z) \\
&= (LR(Y,X)Z)_{N_fM} - \alpha(X, \mathcal{Y}\alpha(Y,Z)) + \alpha(Y, \mathcal{Y}\alpha(X,Z)) \\
&\quad + c\langle Y,Z\rangle (LX)_{N_fM} - c\langle X,Z\rangle (LY)_{N_fM} \\
&= (LA_{\alpha(X,Z)}Y - LA_{\alpha(Y,Z)}X)_{N_fM} \\
&\quad - \alpha(X, \mathcal{Y}\alpha(Y,Z)) + \alpha(Y, \mathcal{Y}\alpha(X,Z)),
\end{aligned}
$$

and this is (4.8).

We have

$$\begin{aligned}(\nabla^\perp_X \mathcal{E})(Y,\eta) &= \nabla^\perp_X \mathcal{E}(Y,\eta) - \mathcal{E}(\nabla_X Y,\eta) - \mathcal{E}(Y,\nabla^\perp_X\eta)\\ &= (\nabla^\perp_X \alpha)(Y,\mathcal{Y}\eta) + (L(\nabla_X A)(Y,\eta))_{N_fM} + \alpha(Y,\nabla_X\mathcal{Y}\eta)\\ &\quad - \alpha(Y,\mathcal{Y}\nabla^\perp_X\eta) - (L\nabla_X A_\eta Y)_{N_fM} + \nabla^\perp_X (LA_\eta Y)_{N_fM}.\end{aligned}$$

Then (4.10) gives

$$\begin{aligned}(\nabla^\perp_X \mathcal{E})(Y,\eta) = (\nabla^\perp_X \alpha)(Y,\mathcal{Y}\eta) &+ (L(\nabla_X A)(Y,\eta))_{N_fM} - \alpha(Y,B_\eta X)\\ &- \alpha(Y,(LA_\eta X)_{TM}) - (L\nabla_X A_\eta Y)_{N_fM}\\ &+ \nabla^\perp_X (LA_\eta Y)_{N_fM} + c\langle \mathcal{T},\eta\rangle \alpha(X,Y).\end{aligned}$$

Using the Codazzi equation, we have

$$\begin{aligned}&(\nabla^\perp_X \mathcal{E})(Y,\eta) - (\nabla^\perp_Y \mathcal{E})(X,\eta)\\ &= \alpha(X,B_\eta Y) - \alpha(Y,B_\eta X) + \alpha(X,(LA_\eta Y)_{TM}) - \alpha(Y,(LA_\eta X)_{TM})\\ &\quad - (L\nabla_X A_\eta Y)_{N_fM} + \nabla^\perp_X (LA_\eta Y)_{N_fM}\\ &\quad + (L\nabla_Y A_\eta X)_{N_fM} - \nabla^\perp_Y (LA_\eta X)_{N_fM}.\end{aligned}$$

From (4.6) and the definition of B we obtain

$$\begin{aligned}&\beta(X,A_\eta Y) - c\langle X,A_\eta Y\rangle (\mathcal{T})_{N_fM}\\ &\quad = \alpha(X,(LA_\eta Y)_{TM}) - (L\nabla_X A_\eta Y)_{N_fM} + \nabla^\perp_X (LA_\eta Y)_{N_fM},\end{aligned}$$

and then (4.9) follows. □

Let $\mathcal{T}$ be an infinitesimal bending of $f\colon M^n \to \mathbb{Q}^m_c$. Let $i\colon \mathbb{Q}^m_c \to \mathbb{E}^{m+1}$ stand for the isometric umbilical inclusion, where $\mathbb{E}^{m+1}$ denotes either the Euclidean space $\mathbb{R}^{m+1}$ ($c>0$) or Lorentzian space $\mathbb{L}^{m+1}$ ($c<0$) with the standard flat metric. Recall that the position vector $\hat f = i\circ f$ in $\mathbb{E}^{m+1}$ lies in $N_{\hat f}M$. Hence, for simplicity we write $f \in \Gamma(N_{\hat f}M)$. Let $\hat\nabla$ denote the Levi-Civita connection on $\mathbb{E}^{m+1}$ and regard $\hat{\mathcal{T}} = i_*\mathcal{T}$ as an infinitesimal bending of $\hat f$. Then the tensor $\hat L X = \hat\nabla_X \hat{\mathcal{T}}$ satisfies

$$\hat L X = i_* L X - c\langle f_* X, \mathcal{T}\rangle f \tag{4.12}$$

for any $X \in \mathfrak{X}(M)$.

Lemma 4.3. *The pair $(\hat\beta,\hat{\mathcal{E}})$ associated to $\hat{\mathcal{T}}$ is given by*

$$\hat\beta(X,Y) = i_*\beta(X,Y) - c\langle X,Y\rangle i_*(\mathcal{T})_{N_fM} - c\langle \alpha(X,Y),\mathcal{T}\rangle f,$$

$$\hat{\mathcal{E}}(X,i_*\eta) = i_*\mathcal{E}(X,\eta) - c\left(\langle X,\mathcal{Y}\eta\rangle + \langle f_* A_\eta X,\mathcal{T}\rangle\right) f$$

and

$$\hat{\mathcal{E}}(X,f) = -i_*\nabla^\perp_X(\mathcal{T})_{N_fM}$$

for any $X,Y\in\mathfrak{X}(M)$ *and* $\eta\in\Gamma(N_fM)$.

Proof. Let $\hat\nabla$ denote the Levi-Civita connection in $\mathbb{E}^{m+1}$. We have that

$$\begin{aligned}\hat\nabla_X\hat LY - \hat L\nabla_XY &= \hat\nabla_X(i_*LY - c\langle f_*Y,\mathcal{T}\rangle f) - i_*L\nabla_XY + c\langle f_*\nabla_XY,\mathcal{T}\rangle f\\ &= i_*(\tilde\nabla_XL)Y - c\langle f_*Y,\mathcal{T}\rangle\hat f_*X - c\langle\alpha(X,Y),\mathcal{T}\rangle f,\end{aligned}$$

where for the last step we used (4.1). Recall that $\hat\beta$ is the normal component of $\hat\nabla\hat L$. It follows from (4.6) that

$$\hat\beta(X,Y) = i_*\beta(X,Y) - c\langle X,Y\rangle i_*(\mathcal{T})_{N_fM} - c\langle\alpha(X,Y),\mathcal{T}\rangle f$$

for any $X,Y\in\mathfrak{X}(M)$.

Let $\hat{\mathcal{Y}}$ associated to $\hat{\mathcal{T}}$ be given by (2.10). We have from (4.12) that

$$\hat{\mathcal{Y}}i_*\eta = \mathcal{Y}\eta$$

for any $\eta\in\Gamma(N_fM)$. It also follows from (4.12) that

$$f_*\hat{\mathcal{Y}}f = (\mathcal{T})_{f_*TM}.$$

Therefore, we obtain that

$$\begin{aligned}\hat{\mathcal{E}}(X,i_*\eta) &= \hat\alpha(X,\mathcal{Y}\eta) + (\hat LA_\eta X)_{N_{\hat f}M}\\ &= i_*\alpha(X,\mathcal{Y}\eta) - c\langle X,\mathcal{Y}\eta\rangle f + i_*(LA_\eta X)_{N_fM} - c\langle f_*A_\eta X,\mathcal{T}\rangle f\\ &= i_*\mathcal{E}(X,\eta) - c(\langle X,\mathcal{Y}\eta\rangle + \langle f_*A_\eta X,\mathcal{T}\rangle)f.\end{aligned}$$

As for the position vector f, we have

$$\begin{aligned}\hat{\mathcal{E}}(X,f) &= \hat\alpha(X,\hat{\mathcal{Y}}f) + (\hat L\hat A_fX)_{N_{\hat f}M}\\ &= \hat\alpha(X,(\mathcal{T})_{TM}) - (\hat LX)_{N_{\hat f}M}\\ &= -i_*\nabla^\perp_X(\mathcal{T})_{N_fM},\end{aligned}$$

and the proof follows. □

An infinitesimal bending $\mathcal{T}$ of an isometric immersion $f\colon M^n\to\mathbb{Q}^m_c$ is said to be a *trivial infinitesimal bending* if it is of the form $\mathcal{T}=\mathcal{Z}+w$, where $\mathcal{Z}$ is the restriction to $f(M)$ of a Killing vector field of $\mathbb{Q}^m_c$ and $w\in\Gamma(N_fM)$ is parallel along f with respect to the ambient connection, that is, if $\tilde\nabla_Xw=0$ for any $X\in\mathfrak{X}(M)$. Notice that if $w\neq0$ then f necessarily reduces codimension.

Proposition 4.4. *An infinitesimal bending $\mathcal{T}$ of $f\colon M^n \to \mathbb{Q}_c^m$ is trivial if and only if there is a skew-symmetric tensor $C \in \Gamma(End(N_fM))$ such that the associated pair $(\beta, \mathcal{E})$ has the form*

$$\beta(X,Y) = C\alpha(X,Y) \ \text{ and } \ \mathcal{E}(X,\eta) = -(\nabla_X^\perp C)\eta. \tag{4.13}$$

Proof. If $\mathcal{T}$ is trivial it has the form $\mathcal{T} = \mathcal{Z} + w$ where $\mathcal{Z}$ is the restriction to the submanifold of a Killing vector field $\mathcal{Z}$ of $\mathbb{Q}_c^m$ and $w \in \Gamma(N_fM)$ is parallel along f. Then the associated tensor L satisfies

$$LX = \mathcal{L}X$$

for any $X \in \mathfrak{X}(M)$, where $\mathcal{L}$ is given by $\mathcal{L}U = \tilde\nabla_U \mathcal{Z}$ for any $U \in \mathfrak{X}(\mathbb{Q}_c^m)$. Since $\mathcal{Z}$ is a Killing vector field it satisfies

$$\langle \mathcal{L}U, V\rangle + \langle \mathcal{L}V, U\rangle = 0$$

for any $U, V \in \mathfrak{X}(\mathbb{Q}_c^m)$. Thus, we have

$$\begin{aligned}(\tilde\nabla_X L)Y &= \tilde\nabla_X LY - L\nabla_X Y = \tilde\nabla_X \tilde\nabla_Y \mathcal{T} - \tilde\nabla_{\nabla_X Y}\mathcal{T}\\ &= \tilde\nabla_X \tilde\nabla_Y \mathcal{Z} - \tilde\nabla_{\tilde\nabla_X Y}\mathcal{Z} + \tilde\nabla_{\alpha(X,Y)}\mathcal{Z}\\ &= \tilde\nabla_X \tilde\nabla_Y \mathcal{Z} - \tilde\nabla_{\tilde\nabla_X Y}\mathcal{Z} + \mathcal{L}\alpha(X,Y)\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. It follows from Exercise 4.1 that

$$(\tilde\nabla_X L)Y = c(f_*X \wedge \mathcal{T})f_*Y + \mathcal{L}\alpha(X,Y)$$

for any $X, Y \in \mathfrak{X}(M)$. Hence, we obtain from (4.6) that

$$c(f_*X \wedge \mathcal{T})f_*Y + \mathcal{L}\alpha(X,Y) = f_*\mathcal{Y}\alpha(X,Y) + \beta(X,Y) + c(f_*X \wedge \mathcal{T})f_*Y.$$

Then, calling $C = (\mathcal{L}|_{N_fM})_{N_fM}$ we have

$$\beta(X,Y) = C\alpha(X,Y).$$

Since $\mathcal{Z}$ is a Killing vector field, then $f_*\mathcal{Y}\eta = (\mathcal{L}\eta)_{f_*TM}$ where $\eta \in \Gamma(N_fM)$. This implies that

$$\begin{aligned}(\tilde\nabla_X \mathcal{L})\eta &= \tilde\nabla_X \mathcal{L}\eta - \mathcal{L}\tilde\nabla_X \eta\\ &= \tilde\nabla_X (f_*\mathcal{Y}\eta + (\mathcal{L}\eta)_{N_fM}) + \mathcal{L}(f_*A_\eta X - \nabla_X^\perp \eta).\end{aligned}$$

Taking normal components, we obtain

$$\begin{aligned}((\tilde\nabla_X \mathcal{L})\eta)_{N_fM} &= \alpha(X, \mathcal{Y}\eta) + \nabla_X^\perp (\mathcal{L}\eta)_{N_fM} + (\mathcal{L}A_\eta X - \nabla_X^\perp \eta)_{N_fM}\\ &= \mathcal{E}(X,\eta) + \nabla_X^\perp(\mathcal{L}\eta)_{N_fM} - (\mathcal{L}\nabla_X^\perp \eta)_{N_fM}.\end{aligned}$$

On the other hand, we have from Exercise 4.1 that

$$(\tilde{\nabla}_X \mathcal{L})\eta = c(f_* X \wedge \mathcal{T})\eta = c\langle \eta, \mathcal{T}\rangle f_* X,$$

which implies that

$$\mathcal{E}(X,\eta) + \nabla^\perp_X (\mathcal{L}\eta)_{N_f M} - (\mathcal{L}\nabla^\perp_X \eta)_{N_f M} = 0.$$

Thus the previous equation is just

$$\mathcal{E}(X,\eta) + (\nabla^\perp_X C)\eta = 0,$$

and hence (4.13) holds.

Assume that $\mathcal{T}$ is an infinitesimal bending of $f\colon M^n \to \mathbb{Q}^m_c$ whose associated pair verifies (4.13) and regard $\hat{\mathcal{T}} = i_*\mathcal{T}$ as an infinitesimal bending of $\hat{f} = i \circ f\colon M^n \to \mathbb{E}^{m+1}$. We claim that that the tensors associated to $\hat{\mathcal{T}}$ have the form (2.19). In fact, let $\hat{C} \in \Gamma(\mathrm{End}(N_{\hat{f}}M))$ be given by

$$\hat{C} i_*\eta = i_* C\eta - c\langle \eta, \mathcal{T}\rangle f \;\text{ and }\; \hat{C} f = i_*(\mathcal{T})_{N_f M}$$

for any $X \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_f M)$. Then, it follows from Lemma 4.3 that

$$\begin{aligned}
\hat{\beta}(X,Y) &= i_*\beta(X,Y) - c\langle X,Y\rangle i_*(\mathcal{T})_{N_f M} - c\langle \alpha(X,Y), \mathcal{T}\rangle f\\
&= i_* C\alpha(X,Y) - c\langle X,Y\rangle \hat{C} f - c\langle \alpha(X,Y), \mathcal{T}\rangle f\\
&= \hat{C}\hat{\alpha}(X,Y)
\end{aligned}$$

for any $X,Y \in \mathfrak{X}(M)$. As for $\hat{\mathcal{E}}$, we also have from Lemma 4.3 that

$$\begin{aligned}
\hat{\mathcal{E}}(X, i_*\eta) &= i_*\mathcal{E}(X,\eta) - c(\langle X, \mathcal{Y}\eta\rangle + \langle f_* A_\eta X, \mathcal{T}\rangle) f\\
&= -i_*(\nabla^\perp_X C)\eta - c(\langle X, \mathcal{Y}\eta\rangle + \langle f_* A_\eta X, \mathcal{T}\rangle) f\\
&= i_*(C\nabla^\perp_X \eta - \nabla^\perp_X C\eta) - c(\langle X, \mathcal{Y}\eta\rangle + \langle f_* A_\eta X, \mathcal{T}\rangle) f\\
&= i_*(C\nabla^\perp_X \eta - \nabla^\perp_X C\eta) + c(\langle LX, \eta\rangle + \langle \tilde{\nabla}_X \eta, \mathcal{T}\rangle - \langle \nabla^\perp_X \eta, \mathcal{T}\rangle) f\\
&= \hat{C} i_* \nabla^\perp_X \eta - i_*\nabla^\perp_X C\eta + cX\langle \eta, \mathcal{T}\rangle f\\
&= \hat{C} i_* \nabla^\perp_X \eta - \hat{\nabla}^\perp_X \hat{C} i_*\eta\\
&= -(\hat{\nabla}^\perp_X \hat{C}) i_*\eta,
\end{aligned}$$

where $\hat{\nabla}^\perp$ denotes the normal connection of $\hat{f}$. From the definition of $\hat{C}$ it is not hard to see that

$$(\hat{\nabla}^\perp_X \hat{C}) f = i_* \nabla^\perp_X (\mathcal{T})_{N_f M}.$$

Hence, we also have from Lemma 4.3 that $\hat{\mathcal{E}} = -\hat{\nabla}^\perp \hat{C}$, and this proves the claim.

It now follows from Proposition 2.7 that $\hat{\mathcal{T}}$ is a trivial infinitesimal bending of $\hat{f}$, that is, that

$$\hat{\mathcal{T}}(x) = \mathcal{D}\hat{f}(x) + w,$$

where $\mathcal{D}$ is a skew-symmetric linear map of $\mathbb{E}^{m+1}$ and $w \in \mathbb{E}^{m+1}$ is constant. Since $\hat{\mathcal{T}} = i_*\mathcal{T}$ is tangent to $\mathbb{Q}_c^m$, then w is orthogonal to the position vector $\hat{f}$, hence it determines a parallel vector field normal to f and therefore $\mathcal{T}$ is trivial. In particular, if f does not reduce codimension then $w = 0$. $\square$

From now on, we identify two infinitesimal bendings $\mathcal{T}_1$ and $\mathcal{T}_2$ of an isometric immersion $f\colon M^n \to \mathbb{Q}_c^m$ if there exist $0 \neq k \in \mathbb{R}$ and a trivial infinitesimal bending $\mathcal{T}_0$ such that $\mathcal{T}_2 = \mathcal{T}_0 + k\mathcal{T}_1$. Accordingly, we also identify pairs of tensors $(\beta_1, \mathcal{E}_1)$ and $(\beta_2, \mathcal{E}_2)$ if there is $0 \neq k \in \mathbb{R}$ such that $(\beta_1 - k\beta_2, \mathcal{E}_1 - k\mathcal{E}_2)$ has the form (4.13).

4.1 The Fundamental Theorem

In this section we give the Fundamental Theorem for infinitesimal variations of submanifolds in nonflat ambient spaces.

Theorem 4.5. *Let $f\colon M^n \to \mathbb{Q}_c^m$ be an isometric immersion of a simply connected Riemannian manifold. Let $\beta\colon TM \times TM \to N_fM$ be a symmetric tensor and let the tensor $\mathcal{E}\colon TM \times N_fM \to N_fM$ satisfy the compatibility condition* (4.5)*. If the pair $0 \neq (\beta, \mathcal{E})$ satisfies* (4.7)*,* (4.8) *and* (4.9)*, then there is a unique infinitesimal bending $\mathcal{T}$ of f having $(\beta, \mathcal{E})$ as associated pair.*

Proof. Set $\hat{f} = i \circ f\colon M^n \to \mathbb{E}^{m+1}$ and define the tensors $\hat{\beta}\colon TM \times TM \to N_{\hat{f}}M$ and $\hat{\mathcal{E}}\colon TM \times N_{\hat{f}}M \to N_{\hat{f}}M$ by

$$\hat{\beta}(X,Y) = i_*\beta(X,Y), \;\; \hat{\mathcal{E}}(X, i_*\eta) = i_*\mathcal{E}(X,\eta) \;\text{ and }\; \hat{\mathcal{E}}(X,f) = 0$$

for any $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$. Notice that at any point both tensors do not have a component in the direction of the position vector f. This already implies that $\hat{\beta}$ verifies (2.12) since β satisfies (4.7). Also notice that $\hat{\nabla}^\perp_X f = 0$ and that

$$\hat{\nabla}^\perp_X i_*\eta = i_*\nabla^\perp_X \eta$$

for any $X \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$. It follows from (4.8) and (4.9) that $\hat{\beta}$ and $\hat{\mathcal{E}}$ verify (2.13) and (2.14) for any $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$. Since $\hat{\mathcal{E}}(X,f) = 0$ and $\hat{A}_f = -I$, then (2.14) also holds for f. Therefore the integrability conditions of equation (2.20) are satisfied.

Similarly as in the proof of Theorem 2.8, we take a skew-symmetric solution $\mathcal{D} \in \Gamma(\mathrm{End}(\hat{f}^*T\mathbb{E}^{m+1}))$ of (2.20) along $\hat{f}$. We claim that $\mathcal{T} \in \Gamma(f^*T\mathbb{Q}_c^m)$ given by

$$i_*\mathcal{T}(x) = \mathcal{D}(x)f(x)$$

is an infinitesimal bending of f. For simplicity, from now on we write U instead of i_*U when considering $U \in \Gamma(f^*T\mathbb{Q}_c^m)$ as an element of $\Gamma(\hat{f}^*T\mathbb{E}^{m+1})$ and, similarly, we write X instead f_*X for tangent vector fields. Recall that $\tilde{\nabla}$ denotes the Levi-Civita connection of $\mathbb{Q}_c^m$ and $\hat{\nabla}$ the one of $\mathbb{E}^{m+1}$. Also recall that

$$\langle \mathcal{D}U, f\rangle + \langle \mathcal{T}, U\rangle = 0$$

for any $U \in \Gamma(f^*T\mathbb{Q}_c^m)$. Then

$$\mathcal{D}U = (\mathcal{D}U)_{T\mathbb{Q}_c^m} - c\langle U, \mathcal{T}\rangle f. \tag{4.14}$$

Thus we have

$$\begin{aligned}
\tilde{\nabla}_X\mathcal{T} &= \hat{\nabla}_X\mathcal{T} + c\langle X, \mathcal{T}\rangle f\\
&= \hat{\nabla}_X\mathcal{D}f + c\langle X, \mathcal{T}\rangle f\\
&= (\hat{\nabla}_X\mathcal{D})f + \mathcal{D}X + c\langle X, \mathcal{T}\rangle f.
\end{aligned}$$

We obtain from (2.20) and (4.14) that

$$\tilde{\nabla}_X\mathcal{T} = -\hat{B}_fX + \hat{\mathcal{E}}(X, f) + (\mathcal{D}X)_{T\mathbb{Q}_c^m}.$$

Notice that $\langle \hat{B}_fX, Y\rangle = \langle \hat{\beta}(X, Y), f\rangle = 0$ and that $\hat{\mathcal{E}}(X, f) = 0$ by the definition of the tensors $\hat{\beta}$ and $\hat{\mathcal{E}}$. Hence

$$\tilde{\nabla}_X\mathcal{T} = (\mathcal{D}X)_{T\mathbb{Q}_c^m}.$$

Since $\mathcal{D}$ is skew symmetric we have from the previous equation that $\mathcal{T}$ verifies (4.1), and the claim follows. Also observe that from the previous equation the tensor L associated to $\mathcal{T}$ is $(\mathcal{D}|_{TM})_{T\mathbb{Q}_c^m}$. Then

$$\begin{aligned}
\hat{\nabla}_X\mathcal{D}Y &= \hat{\nabla}_X(LY - c\langle Y, \mathcal{T}\rangle f)\\
&= \tilde{\nabla}_XLY - c\langle X, LY\rangle f - c\langle\tilde{\nabla}_XY, \mathcal{T}\rangle f - c\langle Y, LX\rangle f - c\langle Y, \mathcal{T}\rangle f_*X\\
&= \tilde{\nabla}_XLY - c\langle\tilde{\nabla}_XY, \mathcal{T}\rangle f - c\langle Y, \mathcal{T}\rangle f_*X,
\end{aligned}$$

where in the last step we made use of (4.1). Using (4.14) we have that

$$
\begin{aligned}
(\hat{\nabla}_X \mathcal{D})Y &= \hat{\nabla}_X \mathcal{D}Y - \mathcal{D}\hat{\nabla}_X Y \\
&= \tilde{\nabla}_X LY - c\langle \tilde{\nabla}_X Y, \mathcal{T}\rangle f - c\langle Y, \mathcal{T}\rangle f_* X - \mathcal{D}\tilde{\nabla}_X Y + c\langle X, Y\rangle \mathcal{T} \\
&= \tilde{\nabla}_X LY - c\langle \tilde{\nabla}_X Y, \mathcal{T}\rangle f - c\langle Y, \mathcal{T}\rangle f_* X \\
&\quad - \mathcal{D}\nabla_X Y - \mathcal{D}\alpha(X,Y) + c\langle X, Y\rangle \mathcal{T} \\
&= (\tilde{\nabla}_X L)Y - c\langle \tilde{\nabla}_X Y, \mathcal{T}\rangle f - c\langle Y, \mathcal{T}\rangle f_* X \\
&\quad + c\langle \nabla_X Y, \mathcal{T}\rangle f - \mathcal{D}\alpha(X,Y) + c\langle X, Y\rangle \mathcal{T} \\
&= (\tilde{\nabla}_X L)Y - c\langle \alpha(X,Y), \mathcal{T}\rangle f - c\langle Y, \mathcal{T}\rangle f_* X \\
&\quad - \mathcal{D}\alpha(X,Y) + c\langle X, Y\rangle \mathcal{T} \\
&= (\tilde{\nabla}_X L)Y - (\mathcal{D}\alpha(X,Y))_{T\mathbb{Q}_c^m} - c\langle Y, \mathcal{T}\rangle f_* X + c\langle X, Y\rangle \mathcal{T}.
\end{aligned}
$$

It follows from (2.20) and the above that

$$
(\tilde{\nabla}_X L)Y = \beta(X,Y) + (\mathcal{D}\alpha(X,Y))_{T\mathbb{Q}_c^m} + c\langle Y, \mathcal{T}\rangle f_* X - c\langle X, Y\rangle \mathcal{T}.
$$

Thus, we have that the symmetric tensor $\tilde{\beta}$ associated to $\mathcal{T}$ is given by

$$
\tilde{\beta} = \beta + C\alpha,
$$

where $C \in \Gamma(\mathrm{End}(N_f M))$ is the skew-symmetric tensor defined as $C\eta = (\mathcal{D}\eta)_{N_f M}$ for any $\eta \in \Gamma(N_f M)$. As for the tensor $\tilde{\mathcal{E}}$ associated to $\mathcal{T}$, we have

$$
\begin{aligned}
\tilde{\mathcal{E}}(X,\eta) &= \alpha(X, \mathcal{Y}\eta) + (LA_\eta X)_{N_f M} \\
&= (\tilde{\nabla}_X (\mathcal{D}\eta)_{TM})_{N_f M} + (\mathcal{D}A_\eta X)_{N_f M} \\
&= (\tilde{\nabla}_X (\mathcal{D}\eta)_{T\mathbb{Q}_c^m})_{N_f M} - \nabla_X^\perp C\eta - (\mathcal{D}\tilde{\nabla}_X \eta)_{N_f M} + C\nabla_X^\perp \eta \\
&= (\hat{\nabla}_X \mathcal{D}\eta)_{N_f M} - (\mathcal{D}\hat{\nabla}_X \eta)_{N_f M} - (\nabla_X^\perp C)\eta \\
&= \mathcal{E}(X,\eta) - (\nabla_X^\perp C)\eta,
\end{aligned}
$$

where for the last step we used (2.20). Therefore $\tilde{\mathcal{E}} = \mathcal{E} - \nabla^\perp C$, and this concludes the proof. □

4.2 The hypersurfaces case

Let $f\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion with second fundamental form $A \in \Gamma(\mathrm{End}(TM))$ with respect to a unit normal map $N \in \Gamma(N_f M)$. Associated to an infinitesimal bending $\mathcal{T}$ of f there is the symmetric tensor $\mathcal{B} \in \Gamma(\mathrm{End}(TM))$ defined by

$$
\langle \mathcal{B}X, Y\rangle N = \beta(X,Y).
$$

In codimension one, a tensor $\mathcal{E}\colon TM \times N_fM \to N_fM$ satisfying (4.5) vanishes. Hence, the fundamental equations of an infinitesimal bending take the form

$$\mathcal{B}X \wedge AY - \mathcal{B}Y \wedge AX = 0 \tag{4.15}$$

and

$$(\nabla_X \mathcal{B})Y = (\nabla_Y \mathcal{B})X$$

for any $X, Y \in \mathfrak{X}(M)$.

Analogously to the case of the flat ambient space we have the following result.

Proposition 4.6. *An infinitesimal bending $\mathcal{T}$ of an hypersurface $f\colon M^n \to \mathbb{Q}_c^{n+1}$ is trivial if and only if its associated tensor $\mathcal{B}$ vanishes.*

For hypersurfaces the Fundamental Theorem takes the following form.

Theorem 4.7. *Let $f\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold. Let $0 \neq \mathcal{B} \in \Gamma(End(TM))$ be a symmetric Codazzi tensor that satisfies* (4.15)*. Then there exists a unique infinitesimal bending $\mathcal{T}$ of f having $\mathcal{B}$ as associated tensor.*

4.3 Infinitesimal rigidity

In this section, we give rigidity results analogous to the ones in Chapter 2 for the flat ambient space.

Proposition 4.8. *Let $\mathcal{T}$ be an infinitesimal bending of a given isometric immersion $f\colon M^n \to \mathbb{Q}_c^m$. Then let $G_t\colon M^n \to \mathbb{E}^{m+1}$ for $t \in I \subset \mathbb{R}$ be the map defined by*

$$G_t(x) = (1 + ct^2\|\mathcal{T}(x)\|^2)^{-1/2}(\hat{f}(x) + ti_*\mathcal{T}(x)), \tag{4.16}$$

where $\hat{f} = i \circ f\colon M^n \to \mathbb{E}^{m+1}$. The following assertions hold:

(i) The maps G_t and G_{-t} determine (locally if $c < 0$ and I is small enough) immersions in $\mathbb{Q}_c^m$ that induce the same metric.

(ii) If f is substantial and there is $0 \neq t_0 \in I$ such that G_{t_0} and G_{-t_0} are congruent as immersions in $\mathbb{Q}_c^m$ then $\mathcal{T}$ is trivial.

Proof. We compute in $\mathbb{E}^{m+1}$ and write $\mathcal{T}$ instead of $i_*\mathcal{T}$ for simplicity. Since the position vector $\hat{f}$ is orthogonal to $\mathbb{Q}_c^m$, then

$$\|\hat{f}(x) + t\mathcal{T}(x)\|^2 = \|\hat{f}(x)\|^2 + t^2\|\mathcal{T}(x)\|^2 \tag{4.17}$$

$$= \frac{1}{c} + t^2\|\mathcal{T}(x)\|^2. \tag{4.18}$$

If $c<0$, we restrict ourselves to open subsets U of M^n where there is a constant $K>0$ such that $K^2>-c\|\mathfrak{T}\|^2$. Therefore $1+ct^2\|\mathfrak{T}(x)\|^2>0$ on U when taking $t\in I=(-K^{-1},K^{-1})$. Hence

$$\|G_t(x)\|^2=\frac{1}{1+ct^2\|\mathfrak{T}(x)\|^2}\left(1/c+t^2\|\mathfrak{T}(x)\|^2\right)=\frac{1}{c}$$

and thus $G_t(x)\in\mathbb{Q}_c^m$ for any $x\in M^n$ ($x\in U$ if $c<0$).

Setting

$$\phi_t(x)=(1+ct^2\|\mathfrak{T}(x)\|^2)^{-1/2},$$

we have

$$G_{t*}X=X(\phi_t)(\hat{f}+t\mathfrak{T})+\phi_t(\hat{f}_*X+t\hat{\nabla}_X\mathfrak{T})$$

for any $X\in\mathfrak{X}(M)$. Using (4.1) we obtain

$$\begin{aligned}
&\|G_{t*}X\|^2\\
&=\|X(\phi_t)(\hat{f}+t\mathfrak{T})+\phi_t(\hat{f}_*X+t\hat{\nabla}_X\mathfrak{T})\|^2\\
&=X(\phi_t)^2\|\hat{f}+t\mathfrak{T}\|^2+2\phi_tX(\phi_t)\langle\hat{f}+t\mathfrak{T},\hat{f}_*X+t\hat{\nabla}_X\mathfrak{T}\rangle+\phi_t^2\|\hat{f}_*X+t\hat{\nabla}_X\mathfrak{T}\|^2\\
&=X(\phi_t)^2(1/c+t^2\|\mathfrak{T}\|^2)+2\phi_tX(\phi_t)(t\langle\hat{f},\hat{\nabla}_X\mathfrak{T}\rangle+t\langle f_*X,\mathfrak{T}\rangle+t^2\langle\tilde{\nabla}_X\mathfrak{T},\mathfrak{T}\rangle)\\
&\quad+\phi_t^2(\|f_*X\|^2+t^2\|\hat{\nabla}_X\mathfrak{T}\|^2)\\
&=(1/c)X(\phi_t)^2\phi_t^{-2}+2\phi_tX(\phi_t)t^2\langle\tilde{\nabla}_X\mathfrak{T},\mathfrak{T}\rangle+\phi_t^2(\|f_*X\|^2+t^2\|\hat{\nabla}_X\mathfrak{T}\|^2)
\end{aligned}$$

for any $X\in\mathfrak{X}(M)$. Since

$$X(\phi_t)=-ct^2\phi_t^3\langle\tilde{\nabla}_X\mathfrak{T},\mathfrak{T}\rangle,$$

then

$$\|G_{t*}X\|^2=\phi_t^2(\|f_*X\|^2+t^2\|\hat{\nabla}_X\mathfrak{T}\|^2-ct^4\phi_t^2\langle\tilde{\nabla}_X\mathfrak{T},\mathfrak{T}\rangle^2)$$

for any $X\in\mathfrak{X}(M)$, and this proves part (i).

Now assume that the immersions G_{t_0} and G_{-t_0} are congruent in $\mathbb{Q}_c^m$ for $t_0\in I$, that is, there is a linear isometry S of $\mathbb{E}^{m+1}$ such that $G_{t_0}=S\circ G_{-t_0}$. Thus

$$\phi_{t_0}(\hat{f}+t\mathfrak{T})=\phi_{-t_0}S(\hat{f}-t_0\mathfrak{T}),$$

and since $\phi_{t_0}=\phi_{-t_0}$, we have

$$(S-Id)\hat{f}=t_0(S+Id)\mathfrak{T},\tag{4.19}$$

where Id is the identity map in $\mathbb{E}^{m+1}$.

We claim that $S+Id$ is invertible. If otherwise, there is $0\neq\delta\in\ker(S+Id)$. Since $S^*=S^{-1}$, where S^* denotes the adjoint operator of

S, then $\delta \in \ker(S^* + Id)$ and hence $(S - Id)^*\delta = -2\delta$. Then the inner product of (4.19) with δ gives

$$\langle \hat{f}(x), \delta \rangle = 0 \text{ for any } x \in M^n.$$

For $c < 0$ since we have $\langle \hat{f}, \hat{f} \rangle < 0$, then $\langle \delta, \delta \rangle > 0$. It follows that $\hat{f}$ is contained in an hyperplane orthogonal to δ in contradiction to the assumption that f is substantial, and this proves the claim.

We have that $S + Id$ is invertible, and hence $\mathcal{T} = \mathcal{D}\hat{f}$ where

$$\mathcal{D} = \frac{1}{t_0}(S + Id)^{-1}(S - Id).$$

To conclude the proof of part (ii) it remains to show that $\mathcal{D}$ is skew-symmetric. For this, first notice that

$$(S - Id)(S^* + Id) = S - S^* = -(S + Id)(S^* - Id).$$

Then

$$\begin{aligned}(S + Id)^{-1}(S - Id) &= (S + Id)^{-1}(S - Id)(S^* + Id)(S^* + Id)^{-1}\\ &= -(S + Id)^{-1}(S + Id)(S^* - Id)(S^* + Id)^{-1}\\ &= -(S^* - Id)(S^* + Id)^{-1}\\ &= -((S + Id)^{-1}(S - Id))^*\end{aligned}$$

as we wished. □

A submanifold $f\colon M^n \to \mathbb{Q}_c^m$ is said to be *infinitesimally rigid* if it admits only trivial infinitesimal bendings. Since Propositions 2.14 and 2.15 also hold when the ambient space is $\mathbb{Q}_c^m$, then using the above we have the following result.

Theorem 4.9. *Let $f\colon M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion such that either $p \le 5$ and the s-nullities satisfy $\nu_s \le n - 2s - 1$ for all $1 \le s \le p$ or the type number satisfies $\tau \ge 3$. Then f is infinitesimally rigid.*

Proof. Let $\mathcal{T}$ be an infinitesimal bending of f. Define the maps G_t by (4.16). Then any point of M^n is contained in a neighborhood U such that for t is small enough the immersions $G_t|_U$ and $G_{-t}|_U$ are congruent. Hence $\mathcal{T}$ is trivial on U and the tensor β associated to $\mathcal{T}$ satisfies $\beta = C_U\alpha$ for some skew symmetric endomorphism $C_U \in \Gamma(\mathrm{End}(N_fU))$. In particular, from the assumptions we have that f has full first normal spaces. Thus, if two such open subsets U and V intersect then $C_U = C_V$. Therefore, the pair $(\beta, \mathcal{E})$ associated to $\mathcal{T}$ has everywhere the form (4.13), and hence $\mathcal{T}$ is globally trivial. □

4.4 The genuine case

In this section, it is shown that several results from the previous chapter also hold when the ambient space is a space form of nonflat sectional curvature.

Several definitions given previously for the Euclidean ambient space extend to the new situation with minor adaptations.

A smooth map $F\colon \tilde{M}^{n+\ell} \to \mathbb{Q}_c^{n+p}$, $0 < \ell < p$, from a differentiable manifold $\tilde{M}^{n+\ell}$ is said to be a *singular extension* of a given isometric immersion $f\colon M^n \to \mathbb{Q}_c^{n+p}$ if there is an embedding $j\colon M^n \to \tilde{M}^{n+\ell}$, $0 < \ell < p$, such that F is an immersion along $\tilde{M}^{n+\ell} \setminus j(M)$ and $f = F \circ j$. Hence, the map F may fail (but not necessarily) to be an immersion along points of $j(M)$.

An infinitesimal bending $\mathcal{T}$ of an isometric immersion $f\colon M^n \to \mathbb{Q}_c^{n+p}$ *extends in the singular sense* if there is a singular extension $F\colon \tilde{M}^{n+\ell} \to \mathbb{Q}_c^{n+p}$ of f and a smooth map $\tilde{\mathcal{T}}\colon \tilde{M}^{n+\ell} \to \mathbb{E}^{m+1}$ such that $\tilde{\mathcal{T}}$ is tangent to $\mathbb{Q}_c^m$ and is an infinitesimal bending of $F|_{\tilde{M}\setminus j(M)}$ with $\mathcal{T} = \tilde{\mathcal{T}}|_{j(M)}$.

That an isometric immersion $f\colon M^n \to \mathbb{Q}_c^m$ is r*-ruled* means that there is a smooth r-dimensional totally geodesic tangent distribution whose leaves are mapped diffeomorphically by f to open subsets of totally geodesic submanifolds of $\mathbb{Q}_c^m$.

An infinitesimal bending $\mathcal{T}$ of an isometric immersion $f\colon M^n \to \mathbb{Q}_c^{n+p}$, $p \geq 2$, is called a *genuine infinitesimal bending* if $\mathcal{T}$ does not extend in the singular sense when restricted to any open subset of M^n. If f admits such a bending we say that it is *genuinely infinitesimally bendable*. Finally, we say that f is *genuinely infinitesimally rigid* if given any infinitesimal bending $\mathcal{T}$ of f there is an open dense subset of M^n such that $\mathcal{T}$ restricted to any connected component extends in the singular sense.

A key ingredient in this section is the following result obtained as a consequence of Proposition 3.5. In what follows $\hat{f} = i \circ f$ where $f\colon M^n \to \mathbb{Q}_c^m$ is an isometric immersion and $i\colon \mathbb{Q}_c^m \to \mathbb{E}^{m+1}$ is the usual umbilical inclusion. Moreover, $\exp_p\colon T_p\mathbb{Q}_c^m \to \mathbb{Q}_c^m$ denotes the exponential map at $p \in \mathbb{Q}_c^m$.

Proposition 4.10. *Let $f\colon M^n \to \mathbb{Q}_c^m$ be an isometric immersion and let D be a smooth tangent distribution of dimension $d > 0$. Assume that there is no open subset $U \subset M^n$ and $Z \in \Gamma(D|_U)$ with $\|Z\|^2 = 1/|c|$ such that the map $F\colon U \times I \to \mathbb{Q}_c^m \subset \mathbb{E}^{m+1}$ given by*

$$F(x,t) = \begin{cases} \cos t\hat{f}(x) + \sin t\hat{f}_*Z & \text{if } c > 0, \\ \cosh t\hat{f}(x) + \sinh t\hat{f}_*Z & \text{if } c < 0 \end{cases} \tag{4.20}$$

is a singular extension of f on an open neighborhood of $U \times \{0\}$. Then, for any $x \in M^n$ there is an open neighborhood V of the origin in $D(x)$ such that $\exp_{f(x)}(f_*V) \subset f(M)$. Hence f is d-ruled along each connected component of an open dense subset of M^n.

Proof. Given a smooth map $g\colon N^n \to \mathbb{Q}_c^m$ from a differentiable manifold N^n, by the *cone* over g we mean the map $\tilde{g}\colon N \times \mathbb{R}_+ \to \mathbb{E}^{m+1}$ given by

$$\tilde{g}(y,s) = sg(y).$$

Observe that if g is an immersion then also is $\tilde{g}$.

On an open subset $U \subset M$ take $Z \in \mathfrak{X}(U)$ with $\|Z\| = 1/|c|$. Then let $\tilde{f}$ be the cone over f and let $\bar{F}\colon U \times \mathbb{R}_+ \times I \to \mathbb{E}^{m+1}$ be given by

$$\bar{F}(x,s,t) = s\hat{f}(x) + t\hat{f}_*Z. \tag{4.21}$$

Assume that there is an open neighborhood of $(x,1,0)$ in $U \times \mathbb{R}_+ \times I$ where $\bar{F}$ is a singular extension of $\tilde{f}$. Then the intersection of its image with $\mathbb{Q}_c^m$ determines a singular extension of f. In fact, taking $s \neq 1$ and t such that $s^2 + \operatorname{sign}(c)t^2 = 1$ we have that $\bar{F}(x,s,t) \in \mathbb{Q}_c^m$. Moreover, for such pair (s,t) if close enough to $(1,0)$, then $\bar{F}$ is transversal to $\mathbb{Q}_c^m$ and thus $\bar{F}(U \times I \times I) \cap \mathbb{Q}_c^m$ is a singular extension of f.

Take $Z \in \Gamma(D)$ with $\|Z\|^2 = 1/|c|$ and let $F\colon U \times I \to \mathbb{Q}_c^m \subset \mathbb{E}^{m+1}$ be given by (4.20). Notice that the cone over F, say $\tilde{F}$, can be parametrized by (4.21). Therefore, our assumptions and the discussion above imply that there does not exist an open subset U and $Z \in \Gamma(D|_U)$, $\|Z\|^2 = 1/|c|$, such that $\tilde{F}$ is a singular extension of $\tilde{f}$. Hence, it follows from Proposition 3.5 that for any $x \in M^n$ there is an open neighborhood V of the origin in $D(x)$ such that $\tilde{f}_*V \subset \tilde{f}(M \times \mathbb{R}_+)$. In other words, locally the cone over F is contained in the cone over f. Thus the piece of geodesic in $\mathbb{Q}_c^m$ passing through $f(x)$ in the direction of Z is contained in $f(M)$, and the proof follows. ☐

Proposition 4.11. *Let $f\colon M^n \to \mathbb{Q}_c^m$ be an isometric immersion and let $\mathcal{T}$ be an infinitesimal bending with associated pair $(\beta, \mathcal{E})$. Then, at any point of M^n the bilinear form $\theta\colon TM \times TM \to N_fM \oplus N_fM$ defined by*

$$\theta(X,Y) = (\alpha(X,Y) + \beta(X,Y), \alpha(X,Y) - \beta(X,Y))$$

is flat with respect to the inner product in $N_fM \oplus N_fM$ given by

$$\langle\!\langle (\xi_1,\eta_1),(\xi_2,\eta_2)\rangle\!\rangle_{N_fM\oplus N_fM} = \langle \xi_1,\xi_2\rangle_{N_fM} - \langle \eta_1,\eta_2\rangle_{N_fM}.$$

Proof. Follows from (4.7). ☐

Similarly to the case of the Euclidean ambient space, we have the following fact.

Proposition 4.12. *Let* $f\colon M^n \to \mathbb{Q}_c^m$ *be* 1*-regular and let* $\beta_1\colon TM \times TM \to N_1$ *be the* N_1*-component of* β*. Then the bilinear form* $\hat\theta\colon TM \times TM \to N_1 \oplus N_1$ *defined at any point by*

$$\hat\theta(X,Y) = (\alpha(X,Y) + \beta_1(X,Y), \alpha(X,Y) - \beta_1(X,Y))$$

is flat with respect to the inner product induced on $N_1 \oplus N_1$*.*

Theorem 4.13. *Let* $f\colon M^n \to \mathbb{Q}^{n+p}$*,* $n > 2p \geq 4$*, be an isometric immersion and let* $\mathcal{T}$ *be an infinitesimal bending of* f*. Then along each connected component of an open and dense subset either* $\mathcal{T}$ *extends in the singular sense or* f *is* r*-ruled with* $r \geq n-2p$*.*

Proof. By Proposition 4.11 the symmetric tensor θ is flat at any point of M^n. Given $Y \in RE(\theta)$ denote $D = \ker\theta_Y$ where $\theta_Y(X) = \theta(Y,X)$. Notice that $Z \in D$ means that $\alpha(Y,Z) = 0 = \beta(Y,Z)$.

Let $U \subset M^n$ be an open subset where $Y \in \mathfrak{X}(U)$ satisfies $Y \in RE(\theta)$ and D has dimension d at any point. Proposition 1.10 gives

$$\langle\!\langle \theta(X,Z), \theta(X,Z)\rangle\!\rangle = 0$$

for any $X \in \mathfrak{X}(U)$ and $Z \in \Gamma(D)$. Equivalently, we have

$$\langle \alpha(X,Z), \beta(X,Z)\rangle = 0 \tag{4.22}$$

for any $X \in \mathfrak{X}(U)$ and $Z \in \Gamma(D)$.

Assume that there is $Z \in \Gamma(D)$, $\|Z\|^2 = 1/|c|$, defined in an open subset V of U such that $F\colon V \times (-\epsilon,\epsilon)$ given by (4.20) is a singular extension of f. Let $\hat L$ be given by (4.12) and define $\tilde{\mathcal{T}}\colon M \times (-\epsilon,\epsilon) \to \mathbb{E}^{m+1}$ by

$$\tilde{\mathcal{T}}(x,t) = \begin{cases} \cos t i_*\mathcal{T}(x) + \sin t\hat L(x)Z(x) & \text{if } c>0,\\ \cosh t i_*\mathcal{T}(x) + \sinh t\hat L(x)Z(x) & \text{if } c<0.\end{cases}$$

We claim that $\tilde{\mathcal{T}}$ is an infinitesimal bending of F on the open subset where F is an immersion. In what follows $\hat\nabla$ denotes the Levi-Civita connection of $\mathbb{E}^{m+1}$. We only argue the case $c>0$ since the computations for $c<0$ are similar. First notice that

$$\begin{aligned}\langle F(x,t), \tilde{\mathcal{T}}(x,t)\rangle &= \langle \cos t\hat f + \sin t\hat f_* Z, \cos t i_*\mathcal{T} + \sin t\hat L Z\rangle\\ &= \cos t\sin t\langle \hat f, \hat L Z\rangle + \cos t\sin t\langle f_*Z, \mathcal{T}\rangle\\ &= -\cos t\sin t\langle f_*Z,\mathcal{T}\rangle + \cos t\sin t\langle f_*Z,\mathcal{T}\rangle\\ &= 0,\end{aligned}$$

and hence $\tilde{\mathcal{T}}$ is tangent to $\mathbb{Q}_c^m$.

We have that

$$\begin{aligned}\langle F_*\partial/\partial t, \hat\nabla_{\partial/\partial t}\tilde{\mathcal{T}}\rangle &= \langle -\sin t\hat f + \cos t\hat f_* Z, -\sin t i_*\mathcal{T} + \cos t\hat L Z\rangle\\ &= 0.\end{aligned}$$

We obtain using (4.1) that

$$\begin{aligned}&\langle F_*\partial/\partial t, \hat\nabla_X\tilde{\mathcal{T}}\rangle + \langle F_*X, \hat\nabla_{\partial/\partial t}\tilde{\mathcal{T}}\rangle\\
&\quad= \langle -\sin t\hat f + \cos t\hat f_* Z, \cos t\hat L X + \sin t\hat\nabla_X\hat L Z\rangle\\
&\qquad+ \langle \cos t\hat f_* X + \sin t\hat\nabla_X\hat f_* Z, -\sin t i_*\mathcal{T} + \cos t\hat L Z\rangle\\
&\quad= \cos t\sin t\langle f_*X, \mathcal{T}\rangle - \sin^2 t\langle \hat f, \hat\nabla_X\hat L Z\rangle + \cos^2 t\langle f_*Z, LX\rangle\\
&\qquad+ \cos t\sin t\langle \hat f_* Z, \hat\nabla_X\hat L Z\rangle - \cos t\sin t\langle f_*X, \mathcal{T}\rangle + \cos^2 t\langle f_*X, LZ\rangle\\
&\qquad- \sin^2 t\langle \hat f_*\nabla_X Z + \alpha(X,Z), \mathcal{T}\rangle + \cos t\sin t\langle \hat\nabla_X\hat f_* Z, \hat L Z\rangle\\
&\quad= -\sin^2 t\langle \hat f, \hat\nabla_X\hat L Z\rangle + \cos t\sin t\langle \hat f_* Z, \hat\nabla_X\hat L Z\rangle\\
&\qquad- \sin^2 t\langle \hat f_*\nabla_X Z + \alpha(X,Z), \mathcal{T}\rangle + \cos t\sin t\langle \hat\nabla_X\hat f_* Z, \hat L Z\rangle\\
&\quad= -\sin^2 t\langle \hat f, (\hat\nabla_X\hat L)Z\rangle - \sin^2 t\langle \alpha(X,Z), \mathcal{T}\rangle + \cos t\sin t X\langle \hat f_* Z, \hat L Z\rangle\\
&\quad= 0\end{aligned}$$

for any $X \in \mathfrak{X}(M)$, where for the last step we used Lemma 4.3. In addition, we have using (2.11), (2.10) and Lemma 4.3 that

$$\begin{aligned}&\langle F_*X, \hat\nabla_X\tilde{\mathcal{T}}\rangle\\
&\quad= \langle \cos t\hat f_* X + \sin t\hat\nabla_X\hat f_* Z, \cos t\hat L X + \sin t\hat\nabla_X\hat L Z\rangle\\
&\quad= \cos t\sin t(\langle \hat f_* X, \hat\nabla_X\hat L Z\rangle + \langle \hat\nabla_X\hat f_* Z, \hat L X\rangle) + \sin^2 t\langle \hat\nabla_X\hat f_* Z, \hat\nabla_X\hat L Z\rangle\\
&\quad= \cos t\sin t(\langle \hat f_* X, (\hat\nabla_X\hat L)Z\rangle + \langle \hat\alpha(X,Z), \hat L X\rangle)\\
&\qquad+ \sin^2 t(\langle \hat f_*\nabla_X Z, (\hat\nabla_X\hat L)Z\rangle + \langle \hat\alpha(X,Z), \hat\nabla_X\hat L Z\rangle)\\
&\quad= \sin^2 t(\langle \hat f_*\nabla_X Z, \mathfrak{Y}\hat\alpha(Y,Z)\rangle + \langle \hat\alpha(X,Z), \hat\nabla_X\hat L Z\rangle)\\
&\quad= \sin^2 t\langle \hat\alpha(X,Z), (\hat\nabla_X\hat L)Z\rangle\\
&\quad= \sin^2 t\langle \hat\alpha(X,Z), \hat\beta(X,Z)\rangle\\
&\quad= \sin^2 t\langle \alpha(X,Z), \beta(X,Z)\rangle\\
&\quad= 0,\end{aligned}$$

where the last steps follows from (4.22), and this proves the claim.

Let $W \subset U$ be an open subset such that $Z \in \Gamma(D)$ as above does not exist along any open subset of W. By Proposition 4.10 the immersion is d-ruled along any connected component of an open dense subset of W. Moreover, we have that $d = \dim D = n - \dim \mathrm{Im}(\theta_Y) \geq n - 2p$. ☐

Let $F\colon \tilde{M}^{n+1} \to \mathbb{Q}_c^{n+p}$ be an isometric immersion and let $\tilde{\mathcal{T}}$ be an infinitesimal bending of F. Given an isometric embedding $j\colon M^n \to \tilde{M}^{n+1}$ consider the composition of isometric immersions $f = F\circ j\colon M^n \to \mathbb{Q}_c^{n+p}$. Then $\mathcal{T} = \tilde{\mathcal{T}}|_{j(M)}$ is an infinitesimal bending of f. It is easy to see that the corresponding tensors B and $\tilde{B}$ given by (4.2) satisfy along f that

$$B(X,Y) = \tilde{B}(X,Y) + \langle \tilde{\nabla}_X Y, F_*\eta\rangle \tilde{L}\eta$$

for $\eta \in \Gamma(N_jM)$ of unit length and any $X,Y \in \mathfrak{X}(M)$. It follows from (4.6) that

$$\langle B(X,Y), F_*\eta\rangle = \langle \beta(X,Y), F_*\eta\rangle - c\langle X,Y\rangle\langle \mathcal{T}, F_*\eta\rangle$$

and similarly using (4.4) that

$$\langle \tilde{B}(X,Y), F_*\eta\rangle = -\langle \alpha^F(X,Y), \tilde{L}\eta\rangle - c\langle X,Y\rangle\langle \mathcal{T}, F_*\eta\rangle.$$

Hence, it follows from (4.1) and the equations above that

$$\langle \beta(X,Y), F_*\eta\rangle + \langle \alpha(X,Y), \tilde{L}\eta\rangle = 0$$

for all $X,Y \in \mathfrak{X}(M)$. As in the case of the Euclidean ambient space, satisfying a condition of this type may guarantee that the infinitesimal bending is not genuine.

We say that an infinitesimal bending of an isometric immersion $f\colon M^n \to \mathbb{Q}_c^{n+p}$, $p \geq 2$, satisfies the *condition* $(*)$ if there is $\eta \in \Gamma(N_fM)$ of unit length and $\xi \in \Gamma(R)$, where R is determined by the orthogonal splitting $N_fM = P \oplus R$ and $P = \operatorname{span}\{\eta\}$, such that

$$B_\eta + A_\xi = 0, \tag{4.23}$$

where $B_\eta = \langle \beta, \eta\rangle$. Thus, that (3.6) holds means that

$$\langle \beta(X,Y), \eta\rangle + \langle \alpha(X,Y), \xi\rangle = 0 \tag{4.24}$$

for any $X,Y \in \mathfrak{X}(M)$.

Assume that $\mathcal{T}$ satisfies the condition $(*)$ and extend the tensor L to a tensor $\bar{L} \in \Gamma(\operatorname{End}(TM \oplus P, f^*T\mathbb{Q}_C^{n+p}))$ by defining

$$\bar{L}\eta = f_*\mathcal{Y}\eta + \xi.$$

Then $\bar{L}$ satisfies

$$\langle \bar{L}X, \eta\rangle + \langle f_*X, \bar{L}\eta\rangle = 0$$

for any $X \in \mathfrak{X}(M)$. Define $\bar{\mathcal{Y}}\colon R \to TM \oplus P$ by

$$\bar{\mathcal{Y}}\delta = \mathcal{Y}\delta - \langle \delta, \xi\rangle\eta,$$

where $\delta \in \Gamma(R)$. Then

$$\langle \bar{\mathcal{Y}}\delta, \lambda\rangle + \langle \bar{L}\lambda, \delta\rangle = 0$$

for any $\lambda \in \Gamma(TM \oplus P)$ and $\delta \in \Gamma(N_f M)$. Let

$$(\nabla_X \bar{L})\lambda = \tilde{\nabla}_X \bar{L}\lambda - \bar{L}\nabla'_X\lambda,$$

where $X \in \mathfrak{X}(M)$, $\lambda \in \Gamma(TM \oplus P)$ and ∇' is the connection induced on $TM \oplus P$. Then let $\bar{\beta}\colon TM \times (TM \oplus P) \to R$ be given by

$$\bar{\beta}(X, \lambda) = ((\tilde{\nabla}_X \bar{L})\lambda)_R + c\langle X, \lambda\rangle \mathcal{T}_R.$$

Proposition 4.14. *We have that*

$$(\tilde{\nabla}_X \bar{L})\lambda = \bar{\mathcal{Y}}(\tilde{\nabla}_X \lambda)_R + \bar{\beta}(X, \lambda) + c(f_* X \wedge \mathcal{T})\lambda \tag{4.25}$$

for any $X \in \mathfrak{X}(M)$ and $\lambda \in \Gamma(TM \oplus P)$.

Proof. Observe that

$$(\tilde{\nabla}_X \bar{L})Y = (\tilde{\nabla}_X L)Y - \langle \alpha(X, Y), \eta\rangle \bar{L}\eta,$$

where $X, Y \in \mathfrak{X}(M)$. Then (4.4) and (4.6) give

$$\begin{aligned}\langle (\tilde{\nabla}_X \bar{L})Y, Z\rangle &= \langle \mathcal{Y}\alpha(X, Y), Z\rangle + c\langle (X \wedge \mathcal{T})Y, Z\rangle - \langle \alpha(X, Y), \eta\rangle\langle \bar{L}\eta, Z\rangle \\ &= \langle \bar{\mathcal{Y}}\alpha(X, Y)_R, Z\rangle + c\langle (X \wedge \mathcal{T})Y, Z\rangle.\end{aligned}$$

Since $\mathcal{T}$ satisfies the condition $(*)$ it follows from (4.6) that

$$\begin{aligned}\langle (\tilde{\nabla}_X \bar{L})Y, \eta\rangle &= \langle \beta(X, Y), \eta\rangle - c\langle X, Y\rangle\langle \mathcal{T}, \eta\rangle \\ &= -\langle \alpha(X, Y), \xi\rangle - c\langle X, Y\rangle\langle \mathcal{T}, \eta\rangle \\ &= \langle \bar{\mathcal{Y}}\alpha(X, Y)_R, \eta\rangle - c\langle X, Y\rangle\langle \mathcal{T}, \eta\rangle.\end{aligned}$$

Then (4.25) holds when $\lambda = Y \in \mathfrak{X}(M)$.

Taking the derivative of

$$\langle \bar{L}\eta, Y\rangle + \langle \eta, \bar{L}Y\rangle = 0$$

in the direction of X we obtain

$$\langle (\tilde{\nabla}_X \bar{L})\eta, Y\rangle + \langle \bar{L}\eta, \alpha(X, Y)_R\rangle + \langle \nabla^\perp_X \eta, \bar{L}Y\rangle + \langle \eta, (\tilde{\nabla}_X \bar{L})Y\rangle = 0.$$

Using that (4.25) holds for $\lambda = Y$ gives

$$\langle (\tilde{\nabla}_X \bar{L})\eta, Y\rangle + \langle \nabla^\perp_X \eta, \bar{L}Y\rangle - c\langle X, Y\rangle\langle \mathcal{T}, \eta\rangle = 0.$$

Then

$$\langle (\tilde{\nabla}_X \bar{L})\eta, Y\rangle = \langle \bar{\mathcal{Y}}\nabla^\perp_X \eta, Y\rangle + c\langle \mathcal{T}, \eta\rangle\langle X, Y\rangle.$$

Finally, taking the derivative of $\langle \bar{L}\eta, \eta\rangle = 0$ yields

$$\langle (\tilde{\nabla}_X \bar{L})\eta, \eta\rangle = \langle \bar{\mathcal{Y}} \nabla^{\perp}_X \eta, \eta\rangle,$$

and this concludes the proof. $\square$

Assume that $\mathcal{T}$ satisfies the condition $(*)$. Given $\lambda \in \Gamma(TU \oplus P)$, $\|\lambda\|^2 = 1/|c|$, where U is an open subset of M^n, define the map $F\colon U \times (-\epsilon, \epsilon) \to \mathbb{Q}_c^{n+p}$ by

$$F(x,t) = \begin{cases} \cos t \hat{f}(x) + \sin t i_* \lambda(x) & \text{if } \ c > 0, \\ \cosh t \hat{f}(x) + \sinh t i_* \lambda(x) & \text{if } \ c < 0 \end{cases} \tag{4.26}$$

and the map $\tilde{\mathcal{T}}\colon U \times (-\epsilon, \epsilon) \to \mathbb{E}^{m+1}$ by

$$\tilde{\mathcal{T}}(x,t) = \begin{cases} \cos t i_* \mathcal{T}(x) + \sin t (i_* \bar{L}(x)\lambda(x) - c\langle \lambda(x), \mathcal{T}(x)\rangle \hat{f}(x)) & \text{if } c > 0, \\ \cosh t i_* \mathcal{T}(x) + \sinh t (i_* \bar{L}(x)\lambda(x) - c\langle \lambda(x), \mathcal{T}(x)\rangle \hat{f}(x)) & \text{if } c < 0. \end{cases} \tag{4.27}$$

Observe that $\langle F(x,t), \tilde{\mathcal{T}}\rangle = 0$ and hence $\tilde{T}$ is tangent to $\mathbb{Q}_c^m$. Assume that $c > 0$ being the computations when $c < 0$ similar. Since $\langle \bar{L}\lambda, \lambda\rangle = 0$ then

$$\langle F_* \partial/\partial t, \hat{\nabla}_{\partial/\partial t} \tilde{\mathcal{T}}\rangle = 0.$$

We have

$$\begin{aligned} &\langle F_* X, \hat{\nabla}_{\partial/\partial t} \tilde{\mathcal{T}}\rangle \\ &\quad = \langle \cos t \hat{f}_* X + \sin t \hat{\nabla}_X i_* \lambda, -\sin t i_* \mathcal{T} + \cos t (i_* \bar{L}\lambda - c\langle \lambda, \mathcal{T}\rangle \hat{f})\rangle \\ &\quad = -\cos t \sin t \langle f_* X, \mathcal{T}\rangle + \cos^2 t \langle f_* X, \bar{L}\lambda\rangle - \sin^2 t \langle \tilde{\nabla}_X \lambda, \mathcal{T}\rangle \\ &\qquad + \cos t \sin t \langle \tilde{\nabla}_X \lambda, \bar{L}\lambda\rangle + c \cos t \sin t \langle \lambda, \mathcal{T}\rangle \langle X, \lambda\rangle \end{aligned}$$

and

$$\begin{aligned} &\langle F_* \partial/\partial t, \hat{\nabla}_X \tilde{\mathcal{T}}\rangle \\ &\quad = \langle -\sin t \hat{f} + \cos t i_* \lambda, \cos t \hat{\nabla}_X i_* \mathcal{T} + \sin t (\hat{\nabla}_X i_* \bar{L}\lambda - c\langle \lambda, \mathcal{T}\rangle \hat{f}_* X - cX\langle \lambda, \mathcal{T}\rangle \hat{f})\rangle \\ &\quad = \cos t \sin t \langle f_* X, \mathcal{T}\rangle + \sin^2 t \langle f_* X, \bar{L}\lambda\rangle + \sin^2 t X\langle \lambda, \mathcal{T}\rangle + \cos^2 t \langle \lambda, \tilde{\nabla}_X \mathcal{T}\rangle \\ &\qquad + \cos t \sin t \langle \lambda, \tilde{\nabla}_X \bar{L}\lambda\rangle - c \cos t \sin t \langle \lambda, \mathcal{T}\rangle \langle X, \lambda\rangle \\ &\quad = \cos t \sin t \langle f_* X, \mathcal{T}\rangle + \sin^2 t \langle \tilde{\nabla}_X \lambda, \mathcal{T}\rangle + \cos^2 t \langle \lambda, LX\rangle \\ &\qquad + \cos t \sin t \langle \lambda, \tilde{\nabla}_X \bar{L}\lambda\rangle - c \cos t \sin t \langle \lambda, \mathcal{T}\rangle \langle X, \lambda\rangle \end{aligned}$$

for any $X \in \mathfrak{X}(U)$. Adding the above expressions gives

$$\begin{aligned} \langle F_* X, \hat{\nabla}_{\partial/\partial t} \tilde{\mathcal{T}}\rangle + \langle F_* \partial/\partial t, \hat{\nabla}_X \tilde{\mathcal{T}}\rangle &= \cos t \sin t \langle \tilde{\nabla}_X \lambda, \bar{L}\lambda\rangle + \cos t \sin t \langle \lambda, \tilde{\nabla}_X \bar{L}\lambda\rangle \\ &= \cos t \sin t X\langle \lambda, \bar{L}\lambda\rangle \\ &= 0. \end{aligned}$$

Therefore, if F is an immersion on some open subset of $U \times (-\epsilon, \epsilon)$, we have that $\tilde{\mathcal{T}}$ is an infinitesimal bending of F if and only if

$$\langle F_*X, \hat{\nabla}_X \tilde{\mathcal{T}} \rangle = 0 \tag{4.28}$$

for any $X \in \mathfrak{X}(M)$.

Lemma 4.15. *Assume that $\mathcal{T}$ satisfies the condition $(*)$. Then*

$$\langle F_*X, \hat{\nabla}_X \tilde{\mathcal{T}} \rangle = \begin{cases} \sin^2 t \langle (\tilde{\nabla}_X \lambda)_R, \bar{\beta}(X, \lambda) \rangle & \text{if } c > 0, \\ \sinh^2(t) \langle (\tilde{\nabla}_X \lambda)_R, \bar{\beta}(X, \lambda) \rangle & \text{if } c < 0 \end{cases} \tag{4.29}$$

for any $X \in \mathfrak{X}(M)$ and $\lambda \in \Gamma(TM \oplus P)$ with $\|\lambda\|^2 = 1/|c|$.

Proof. Again, we only argue the case $c > 0$. We have

$$\begin{aligned}
&\langle F_*X, \hat{\nabla}_X i_* \mathcal{T} \rangle \\
&= \langle \cos t \hat{f}_* X + \sin t \hat{\nabla}_X i_* \lambda, \cos t \hat{\nabla}_X i_* \mathcal{T} + \sin t (\hat{\nabla}_X i_* \bar{L} \lambda \\
&\quad - c \langle \lambda, \mathcal{T} \rangle \hat{f}_* X - c X \langle \lambda, \mathcal{T} \rangle \hat{f}) \rangle \\
&= \cos t \sin t \langle f_* X, \tilde{\nabla}_X \bar{L} \lambda \rangle - c \cos t \sin t \langle \lambda, \mathcal{T} \rangle \|X\|^2 + \cos t \sin t \langle \tilde{\nabla}_X \lambda, LX \rangle \\
&\quad + c \cos t \sin t \langle X, \lambda \rangle \langle f_* X, \mathcal{T} \rangle + \sin^2 t \langle \tilde{\nabla}_X \lambda, \tilde{\nabla}_X \bar{L} \lambda \rangle + c \sin^2 t \langle X, \lambda \rangle \langle f_* X, \bar{L} \lambda \rangle \\
&\quad - c \sin^2 t \langle \lambda, \mathcal{T} \rangle \langle f_* X, \tilde{\nabla}_X \lambda \rangle + c \sin^2 t X \langle \lambda, \mathcal{T} \rangle \langle X, \lambda \rangle \\
&= \cos t \sin t (\langle f_* X, (\tilde{\nabla}_X \bar{L}) \lambda \rangle + \langle (\tilde{\nabla}_x \lambda)_R, LX \rangle - c \langle \lambda, \mathcal{T} \rangle \|X\|^2 + c \langle X, \lambda \rangle \langle f_* X, \mathcal{T} \rangle) \\
&\quad + \sin^2 t (\langle \nabla'_X \lambda, (\tilde{\nabla}_X \bar{L}) \lambda \rangle + \langle (\tilde{\nabla}_X \lambda)_R, \tilde{\nabla}_X \bar{L} \lambda \rangle - c \langle \lambda, \mathcal{T} \rangle \langle f_* X, \tilde{\nabla}_X \lambda \rangle \\
&\quad + c \langle X, \lambda \rangle \langle \tilde{\nabla}_X \lambda, \mathcal{T} \rangle).
\end{aligned}$$

From (4.25) we obtain that

$$\begin{aligned}
&\langle F_*X, \hat{\nabla}_X i_* \mathcal{T} \rangle \\
&\quad = \cos t \sin t (\langle f_* X, \bar{\mathcal{Y}}(\tilde{\nabla}_X \lambda)_R + c (f_* X \wedge \mathcal{T}) \lambda \rangle + \langle (\tilde{\nabla}_X \lambda)_R, LX \rangle \\
&\qquad - c \langle \lambda, \mathcal{T} \rangle \|X\|^2 + c \langle X, \lambda \rangle \langle f_* X, \mathcal{T} \rangle) \\
&\qquad + \sin^2 t (\langle \nabla'_X \lambda, \bar{\mathcal{Y}}(\tilde{\nabla}_X \lambda)_R + c (f_* X \wedge \mathcal{T}) \lambda \rangle + \langle (\tilde{\nabla}_X \lambda)_R, \tilde{\nabla}_X \bar{L} \lambda \rangle \\
&\qquad - c \langle \lambda, \mathcal{T} \rangle \langle f_* X, \tilde{\nabla}_X \lambda \rangle + c \langle X, \lambda \rangle \langle \tilde{\nabla}_X \lambda, \mathcal{T} \rangle) \\
&\quad = \sin^2 t (- \langle \bar{L} \nabla'_X \lambda, (\tilde{\nabla}_X \lambda)_R \rangle + \langle (\tilde{\nabla}_X \lambda)_R, \tilde{\nabla}_X \bar{L} \lambda \rangle + c \langle X, \lambda \rangle \langle (\tilde{\nabla}_X \lambda)_R, \mathcal{T} \rangle) \\
&\quad = \sin^2 t (\langle (\tilde{\nabla}_X \lambda)_R, (\tilde{\nabla}_X \bar{L}) \lambda + c \langle X, \lambda \rangle \mathcal{T} \rangle) \\
&\quad = \sin^2 t \langle (\tilde{\nabla}_X \lambda)_R, \bar{\beta}(X, \lambda) \rangle,
\end{aligned}$$

and this concludes the proof. ∎

Lemma 4.16. *Assume that $\mathcal{T}$ satisfies the condition* $(*)$. *Then, the bilinear form* $\varphi\colon TM \times f_*TM \oplus P \to R \oplus R$ *defined by*

$$\varphi(X,\lambda) = ((\tilde\nabla_X\lambda)_R + \bar\beta(X,\lambda), (\tilde\nabla_X\lambda)_R - \bar\beta(X,\lambda))$$

is flat with respect to the indefinite inner product given by

$$\langle\!\langle (\xi_1,\mu_1),(\xi_2,\mu_2)\rangle\!\rangle_{R\oplus R} = \langle \xi_1,\xi_2\rangle_R - \langle \mu_1,\mu_2\rangle_R.$$

Proof. We need to show that

$$\Theta = \langle\!\langle \varphi(X,\lambda),\varphi(Y,\delta)\rangle\!\rangle - \langle\!\langle \varphi(X,\delta),\varphi(Y,\lambda)\rangle\!\rangle = 0$$

for any $X,Y \in \mathfrak{X}(M)$ and $\lambda,\delta \in f_*TM \oplus P$. We have

$$\begin{aligned}\frac{1}{2}\Theta &= \langle(\tilde\nabla_X\lambda)_R, \bar\beta(Y,\delta)\rangle + \langle(\tilde\nabla_Y\delta)_R, \bar\beta(X,\lambda)\rangle\\ &\quad -\langle(\tilde\nabla_X\delta)_R, \bar\beta(Y,\lambda)\rangle - \langle(\tilde\nabla_Y\lambda)_R, \bar\beta(X,\delta)\rangle.\end{aligned}$$

Clearly $\Theta = 0$ if $\lambda,\delta \in \Gamma(P)$.

From the definitions of $\bar\beta$ and β we obtain

$$\begin{aligned}\bar\beta(X,Y) &= ((\tilde\nabla_X\bar L)Y)_R + c\langle X,Y\rangle\mathcal{T}_R\\ &= ((\tilde\nabla_X L)Y)_R - \langle A_\eta X,Y\rangle(\bar L\eta)_R + c\langle X,Y\rangle\mathcal{T}_R\\ &= \beta(X,Y)_R - \langle A_\eta X,Y\rangle\xi\end{aligned}$$

for any $X,Y \in \mathfrak{X}(M)$. Then

$$\begin{aligned}\frac{1}{2}\Theta &= \langle\alpha(X,\lambda)_R, \beta(Y,\delta)_R - \langle A_\eta Y,\delta\rangle\xi\rangle + \langle\alpha(Y,\delta)_R, \beta(X,\lambda)_R - \langle A_\eta X,\lambda\rangle\xi\rangle\\ &\quad - \langle\alpha(X,\delta)_R, \beta(Y,\lambda)_R - \langle A_\eta Y,\lambda\rangle\xi\rangle - \langle\alpha(Y,\lambda)_R, \beta(X,\delta)_R - \langle A_\eta X,\delta\rangle\xi\rangle\end{aligned}$$

if $\delta,\lambda \in \mathfrak{X}(M)$. Then (4.7) and (4.24) give

$$\begin{aligned}\frac{1}{2}\Theta &= \langle\alpha(X,\lambda),\beta(Y,\delta)\rangle + \langle\alpha(Y,\delta),\beta(X,\lambda)\rangle\\ &\quad - \langle\alpha(X,\delta),\beta(Y,\lambda)\rangle - \langle\alpha(Y,\lambda),\beta(X,\delta)\rangle\\ &= 0.\end{aligned}$$

Finally, if $\lambda = \eta$ and $\delta = Z \in \mathfrak{X}(M)$ then

$$\begin{aligned}\frac{1}{2}\Theta &= \langle\nabla^\perp_X\eta, \bar\beta(Y,Z)\rangle + \langle\alpha(Y,Z), \bar\beta(X,\eta)\rangle - \langle\alpha(X,Z), \bar\beta(Y,\eta)\rangle - \langle\nabla^\perp_Y\eta, \bar\beta(X,Z)\rangle\\ &= \langle\nabla^\perp_X\eta, \beta(Y,Z)\rangle - \langle A_\eta Y,Z\rangle\langle\nabla^\perp_X\eta,\xi\rangle + \langle\alpha(Y,Z)_R, (\tilde\nabla_X\bar L\eta + LA_\eta X)_R\rangle\\ &\quad - \langle\alpha(X,Z)_R, (\tilde\nabla_Y\bar L\eta + LA_\eta Y)_R\rangle - \langle\nabla^\perp_Y\eta, \beta(X,Z)\rangle + \langle A_\eta X,Z\rangle\langle\nabla^\perp_Y\eta,\xi\rangle\\ &= X\langle\eta,\beta(Y,Z)\rangle - \langle\eta,\nabla^\perp_X\beta(Y,Z)\rangle + \langle A_\eta Y,Z\rangle\langle\eta,\nabla^\perp_X\xi\rangle\\ &\quad + \langle\alpha(Y,Z)_R, \mathcal{E}(X,\eta)_R + (\nabla^\perp_X\xi)_R\rangle - \langle\alpha(X,Z)_R, \mathcal{E}(Y,\eta)_R + (\nabla^\perp_Y\xi)_R\rangle\\ &\quad - Y\langle\eta,\beta(X,Z)\rangle + \langle\eta,\nabla^\perp_Y\beta(X,Z)\rangle - \langle A_\eta X,Z\rangle\langle\eta,\nabla^\perp_Y\xi\rangle\\ &= X\langle\eta,\beta(Y,Z)\rangle - \langle\eta,\nabla^\perp_X\beta(Y,Z)\rangle + \langle\alpha(Y,Z)_R, \mathcal{E}(X,\eta)_R\rangle + \langle\alpha(Y,Z),\nabla^\perp_X\xi\rangle\\ &\quad - \langle\alpha(X,Z)_R, \mathcal{E}(Y,\eta)_R\rangle - \langle\alpha(X,Z),\nabla^\perp_Y\xi\rangle - Y\langle\eta,\beta(X,Z)\rangle + \langle\eta,\nabla^\perp_Y\beta(X,Z)\rangle.\end{aligned}$$

Now using (4.5) and (4.24) we obtain

$$\begin{aligned}\frac{1}{2}\Theta = &-X\langle\xi,\alpha(Y,Z)\rangle-\langle\eta,\nabla^\perp_X\beta(Y,Z)\rangle+\langle\alpha(Y,Z),\mathcal{E}(X,\eta)\rangle+\langle\alpha(Y,Z),\nabla^\perp_X\xi\rangle\\&-\langle\alpha(X,Z),\mathcal{E}(Y,\eta)\rangle-\langle\alpha(X,Z),\nabla^\perp_Y\xi\rangle+Y\langle\xi,\alpha(X,Z)\rangle+\langle\eta,\nabla^\perp_Y\beta(X,Z)\rangle\\=&-\langle\xi,(\nabla^\perp_X\alpha)(Y,Z))\rangle-\langle\eta,(\nabla^\perp_X\beta)(Y,Z)\rangle+\langle\alpha(Y,Z),\mathcal{E}(X,\eta)\rangle\\&-\langle\alpha(X,Z),\mathcal{E}(Y,\eta)\rangle+\langle\xi,(\nabla^\perp_Y\alpha)(X,Z)\rangle+\langle\eta,(\nabla^\perp_Y\beta)(X,Z)\rangle\\=&\,0,\end{aligned}$$

where the last equality follows from (4.5), (4.8) and the Codazzi equation. □

Theorem 4.17. *Let $f\colon M^n\to\mathbb{Q}_c^{n+p}$, $p\geq 2$, be an isometric immersion and let $\mathcal{T}$ be an infinitesimal bending of f that satisfies the condition $(*)$. Then along each connected component of an open and dense subset of M^n either $\mathcal{T}$ extends in the singular sense or f is r-ruled with $r\geq n-2p+3$.*

Proof. From Lemma 4.16 the bilinear form φ is flat. Let $U\subset M^n$ be an open subset where there is $Y\in\mathfrak{X}(U)$ such that $Y\in RE(\varphi)$ and $D=\ker\varphi_Y$ has dimension d at any point. Then Proposition 1.10 gives

$$\langle\!\langle\varphi(X,\lambda),\varphi(X,\lambda)\rangle\!\rangle=0$$

for any $X\in\mathfrak{X}(U)$ and $\lambda\in\Gamma(D)$. This and (4.29) imply that (4.28) holds for any $\lambda\in\Gamma(D)$. Whenever there is $\lambda\in\Gamma(D)$, $\|\lambda\|^2=1/\|c\|$, on an open subset $V\subset U$ such that (4.26) defines a singular extension of $f|_V$, then $\mathcal{T}|_V$ extends in the singular sense by means of (4.27).

Let $W\subset U$ be an open subset where $\lambda\in\Gamma(D)$ as above does not exist along any open subset of W. Hence D must be a tangent distribution on W, and from Proposition4.10 it follows that $f|_W$ is d-ruled on connected components of an open dense subset of W. Moreover, we have that the dimension of the rulings is bounded from below by $n+1-\dim\operatorname{Im}(\varphi_Y)\geq n-2p+3$. □

Corollary 4.18. *Let $f\colon M^n\to\mathbb{Q}_c^{n+p}$, $p\geq 2$, be an isometric immersion and let $\mathcal{T}$ be a genuine infinitesimal bending of f that satisfies the condition $(*)$. Then f is r-ruled with $r\geq n-2p+3$ on connected components of an open dense subset of M^n.*

Finally, we have the following result.

Theorem 4.19. *Let $f\colon M^n\to\mathbb{Q}_c^{n+p}$, $n>2p$, be a genuinely infinitesimally bendable isometric immersion. If $2\leq p\leq 5$, then one of the following facts holds along any connected component, say U, of an open dense subset of M^n:*

(i) $f|_U$ is ν-ruled by leaves of relative nullity with $\nu\geq n-2p$.

(ii) *$f|_U$ has index of relative nullity $\nu < n - 2p$ at any point of U and is r-ruled with $r \geq n - 2p + 3$.*

Proof. The proof follows along similar arguments as the proof of Theorem 3.9 using Corollary 4.12 and Corollary 4.18. □

4.5 Exercises

Exercise 4.1. Let $\mathcal{Z}$ be a Killing vector field of a Riemannian manifold M^n and let $\mathcal{L} \in \Gamma(\mathrm{End}(TM))$ be the tensor defined by $\mathcal{L}X = \nabla_X \mathcal{Z}$. Prove that $\mathcal{L}$ satisfies

$$(\nabla_X \mathcal{L})Y = R(X, \mathcal{Z})Y$$

for any $X, Y \in \mathfrak{X}(M)$.

Hint: First prove that the 1-form defined by $\omega(X) = \langle X, \mathcal{Z}\rangle$ satisfies

$$d\omega(X, Y) = 2\langle \mathcal{L}X, Y\rangle$$

for any $X, Y \in \mathfrak{X}(M)$ and conclude that the 2-form $\langle \mathcal{L}X, Y\rangle$ is closed. Then show that

$$\langle (\nabla_X \mathcal{L})Y, Z\rangle - \langle (\nabla_Y \mathcal{L})X, Z\rangle + \langle (\nabla_Z \mathcal{L})X, Y\rangle = 0.$$

Finally, prove and use that

$$(\nabla_X \mathcal{L})Y - (\nabla_Y \mathcal{L})X = R(X, Y)\mathcal{Z}$$

for any $X, Y \in \mathfrak{X}(M)$.

Exercise 4.2. Let $f\colon M^n \to \mathbb{Q}_c^{n+1}$, $n \geq 4$, be an isometric immersion of a compact Riemannian manifold M^n. Assume that there are no open subsets of M^n where f is totally geodesic. Prove that f is infinitesimally rigid.

Hint: Use Proposition 4.8 and Theorem 13.2 in [21].

Chapter 5

Variations of product manifolds

This chapter is about infinitesimal variations of submanifolds that are intrinsically a Riemannian product of manifolds. The study of such variations is done analyzing the structure of the possible infinitesimal bendings. The results obtained provide local and global conditions under which the submanifold splits as an extrinsic product of immersions and any infinitesimal bending of the submanifold has to be the sum of infinitesimal bendings of each of the factors.

Let $M^n = M_1^{n_1} \times \cdots \times M_r^{n_r}$ be a Riemannian product of Riemannian manifolds of dimensions $n_i \geq 2$, $1 \leq i \leq r$. The *extrinsic product* $f\colon M^n \to \mathbb{R}^m$ of the set of isometric immersions $f_i\colon M_i^{n_i} \to \mathbb{R}^{m_i}$, $1 \leq i \leq r$, is the isometric immersion given by

$$f(x) = (f_1(x_1), \ldots, f_r(x_r)),$$

where $x = (x_1, \ldots, x_r) \in M^n$ and $\mathbb{R}^m = \oplus_{i=1}^r \mathbb{R}^{m_i}$.

Let $\iota_i^{\bar{x}}\colon M_i^{n_i} \to M^n$ denote the inclusion map for $\bar{x} = (\bar{x}_1 \ldots, \bar{x}_r)$, that is,

$$\iota_i^{\bar{x}}(x_i) = (\bar{x}_1, \ldots, x_i, \ldots, \bar{x}_r).$$

Then let $\tilde{\iota}_i^y$ be the inclusion of $\mathbb{R}^{m_i}$ into $\mathbb{R}^m$ defined in a similar manner. The normal space of f at $x = (x_1, \ldots, x_r) \in M^m$ is

$$N_f M(x) = \oplus_{i=1}^r N_{f_i} M_i(x_i),$$

where $N_{f_i} M_i(x_i)$ is the normal space of f_i at $x_i \in M_i^{n_i}$, $1 \leq i \leq r$. If α_i is the second fundamental form of f_i at $x_i \in M_i^{n_i}$, $1 \leq i \leq r$, then the second fundamental form α of f at $x=(x_1,\dots,x_r)\in M^n$ is given by

$$\alpha(\iota^x_{i*}X,\iota^x_{j*}Y)=\begin{cases}\tilde{\iota}^{f(x)}_{i*}\alpha_i(X,Y) & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases} \tag{5.1}$$

where $X\in\mathfrak{X}(M_i)$ and $Y\in\mathfrak{X}(M_j)$, $1\le i,j\le r$.

Let $\mathcal{T}_i$ be an infinitesimal bending of f_i in $\mathbb{R}^{m_i}$, $1\le i\le r$, and let $(\beta_i,\mathcal{E}_i)$ be its associated pair. Then $\mathcal{T}(x)=\sum_{i=1}^r\tilde{\iota}^{f(x)}_{i*}\mathcal{T}_i(x_i)$ is an infinitesimal bending of f in $\mathbb{R}^m$. Let L be associated to $\mathcal{T}$ and let L_i be associated to $\mathcal{T}_i$. Then

$$L\iota^x_{i*}X=\tilde{\iota}^{f(x)}_{i*}L_iX \tag{5.2}$$

for any $X\in\mathfrak{X}(M_i)$. If B_i is associated to $\mathcal{T}_i$, it follows that

$$B(\iota^x_{i*}X,\iota^x_{j*}Y)=(\tilde{\nabla}_{\iota^x_{i*}X}L)\iota^x_{j*}Y=\begin{cases}\tilde{\iota}^{f(x)}_{i*}B_i(X,Y) & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases}$$

where $X\in\mathfrak{X}(M_i)$ and $Y\in\mathfrak{X}(M_j)$. In particular,

$$\beta(\iota^x_{i*}X,\iota^x_{j*}Y)=\begin{cases}\tilde{\iota}^{f(x)}_{i*}\beta_i(X,Y) & \text{if } i=j,\\ 0 & \text{if } i\neq j\end{cases} \tag{5.3}$$

and

$$\mathcal{E}(\iota^x_{i*}X,\tilde{\iota}^{f(x)}_{j*}\eta)=\begin{cases}\tilde{\iota}^{f(x)}_{i*}\mathcal{E}_i(X,\eta) & \text{if } i=j,\\ 0 & \text{if } i\neq j\end{cases} \tag{5.4}$$

for any $X\in\mathfrak{X}(M_i)$, $Y\in\mathfrak{X}(M_j)$ and $\eta\in\Gamma(N_{f_j}M_j)$, where (5.4) follows from (5.1), (5.2) and the definition (2.10) of $\mathcal{Y}$ in terms of which $\mathcal{E}$ is given.

If $(\beta,\mathcal{E})$ is the pair associated to an infinitesimal bending $\mathcal{T}$ of an extrinsic product $f=(f_1,\dots,f_r)$, we say that β is *adapted* to the product structure if

$$\beta(\iota^x_{i*}X,\iota^x_{j*}Y)=0$$

for any $X\in\mathfrak{X}(M_i)$ and $Y\in\mathfrak{X}(M_j)$ with $i\neq j$.

Proposition 5.1. *Let $f\colon M^n\to\mathbb{R}^m$ be an extrinsic product of isometric immersions $f_i\colon M_i^{n_i}\to\mathbb{R}^{m_i}$, $n_i\ge 2$, $1\le i\le r$, with full first normal spaces. If the tensor β in the pair associated to an infinitesimal bending $\mathcal{T}$ of f is adapted, then there exist locally infinitesimal bendings $\mathcal{T}_i$ of f_i, $1\le i\le r$, such that $\mathcal{T}(x)=\sum_{i=1}^r\tilde{\iota}^{f(x)}_{i*}\mathcal{T}_i(x_i)$.*

Proof. From (2.12) we obtain

$$\langle\beta(\iota^x_{i*}X,\iota^x_{i*}Y),\alpha(\iota^x_{j*}Z,\iota^x_{j*}W)\rangle+\langle\alpha(\iota^x_{i*}X,\iota^x_{i*}Y),\beta(\iota^x_{j*}Z,\iota^x_{j*}W)\rangle=0 \tag{5.5}$$

for any $X,Y\in\mathfrak{X}(M_i)$ and $Z,W\in\mathfrak{X}(M_j)$ with $i\neq j$.

Let $\alpha_i(X_k,Y_k)$, $1\leq k\leq\dim N_{f_i}M_i$, with $X_k,Y_k\in\mathfrak{X}(M_i)$ be a basis of $N_{f_i}M_i$. Set

$$C_{ij}\alpha_i(X_k,Y_k)=\beta_{N_{f_j}M_j}(\iota^x_{i*}X_k,\iota^x_{i*}Y_k),\ i\neq j,$$

where $\beta_{N_{f_j}M_j}$ denotes the $N_{f_j}M_j$-component of β. We claim that the linear extension to a map $C_{ij}\colon N_{f_i}M_i\to N_{f_j}M_j$, $i\neq j$, satisfies

$$C_{ij}\alpha_i(X,Y)=\beta_{N_{f_j}M_j}(\iota^x_{i*}X,\iota^x_{i*}Y)$$

for any $X,Y\in\mathfrak{X}(M_i)$. In fact, if

$$\alpha_i(X,Y)=\sum_k c_k\alpha_i(X_k,Y_k)$$

for $X,Y\in\mathfrak{X}(M_i)$ and $c_k\in\mathbb{R}$, $1\leq k\leq\dim N_{f_i}M_i$, we obtain from (5.5) that

$$\langle\beta(\iota^x_{i*}X,\iota^x_{i*}Y)-\sum_k c_k\beta(\iota^x_{i*}X_k,\iota^x_{i*}Y_k),\alpha(\iota^x_{j*}Z,\iota^x_{j*}W)\rangle=0$$

for any $Z,W\in\mathfrak{X}(M_j)$, $i\neq j$, and the claim follows.

We have from (5.5) that the map $C\in\Gamma(\mathrm{End}(N_fM))$ defined by

$$C\tilde{\iota}^{f(x)}_{i*}\eta_i=\sum_{j\neq i}\tilde{\iota}^{f(x)}_{j*}C_{ij}\eta_i,$$

where $\eta_i\in\Gamma(N_{f_i}M_i)$, $1\leq i\leq r$, is skew-symmetric. Then, we obtain that $\beta(\iota^x_{i*}X,\iota^x_{i*}Y)$ decomposes orthogonally as

$$\beta(\iota^x_{i*}X,\iota^x_{i*}Y)=\beta_{N_{f_i}M_i}(\iota^x_{i*}X,\iota^x_{i*}Y)+C\alpha(\iota^x_{i*}X,\iota^x_{i*}Y)\tag{5.6}$$

for any $X,Y\in\mathfrak{X}(M_i)$.

Let $L_i\colon TM_i\to f_i^*T\mathbb{R}^{m_i}$ be given by

$$L_iX=(L\iota^x_{i*}X)_{\mathbb{R}^{m_i}}.$$

Since f is an extrinsic product of immersions, we have

$$\begin{aligned}\tilde{\nabla}_{\iota^x_{j*}Y}\tilde{\iota}^{f(x)}_{i*}L_iX&=\tilde{\nabla}_{\iota^x_{j*}Y}(L\iota^x_{i*}X)_{\mathbb{R}^{m_i}}\\&=(\tilde{\nabla}_{\iota^x_{j*}Y}L\iota^x_{i*}X)_{\mathbb{R}^{m_i}}\\&=(B(\iota^x_{j*}Y,\iota^x_{i*}X))_{\mathbb{R}^{m_i}}\\&=(f_*\mathcal{Y}\alpha(\iota^x_{j*}Y,\iota^x_{i*}X)+\beta(\iota^x_{j*}Y,\iota^x_{i*}X))_{\mathbb{R}^{m_i}}\\&=0\end{aligned}$$

for any $X \in \mathfrak{X}(M_i)$ and $Y \in \mathfrak{X}(M_j)$ with $i \neq j$, where the last steps follow using (2.11) and the assumption on β. Thus the tensors L_i are well defined on $M_i^{n_i}$, $1 \leq i \leq r$. Moreover, since B is symmetric, these tensors verify

$$(\tilde{\nabla}^i_X L_i)Y = (\tilde{\nabla}^i_Y L_i)X$$

for any $X, Y \in \mathfrak{X}(M_i)$, where $\tilde{\nabla}^i$ is the connection in $\mathbb{R}^{m_i}$. Thus, there exist locally vector fields $\mathcal{T}_i \in \Gamma(f_i^* T\mathbb{R}^{m_i})$ with $\tilde{\nabla}^i_X \mathcal{T}_i = L_i X$ for any $X \in \mathfrak{X}(M_i)$, $1 \leq i \leq r$. In particular, since L_i verifies (2.8), then $\mathcal{T}_i$ is an infinitesimal bending of f_i and, if β_i belongs to the pair associated to $\mathcal{T}_i$, we have

$$\tilde{\iota}^{f(x)}_{i*}\beta_i(X, Y) = \beta_{N_{f_i}M_i}(\iota^x_{i*}X, \iota^x_{i*}Y) \tag{5.7}$$

for any $X, Y \in \mathfrak{X}(M_i)$, $1 \leq i \leq r$.

Define an infinitesimal bending $\tilde{\mathcal{T}}$ of f by $\tilde{\mathcal{T}} = \sum_{i=1}^r \tilde{\iota}^{f(x)}_{i*}\mathcal{T}_i$. We have from (5.3), (5.6) and (5.7) that $\mathcal{T} - \tilde{\mathcal{T}}$ has the associated tensor $\beta - \tilde{\beta} = C\alpha$. Since C is skew-symmetric the tensor $\nabla^\perp C$ satisfies (2.7). Moreover, we have

$$\begin{aligned}(\nabla^\perp_X \beta - \nabla^\perp_X \tilde{\beta})(Y, Z) &= (\nabla^\perp_X C\alpha)(Y, Z)\\ &= \nabla^\perp_X C\alpha(Y, Z) - C\alpha(\nabla_X Y, Z) - C\alpha(Y, \nabla_X Z)\\ &= (\nabla^\perp_X C)\alpha(Y, Z) + C(\nabla^\perp_X \alpha)(Y, Z)\end{aligned}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Using the Codazzi equation, we obtain

$$(\nabla^\perp_X \beta - \nabla^\perp_X \tilde{\beta})(Y, Z) - (\nabla^\perp_Y \beta - \nabla^\perp_Y \tilde{\beta})(X, Z) = (\nabla^\perp_X C)\alpha(Y, Z) - (\nabla^\perp_Y C)\alpha(X, Z)$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Now Proposition 2.6 gives $\mathcal{E} - \tilde{\mathcal{E}} = -(\nabla^\perp C)$, and the proof follows from Proposition 2.7. $\square$

The assumption in the above result that f has full first normal spaces cannot be dropped. In fact, we observed in Examples 2.1 that if this is not the case then a smooth normal vector field in $N_1^\perp$ is an infinitesimal bending.

Proposition 5.2. *Let $f \colon M^n \to \mathbb{R}^{n+p}$, $p < n$, be an extrinsic product of isometric immersions $f_i \colon M_i^{n_i} \to \mathbb{R}^{n_i+p_i}$, $n_i \geq 2$ and $1 \leq i \leq r$. Assume that the s-nullities of f satisfy $\nu_s(x) < n - s$, $1 \leq s \leq p$, at any $x \in M^n$. Then any infinitesimal bending $\mathcal{T}$ of f is locally of the form $\mathcal{T}(x) = \sum_{i=1}^r \tilde{\iota}^{f(x)}_{i*}\mathcal{T}_i(x_i)$, where $\mathcal{T}_i$, $1 \leq i \leq r$, is an infinitesimal bending of f_i.*

Proof. Since f is an extrinsic product of immersions, then $\sum_{j \neq i} \iota^x_{j*}TM_j \subset \mathcal{N}(\alpha_{N_{f_i}M_i})$. Thus the assumption on the s-nullities yields $\sum_{j \neq i} n_j < n - p_i$, that is,

$$p_i < n_i, \quad 1 \leq i \leq r. \tag{5.8}$$

Take $X \in \mathfrak{X}(M_i)$, $W \in \mathfrak{X}(M_j)$ and $Y, Z \in \mathfrak{X}(M_k)$ with $i \neq j$. If also $k \neq i, j$, we obtain from (2.12) that

$$\langle \beta(\iota^x_{i*}X, \iota^x_{j*}W), \alpha(\iota^x_{k*}Y, \iota^x_{k*}Z)\rangle = 0. \tag{5.9}$$

On the other hand, for $k = i$ it follows from (2.12) that

$$\langle \beta(\iota^x_{i*}X, \iota^x_{j*}W), \alpha(\iota^x_{i*}Y, \iota^x_{i*}Z)\rangle - \langle \beta(\iota^x_{i*}Y, \iota^x_{j*}W), \alpha(\iota^x_{i*}X, \iota^x_{i*}Z)\rangle = 0. \tag{5.10}$$

Let $\beta^i_W \colon TM_i \to N_{f_i}M_i$ be given by

$$\beta^i_W X = \beta(\iota^x_{i*}X, \iota^x_{j*}W)_{N_{f_i}M_i}.$$

Suppose that $\dim \mathrm{Im}\beta^i_W = s > 0$. Then (5.8) gives $\dim\ker\beta^i_W = n_i - s > 0$. It follows from (5.10) that

$$\langle \beta(\iota^x_{i*}X, \iota^x_{j*}W), \alpha(\iota^x_{i*}T, \iota^x_{i*}Z)\rangle = 0$$

for any T in $\ker\beta^i_W$. This implies that $\nu_s \geq n - s$, which contradicts our assumption and proves that $\beta(\iota^x_{i*}X, \iota^x_{j*}W)_{N_{f_i}M_i} = 0$ for any $X \in \mathfrak{X}(M_i)$ and $W \in \mathfrak{X}(M_j)$. This together with (5.9) imply that

$$\beta(\iota^x_{i*}X, \iota^x_{j*}W) = 0 \ \text{ if } \ i \neq j.$$

Thus β is adapted, and the proof now follows from Proposition 5.1. $\square$

For the proof of the next theorem we need the following result on isometric immersions from [22]. It can also be seen as Theorem 8.14 in [21].

Proposition 5.3. *Let $f\colon M^n = M_1^{n_1} \times \cdots \times M_r^{n_r} \to \mathbb{R}^{n+p}$, $2p < n$, be an isometric immersion such that the s-nullities of f satisfy $\nu_s(x) < n - 2s$, $1 \leq s \leq p$, at any $x \in M^n$. Then f is an extrinsic product of isometric immersions.*

The following is the main local result of this chapter.

Theorem 5.4. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $2p < n$, be an isometric immersion of a Riemannian product $M^n = M_1^{n_1} \times \cdots \times M_r^{n_r}$ with $n_j \geq 2$, $1 \leq j \leq r$. Assume that the s-nullities of f satisfy $\nu_s(x) < n - 2s$, $1 \leq s \leq p$, at any $x \in M^n$. Then f is an extrinsic product of isometric immersions $f = (f_1, \ldots, f_r)$ and any infinitesimal bending $\mathcal{T}$ of f is locally of the form $\mathcal{T}(x) = \sum_{i=1}^r \tilde{\iota}^{f(x)}_{i*}\mathcal{T}_i(x_i)$, where $\mathcal{T}_i$ is an infinitesimal bending of $f_i\colon M_i^{n_i} \to \mathbb{R}^{m_i}$, $1 \leq i \leq r$.*

Proof. From Proposition 5.3 we obtain that f is an extrinsic product of isometric immersions, and the proof follows from Proposition 5.2. $\square$

Concerning isometric immersions of Riemannian products there is the following basic rigidity result from [30]. It can also be seen as Theorem 8.10 in [21].

Proposition 5.5. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a Riemannian product $M^n = M_1^{n_1} \times \cdots \times M_p^{n_p}$, with $n_i \geq 2$ for all $1 \leq i \leq p$. If the subset of points of $M_i^{n_i}$, $1 \leq i \leq p$, at which all the sectional curvatures vanish has empty interior then f is an extrinsic product of hypersurfaces $f = (f_1, \ldots, f_p)$.*

The following is the corresponding version of the above result for infinitesimal variations.

Theorem 5.6. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a Riemannian product $M^n = M_1^{n_1} \times \cdots \times M_p^{n_p}$, $n_i \geq 2$ and $1 \leq i \leq p$. Assume that the subset of points of $M_i^{n_i}$, $1 \leq i \leq p$, at which all the sectional curvatures vanish has empty interior. Then f is an extrinsic product of hypersurfaces $f_i\colon M_i^{n_i} \to \mathbb{R}^{n_i+1}$, $1 \leq i \leq p$, and any infinitesimal $\mathcal{T}$ bending of f is locally of the form $\mathcal{T}(x) = \sum_{i=1}^p \tilde{\iota}_{i*}^{f(x)} \mathcal{T}_i(x_i)$, where $\mathcal{T}_i$ is an infinitesimal bending of f_i, $1 \leq i \leq p$.*

Proof. The first statement follows from Proposition 5.5. Since each $M_i^{n_i}$ has no flat open subset then, in an open and dense subset $\tilde{M} \subset M^n$, the index of relative nullity of f_i satisfies $\nu^i = \dim \Delta_i \leq n_i - 2$ for each $1 \leq i \leq p$.

Fix $x = (x_1, \ldots, x_p) \in \tilde{M}$ and let $U^s \subset N_f M(x) = \oplus_{i=1}^p N_{f_i} M_i(x_i)$ be a subspace. Assume that $X = (X_1, \ldots, X_p) \in \mathcal{N}(\pi_{U^s} \circ \alpha)(x)$, where $X_i \in T_{x_i} M_i$, $1 \leq i \leq p$, and $\pi_{U^s}\colon N_f M(x) \to U$ is the orthogonal projection. Let $\eta = (\eta_1, \ldots, \eta_p) \in U^s$ where $\eta_i \in N_{f_i} M_i(x_i)$. Then $\langle \alpha(X,Y), \eta \rangle = 0$ for any $Y \in T_x M$. If $\eta_i \neq 0$, then taking $Y = (0, \ldots, Y_i, \ldots, 0)$ for $Y_i \in T_{x_i} M_i$, we obtain that $X_i \in \Delta_i$. Therefore, we have that $(\mathcal{N}(\pi_{U^s} \circ \alpha))_{T_{x_i} M_i} \subset \Delta_i$.

If $U^s \subset N_{f_j} M_j^{\perp}$ for some j, we have that $T_{x_j} M_j \subset \mathcal{N}(\pi_{U^s} \circ \alpha)(x)$. Notice that

$$\dim \mathcal{N}(\pi_{U^s} \circ \alpha) \leq \sum_{i=1}^p \dim(\mathcal{N}(\pi_{U^s} \circ \alpha))_{T_{x_i} M_i}.$$

By rearranging the factors, if necessary, we can assume that $U^s \subset \oplus_{i=1}^k N_{f_i} M_i$ with $1 \leq k \leq p$. Hence

$$\begin{aligned}
\dim \mathcal{N}(\pi_{U^s} \circ \alpha) &\leq \sum_{i=1}^k \nu^i + \sum_{j>k} n_j \\
&\leq \sum_{i=1}^k (n_i - 2) + \sum_{j>k} n_j \leq n - 2k \leq n - 2s \\
&< n - s.
\end{aligned}$$

Thus $\nu_s < n - s$, $1 \leq s \leq p$, on M^n, and the proof follows from Proposition 5.2. ∎

The following result on isometric immersions is Corollary 8.24 in [21].

Proposition 5.7. *Let $M_1^{n_1},\ldots,M_p^{n_p}$, $n_i \geq 2$, $1 \leq i \leq p$, be compact and nonflat Riemannian manifolds. Then any isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ of the Riemannian product manifold $M^n = M_1^{n_1} \times \cdots \times M_p^{n_p}$ is an extrinsic product of hypersurfaces $f_i\colon M_i^{n_i} \to \mathbb{R}^{n_i+1}$, $1 \leq i \leq p$.*

Finally, there is the following global version of Theorem 5.6.

Theorem 5.8. *Let $M_1^{n_1},\ldots,M_p^{n_p}$, $n_i \geq 2$, $1 \leq i \leq p$, be compact Riemannian manifolds and let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of the Riemannian product $M^n = M_1^{n_1} \times \ldots \times M_p^{n_p}$. Assume that the subset of points of $M_i^{n_i}$, $1 \leq i \leq p$, at which all the sectional curvatures vanish has empty interior. Then f is an extrinsic product of hypersurfaces $f_i\colon M_i^{n_i} \to \mathbb{R}^{n_i+1}$, $1 \leq i \leq p$, and f is infinitesimally rigid.*

Proof. The first statement follows from Proposition (5.7). Let $\mathcal{T}$ be an infinitesimal bending of f. Fix $1 \leq i \leq p$ as well as points $y_j \in M_j^{n_j}$ for any $j \neq i$. Then the vector field $\hat{\mathcal{T}}_i(x_i) = (\mathcal{T}(y_1,\ldots,x_i,\ldots,y_p))_{\mathbb{R}^{n_i+1}}$ is an infinitesimal bending of f_i. Since $M_i^{n_i}$ is compact and posses no flat open subset, then by Theorem 2.13 we have that f_i is infinitesimally rigid. Hence $\hat{\mathcal{T}}_i$ is trivial for each choice of y_j.

On the other hand, we have from Theorem 5.6 that $\mathcal{T}(x) = \sum_{i=1}^{p} \tilde{\iota}_{i*}^{f(x)}\mathcal{T}_i(x_i)$ locally. Thus, we necessarily have that $\mathcal{T}_i = \hat{\mathcal{T}}_i$ locally for $1 \leq i \leq p$, and thus $\mathcal{T}_i$ is trivial. Hence the associated tensors of $\mathcal{T}$ have the form 2.19 at every point showing that $\mathcal{T}$ is trivial. $\square$

Chapter 6

Variations of complete hypersurfaces

This chapter gives a classification of the complete Euclidean hypersurfaces of dimension at least four that admit nontrivial infinitesimal variations. If the hypersurface is compact, it does not admit even isometric variations due to the classical result of Sacksteder [32]. Dajczer-Gromoll [13] proved that if the hypersurface is a complete manifold that does not contain a cylinder of a certain type as an open subset, then it allows isometric variations only along ruled strips. Before we state and prove the infinitesimal analogue of the latter result that is due to Jimenez [28], we hold a discussion on ruled hypersurfaces whose rulings are complete Euclidean spaces.

6.1 Ruled hypersurfaces

A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, is said to be *ruled* if it is $(n-1)$-ruled. A hypersurface with possible boundary is said to be ruled if, in addition, the rulings are tangent to the boundary. A connected component of the subset of M^n where all the rulings are complete manifolds is called a *ruled strip*. Therefore, a ruled strip is an affine vector bundle over a curve with or without end points.

From now on the quantity $\dim \Delta^\perp(x) = n - \nu(x)$ is called the *rank* of the hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ at $x \in M^n$. Thus, the rank is just the number of nonzero principal curvatures. The second fundamental form A of a ruled hypersurface has rank at most two and, outside totally geodesic points, the leaves of relative nullity are contained in the rulings.

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a ruled hypersurface and let $c\colon I \to M^n$, $c = c(s)$ and $s \in I \subset \mathbb{R}$, be a unit speed curve orthogonal to the rulings. The rulings form an affine vector bundle over $\tilde{c} = f \circ c$ in $\mathbb{R}^{n+1}$. Let $T_i(s)$, $1 \le i \le n-1$, be orthonormal tangent fields on the corresponding bundle along c which are parallel with respect to the induced connection. Set $f_* dc/ds = \tilde{T}_0$, $\tilde{T}_i = f_* T_i$ and let N be a unit vector field along c normal to f. Then

$$\begin{cases} \tilde{\nabla}_{\partial/\partial s}\tilde{T}_0 = -\Sigma_i \varphi_i \tilde{T}_i + \theta N, \\ \tilde{\nabla}_{\partial/\partial s}\tilde{T}_i = \varphi_i \tilde{T}_0 + \beta_i N, \end{cases}$$

where $\theta = \langle AT_0, T_0 \rangle$, $\varphi_i = \langle \nabla_{T_0} T_i, T_0 \rangle$ and $\beta_i = \langle AT_i, T_0 \rangle$, $1 \le i \le n-1$.

We parametrize a neighborhood of $\tilde{c}(I)$ in $f(M)$ by $\tilde{f}\colon W \subset I \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}$ given by

$$\tilde{f}(s, u_1, \ldots, u_{n-1}) = \tilde{c}(s) + \Sigma_i u_i \tilde{T}_i(s). \tag{6.1}$$

We have at $(s, u_1, \ldots, u_{n-1})$ that

$$\tilde{f}_* \partial/\partial s = (1 + \Sigma_i u_i \varphi_i)\tilde{T}_0 + \Sigma_i u_i \beta_i N.$$

Thus, the map $\tilde{f}$ has maximal rank if and only if

$$\|\tilde{f}_* \partial/\partial s\|^2 = (1 + \Sigma_i u_i \varphi_i)^2 + (\Sigma_i u_i \beta_i)^2 \ne 0.$$

Note that the directions $\Sigma_i u_i T_i(s)$ for which $\Sigma_i u_i \beta_i = 0$ are in the relative nullity of f at $c(s)$.

Proposition 6.1. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \ge 3$, be an isometric immersion and let $U \subset M^n$ be an open subset where f has rank two. Assume that $f|_U$ is ruled and that the relative nullity leaves are complete. Let $\delta\colon [0,a] \to M^n$ be a unit speed geodesic orthogonal to Δ such that $\delta([0,a)) \subset U$ is contained on a ruling. Then the rank of f at $\delta(a)$ is two. Moreover, every point in U has a neighborhood V such that $f|_V$ extends to a ruled strip of constant rank two.*

Proof. Let $W \subset I \times \mathbb{R}^{n-1}$ be an open subset where the parametrization (6.1) is defined and write $W_s = W \cap (\{s\} \times \mathbb{R}^{n-1})$. Assume that the geodesic δ is contained on the ruling determined by $\tilde{f}|_{W_s}$ and has T_{n-1} as its tangent vector field. Notice that $r \mapsto \tilde{f}(s, 0, \ldots, 0, r)$ is a parametrization of δ. Since $\beta_{n-1}(s) \ne 0$, then the map $\tilde{f}$ has maximal rank along δ and we have at $\delta(r)$ that

$$\tilde{f}_*(\partial/\partial s) = (1 + r\varphi_{n-1})\tilde{T}_0 + r\beta_{n-1} N.$$

Let $\tilde{N}(\delta(r)) = \alpha(r)\tilde{T}_0 + N$ be a vector field normal to f along δ (not necessarily unitary). Then

$$0 = \langle \tilde{f}_*(\partial/\partial s), \alpha\tilde{T}_0 + N \rangle = \alpha(r)(1 + r\varphi_{n-1}) + r\beta_{n-1}.$$

Taking $r \in (0, a]$ we have that $1 + r\varphi_{n-1} \neq 0$, then

$$\alpha(r) = -\frac{r\beta_{n-1}}{(1 + r\varphi_{n-1})}.$$

Then, we obtain that

$$\begin{aligned}\langle \tilde{\nabla}_{\delta'(r)} \tilde{f}_*(\partial/\partial s), \tilde{N}(\delta(r))\rangle &= \langle \varphi_{n-1}\tilde{T}_0 + \beta_{n-1}N, \alpha(r)\tilde{T}_0 + N\rangle \\ &= \frac{\beta_{n-1}}{1 + r\varphi_{n-1}},\end{aligned}$$

which does not vanish. Thus the rank of $\tilde{f}$ at $\delta(a)$ is two, and therefore the same holds for f.

It remains to prove that $f|_U$ extends locally to a ruled strip. Fix $x \in U$ and let $V \subset U$ be a neighborhood of x parametrized by (6.1). Extend $\tilde{f}$ to $I \times \mathbb{R}^{n-1}$ with the same expression. We claim that this extension defines a ruled strip of constant rank two. We first prove that $\tilde{f}$ has no singular points. As seen previously, we have that $\tilde{f}$ is singular at points where

$$(1 + \Sigma_i u_i\varphi_i)^2 + (\Sigma_i u_i\beta_i)^2 = 0.$$

Then, it suffices to show that $\Sigma_i u_i\varphi_i = 0$ for any $T = \Sigma_i u_i T_i(s) \in \Delta(c(s))$.

Given $T \in \Delta(c(s))$, we have

$$\Sigma_i u_i\varphi_i = \langle \nabla_{T_0} T, T_0\rangle = -\langle C_T T_0, T_0\rangle,$$

where C_T is the splitting tensor of Δ with respect to T. If C_T vanishes there is nothing to prove. Otherwise, let X be a unit vector field on V tangent to a ruling and orthogonal to the relative nullity. Since each ruling is totally geodesic and the only real eigenvalue of C_T is zero by Proposition 1.8, then we have that $C_T X = 0$ for any $T \in \Gamma(\Delta)$. Finally, using Proposition 1.8 once more, we have that $\langle C_T T_0, T_0\rangle = 0$, and therefore $\tilde{f}$ has no singular points.

It follows from Proposition 1.7 that the open subset where $\tilde{f}$ has rank two is a union of complete relative nullity leaves. From the previous discussion we have that the rank of $\tilde{f}$ along any ruling is two, and the claim follows. $\square$

6.2 The classification

The following is a classification of the complete hypersurfaces that admit nontrivial infinitesimal variations.

Theorem 6.2. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, be an isometric immersion of a complete Riemannian manifold. Assume that there is no open subset* *of M^n where f is either totally geodesic or a cylinder over a hypersurface in $\mathbb{R}^4$ with complete one-dimensional leaves of relative nullity. Then f admits nontrivial infinitesimal variations only along ruled strips.*

If the hypersurface contains a ruled strip, a rather simple argument given in [13] shows that there is a one-to-one correspondence between the set of smooth functions on an open interval and the set of isometric deformations of the hypersurface that act only along the ruled strip. Then the same is true for the isometric variations obtained multiplying such a function by a parameter. We show below that any infinitesimal bending of the hypersurface is just the variational vector field of such an isometric variation. Thus, the classification result for infinitesimal variations is the same than for isometric variations.

Remark 6.3. Notice that Theorem 2.13 for $n \geq 4$ is a corollary of the above result.

For the proof of Theorem 6.2 we need several lemmas.

Lemma 6.4. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion. If $U \subset M^n$ is an open subset where f has constant rank two and the leaves of the relative nullity are complete, then the codimension of*

$$C_0 = \{T \in \Delta : C_T = 0\} \tag{6.2}$$

is at most one. Moreover, if $\dim C_0^\perp = 1$ and C_T is invertible for $T \in \Gamma(C_0^\perp)$, then $f|_U$ is a cylinder over a hypersurface $g\colon L^3 \to \mathbb{R}^4$ that carries a one-dimensional relative nullity distribution with complete leaves.

Proof. Assume that $C_0^\perp \subset \Delta$ has dimension at least two. Then, for dimension reasons there is $T \in \Delta$ such that $C_T \neq 0$ is self adjoint, which contradicts Proposition 1.8.

Now assume that $\dim C_0^\perp = 1$ and that C_T is invertible for $T \in \Gamma(C_0^\perp)$. We have from (1.8) that

$$\langle \nabla_X S, T\rangle C_T Y = \langle \nabla_Y S, T\rangle C_T X$$

for all $S \in \Gamma(C_0)$ and $X, Y \in \Gamma(\Delta^\perp)$. Then

$$\langle \nabla_X S, T\rangle Y - \langle \nabla_Y S, T\rangle X \in \Gamma(\ker C_T).$$

Since C_T is invertible, we necessarily have that

$$\langle \nabla_X S, T\rangle Y - \langle \nabla_Y S, T\rangle X = 0.$$

Thus $\langle \nabla_X S, T\rangle = 0$ for any $X \in \Gamma(\Delta^\perp)$. Note that

$$\langle \nabla_X S, Y\rangle = -\langle C_S X, Y\rangle = 0$$

for any $X, Y \in \Delta^\perp$, and thus $\nabla_X S \in \Gamma(C_0)$.

On the other hand, it follows from (1.7) that

$$C_{\nabla_R S} = \nabla_R C_S - C_S C_R = 0$$

for any $R \in \Gamma(\Delta)$. Therefore C_0 is a parallel distribution on U. Since $C_0 \subset \Delta$ the proof follows from Proposition 1.9. $\square$

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion, and let $\mathcal{T}$ be an infinitesimal bending of f with associated tensor $\mathcal{B}$. Set $\Delta^* = \Delta \cap \ker \mathcal{B}$. Assume that the second fundamental form of f has rank $A = 2$. Then (2.21) gives $\Delta \subset \ker \mathcal{B}$, hence rank $\mathcal{B} \leq 2$ and, in particular, we have $\Delta^* = \Delta$. Since $\mathcal{B}$ is a Codazzi tensor, we obtain that

$$\nabla_T \mathcal{B} = \mathcal{B} C_T = C_T' \mathcal{B} \tag{6.3}$$

for any $T \in \Gamma(\Delta^*)$, where C is the splitting tensor of $\Delta^* = \Delta$.

Lemma 6.5. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a ruled hypersurface of constant rank two with complete relative nullity leaves. Assume that the splitting tensor of the relative nullity foliation does not vanish on any open subset. If $\mathcal{T}$ is an infinitesimal bending of f, then its associated symmetric tensor $\mathcal{B}$ satisfies*

$$\mathcal{B}|_{\Delta^\perp} = \begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix} \tag{6.4}$$

with respect to a local orthonormal basis $\{Y, X\}$ of $\Delta^\perp$ such that Y is orthogonal to the rulings. Moreover, the smooth function θ verifies that

$$X(\theta) = \langle \nabla_Y Y, X \rangle \theta. \tag{6.5}$$

Proof. On the open dense subset where $C \neq 0$, let $T \in \Gamma(C_0^\perp)$ be unitary, where C_0 is given by (6.2). Locally take $X, Y \in \Gamma(\Delta^\perp)$ orthonormal such that Y is orthogonal to the rulings. We have seen in the proof of Proposition 6.1 that $X \in \Gamma(\ker C_T)$. Moreover, Proposition 1.8 implies that $C_T = \mu J$ for some smooth function μ, where $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ is defined by $JX = 0$ and $JY = X$.

The restrictions of A and $\mathcal{B}$ to $\Delta^\perp$ are denoted by the same letters and let $D \in \Gamma(\mathrm{End}(\Delta^\perp))$ be given by $D = A^{-1}\mathcal{B}$. From (1.9) and (6.3) we have

$$ADC_T = C_T' AD = AC_T D.$$

Hence $A[D, C_T] = 0$, and thus D commutes with J. This gives $D = \phi_1 I + \phi_2 J$ and

$$\mathcal{B} = \phi_1 A + \phi_2 AJ.$$

Since the immersion is ruled, then A has the form

$$A = \begin{bmatrix} \lambda & \nu \\ \nu & 0 \end{bmatrix}$$

with respect to $\{Y, X\}$. We easily have from (2.21) that $\phi_1 = 0$, and therefore $\mathcal{B}$ has the form (6.4). Finally (6.5) follows from $\mathcal{B}$ being a Codazzi tensor. $\square$

The following fact is essential in the proof of Theorem 6.2.

Lemma 6.6. *Let $\nu^* > 0$ be constant on an open subset $U \subset M^n$. If $\gamma\colon [0, b] \to M^n$ is a unit speed geodesic such that $\gamma([0, b))$ is contained in a leaf of Δ^* in U, then* (6.3) *holds on* $[0, b]$.

Proof. The proof follows immediately from Proposition 1.7. $\square$

Proof of Theorem 6.2: Let $\mathcal{T}$ be a nontrivial infinitesimal bending of f and let $\mathcal{B}$ be its associated symmetric tensor. We consider the subsets of M^n defined by

$$M_i = \{x \in M^n : \text{rank } A(x) \geq i\}.$$

We have that $M_2 \neq \emptyset$. If otherwise, we have from Proposition 1.8 that the splitting tensor of Δ vanishes. Then Proposition 1.9 gives that f is a cylinder over a curve, but this is ruled out by assumption. From Proposition 2.10 and Theorem 2.17 we have that $\mathcal{B}|_{M_3} = 0$. Let $V \subset W_2 = M_2 \setminus \bar{M}_3$ be the open subset of M^n defined by

$$V = \{x \in W_2 : \mathcal{B}(x) \neq 0\}.$$

Since rank $A = 2$ on V it follows from (2.21) that $\Delta = \Delta^*$.

We claim that the leaves of relative nullity in V are complete. Otherwise, there is a geodesic $\gamma\colon [0, b] \to M^n$ contained in a leaf of the relative nullity foliation such that $\gamma([0, b)) \subset V$ and $\gamma(b) \notin V$. From Lemma 6.6 we obtain that $\mathcal{B}$ satisfies

$$\nabla_{\gamma'(s)}\mathcal{B} = C'_{\gamma'(s)}\mathcal{B} \tag{6.6}$$

on $[0, b]$ with $\mathcal{B}(b) = 0$, where $C'_{\gamma'}$ denotes the transpose of $C_{\gamma'}$. Take a parallel orthonormal basis of $\Delta^\perp$ along γ and regard (6.6) as a differential equation of matrices. Since $\mathcal{B}(b) = 0$ then $\mathcal{B}$ necessarily vanishes along γ (see Exercise 6.1). This is a contradiction, and proves the claim.

We show next that $f|_V$ is ruled. By Lemma 6.4 the codimension of C_0 in Δ is at most one. The assumption that $f(M)$ does not contain a cylinder gives that the subset

$$V_0 = \{x \in V : C(x) = 0\}$$

has empty interior. Let $T \in \Gamma(\Delta)$ be a local unit vector field on the open subset $V_1 = V \setminus V_0$ spanning the orthogonal complement of C_0. Using again Lemma 6.4 it follows that rank $C_T = 1$. Moreover, we have from

Proposition 1.8 that V_1 and V_0 are both union of complete relative nullity leaves.

We claim that the smooth distribution $\Delta \oplus \ker C_T$ on V_1 is totally geodesic. If $\ker C_T$ is locally spanned by a unit vector field X, then

$$(\nabla_X T)_{\Delta^\perp} = -C_T X = 0.$$

Similarly $(\nabla_X S)_{\Delta^\perp} = 0$ for any $S \in \Gamma(C_0)$. We have from (1.7) that $\nabla_R S \in \Gamma(C_0)$ for any $S \in \Gamma(C_0)$ and $R \in \Gamma(\Delta)$, thus the integral curves of T are geodesics. Then Proposition 1.8 gives that that $\nabla_T X = 0$. Moreover, we have from equation (1.7) that $C_T(\nabla_S X)_{\Delta^\perp} = 0$ and then $(\nabla_S X)_{\Delta^\perp} = 0$ for any $S \in \Gamma(C_0)$. It remains to show that $\langle \nabla_X X, Y\rangle = 0$ where $Y \in \Gamma(\Delta^\perp)$ is a unit vector field orthogonal to X. Since the only real eigenvalue of C_T is zero, then $C_T Y = \mu X$ for a smooth non vanishing function μ. Equation 1.8 yields

$$(\nabla^h_X C_T)Y = (\nabla^h_Y C_T)X,$$

which is equivalent to

$$X(\mu) = \langle \nabla_Y Y, X\rangle \mu \tag{6.7}$$

and

$$\mu \langle \nabla_X X, Y\rangle = 0.$$

The last equation proves the claim.

Since C_T is nilpotent, we have that $\ker C'_T = \mathrm{Im} C'_T$. From (1.9) we obtain that $C'_T A = A C_T$, which implies that $C'_T A X = 0$, and then that

$$\langle AX, X\rangle = 0.$$

Thus the leaves of $\Delta \oplus \ker C_T$ are totally geodesic submanifolds of $\mathbb{R}^{n+1}$, that is, $f|_{V_1}$ is ruled.

Recall that the leaves of relative nullity in V_1 are complete. Next we prove that the rulings contained in V_1 are also complete. Assume, on the contrary, that there is an incomplete ruling in V_1. Thus, there is a geodesic $\delta\colon [0,a] \to M^n$ in the direction of X such that $\delta(a) \notin V_1$. We have from Proposition 6.1 that the rank of f at $\delta(a)$ is two. Moreover, from the second statement on that result, it follows that (6.7) extends to $\delta(a)$ where $Y \in \Gamma(\Delta^\perp)$ is as before. Since μ is not zero along δ we have that $\delta(a) \notin V_0$, and hence $\delta(a) \notin V$. On the other hand, Lemma 6.5 yields that $\mathcal{B}$ has the form (6.4) with respect to $\{Y, X\}$ and that $\theta \in C^\infty(M)$ verifies (6.5). Using again Proposition 6.1 we obtain that (6.5) extends smoothly to $[0,a]$ with $X = \delta'$. But then $\mathcal{B}$ has to vanish along δ, which is a contradiction proving that the rulings on V_1 are complete.

Let S be a connected component of V_1 and let $x \in \partial \bar{S}$ together with a sequence $x_j \in S$ be such that $x_j \to x$. Let L_j be the affine subspace of $\mathbb{R}^{n+1}$ determined by the ruling through $f(x_j)$. Since the rulings are complete, there is an affine subspace L through $f(x)$ which is the limit of the sequence determined by L_j. In fact, suppose that there are two subsequences L'_j and L''_j converging to different subspaces L' and L'' that intersect at $f(x)$. Then, in a neighborhood of x different subspaces L'_j and L''_j would intersect, and this is a contradiction. Clearly $L \subset f(\partial\bar{S})$, and thus $f|_{\bar{S}}$ is a ruled strip.

Notice that if two ruled strips have common boundary then their union is also a ruled strip. Take $x \in V_0$. Since V_1 is dense in V, then $f(x) \in L \subset f(M)$ where L is an affine $(n-1)$-dimensional subspace of $\mathbb{R}^{n+1}$ that is the limit of a sequence of rulings of V_1. Suppose that there exist two sequences of rulings $L'_j \subset f(V_1)$ and $L''_j \subset f(V_1)$ converging to affine subspaces $L' \neq L''$ that intersect at $f(x)$. Then L'_j intersects L'' in a hyperplane for large values of j. Fixing j large enough, the same holds for any ruling in a neighborhood of rulings of L'_j. Let Z' and Z'' be vector fields tangent to L'_j and L'', respectively, and let R be a vector field tangent to $L'' \cap L'_j$. Since $\tilde{\nabla}_R Z'$ and $\tilde{\nabla}_R Z''$ have no normal components, it follows that $L'' \cap L'_j$ is a complete relative nullity leaf. The same holds for the nearby rulings. In a neighborhood of $y \in L'' \cap L'_j$, as before take unit vector fields $T \in \Gamma(C_0^\perp)$, $X \in \Gamma(\ker C_T)$ and Y such that $C_T Y = \mu X$ with $\mu \neq 0$. Let γ be the unit speed geodesic of M^n such that $f \circ \gamma$ lies in L'', $f(\gamma(0)) = y$ and is orthogonal to Δ. Then $\gamma' = aX + bY$ with $b \neq 0$. Hence

$$\langle C_T\gamma', \gamma'\rangle = \langle T, \nabla_{\gamma'}\gamma'\rangle = 0$$

is equivalent to $ab\mu = 0$. This yields $a = 0$, and thus $\gamma' = Y$ is orthogonal to X. Since $f_*\nabla_{\gamma'}T$ is tangent to L'' and f_*X is orthogonal to L'', then $C_T Y = 0$, and this is a contradiction. Hence, we have seen that any sequence of points in V_1 converging to x, determines the same affine subspace L as the limit of the correspondent rulings. Moreover, we have shown that L does not intersect $f(V_1)$. Take a neighborhood U_0 of x where f is an embedding, then $L \cap f(U_0)$ determines a ruling through $f(x)$. Hence we have that $f|_V$ is ruled and has complete relative nullity leaves. Using Lemma 6.5 as above, we obtain that the affine subspace L is in fact contained in $f(V_0)$. Thus, every connected component of V defines a ruled strip.

To conclude the proof of the theorem it remains to show that $\mathcal{B} = 0$ on the open subset $W_1 = M_1 \setminus \bar{M}_2$, that is, that $\mathcal{B}$ vanishes outside ruled strips. It follows from (2.21) that $\mathcal{B}(\Delta) \subset \mathrm{Im} A$ on W_1, hence rank $\mathcal{B} \leq 2$. Let V' be the open subset of W_1 defined as

$$V' = \{x \in W_1 : \mathcal{B}(x) \neq 0\},$$

then $\Delta^* = \ker \mathcal{B}$ on V'.

We claim that V' is empty. Suppose otherwise. Let $V'' \subset V'$ be the open subset where $\nu^* = \dim \Delta^*$ attains its minimum in V', say ν_0^*. We see next that the leaves of Δ^* are complete on V''. Suppose, on the contrary, that there is a geodesic $\gamma\colon [0,b] \to M^n$ such that $\gamma([0,b)) \subset V'$ is contained on a leaf of Δ^* and that $\gamma(b) \notin V''$. By Propositions 1.8 and 1.7 we know that $\nu(\gamma(b)) = n-1$ and $\nu^*(\gamma(b)) = \nu_0^*$. Then $\mathcal{B}(\gamma(b)) \neq 0$ and $\gamma(b) \in \bar{M}_2$. Take a neighborhood of $\gamma(b)$ where $\mathcal{B} \neq 0$. Since $\gamma(b) \in \bar{M}_2$, there is a sequence $x_k \in V$ such that $x_k \to \gamma(b)$. Recall that each connected component of V defines a ruled strip. Let L_k be the affine subspace of $\mathbb{R}^{n+1}$ given by the ruling through $f(x_k)$. As before, there is an affine subspace L of dimension $n-1$ which is the limit of the sequence L_k and determines a ruling through $f(\gamma(b))$. Since $A\gamma'(b) = 0$ and the geodesic $f \circ \gamma$ is transversal to L, we have that $A(\gamma(b)) = 0$, and this is a contradiction. Hence Δ^* has complete leaves in V''.

The leaves of the relative nullity foliation cannot be complete on any open subset of W_1. This follows easily from Propositions 1.8 and 1.9 together with the assumptions on f. Hence necessarily $\nu_0^* = n-2$.

Take local orthonormal vector fields X and Y in V'' orthogonal to $\ker \mathcal{B}$ such that X is an eigenfield of A. Then A and $\mathcal{B}$ have the expressions

$$A|_{\ker \mathcal{B}^\perp} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{B}|_{\ker \mathcal{B}^\perp} = \begin{bmatrix} \mu & \rho \\ \rho & 0 \end{bmatrix}$$

with respect to the frame $\{X, Y\}$ and $\lambda \neq 0 \neq \rho$.

Given $T \in \Gamma(\Delta)$ let c_T be defined by $C_T X = c_T X$. Since X is parallel along the relative nullity leaves, we have

$$(\nabla_X \mathcal{B})Y = (X(\rho) + c_Y \mu)X + 2c_Y \rho Y + \rho(\nabla_X X)_{\ker \mathcal{B}}$$

and

$$(\nabla_Y \mathcal{B})X = Y(\mu)X + Y(\rho)Y + \rho \nabla_Y Y.$$

In particular, the Codazzi equation for $\mathcal{B}$ yields

$$Y(\rho) = 2c_Y \rho. \tag{6.8}$$

Recall that each leaf of Δ in V'' is foliated by leaves of Δ^* that are complete. Hence the integral curves of Y are geodesics. Let $W_1' \subset W_1$ be the dense subset where the relative nullity leaves are not complete. Take a point $x \in V'' \cap W_1'$. Since the leaf of the relative nullity foliation through x is not complete, there is a geodesic $\delta\colon [0,b] \to M^n$ contained in that leaf tangent to Y such that $\delta([0,b)) \subset V''$ and $\delta(b) \notin V''$. Then, either $\delta(b) \in V'$ and rank $\mathcal{B}(\delta(b)) = 1$ or $\delta(b) \notin V'$. In the former case we have that $\rho(\delta(b)) = 0$ and it follows from (6.8) that $\rho = 0$ along δ, which is a contradiction. In the latter case, by the same transversality argument as above we have that $\mathcal{B}(\delta(b))$=0, and hence $\rho(\delta(b)) = 0$ leading again to a contradiction. This proves the claim that V' is empty and concludes the proof. $\square$

Proposition 6.7. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold M^n satisfying the hypothesis of Theorem 6.2. If $\mathcal{T}$ is a nontrivial infinitesimal bending of f, then $\mathcal{T}$ is the variational field of an isometric bending.*

Proof. Let $\mathcal{B}$ be the symmetric tensor associated to the infinitesimal bending $\mathcal{T}$. The symmetric tensors $A + t\mathcal{B}$, $t \in \mathbb{R}$, satisfy the Gauss and Codazzi equations (see Exercise 6.2). Then, they give rise to an isometric variation of f whose variational field $\mathcal{T}'$ has $\mathcal{B}$ as associated tensor. Thus $\mathcal{T} - \mathcal{T}'$ is trivial, and this concludes the proof. $\square$

6.3 Exercises

Exercise 6.1. Let $U\colon [0, b] \to M_n(\mathbb{R})$ be a solution of the ODE $U'(s) = T(s)U(s)$ where $T\colon [0, b] \to M_n(\mathbb{R})$ is continuous. Show that the rank $U(s)$ is constant on $[0, b]$.

Hint: Take $v \in \mathbb{R}^n$ and define $v(s) = U(s)v$ for $s \in [0, b]$. Observe that $v(s)$ satisfies

$$v'(s) = U'(s)v = T(s)v(s).$$

From that conclude that the dimension of the kernel of $U(s)$ is constant on $[0, b]$.

Exercise 6.2. Fill the details in the proof of Corollary 6.7.

Hint: Use (2.21), (6.4) and the Gauss equation for A to show that the symmetric tensors $A+t\mathcal{B}$, $t \in \mathbb{R}$, satisfy the Gauss equation. That $A+t\mathcal{B}$ satisfies the Codazzi equation follows from the Codazzi equations for A and $\mathcal{B}$.

Chapter 7

Conformal infinitesimal variations

This chapter is about smooth variations of an Euclidean submanifold by immersions that are infinitesimally conformal. This concept belongs to conformal geometry since the class of conformal infinitesimal variations is invariant by conformal transformations of the ambient space. The main contents of this chapter are a Fundamental Theorem for conformal infinitesimal variations and a rigidity theorem, both results due to Dajczer-Jimenez [17].

7.1 Conformal infinitesimal variations

In this section, the notions of conformal infinitesimal variation and conformal infinitesimal bending of an Euclidean submanifold are introduced and shown that they belong to the realm of conformal geometry.

A *conformal variation* of a given isometric immersion $f\colon M^n \to \mathbb{R}^m$ is a smooth variation $\mathcal{F}\colon I \times M^n \to \mathbb{R}^m$, where $0 \in I \subset \mathbb{R}$ is an open interval and each $f_t = \mathcal{F}(t, \cdot)$ with $f_0 = f$ is a conformal immersion for any $t \in I$. Hence, there is a positive function $\gamma \in C^\infty(I \times M)$ with $\gamma(0, x) = 1$ such that

$$\gamma(t, x)\langle f_{t*}X, f_{t*}Y\rangle = \langle X, Y\rangle \tag{7.1}$$

for any $X, Y \in \mathfrak{X}(M)$. The derivative of (7.1) with respect to t computed at $t = 0$ gives that the *variational vector field* $\mathcal{T} = \mathcal{F}_*\partial/\partial t|_{t=0}$ of $\mathcal{F}$ satisfies the condition

$$\langle \tilde{\nabla}_X \mathcal{T}, f_*Y\rangle + \langle f_*X, \tilde{\nabla}_Y \mathcal{T}\rangle = 2\rho\langle X, Y\rangle, \tag{7.2}$$

where $\rho \in C^\infty(M)$ is given by $\rho(x) = -(1/2)\partial\gamma/\partial t(0,x)$.

A *trivial conformal variation* of an isometric immersion is the composition of the immersion with a smooth family of conformal transformations of the Euclidean ambient space. Recall that conformal transformations of Euclidean space are characterized by Liouville's classical theorem. In this case, the variational vector field is, at least locally, the restriction of a conformal Killing vector field of the ambient Euclidean space to the submanifold.

A smooth variation $\mathcal{F}\colon I \times M^n \to \mathbb{R}^m$ of an isometric immersion $f\colon M^n \to \mathbb{R}^m$ is called a *conformal infinitesimal variation* if there is a function $\gamma \in C^\infty(I \times M)$ satisfying $\gamma(0,x) = 1$ and

$$\frac{\partial}{\partial t}|_{t=0}\gamma(t,x)\langle f_{t*}X, f_{t*}Y\rangle = 0 \tag{7.3}$$

for any $X, Y \in \mathfrak{X}(M)$. This concept is just the infinitesimal analogue of a conformal variation.

The notion of conformal infinitesimal variation is indeed a concept in conformal geometry. In fact, let $\mathcal{F}\colon I \times M^n \to \mathbb{R}^m$ be a conformal infinitesimal variation of $f\colon M^n \to \mathbb{R}^m$. Then, let $\mathcal{G}\colon I \times M^n \to \mathbb{R}^m$ be the variation given by $\mathcal{G} = \psi \circ \mathcal{F}$ where ψ is a conformal transformation of $\mathbb{R}^m$ with positive conformal factor $\lambda \in C^\infty(\mathbb{R}^m)$. We argue that $\mathcal{G}$ is a conformal infinitesimal variation of $g = \psi \circ f$ where

$$\tilde\gamma(t,x) = \gamma(t,x) - 2t\langle \mathcal{T}(x), \tilde\nabla \log\lambda(f(x))\rangle.$$

In fact, we have using (7.3) that

$$\begin{aligned}
&\frac{\partial}{\partial t}|_{t=0}\tilde\gamma\langle g_{t*}X, g_{t*}Y\rangle\\
&\qquad = \frac{\partial}{\partial t}|_{t=0}((\gamma(t,x) - 2t\langle \mathcal{T}(x), \tilde\nabla \log\lambda(f(x))\rangle)\lambda^2(\mathcal{F}(t,x))\langle f_{t*}X, f_{t*}Y\rangle)\\
&\qquad = \langle f_*X, f_*Y\rangle\frac{\partial}{\partial t}|_{t=0}\lambda^2(\mathcal{F}(t,x)) - 2\frac{\partial}{\partial t}|_{t=0}t\lambda^2\langle \mathcal{T}, \tilde\nabla\log\lambda\rangle\langle f_{t*}X, f_{t*}Y\rangle\\
&\qquad = 2\lambda\langle \mathcal{T}, \tilde\nabla\lambda\rangle\langle f_*X, f_*Y\rangle - 2\lambda^2\langle \mathcal{T}, \tilde\nabla\log\lambda\rangle\langle f_*X, f_*Y\rangle\\
&\qquad = 0,
\end{aligned}$$

as we wished.

As already seen in the case of infinitesimal variations, in order to study conformal infinitesimal variations one has to deal with the variational vector field. That the variational field satisfies (7.2) leads to the following definition.

A *conformal infinitesimal bending* $\mathcal{T}$ with *conformal factor* $\rho \in C^\infty(M)$ of an isometric immersion $f\colon M^n \to \mathbb{R}^m$ is a smooth section $\mathcal{T} \in$

$\Gamma(f^*T\mathbb{R}^m)$ that satisfies the condition

$$\langle\tilde{\nabla}_X\mathcal{T}, f_*Y\rangle + \langle f_*X, \tilde{\nabla}_Y\mathcal{T}\rangle = 2\rho\langle X, Y\rangle \tag{7.4}$$

for any $X, Y \in \mathfrak{X}(M)$.

On one hand, there is a conformal infinitesimal bending associated to any conformal infinitesimal variation. On the other hand, associated to a conformal infinitesimal bending we have the variation $\mathcal{F}\colon \mathbb{R}\times M^n \to \mathbb{R}^m$ given by

$$\mathcal{F}(t,x) = f(x) + t\mathcal{T}(x). \tag{7.5}$$

This is a conformal infinitesimal variation with variational vector field $\mathcal{T}$ since (7.3) is satisfied for $\gamma(t,x) = e^{-2t\rho(x)}$. By no means (7.5) is unique with this property, although it may be seen as the simplest one. In fact, new conformal infinitesimal variations with variational vector field $\mathcal{T}$ are obtained by adding to (7.5) terms of the type $t^k\delta$, $k>1$, where $\delta\in\Gamma(f^*T\mathbb{R}^m)$ and, maybe, for restricted values of the parameter t.

In view of the above, we call a conformal infinitesimal variation of $f\colon M^n\to\mathbb{R}^m$ a *trivial conformal infinitesimal variation* if the associated conformal infinitesimal bending is trivial. In turn, that a conformal infinitesimal bending is *trivial* means that at least locally it is the restriction of a conformal Killing vector field of the Euclidean ambient space to the submanifold. Finally, if any conformal infinitesimal variation of f is trivial we say that the submanifold is *conformally infinitesimally rigid*.

We conclude this section with some nontrivial examples of conformal infinitesimal variations that are of rather simple geometric nature.

Examples 7.1. (i) If $f\colon M^n\to\mathbb{R}^m$ is an isometric immersion then a conformal Killing vector field of M^n is a conformal infinitesimal bending of f.

(ii) Let $g\colon M^n\to\mathbb{S}^m$ be an isometric immersion. Then $\mathcal{T}=\varphi f$ is a conformal infinitesimal bending of $f = i\circ g\colon M^n\to\mathbb{R}^{m+1}$ where $\varphi\in C^\infty(M)$ and $i\colon\mathbb{S}^m\to\mathbb{R}^{m+1}$ is the inclusion.

7.2 The associated pair

In this section, given a conformal infinitesimal bending $\mathcal{T}\in\Gamma(f^*T\mathbb{R}^m)$ with conformal factor $\rho\in C^\infty(M)$ of an isometric immersion $f\colon M^n\to\mathbb{R}^m$, we show that the bending together with the second fundamental form of f determine an *associate pair* of tensors $(\beta,\mathcal{E})$ where $\beta\colon TM\times TM\to N_fM$ is symmetric and $\mathcal{E}\colon TM\times N_fM\to N_fM$ satisfies the compatibility condition

$$\langle\mathcal{E}(X,\eta),\xi\rangle + \langle\mathcal{E}(X,\xi),\eta\rangle = 0 \tag{7.6}$$

for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_f M)$.

Let $L \in \Gamma(\mathrm{Hom}(TM, f^*T\mathbb{R}^m))$ be the tensor defined by

$$LX = \tilde{\nabla}_X \mathcal{T} - \rho f_* X = \mathcal{T}_* X - \rho f_* X$$

for any $X \in \mathfrak{X}(M)$. Notice that (7.4) in terms of L has the form

$$\langle LX, f_* Y\rangle + \langle f_* X, LY\rangle = 0 \tag{7.7}$$

for any $X, Y \in \mathfrak{X}(M)$. Let $B\colon TM \times TM \to f^*T\mathbb{R}^m$ be the tensor given by

$$B(X,Y) = (\tilde{\nabla}_X L)Y = \tilde{\nabla}_X LY - L\nabla_X Y$$

for any $X, Y \in \mathfrak{X}(M)$. Then, the tensor $\beta\colon TM \times TM \to N_f M$ is defined by

$$\beta(X,Y) = (B(X,Y))_{N_f M}$$

for any $X, Y \in \mathfrak{X}(M)$. Flatness of the ambient space and that

$$\beta(X,Y) = (\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{T} - \tilde{\nabla}_{\nabla_X Y} \mathcal{T})_{N_f M} - \rho\alpha(X,Y)$$

give that β is symmetric.

Let $\mathcal{Y} \in \Gamma(\mathrm{Hom}(N_f M, TM))$ be defined by

$$\langle \mathcal{Y}\eta, X\rangle + \langle \eta, LX\rangle = 0 \tag{7.8}$$

for any $X \in \mathfrak{X}(M)$. Then, let $\mathcal{E}\colon TM \times N_f M \to N_f M$ be the tensor given by

$$\mathcal{E}(X,\eta) = \alpha(X, \mathcal{Y}\eta) + (LA_\eta X)_{N_f M}.$$

We have

$$\begin{aligned}
\langle \mathcal{E}(X,\eta), \xi\rangle &= \langle \alpha(X, \mathcal{Y}\eta) + LA_\eta X, \xi\rangle \\
&= \langle A_\xi X, \mathcal{Y}\eta\rangle - \langle \mathcal{Y}\xi, A_\eta X\rangle \\
&= -\langle LA_\xi X, \eta\rangle - \langle \alpha(X, \mathcal{Y}\xi), \eta\rangle \\
&= -\langle \mathcal{E}(X,\xi), \eta\rangle,
\end{aligned}$$

and thus condition (7.6) is satisfied.

Proposition 7.2. *It holds that*

$$(B(X,Y))_{f_* TM} = f_*(\mathcal{Y}\alpha(X,Y) + (X \wedge \nabla\rho)Y)$$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. We have to show that

$$C(X,Y,Z) = \langle (B - f_*\mathcal{Y}\alpha)(X,Y), f_*Z\rangle + \langle X,Y\rangle\langle Z,\nabla\rho\rangle - \langle Y,\nabla\rho\rangle\langle X,Z\rangle$$

vanishes for any $X,Y,Z \in \mathfrak{X}(M)$. The derivative of (7.7) gives

$$\begin{aligned}
0 &= \langle \tilde\nabla_Z LX, f_*Y\rangle + \langle LX, \tilde\nabla_Z f_*Y\rangle + \langle \tilde\nabla_Z LY, f_*X\rangle + \langle LY, \tilde\nabla_Z f_*X\rangle \\
&= \langle B(Z,X), f_*Y\rangle + \langle L\nabla_Z X, f_*Y\rangle + \langle LX, f_*\nabla_Z Y + \alpha(Z,Y)\rangle \\
&\quad + \langle B(Z,Y), f_*X\rangle + \langle L\nabla_Z Y, f_*X\rangle + \langle LY, f_*\nabla_Z X + \alpha(Z,X)\rangle \\
&= \langle B(Z,X), f_*Y\rangle + \langle LX, \alpha(Z,Y)\rangle + \langle B(Z,Y), f_*X\rangle + \langle LY, \alpha(Z,X)\rangle \\
&= \langle (B - f_*\mathcal{Y}\alpha)(Z,X), f_*Y\rangle + \langle (B - f_*\mathcal{Y}\alpha)(Z,Y), f_*X\rangle.
\end{aligned}$$

On the other hand,

$$\langle B(X,Y), f_*Z\rangle = \langle \tilde\nabla_X\tilde\nabla_Y\mathcal{T} - \tilde\nabla_{\nabla_X Y}\mathcal{T}, f_*Z\rangle - \langle X,\nabla\rho\rangle\langle Y,Z\rangle.$$

It follows that

$$C(X,Y,Z) = C(Y,X,Z) \quad \text{and} \quad C(Z,X,Y) = -C(Z,Y,X)$$

for any $X,Y,Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
C(X,Y,Z) &= -C(X,Z,Y) = -C(Z,X,Y) = C(Z,Y,X) \\
&= C(Y,Z,X) = -C(Y,X,Z) = -C(X,Y,Z) \\
&= 0,
\end{aligned}$$

as we wished. $\square$

7.3 The fundamental equations

In this section, we determine the set of fundamental equations for a conformal infinitesimal variation.

Proposition 7.3. *The pair $(\beta,\mathcal{E})$ associated to a conformal infinitesimal bending satisfies the system of equations*

$$(S)\begin{cases}
A_{\beta(Y,Z)}X + B_{\alpha(Y,Z)}X - A_{\beta(X,Z)}Y - B_{\alpha(X,Z)}Y \\
\qquad + (X\wedge HY - Y\wedge HX)Z = 0 & (7.9)\\
(\nabla^\perp_X\beta)(Y,Z) - (\nabla^\perp_Y\beta)(X,Z) = \mathcal{E}(Y,\alpha(X,Z)) - \mathcal{E}(X,\alpha(Y,Z)) \\
\qquad + \langle Y,Z\rangle\alpha(X,\nabla\rho) - \langle X,Z\rangle\alpha(Y,\nabla\rho) & (7.10)\\
(\nabla^\perp_X\mathcal{E})(Y,\eta) - (\nabla^\perp_Y\mathcal{E})(X,\eta) = \beta(X,A_\eta Y) - \beta(A_\eta X,Y) \\
\qquad + \alpha(X,B_\eta Y) - \alpha(B_\eta X,Y), & (7.11)
\end{cases}$$

where $X,Y,Z \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$. Moreover, we have that $B_\eta, H \in \Gamma(End(TM))$ are given, respectively, by

$$\langle B_\eta X,Y\rangle = \langle \beta(X,Y),\eta\rangle \quad \textit{and} \quad HX = \nabla_X\nabla\rho.$$

Proof. We first show that

$$(\tilde{\nabla}_X \mathcal{Y})\eta = -f_* B_\eta X - LA_\eta X + \mathcal{E}(X,\eta), \tag{7.12}$$

where

$$(\tilde{\nabla}_X \mathcal{Y})\eta = \tilde{\nabla}_X f_* \mathcal{Y}\eta - f_* \mathcal{Y}\nabla^\perp_X \eta.$$

Taking the derivative of (7.8), we have from (7.7) and (7.8) that

$$\begin{aligned} 0 &= \langle \tilde{\nabla}_X f_* \mathcal{Y}\eta, f_* Y\rangle + \langle \mathcal{Y}\eta, \nabla_X Y\rangle + \langle \tilde{\nabla}_X LY, \eta\rangle + \langle LY, \tilde{\nabla}_X \eta\rangle \\ &= \langle (\tilde{\nabla}_X \mathcal{Y})\eta, f_* Y\rangle + \langle B_\eta X, Y\rangle + \langle LA_\eta X, f_* Y\rangle. \end{aligned}$$

Since $\langle f_* \mathcal{Y}\eta, \xi\rangle = 0$, then

$$\begin{aligned} 0 &= \langle \tilde{\nabla}_X f_* \mathcal{Y}\eta, \xi\rangle + \langle f_* \mathcal{Y}\eta, \tilde{\nabla}_X \xi\rangle \\ &= \langle (\tilde{\nabla}_X \mathcal{Y})\eta, \xi\rangle - \langle \alpha(X, \mathcal{Y}\eta), \xi\rangle \\ &= \langle (\tilde{\nabla}_X \mathcal{Y})\eta, \xi\rangle + \langle LA_\eta X - \mathcal{E}(X,\eta), \xi\rangle \end{aligned}$$

for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_f M)$, and hence (7.12) follows.

Since

$$(\tilde{\nabla}_X B)(Y,Z) = \tilde{\nabla}_X (\tilde{\nabla}_Y L)Z - (\tilde{\nabla}_{\nabla_X Y} L)Z - (\tilde{\nabla}_Y L)\nabla_X Z, \tag{7.13}$$

it is easy to see that

$$(\tilde{\nabla}_X B)(Y,Z) - (\tilde{\nabla}_Y B)(X,Z) = -LR(X,Y)Z \tag{7.14}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. It follows using Proposition 7.2 that

$$\begin{aligned} &\langle (\tilde{\nabla}_X B)(Y,Z), f_* W\rangle \\ &\qquad = \langle (\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) + f_* \mathcal{Y}(\nabla^\perp_X \alpha)(YZ) - f_* A_{\beta(Y,Z)} X, f_* W\rangle \\ &\qquad\quad + \langle Y, W\rangle \mathit{Hess}\,\rho(Z,X) - \langle Y, Z\rangle \mathit{Hess}\,\rho(X,W) \end{aligned}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$. Then (7.14) and the Gauss and Codazzi equations give

$$\begin{aligned} &\langle (\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) - (\tilde{\nabla}_Y \mathcal{Y})\alpha(X,Z), f_* W\rangle \\ &\qquad = \langle LA_{\alpha(X,Z)} Y - LA_{\alpha(Y,Z)} X + f_* A_{\beta(Y,Z)} X - f_* A_{\beta(X,Z)} Y, f_* W\rangle \\ &\qquad\quad + \langle Y, Z\rangle \mathit{Hess}\,\rho(X,W) - \langle Y, W\rangle \mathit{Hess}\,\rho(Z,X) \\ &\qquad\quad + \langle X, W\rangle \mathit{Hess}\,\rho(Y,Z) - \langle X, Z\rangle \mathit{Hess}\,\rho(Y,W). \end{aligned}$$

On the other hand, it follows from (7.12) that

$$\begin{aligned} &\langle (\tilde{\nabla}_X \mathcal{Y})\alpha(Y,Z) - (\tilde{\nabla}_Y \mathcal{Y})\alpha(X,Z), f_* W\rangle \\ &\qquad = \langle f_* B_{\alpha(X,Z)} Y + LA_{\alpha(X,Z)} Y - f_* B_{\alpha(Y,Z)} X - LA_{\alpha(Y,Z)} X, f_* W\rangle. \end{aligned}$$

The last two equations give

$$\begin{aligned}&\langle B_{\alpha(X,Z)}Y - B_{\alpha(Y,Z)}X, f_*W\rangle\\ &= \langle A_{\beta(Y,Z)}X - A_{\beta(X,Z)}Y, W\rangle + \langle Y,Z\rangle Hess\,\rho(X,W) - \langle Y,W\rangle Hess\,\rho(Z,X)\\ &\ + \langle X,W\rangle Hess\,\rho(Y,Z) - \langle X,Z\rangle Hess\,\rho(Y,W),\end{aligned}$$

and this is (7.9).

Using (7.13) we obtain

$$\begin{aligned}((\tilde\nabla_X B)(Y,Z))_{N_fM} =& \alpha(X, \mathcal{Y}\alpha(Y,Z)) + (\nabla^\perp_X\beta)(Y,Z)\\ &+ \langle Z,\nabla\rho\rangle\alpha(X,Y) - \langle Y,Z\rangle\alpha(X,\nabla\rho).\end{aligned}$$

Then, we have from (7.14) and the Gauss equation that

$$\begin{aligned}&(\nabla^\perp_X\beta)(Y,Z) - (\nabla^\perp_Y\beta)(X,Z)\\ &= (LR(Y,X)Z)_{N_fM} - \alpha(X,\mathcal{Y}\alpha(Y,Z)) + \alpha(Y,\mathcal{Y}\alpha(X,Z))\\ &\quad + \langle Y,Z\rangle\alpha(X,\nabla\rho) - \langle X,Z\rangle\alpha(Y,\nabla\rho)\\ &= (LA_{\alpha(X,Z)}Y - LA_{\alpha(Y,Z)}X)_{N_fM} - \alpha(X,\mathcal{Y}\alpha(Y,Z)) + \alpha(Y,\mathcal{Y}\alpha(X,Z))\\ &\quad + \langle Y,Z\rangle\alpha(X,\nabla\rho) - \langle X,Z\rangle\alpha(Y,\nabla\rho),\end{aligned}$$

and this is (7.10).

We have

$$\begin{aligned}(\nabla^\perp_X\mathcal{E})(Y,\eta) &= \nabla^\perp_X\mathcal{E}(Y,\eta) - \mathcal{E}(\nabla_XY,\eta) - \mathcal{E}(Y,\nabla^\perp_X\eta)\\ &= (\nabla^\perp_X\alpha)(Y,\mathcal{Y}\eta) + (L(\nabla_XA)(Y,\eta))_{N_fM} + \alpha(Y,\nabla_X\mathcal{Y}\eta)\\ &\quad - \alpha(Y,\mathcal{Y}\nabla^\perp_X\eta) - (L\nabla_XA_\eta Y)_{N_fM} + \nabla^\perp_X(LA_\eta Y)_{N_fM}.\end{aligned}$$

Then (7.12) yields

$$\begin{aligned}(\nabla^\perp_X\mathcal{E})(Y,\eta) = {}& (\nabla^\perp_X\alpha)(Y,\mathcal{Y}\eta) + (L(\nabla_XA)(Y,\eta))_{N_fM} - \alpha(Y,B_\eta X)\\ &- \alpha(Y,(LA_\eta X)_{TM}) - (L\nabla_XA_\eta Y)_{N_fM} + \nabla^\perp_X(LA_\eta Y)_{N_fM}.\end{aligned}$$

Using the Codazzi equation, we obtain

$$\begin{aligned}&(\nabla^\perp_X\mathcal{E})(Y,\eta) - (\nabla^\perp_Y\mathcal{E})(X,\eta)\\ &\qquad = \alpha(X,B_\eta Y) - \alpha(Y,B_\eta X) + \alpha(X,(LA_\eta Y)_{TM})\\ &\qquad\quad - \alpha(Y,(LA_\eta X)_{TM}) - (L\nabla_XA_\eta Y)_{N_fM} + \nabla^\perp_X(LA_\eta Y)_{N_fM}\\ &\qquad\quad + (L\nabla_YA_\eta X)_{N_fM} - \nabla^\perp_Y(LA_\eta X)_{N_fM}.\end{aligned}$$

Since

$$\beta(X,A_\eta Y) = \alpha(X,(LA_\eta Y)_{TM}) - (L\nabla_XA_\eta Y)_{N_fM} + \nabla^\perp_X(LA_\eta Y)_{N_fM},$$

then (7.11) follows. $\square$

7.4 Trivial infinitesimal variations

In this section, we characterize the trivial conformal infinitesimal bendings in terms of the associated pair of tensors.

It is well-known that any trivial conformal infinitesimal bending of an isometric immersion $f\colon M^n \to \mathbb{R}^m$ is, at least locally, of the form

$$\mathcal{T}(x) = (\langle f(x), v\rangle + \lambda)f(x) - 1/2\|f(x)\|^2 v + \mathcal{D}f(x) + w,$$

where $\lambda \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $\mathcal{D} \in \mathrm{End}(\mathbb{R}^m)$ is skew-symmetric. Moreover, the conformal factor is $\rho(x) = \langle f(x), v\rangle + \lambda$; cf. [35] for details.

Then

$$LX = \langle f_*X, v\rangle f(x) - \langle f_*X, f(x)\rangle v + \mathcal{D}f_*X.$$

Hence

$$(\tilde\nabla_X L)Y = \langle f_*Y, v\rangle f_*X - \langle X, Y\rangle v + \langle \alpha(X,Y), v\rangle f(x) - \langle \alpha(X,Y), f(x)\rangle v + \mathcal{D}\alpha(X,Y).$$

If $\mathcal{D}' \in \Gamma(\mathrm{End}(f^*T\mathbb{R}^m))$ is the skew-symmetric map given by

$$\mathcal{D}'\sigma = \langle \sigma, v\rangle f(x) - \langle \sigma, f(x)\rangle v + \mathcal{D}\sigma,$$

then $LX = \mathcal{D}'f_*X$. Moreover, we have $f_*\mathcal{Y}\eta = (\mathcal{D}'\eta)_{f_*TM}$ and

$$(\tilde\nabla_X L)Y = \langle f_*Y, v\rangle f_*X - \langle X, Y\rangle v + \mathcal{D}'\alpha(X,Y).$$

Let $\mathcal{D}^N \in \Gamma(\mathrm{End}(N_fM))$ be given by $\mathcal{D}^N\xi = (\mathcal{D}'\xi)_{N_fM}$. Then

$$\beta(X,Y) = \mathcal{D}^N\alpha(X,Y) - \langle X, Y\rangle v_N,$$

where $v_N = (v)_{N_fM}$.

We have

$$\begin{aligned}(\tilde\nabla_X \mathcal{D}')\sigma &= \tilde\nabla_X \mathcal{D}'\sigma - \mathcal{D}'\tilde\nabla_X\sigma\\ &= \langle \sigma, v\rangle f_*X - \langle \sigma, f_*X\rangle v\end{aligned}$$

for any $X \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(f^*T\mathbb{R}^m)$. Thus

$$\begin{aligned}\mathcal{E}(X,\xi) &= \alpha(X, \mathcal{Y}\xi) + (LA_\xi X)_{N_fM}\\ &= (\tilde\nabla_X \mathcal{D}'\xi - \tilde\nabla_X \mathcal{D}^N\xi)_{N_fM} + (LA_\xi X)_{N_fM}\\ &= ((\tilde\nabla_X \mathcal{D}')\xi + \mathcal{D}'\tilde\nabla_X\xi - \tilde\nabla_X\mathcal{D}^N\xi + LA_\xi X)_{N_fM}\\ &= -(\nabla^\perp_X \mathcal{D}^N)\xi\end{aligned}$$

for any $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_fM)$.

Proposition 7.4. *A conformal infinitesimal bending $\mathcal{T}$ of $f\colon M^n \to \mathbb{R}^m$, $n \geq 3$, is trivial if and only if there exist $\delta \in \Gamma(N_fM)$ and $C \in \Gamma(End(N_fM))$ skew-symmetric such that the associated pair has the form*

$$\beta(X,Y) = C\alpha(X,Y) - \langle X,Y\rangle\delta \ \text{ and } \ \mathcal{E}(X,\xi) = -(\nabla^{\perp}_X C)\xi. \tag{7.15}$$

Proof. For $(\beta,\mathcal{E})$ as in (7.15) and if ρ is the conformal factor of $\mathcal{T}$, then (7.9) gives

$$\begin{aligned}&\langle X,Z\rangle(\langle\alpha(Y,W),\delta\rangle - Hess\,\rho(Y,W)) + \langle Y,W\rangle(\langle\alpha(X,Z),\delta\rangle - Hess\,\rho(X,Z))\\ &-\langle X,W\rangle(\langle\alpha(Y,Z),\delta\rangle - Hess\,\rho(Y,Z)) - \langle Y,Z\rangle(\langle\alpha(X,W),\delta\rangle - Hess\,\rho(X,W)) = 0\end{aligned}$$

for any $X,Y,Z,W \in \mathfrak{X}(M)$. For X,Y,W orthonormal and $Z = X$ it follows that

$$\langle\alpha(Y,W),\delta\rangle = Hess\,\rho(Y,W)$$

whereas for $X = Z$ and $Y = W$ orthonormal we have

$$\langle\alpha(X,X),\delta\rangle - Hess\,\rho(X,X) = -\langle\alpha(Y,Y),\delta\rangle + Hess\,\rho(Y,Y) = 0.$$

Thus

$$\langle\alpha(X,Y),\delta\rangle = Hess\,\rho(X,Y) \tag{7.16}$$

for any $X,Y \in \mathfrak{X}(M)$.

Since β and $\mathcal{E}$ have the form (7.15) we obtain from (7.10) and the Codazzi equation that

$$\langle X,Z\rangle(\nabla^{\perp}_Y\delta + \alpha(Y,\nabla\rho)) = \langle Y,Z\rangle(\nabla^{\perp}_X\delta + \alpha(X,\nabla\rho).$$

Hence

$$\nabla^{\perp}_X\delta + \alpha(X,\nabla\rho) = 0 \tag{7.17}$$

for any $X \in \mathfrak{X}(M)$. Equations (7.16) and (7.17) are equivalent to $f_*\nabla\rho + \delta = v$ being constant along f. In particular $\rho(x) = \langle f(x),v\rangle + \lambda$ for some $\lambda \in \mathbb{R}$.

Let $\mathcal{T}_1 \in \Gamma(f^*T\mathbb{R}^m)$ be the trivial conformal infinitesimal bending

$$\mathcal{T}_1(x) = (\langle f(x),v\rangle + \lambda)f(x) - 1/2\|f(x)\|^2 v.$$

Notice that $\mathcal{T}$ and $\mathcal{T}_1$ have the same conformal factor, then $\mathcal{T}_2 = \mathcal{T} - \mathcal{T}_1$ is an infinitesimal bending. If L and L_1 are associated to $\mathcal{T}$ and $\mathcal{T}_1$, respectively, then the tensor L_2 associated to $\mathcal{T}_2$ is given by

$$L_2X = \tilde{\nabla}_X\mathcal{T}_2 = LX - L_1X.$$

Hence, the tensors $(\beta_2,\mathcal{E}_2)$ associated to $\mathcal{T}_2$ satisfy $\beta_2 = \beta - \beta_1$ and $\mathcal{E}_2 = \mathcal{E} - \mathcal{E}_1$, where the pair $(\beta_1,\mathcal{E}_1)$ is associated to $\mathcal{T}_1$. Recall that $\delta = (v)_{N_fM}$, then $(\beta_2,\mathcal{E}_2)$ is as in (2.19) and thus $\mathcal{T}_2$ is trivial. $\square$

Remark 7.5. Two conformal infinitesimal bendings $\mathcal{T}_i$, $i = 1, 2$, of a submanifold $f\colon M^n \to \mathbb{R}^m$ differ by a trivial one if and only if the associated pairs $(\beta_i, \mathcal{E}_i)$, $i = 1, 2$, differ by tensors as in (7.15).

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface with shape operator A corresponding to the Gauss map $N \in \Gamma(N_f M)$. Associated to a conformal infinitesimal bending we are now reduced to consider the tensor $\mathcal{B} \in \Gamma(\mathrm{End}(TM))$ given by

$$\beta(X, Y) = \langle \mathcal{B}X, Y\rangle N.$$

Then the fundamental system of equations takes the form

$$\mathcal{B}X \wedge AY - \mathcal{B}Y \wedge AX + X \wedge HY - Y \wedge HX = 0 \tag{7.18}$$

and

$$(\nabla_X \mathcal{B})Y - (\nabla_Y \mathcal{B})X + (X \wedge Y)A\nabla\rho = 0 \tag{7.19}$$

for any $X, Y \in \mathfrak{X}(M)$.

Corollary 7.6. *A conformal infinitesimal bending $\mathcal{T}$ of $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, is trivial if and only if its associated tensor $\mathcal{B}$ has the form $\mathcal{B} = \varphi I$ for $\varphi \in C^\infty(M)$.*

Proof. For a hypersurface, the tensor $\mathcal{E}$ vanishes. Then $\mathcal{T}$ is trivial if and only if

$$\beta(X, Y) = -\langle X, Y\rangle \delta$$

for some $\delta \in \Gamma(N_f M)$. This is equivalent to $\mathcal{B} = \varphi I$ for $\varphi = -\langle \delta, N\rangle$. $\square$

7.5 The Fundamental Theorem

This section is devoted to the Fundamental Theorem for conformal infinitesimal variations of Euclidean submanifolds.

Let $\mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ denote the *light cone* of the standard flat Lorentz space form $\mathbb{L}^{m+2}$ defined by

$$\mathbb{V}^{m+1} = \{v \in \mathbb{L}^{m+2} \colon \langle v, v\rangle = 0, v \neq 0\}.$$

Given $w \in \mathbb{V}^{m+1}$, then

$$\mathbb{E}^m = \{v \in \mathbb{V}^{m+1} \colon \langle v, w\rangle = 1\}$$

is a model of Euclidean space $\mathbb{R}^m$ in $\mathbb{L}^{m+2}$. In fact, given $v \in \mathbb{E}^m$ and a linear isometry $C\colon \mathbb{R}^m \to (\mathrm{span}\{v, w\})^\perp \subset \mathbb{L}^{m+2}$, the map $\Psi\colon \mathbb{R}^m \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ given by

$$\Psi(x) = v + Cx - \frac{1}{2}\|x\|^2 w \tag{7.20}$$

is an isometric embedding such that $\Psi(\mathbb{R}^m) = \mathbb{E}^m$. From Exercise 7.4 we have that the normal bundle of Ψ is $N_\Psi\mathbb{R}^m = \text{span}\{\Psi, w\}$ and that its second fundamental form is given by

$$\alpha^\Psi(U,V) = -\langle U,V\rangle w \tag{7.21}$$

for any $U,V \in T\mathbb{R}^m$.

The sum of any two conformal infinitesimal bendings is again a conformal infinitesimal bending. In the following result and afterward, we identify two conformal infinitesimal bendings if they differ by a trivial conformal infinitesimal bending.

Theorem 7.7. *Let $f\colon M^n \to \mathbb{R}^m$, $n \geq 3$, be an isometric immersion of a simply connected Riemannian manifold. A triple $(\beta, \mathcal{E}, \rho) \neq 0$, formed by a symmetric tensor $\beta\colon TM\times TM \to N_fM$, a tensor $\mathcal{E}\colon TM\times N_fM \to N_fM$ for which* (7.6) *holds and $\rho \in C^\infty(M)$, that satisfies system* (S) *determines a unique conformal infinitesimal bending of f.*

Proof. Let $F\colon M^n \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ be the isometric immersion $F = \Psi\circ f$, where Ψ is given by (7.20). By (7.21) the second fundamental form of F satisfies

$$\alpha^F(X,Y) = \Psi_*\alpha(X,Y) - \langle X,Y\rangle w \tag{7.22}$$

for any $X,Y \in \mathfrak{X}(M)$.

Let $\hat\beta\colon TM\times TM \to N_FM$ be the symmetric tensor given by

$$\hat\beta(X,Y) = \Psi_*\beta(X,Y) - \mathit{Hess}\,\rho(X,Y)F$$

for any $X,Y \in \mathfrak{X}(M)$. Then (7.9) is equivalent to

$$A^F_{\hat\beta(Y,Z)}X + \hat B_{\alpha^F(Y,Z)}X - A^F_{\hat\beta(X,Z)}Y - \hat B_{\alpha^F(X,Z)}Y = 0, \tag{7.23}$$

where A^F_ξ is the shape operator of F with respect to $\xi \in \Gamma(N_FM)$ and $\hat B_\xi$ is given by

$$\langle \hat B_\xi X, Y\rangle = \langle \hat\beta(X,Y), \xi\rangle.$$

Let $\hat{\mathcal{E}}\colon TM\times N_FM \to N_FM$ be the tensor defined by

$$\hat{\mathcal{E}}(X,\Psi_*\eta) = \Psi_*\mathcal{E}(X,\eta) - \langle \alpha(X,\nabla\rho),\eta\rangle F,$$

$$\hat{\mathcal{E}}(X,w) = \Psi_*\alpha(X,\nabla\rho) \;\text{ and }\; \hat{\mathcal{E}}(X,F) = 0.$$

Since $\mathcal{E}$ satisfies (7.6) then also does $\hat{\mathcal{E}}$. For simplicity, from now on we just write η for $\eta \in \Gamma(N_fM)$ as well as its image under Ψ_*. We have

$$(\nabla'^\perp_X\hat\beta)(Y,Z) = (\nabla^\perp_X\beta)(Y,Z) - (\nabla_X \mathit{Hess}\,\rho)(Y,Z)F,$$

where $\nabla'^{\perp}$ is the normal connection of F. Then

$$\begin{aligned}(\nabla'^{\perp}_X\hat\beta)(Y,Z) - (\nabla'^{\perp}_Y\hat\beta)(X,Z)&\\ = (\nabla^{\perp}_X\beta)(Y,Z) - (\nabla^{\perp}_Y\beta)(X,Z)&\\ + ((\nabla_Y \mathit{Hess}\,\rho)(X,Z) - (\nabla_X \mathit{Hess}\,\rho)(Y,Z))F.&\end{aligned}$$

It follows from $\mathit{Hess}\,\rho(X,Y) = \langle\nabla_X\nabla\rho, Y\rangle$ and the Gauss equation that

$$\begin{aligned}(\nabla_Y \mathit{Hess}\,\rho)(X,Z) - (\nabla_X \mathit{Hess}\,\rho)(Y,Z)&\\ = \langle R(Y,X)\nabla\rho, Z\rangle&\\ = \langle\alpha(Y,Z),\alpha(X,\nabla\rho)\rangle - \langle\alpha(Y,\nabla\rho),\alpha(X,Z)\rangle,&\end{aligned} \tag{7.24}$$

where R is the curvature tensor of M^n. Thus, from (7.10) and (7.24) we have

$$\begin{aligned}(\nabla'^{\perp}_X\hat\beta)(Y,Z) &- (\nabla'^{\perp}_Y\hat\beta)(X,Z)\\ &= \mathcal{E}(Y,\alpha(X,Z)) - \mathcal{E}(X,\alpha(Y,Z)) + \langle Y,Z\rangle\alpha(X,\nabla\rho) - \langle X,Z\rangle\alpha(Y,\nabla\rho)\\ &\quad + \langle\alpha(Y,Z),\alpha(X,\nabla\rho)\rangle F - \langle\alpha(Y,\nabla\rho),\alpha(X,Z)\rangle F,\end{aligned}$$

and hence

$$(\nabla'^{\perp}_X\hat\beta)(Y,Z) - (\nabla'^{\perp}_Y\hat\beta)(X,Z) = \hat{\mathcal{E}}(Y,\alpha^F(X,Z)) - \hat{\mathcal{E}}(X,\alpha^F(Y,Z)) \tag{7.25}$$

for any $X,Y,Z\in\mathfrak{X}(M)$.

From the definition of $\hat{\mathcal{E}}$, it follows that

$$(\nabla'^{\perp}_X\hat{\mathcal{E}})(Y,\eta) = (\nabla^{\perp}_X\mathcal{E})(Y,\eta) - \langle(\nabla^{\perp}_X\alpha)(Y,\nabla\rho),\eta\rangle F - \langle\alpha(Y,\nabla_X\nabla\rho),\eta\rangle F$$

for any $X,Y\in\mathfrak{X}(M)$ and $\eta\in\Gamma(N_fM)$. Using the Codazzi equation, we obtain

$$\begin{aligned}(\nabla'^{\perp}_X\hat{\mathcal{E}})(Y,\eta) - (\nabla'^{\perp}_Y\hat{\mathcal{E}})(X,\eta) &= (\nabla^{\perp}_X\mathcal{E})(Y,\eta) - (\nabla^{\perp}_Y\mathcal{E})(X,\eta)\\ &\quad + (\langle\alpha(X,\nabla_Y\nabla\rho),\eta\rangle - \langle\alpha(Y,\nabla_X\nabla\rho),\eta\rangle)F.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\hat\beta(X,A^F_\eta Y) &- \hat\beta(A^F_\eta X,Y) + \alpha^F(X,\hat B_\eta Y) - \alpha^F(\hat B_\eta X,Y)\\ &= \beta(X,A_\eta Y) - \mathit{Hess}\,\rho(X,A_\eta Y)F - \beta(A_\eta X,Y) + \mathit{Hess}\,\rho(A_\eta X,Y)F\\ &\quad + \alpha(X,B_\eta Y) - \langle X,B_\eta Y\rangle w - \alpha(B_\eta X,Y) + \langle B_\eta X,Y\rangle w.\end{aligned}$$

From (7.11), the symmetry of β and that

$$\mathit{Hess}\,\rho(X,A_\eta Y) = \langle\alpha(Y,\nabla_X\nabla\rho),\eta\rangle,$$

it follows that

$$\begin{aligned}(\nabla_X{}'^{\perp}\hat{\mathcal{E}})(Y,\eta)-(\nabla_Y'^{\perp}\hat{\mathcal{E}})(X,\eta)\\ =\hat{\beta}(X,A_\eta^F X)-\hat{\beta}(A_\eta^F X,Y)+\alpha^F(X,\hat{B}_\eta Y)-\alpha^F(\hat{B}_\eta X,Y).\end{aligned}\tag{7.26}$$

Using the Codazzi equation again, we obtain

$$(\nabla_X'^{\perp}\hat{\mathcal{E}})(Y,w)-(\nabla_Y'^{\perp}\hat{\mathcal{E}})(X,w)=\alpha(Y,\nabla_X\nabla\rho)-\alpha(X,\nabla_Y\nabla\rho).$$

Notice that $A_w^F=0$ and $\hat{B}_w X=-\nabla_X\nabla\rho$ for any $X\in\mathfrak{X}(M)$. Then

$$\begin{aligned}\hat{\beta}(X,&A_w^F X)-\hat{\beta}(A_w^F X,Y)+\alpha^F(X,\hat{B}_w Y)-\alpha^F(\hat{B}_w X,Y)\\ &=\alpha(Y,\nabla_X\nabla\rho)-\langle Y,\nabla_X\nabla\rho\rangle w-\alpha(X,\nabla_Y\nabla\rho)+\langle X,\nabla_Y\nabla\rho\rangle w,\end{aligned}$$

and hence

$$\begin{aligned}(\nabla_X'^{\perp}\hat{\mathcal{E}})(Y,w)-(\nabla_Y'^{\perp}\hat{\mathcal{E}})(X,w)\\ =\hat{\beta}(X,A_w^F Y)-\hat{\beta}(A_w^F X,Y)+\alpha^F(X,\hat{B}_w Y)-\alpha^F(\hat{B}_w X,Y).\end{aligned}\tag{7.27}$$

Since $\hat{B}_F=0$, $A_F^F X=-X$, $\mathcal{E}(X,F)=0$ and $\nabla_X'^{\perp}F=0$, then

$$\begin{aligned}(\nabla_X'^{\perp}\hat{\mathcal{E}})(Y,F)-(\nabla_Y'^{\perp}\hat{\mathcal{E}})(X,F)\\ =\hat{\beta}(X,A_F^F Y)-\hat{\beta}(A_F^F X,Y)+\alpha^F(X,\hat{B}_F Y)-\alpha^F(\hat{B}_F X,Y)\end{aligned}\tag{7.28}$$

trivially holds.

Summarizing, we have that $\hat{\beta}$ is symmetric, that $\hat{\mathcal{E}}$ satisfies (7.6) and that the pair verifies (7.23),(7.25),(7.26), (7.27) and (7.28). In this situation, we have that Theorem 2.8 applies. Recall that in Remark 2.9 it is observed that this result holds for ambient spaces of any signature, in particular, for the Lorentzian space considered here. We conclude that there is an infinitesimal bending $\tilde{\mathcal{T}}\in\Gamma(F^*(T\mathbb{L}^{m+2}))$ of F whose associated pair $(\tilde{\beta},\tilde{\mathcal{E}})$ satisfies

$$\tilde{\beta}=\hat{\beta}+\tilde{C}\alpha^F\ \text{ and }\ \tilde{\mathcal{E}}=\hat{\mathcal{E}}-\nabla'^{\perp}\tilde{C},\tag{7.29}$$

where $\tilde{C}\in\Gamma(\mathrm{End}(N_F M))$ is skew-symmetric. Moreover, we have that $\tilde{\mathcal{T}}$ is unique up to a trivial infinitesimal bending. Write $\tilde{\mathcal{T}}$ as

$$\tilde{\mathcal{T}}=\Psi_*\mathcal{T}+\langle\tilde{\mathcal{T}},w\rangle F+\langle\tilde{\mathcal{T}},F\rangle w.$$

Since $\tilde{\mathcal{T}}$ is an infinitesimal bending of F, we have

$$\langle\tilde{\nabla}'_X\tilde{\mathcal{T}},F_*Y\rangle+\langle\tilde{\nabla}'_Y\tilde{\mathcal{T}},F_*X\rangle=0$$

for all $X,Y\in\mathfrak{X}(M)$, where $\tilde{\nabla}'$ is the connection in $\mathbb{L}^{m+2}$. Then

$$\langle\tilde{\nabla}_X\mathcal{T},f_*Y\rangle+\langle\tilde{\nabla}_Y\mathcal{T},f_*X\rangle+2\langle\tilde{\mathcal{T}},w\rangle\langle X,Y\rangle=0$$

for all $X, Y \in \mathfrak{X}(M)$. Hence, setting $\rho_1 = -\langle \tilde{\mathcal{T}}, w\rangle$ we have that $\mathcal{T}$ is a conformal infinitesimal bending of f with conformal factor ρ_1.

Let β' and $\mathcal{E}'$ be the tensors associated to the conformal infinitesimal bending $\mathcal{T}$ of f. Notice that

$$(\tilde{\nabla}'_X \tilde{\mathcal{T}})_{\Psi_* T\mathbb{R}^m} = \Psi_* \tilde{\nabla}_X \mathcal{T} - \rho_1 F_* X \tag{7.30}$$

for any $X \in \mathfrak{X}(M)$. Thus $(\tilde{\beta})_{\Psi_* N_f M}$ coincides with β'. Let $C \in \Gamma(\mathrm{End}(N_f M))$ be given by $C\eta = (\tilde{C}\eta)_{\Psi_* N_f M}$ for any $\eta \in \Gamma(N_f M)$. Then C is skew symmetric. It follows from (7.22) and (7.29) that the tensor β' satisfies

$$\beta'(X,Y) = \beta(X,Y) + C\alpha(X,Y) - \langle X, Y\rangle \delta, \tag{7.31}$$

where $\delta = (\tilde{C}w)_{\Psi_* N_f M}$.

Let $\tilde{L}$ be associated to $\tilde{\mathcal{T}}$ and let $\tilde{\mathcal{Y}}$ be given by (2.10) with respect to $\tilde{L}$. Given $\eta \in \Gamma(N_f M)$ we have

$$\langle \tilde{L}X, \eta\rangle = \langle \tilde{\nabla}_X \mathcal{T}, \eta\rangle = \langle LX, \eta\rangle,$$

and then $\tilde{\mathcal{Y}}\eta = \mathcal{Y}\eta$, here L and $\mathcal{Y}$ are associated to $\mathcal{T}$. Notice that (7.30) is just $(\tilde{L}X)_{\Psi_* T\mathbb{R}^m} = \Psi_* LX$. This together with (7.22) imply that $(\tilde{\mathcal{E}}(X,\eta))_{\Psi_* N_f M}$ coincides with $\mathcal{E}'(X,\eta)$ for any $X \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_f M)$. Notice also that $\Psi_* N_f M$ is parallel with respect to $\nabla'^{\perp}$, thus we have from (7.29) that

$$\mathcal{E}' = \mathcal{E} - \nabla^{\perp} C. \tag{7.32}$$

Finally it follows from (7.31), (7.32) and Proposition 7.4 that any other conformal infinitesimal bending arising in this manner differs from $\mathcal{T}$ by a trivial conformal infinitesimal bending, and this concludes the proof. □

Theorem 7.7 takes a rather simpler form in the hypersurface case.

Corollary 7.8. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an isometric immersion of a simply connected Riemannian manifold. Then a symmetric tensor $0 \neq \mathcal{B} \in \Gamma(End(TM))$ and $\rho \in C^\infty(M)$ that satisfy* (7.18) *and* (7.19) *determine a unique conformal infinitesimal bending of f.*

Proof. In this case $\mathcal{E}$ vanishes and $\langle \mathcal{B}X, Y\rangle = \langle \beta(X,Y), N\rangle$. Thus (7.11) holds trivially for β and $\mathcal{E} = 0$. Moreover, by the assumptions on $\mathcal{B}$ we have that $(\beta, 0, \rho)$ satisfies (7.9) and (7.10). Hence, Theorem 7.7 gives that $(\beta, 0, \rho)$ determines a unique conformal infinitesimal bending $\mathcal{T}$ of f. □

7.6 Conformal infinitesimal rigidity

In this section, we give a rigidity theorem for conformal infinitesimal variations of Euclidean submanifolds that lie in low codimension. This result is the infinitesimal version of the conformal rigidity result due to do Carmo-Dajczer given in [5].

The notion of conformal s-nullity given next is a concept in conformal geometry since it is easily seen to be invariant under a conformal change of the metric of the ambient space.

The *conformal s-nullity* $\nu_s^c(x)$, $1 \leq s \leq p$, of an immersion $f\colon M^n \to \mathbb{R}^{n+p}$ at $x \in M^n$ is defined as

$$\nu_s^c(x) = \max\{\dim \mathcal{N}(\alpha_{U^s} - \langle\,,\,\rangle\xi)(x)\colon U^s \subset N_fM(x) \text{ and } \xi \in U^s\},$$

where $\alpha_{U^s} = \pi_{U^s} \circ \alpha$ and $\pi_{U^s}\colon N_fM \to U^s$ is the orthogonal projection onto the normal subspace U^s.

The next is the version of Proposition 2.15 for conformal infinitesimal variations.

Theorem 7.9. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n \geq 2p+3$, be an isometric immersion with codimension $1 \leq p \leq 4$. If the conformal s-nullities of f satisfy $\nu_s^c \leq n-2s-1$ for all $1 \leq s \leq p$ at any point of M^n, then f is conformally infinitesimally rigid.*

By the above result, in the case of a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, the existence of a nontrivial conformal infinitesimal variation requires the presence of a principal curvature of multiplicity at least $n-2$ at any point.

The proof of Theorem 2.17 for infinitesimal variations was quite short due to the use of the classical trick given by Proposition 2.12. But this strategy fails completely in the conformal case and this is why, in sharp contrast, the proof of the above result requires several lemmas.

Lemma 7.10. *Let $\mathcal{T}$ be a conformal infinitesimal bending of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ with conformal factor ρ and associated pair $(\beta, \mathcal{E})$. Then, at any point of M^n the bilinear form $\theta\colon TM \times TM \to N_fM \oplus \mathbb{R} \oplus N_fM \oplus \mathbb{R}$ defined by*

$$\theta = (\alpha + \beta, \langle\,,\,\rangle + \operatorname{Hess}\rho, \alpha - \beta, \langle\,,\,\rangle - \operatorname{Hess}\rho) \tag{7.33}$$

is flat with respect to the inner product $\langle\!\langle\,,\,\rangle\!\rangle$ of signature $(p+1, p+1)$ given by

$$\langle\!\langle(\xi_1, a_1, \eta_1, b_1), (\xi_2, a_2, \eta_2, b_2)\rangle\!\rangle = \langle\xi_1, \xi_2\rangle_{N_fM} + a_1a_2 - \langle\eta_1, \eta_2\rangle_{N_fM} - b_1b_2.$$

Proof. A straightforward computation yields

$$\begin{aligned}\frac{1}{2}(\langle\!\langle\theta(X,W),\theta(Y,Z)\rangle\!\rangle - \langle\!\langle\theta(X,Z),\theta(Y,W)\rangle\!\rangle)\\ =&\langle\beta(X,W),\alpha(Y,Z)\rangle + \langle\alpha(X,W),\beta(Y,Z)\rangle\\ &-\langle\beta(X,Z),\alpha(Y,W)\rangle - \langle\alpha(X,Z),\beta(Y,W)\rangle\\ &+\langle X,W\rangle \mathit{Hess}\,\rho(Y,Z) + \langle Y,Z\rangle \mathit{Hess}\,\rho(X,W)\\ &-\langle X,Z\rangle \mathit{Hess}\,\rho(Y,W) - \langle Y,W\rangle \mathit{Hess}\,\rho(X,Z)\end{aligned}$$

for any $X,Y,Z,W\in\mathfrak{X}(M)$, and the proof follows from (7.9). □

Lemma 7.11. *Let $f\colon M^n\to\mathbb{R}^m$ be an isometric immersion. Let $Z_1,Z_2\in T_xM$ be nonzero vectors satisfying either $Z_1=Z_2$ or $\langle Z_1,Z_2\rangle=0$. If $n\geq 4$ and $\nu_1^c(x)\leq n-3$, then*

$$N_fM(x)=\mathit{span}\{\alpha(X,Y)\colon X,Y\in T_xM;\langle X,Y\rangle=\langle X,Z_1\rangle=\langle Y,Z_2\rangle=0\}.$$

Proof. First assume that $\langle Z_1,Z_2\rangle=0$. Let $U^s\subset N_fM(x)$ be the subspace given by $U^s\perp\alpha(X,Y)$ for any $X,Y\in T_xM$ as in the statement. If, in addition, we have $\langle X,Z_2\rangle=\langle Y,Z_1\rangle=0$ and $\|X\|=\|Y\|$, then that $\alpha(X+Y,X-Y)_{U^s}=0$ gives

$$\alpha_{U^s}(X,X)=\alpha_{U^s}(Y,Y).$$

Thus, there is $\zeta\in U^s$ such that

$$\alpha_{U^s}(X,Y)=\langle X,Y\rangle\zeta$$

for any $X,Y\in\operatorname{span}\{Z_1,Z_2\}^\perp$. By assumption $\alpha_{U^s}(W,Z_1)=\alpha_{U^s}(W,Z_2)=0$ for any $W\in\operatorname{span}\{Z_1,Z_2\}^\perp$. Then

$$\operatorname{span}\{Z_1,Z_2\}^\perp\subset\mathcal{N}(\alpha_{U^s}-\langle\,,\,\rangle\zeta),$$

and this contradicts our assumption on ν_1^c unless $s=0$.

If $Z_1=Z_2$ we again have that there is $\zeta\in U^s$ such that

$$\alpha_{U^s}(X,Y)=\langle X,Y\rangle\zeta$$

for any $X,Y\in\operatorname{span}\{Z_1\}^\perp$. It follows that A_ζ has an eigenspace of multiplicity at least $n-2$ again contradicting the assumption on ν_1^c. □

Lemma 7.12. *Let $f\colon M^n\to\mathbb{R}^m$, $n\geq 4$, be an isometric immersion and let $\mathcal{T}$ be a conformal infinitesimal bending of f with conformal factor ρ and associated pair $(\beta,\mathcal{E})$. If $\nu_1^c(x)\leq n-3$ at any $x\in M^n$, then $\mathcal{E}$ is the unique tensor satisfying* (7.6) *as well as an equation of the form*

$$\begin{aligned}(\nabla^\perp_X\beta)(Y,Z)-(\nabla^\perp_Y\beta)(X,Z)=\;&\mathcal{E}(Y,\alpha(X,Z))-\mathcal{E}(X,\alpha(Y,Z))\\ &+\langle Y,Z\rangle\psi(X)-\langle X,Z\rangle\psi(Y),\end{aligned}\tag{7.34}$$

where $\psi\in\Gamma(\mathit{Hom}(TM,N_fM))$.

Proof. If also $\mathcal{E}_0\colon TM \times N_fM \to N_fM$ satisfies (7.6) and (7.34), then (7.10) gives

$$\begin{aligned}&(\mathcal{E}-\mathcal{E}_0)(X,\alpha(Y,Z))-(\mathcal{E}-\mathcal{E}_0)(Y,\alpha(X,Z))\\&+\langle Y,Z\rangle(\psi(X)-\alpha(X,\nabla\rho))-\langle X,Z\rangle(\psi(Y)-\alpha(Y,\nabla\rho))=0.\end{aligned}$$

Hence

$$(\mathcal{E}-\mathcal{E}_0)(X,\alpha(Y,Z))=(\mathcal{E}-\mathcal{E}_0)(Y,\alpha(X,Z))$$

if Z is orthogonal to X and Y. Writing

$$\langle(\mathcal{E}-\mathcal{E}_0)(X_1,\alpha(X_2,X_3)),\alpha(X_4,X_5)\rangle=(X_1,X_2,X_3,X_4,X_5)$$

and taking $\langle X_1,X_3\rangle=\langle X_2,X_3\rangle=0$, we have symmetry in the pairs $\{X_1,X_2\}$, $\{X_2,X_3\}$ and $\{X_4,X_5\}$. Moreover, since $\mathcal{E}$ and $\mathcal{E}_0$ verify (7.6) we obtain

$$(X_1,X_2,X_3,X_4,X_5)=-(X_1,X_4,X_5,X_2,X_3).$$

Hence, if $\{X_i\}_{1\le i\le 5}$ satisfies

$$\langle X_1,X_3\rangle=\langle X_1,X_4\rangle=\langle X_2,X_3\rangle=\langle X_2,X_5\rangle=\langle X_4,X_5\rangle=0, \tag{7.35}$$

then

$$\begin{aligned}(X_1,X_2,X_3,X_4,X_5)&=-(X_1,X_4,X_5,X_2,X_3)=-(X_5,X_4,X_1,X_2,X_3)\\&=(X_5,X_2,X_3,X_4,X_1)=(X_3,X_2,X_5,X_4,X_1)\\&=-(X_3,X_4,X_1,X_2,X_5)=-(X_4,X_3,X_1,X_2,X_5)\\&=(X_4,X_2,X_5,X_3,X_1)=(X_2,X_4,X_5,X_3,X_1)\\&=-(X_2,X_3,X_1,X_4,X_5)=-(X_1,X_2,X_3,X_4,X_5)\\&=0.\end{aligned}$$

Thus

$$\langle(\mathcal{E}-\mathcal{E}_0)(X_1,\alpha(X_2,X_3)),\alpha(X_4,X_5)\rangle=0$$

if (7.35) holds. We already have that

$$\langle X_1,X_4\rangle=\langle X_2,X_5\rangle=\langle X_4,X_5\rangle=0.$$

Hence, if also $\langle X_1,X_2\rangle=0$, we obtain from Lemma 7.11 that

$$(\mathcal{E}-\mathcal{E}_0)(X_1,\alpha(X_2,X_3))=0$$

for any $X_1,X_2,X_3\in\mathfrak{X}(M)$ with $\langle X_1,X_2\rangle=\langle X_1,X_3\rangle=\langle X_2,X_3\rangle=0$. Now using Lemma 7.11 again, it follows that

$$(\mathcal{E}-\mathcal{E}_0)(X,\eta)=0$$

for any $X\in\mathfrak{X}(M)$ and $\eta\in\Gamma(N_fM)$. $\square$

Lemma 7.13. *Let $S \subset \mathbb{R}^m$ be a vector subspace and let $T_0\colon S \to \mathbb{R}^m$ be a linear map that is an isometry between S and $T_0(S)$. Assume that there is no vector $0 \neq v \in S$ such that $T_0 v = -v$. Then there is an isometry $T \in End(\mathbb{R}^m)$ that extends T_0 and has 1 as the only possible real eigenvalue.*

Proof. Extend T_0 to an isometry T of $\mathbb{R}^m$. Suppose that the eigenspace of the eigenvalue -1 of T satisfies $\dim E_{-1} = k > 0$. By assumption we have that

$$E_{-1} \cap S = E_{-1} \cap T_0(S) = 0.$$

Let $e_1, \ldots, e_k$ be an orthonormal basis of E_{-1} and set

$$P = T_0(S) \oplus \operatorname{span}\{e_2, \ldots, e_k\}.$$

Let $\xi \in P^\perp$ be a unit vector collinear with the $P^\perp$-component of e_1. Let $\eta \in \mathbb{R}^m$ be such that $T\eta = \xi$ and let H be the hyperplane $\{\eta\}^\perp$. If R is the reflection with respect to the hyperplane $\{\xi\}^\perp$, then the isometry $T_1 = RT$ satisfies $T_1 v = Tv$ for any $v \in H$ since $Tv \in \{\xi\}^\perp$.

Since $\langle \eta, e_1\rangle = -\langle \xi, e_1\rangle \neq 0$, there is $v \in H$ such that $\eta + v$ is collinear with e_1. Hence

$$T(\eta + v) = \xi + Tv = -\eta - v.$$

We claim that no vector of the form $\eta + u$, $u \in H$, is an eigenvector of T_1 associated to -1. If otherwise

$$T_1(\eta + u) = -\xi + Tu = -\eta - u$$

for some $u \in H$. We obtain from the last two equations that

$$T(u + v) = -2\eta - (u + v).$$

Then

$$\|T(u + v)\|^2 = 4 + \|u + v\|^2$$

which contradicts that T is an isometry and proves the claim.

We have proved that the eigenspace of T_1 associated to -1 is contained in H, in fact, that it is $\operatorname{span}\{e_2 \ldots ..., e_k\}$. Therefore, by composing T with k appropriate reflections we obtain an isometry as required by the statement. $\square$

Lemma 7.14. *Let $\mathcal{T}$ be a conformal infinitesimal bending of an isometric immersion $f\colon M^n \to \mathbb{R}^m$. If $\mathcal{T}$ is trivial then θ is null. Conversely, if θ is null, $n \geq 4$ and $\nu_1^c(x) \leq n - 3$ at any $x \in M^n$ then $\mathcal{T}$ is trivial.*

Proof. If $\mathcal{T}$ is a trivial conformal infinitesimal bending of f, then

$$\mathcal{T}(x) = (\langle f(x), v\rangle + \lambda) f(x) - 1/2\|f(x)\|^2 v + \mathcal{D} f(x) + w$$

for $\lambda \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $\mathcal{D} \in \mathrm{End}(\mathbb{R}^m)$ skew-symmetric. Since $\rho(x) = \langle f(x), v\rangle + \lambda$, then $f_*\nabla\rho = v_{TM}$. Hence

$$\langle\tilde{\nabla}_X v, f_*Y\rangle = \mathit{Hess}\,\rho(X,Y) - \langle A_{v_{N_fM}}X, Y\rangle = 0 \tag{7.36}$$

for any $X, Y \in \mathfrak{X}(M)$. Moreover, we have seen that

$$\beta(X,Y) = C\alpha(X,Y) - \langle X,Y\rangle v_{N_fM},$$

where $C \in \Gamma(\mathrm{End}(N_fM))$ is skew-symmetric. Using (7.36) and that C is skew-symmetric, we obtain that the bilinear form θ is null. In fact,

$$\begin{aligned}\frac{1}{2}\langle\langle\theta(X,Y),\theta(Z,W)\rangle\rangle &= \langle\alpha(X,Y),\beta(Z,W)\rangle + \langle\beta(X,Y),\alpha(Z,W)\rangle\\ &\quad + \langle X,Y\rangle \mathit{Hess}\,\rho(Z,W) + \langle Z,W\rangle \mathit{Hess}\,\rho(X,Y)\\ &= -\langle Z,W\rangle\langle\alpha(X,Y), v_{N_fM}\rangle - \langle X,Y\rangle\langle\alpha(Z,W), v_{N_fM}\rangle\\ &\quad + \langle X,Y\rangle \mathit{Hess}\,\rho(Z,W) + \langle Z,W\rangle \mathit{Hess}\,\rho(X,Y)\\ &= 0.\end{aligned}$$

For the converse, that θ is null means that

$$\begin{aligned}&\langle\alpha(X,Y),\beta(Z,W)\rangle + \langle\beta(X,Y),\alpha(Z,W)\rangle\\ &+ \langle X,Y\rangle \mathit{Hess}\,\rho(Z,W) + \langle Z,W\rangle \mathit{Hess}\,\rho(X,Y) = 0\end{aligned} \tag{7.37}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$. Let $S \subset N_fM(x) \oplus \mathbb{R}$ be the subspace given by

$$S = \mathrm{span}\{(\alpha(X,Y) + \beta(X,Y), \langle X,Y\rangle + \mathit{Hess}\,\rho(X,Y))\colon X, Y \in T_xM\}.$$

Then, the map T_0 defined by

$$\begin{aligned}T_0(\alpha(X,Y) + \beta(X,Y), &\langle X,Y\rangle + \mathit{Hess}\,\rho(X,Y))\\ &= (\alpha(X,Y) - \beta(X,Y), \langle X,Y\rangle - \mathit{Hess}\,\rho(X,Y))\end{aligned}$$

is an isometry between S and $T(S)$. We claim that -1 is not an eigenvalue of T_0. Suppose that $T_0 v = -v$ where

$$v = \sum_i (\alpha(X_i,Y_i) + \beta(X_i,Y_i), \langle X_i,Y_i\rangle + \mathit{Hess}\,\rho(X_i,Y_i)) \in S.$$

Hence $\sum_i \alpha(X_i,Y_i) = 0$ and $\sum_i\langle X_i,Y_i\rangle = 0$. Now (7.37) gives

$$\sum_i \langle\beta(X_i,Y_i),\alpha(Z,W)\rangle + \langle Z,W\rangle \sum_i \mathit{Hess}\,\rho(X_i,Y_i) = 0$$

for any $Z, W \in \mathfrak{X}(M)$. That is, we have that $A_\eta = -hI$ where $\eta = \sum_i \beta(X_i,Y_i)$ and $h = \sum_i \mathit{Hess}\,\rho(X_i,Y_i)$. From our assumption on ν_1^c we obtain $\eta = h = 0$, hence $v = 0$ proving the claim.

Let T be the isometry of $N_fM(x)\oplus\mathbb{R}$ extending T_0 given by Lemma 7.13. Then

$$(I+T^t)(I-T)=(T^t-T)=-(I-T^t)(I+T),$$

where T^t is the transpose of T. Thus

$$(I-T)(I+T)^{-1}=-(I+T^t)^{-1}(I-T^t)=-((I-T)(I+T)^{-1})^t,$$

that is, $(I-T)(I+T)^{-1}$ is a skew-symmetric endomorphism of $N_fM(x)\oplus\mathbb{R}$.

It is easy to see that

$$(I-T)(I+T)^{-1}(\alpha(X,Y),0)=(\beta(X,Y),\mathit{Hess}\,\rho(X,Y))$$

for any $X,Y\in T_xM$ such that $\langle X,Y\rangle=0$. Thus, there is $C\in\mathrm{End}(N_fM(x))$ skew-symmetric such that

$$\beta(X,Y)=C\alpha(X,Y)$$

for any $X,Y\in T_xM$ with $\langle X,Y\rangle=0$. Since

$$\beta(X+Y,X-Y)=C\alpha(X+Y,X-Y)$$

for any orthonormal vectors $X,Y\in T_xM$, it follows that

$$\beta(X,X)-C\alpha(X,X)=\beta(Y,Y)-C\alpha(Y,Y).$$

Hence, there is $\delta\in N_fM(x)$ such that

$$\beta(X,Y)=C\alpha(X,Y)-\langle X,Y\rangle\delta \tag{7.38}$$

for any $X,Y\in T_xM$.

By Lemma 7.11 there are smooth local vector fields X_i,Y_i, $1\leq i\leq p$, satisfying $\langle X_i,Y_i\rangle=0$ such that the vectors $\alpha(X_i,Y_i)$, $1\leq i\leq p$, span the normal bundle. Thus C and δ are smooth.

Define $\mathcal{E}_0\colon TM\times N_fM\to N_fM$ by

$$\mathcal{E}_0(X,\eta)=-(\nabla^\perp_XC)\eta.$$

It follows from (7.38) that

$$\begin{aligned}(\nabla^\perp_X\beta)(Y,Z)-(\nabla^\perp_Y\beta)(X,Z)&=\mathcal{E}_0(Y,\alpha(X,Z))-\mathcal{E}_0(X,\alpha(Y,Z))\\&\quad-\langle Y,Z\rangle\nabla^\perp_X\delta+\langle X,Z\rangle\nabla^\perp_Y\delta.\end{aligned}$$

Then Lemma 7.12 gives $\mathcal{E}=\mathcal{E}_0$, and thus $\mathcal{T}$ is trivial by Proposition 7.4. $\square$

Proof of Theorem 7.9: Let $\mathcal{T}$ be a conformal infinitesimal bending of f such that the flat bilinear θ given by (7.33) is not null at $x\in M^n$. Since $\mathcal{N}(\theta)=0$, there is an orthogonal decomposition

$$W_0^{2p+2}=N_fM(x)\oplus\mathbb{R}\oplus N_fM(x)\oplus\mathbb{R}=W_1^{\ell,\ell}\oplus W_2^{p-\ell+1,p-\ell+1},\ 1\leq\ell\leq p,$$

such that θ splits as $\theta = \theta_1 + \theta_2$ as in Theorem 1.11. Denoting $\Delta = \mathcal{N}(\theta_2)$, we have $\dim \Delta \geq n - 2(p-\ell+1)$. Thus $\theta(Z,X) = \theta_1(Z,X)$ for any $Z \in \Delta$ and $X \in T_xM$.

Let $S \subset N_fM(x) \oplus \mathbb{R}$ be the vector subspace given by

$$S = \operatorname{span}\{(\alpha(Z,X)+\beta(Z,X), \langle Z,X\rangle + Hess\,\rho(Z,X)) \colon Z \in \Delta \,\text{and}\, X \in T_xM\}.$$

If Π_1 denotes the orthogonal projection from W_0^{2p+2} onto the first copy of $N_fM(x) \oplus \mathbb{R}$, then $S \subset \Pi_1(\mathcal{S}(\theta) \cap \mathcal{S}(\theta)^\perp)$ and, in particular, $\dim S \leq \ell$.

That θ_1 is null means that the map $T \colon S \to N_fM(x) \oplus \mathbb{R}$ defined by

$$\begin{aligned} T(\alpha(Z,X)&+\beta(Z,X), \langle Z,X\rangle + Hess\,\rho(Z,X)) \\ &= (\alpha(Z,X) - \beta(Z,X), \langle Z,X\rangle - Hess\,\rho(Z,X)) \end{aligned}$$

is an isometry between S and $T(S)$. We have that

$$\frac{1}{2}(I+T)(\alpha(Z,X)+\beta(Z,X), \langle Z,X\rangle + Hess\,\rho(Z,X)) = (\alpha(Z,X), \langle Z,X\rangle).$$

If $S_1 = ((I+T)(S))^\perp \subset N_fM \times \mathbb{R}$, then $\dim S_1 \geq p-\ell+1$. For $(\eta, a) \in S_1$ we have that

$$\langle \alpha(Z,X), \eta\rangle + a\langle X,Z\rangle = 0 \tag{7.39}$$

for any $Z \in \Delta$ and $X \in T_xM$. Let $U \subset N_fM$ be the orthogonal projection of S_1 in N_fM. Since S_1 does not posses elements of the form $(0,a)$ with $0 \neq a \in \mathbb{R}$, then

$$\dim U \geq p - \ell + 1.$$

It follows from (7.39) that there exists $\xi \in U$ such that

$$\alpha_U(Z,X) = \langle Z,X\rangle \xi$$

for any $Z \in \Delta$ and $X \in T_xM$. Hence $\alpha_U - \langle\,,\,\rangle\xi$ has a kernel of dimension at least $\dim \Delta \geq n - 2(p-\ell+1)$. But this is in contradiction with the assumption on the conformal s-nullities, and hence θ is necessarily null at any point. Finally, Lemma 7.14 gives that $\mathcal{T}$ is trivial. $\square$

7.7 Exercises

Exercise 7.1. Prove the statements in Examples 7.1.

Exercise 7.2. Let $f, g \colon M^n \to \mathbb{R}^m$ be conformal immersions such that the map $h = f + g \colon M^n \to \mathbb{R}^m$ is also a conformal immersion. Then show that $\mathcal{T} = f - g$ is a conformal infinitesimal bending of h.

Exercise 7.3. Prove that (7.10) is equivalent to the equation

$$\begin{aligned}(\nabla_X B_\eta)Y - &(\nabla_Y B_\eta)X - B_{\nabla^\perp_X \eta} Y + B_{\nabla^\perp_Y \eta} X \\ &= A_{\mathcal{E}(X,\eta)}Y - A_{\mathcal{E}(Y,\eta)}X + \langle A_\eta X, \nabla\rho\rangle Y - \langle A_\eta Y, \nabla\rho\rangle X\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$.

Exercise 7.4. Show that the normal bundle of the map $\Psi\colon \mathbb{R}^m \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ defined by (7.20) is $N_\Psi\mathbb{R}^m = \operatorname{span}\{\Psi, w\}$ and that its second fundamental form is given by (7.21).

Exercise 7.5. Let $f\colon M^n \to \mathbb{R}^m$, $n \geq 3$, be an isometric immersion of a simply connected Riemannian manifold. Let $(\beta, \mathcal{E}, \rho) \neq 0$ be a triple formed by a symmetric tensor $\beta\colon TM \times TM \to N_fM$, a tensor $\mathcal{E}\colon TM \times N_fM \to N_fM$ that verifies (7.6) and $\rho \in C^\infty(M)$ that satisfies system (S). Show that the triple determines a unique conformal infinitesimal bending of f with conformal factor ρ.

Hint: Given a pair $(\beta, \mathcal{E})$ as in the statement, show that there is $\mathcal{D} \in \Gamma(\operatorname{End}(f^*T\mathbb{R}^m))$ satisfying

$$(\tilde\nabla_X \mathcal{D})(Y + \eta) = f_*(\langle Y, \nabla\rho\rangle X - \langle X, Y\rangle\nabla\rho - B_\eta X) + \beta(X,Y) + \mathcal{E}(X,\eta)$$

for any $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$. For that check that its integrability condition

$$(\tilde\nabla_X\tilde\nabla_Y\mathcal{D} - \tilde\nabla_Y\tilde\nabla_X\mathcal{D} - \tilde\nabla_{[X,Y]}\mathcal{D})(Z+\eta) = 0$$

holds for any $X, Y, Z \in \mathfrak{X}(M)$ and $\eta \in \Gamma(N_fM)$. Then, as in the proof of Theorem 2.8, show that $\mathcal{D}$ can be assumed to be skew-symmetric. Define $L \in \Gamma(\operatorname{Hom}(TM, f^*T\mathbb{R}^m))$ by $L(x) = \mathcal{D}(x)|_{T_xM}$. Then prove that there is a vector field $\mathcal{T} \in \Gamma(f^*T\mathbb{R}^m)$ such that $\tilde\nabla_X\mathcal{T} = LX + \rho X$ for any $X \in \mathfrak{X}(M)$. Conclude that $\mathcal{T}$ is a conformal infinitesimal bending of f with conformal factor ρ whose associate pair $(\tilde\beta, \tilde{\mathcal{E}})$ is

$$\tilde\beta(X,Y) = \beta(X,Y) + \mathcal{D}^N\alpha(X,Y) \;\; \textit{and} \;\; \tilde{\mathcal{E}}(X,\eta) = \mathcal{E}(X,\eta) - (\nabla^\perp_X\mathcal{D}^N)\eta,$$

where $\mathcal{D}^N\eta = (\mathcal{D}\eta)_{N_fM}$ for any $\eta \in \Gamma(N_fM)$.

Chapter 8

Conformal variations of hypersurfaces

The main purpose of this chapter is to parametrically classify the hypersurfaces in Euclidean space $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, that admit nontrivial conformal infinitesimal variations. The key ingredient in the classification is a class of surfaces that is discussed in the first section. In the special case of conformal variations, such a classification was first considered by Cartan [7] and by Dajczer-Tojeiro [20] in a modern form. The contains of this chapter are due to Dajczer-Jimenez-Vlachos [18].

8.1 Special surfaces

The classification of the Euclidean hypersurfaces that admit nontrivial conformal infinitesimal variations will be given by means of the conformal Gauss parametrization in terms of a class of spherical surfaces discussed in this section.

Recall that $(\mathbb{E}^{m+1}, \langle\,,\,\rangle)$ stands for either Euclidean space $\mathbb{R}^{m+1}$ or Lorentzian space $\mathbb{L}^{m+1}$ with the standard flat metric. Then $\mathbb{S}^m_\epsilon \subset \mathbb{E}^{m+1}$, $\epsilon = 0, 1$, is either the Euclidean unit sphere $\mathbb{S}^m_0 \subset \mathbb{R}^{m+1}$ or the Lorentzian unit sphere (or de Sitter space) $\mathbb{S}^m_1 \subset \mathbb{L}^{m+1}$, that is,

$$\mathbb{S}^m_\epsilon = \{X \in \mathbb{E}^{m+1} \colon \langle X, X\rangle = 1\}.$$

Moreover, we always denote by $i\colon \mathbb{S}^m_\epsilon \to \mathbb{E}^{m+1}$ the isometric umbilical inclusion.

In the sequel, let $g\colon L^2 \to \mathbb{S}^m_\epsilon$, $m \geq 4$, be a space-like surface with second fundamental form $\alpha^g\colon TL \times TL \to N_gL$. Assume that g has full first normal spaces of dimension two. Hence, given a basis X, Y of T_xL

there exists $0 \neq (a, b, c) \in \mathbb{R}^3$ such that the second fundamental form of g satisfies

$$a\alpha^g(X,X) + 2c\alpha^g(X,Y) + b\alpha^g(Y,Y) = 0.$$

The surface g is said to be *hyperbolic* (respectively, *elliptic*) *at* $x \in L^2$ if $ab - c^2 < 0$ (respectively, $ab - c^2 > 0$). In Exercise 8.2 it is shown that this condition is independent of the given basis. Moreover, the condition is equivalent to the existence of a unique endomorphism J on T_xL satisfying $J \neq I$ and $J^2 = I$ (respectively, $J^2 = -I$) and

$$\alpha^g(JX,Y) = \alpha^g(X,JY) \tag{8.1}$$

for all $X, Y \in T_xL$.

The surface g is said to be *hyperbolic* (respectively, *elliptic*) if it is hyperbolic (respectively, elliptic) at every point of L^2. In this case, the endomorphisms J on each tangent space give rise to a tensor $J \in \Gamma(\mathrm{End}(TL))$ such that (8.1) holds for all $X, Y \in \mathfrak{X}(L)$.

A local system of coordinates (u, v) on L^2 is said to be *real conjugate* for a given surface $g\colon L^2 \to \mathbb{S}^m_\epsilon$ if the condition

$$\alpha^g(\partial_u, \partial_v) = 0$$

holds for the coordinate vector fields $\partial_u = \partial/\partial u$ and $\partial_v = \partial/\partial v$. The local coordinate system (u, v) is said to be *complex conjugate* for g if

$$\alpha^g(\partial_z, \partial_{\bar z}) = 0,$$

where $z = u + iv$ and $\partial_z = (1/2)(\partial_u - i\partial_v)$, that is, if

$$\alpha^g(\partial_u, \partial_u) + \alpha^g(\partial_v, \partial_v) = 0.$$

In the case of real conjugate coordinates, we denote $F = \langle \partial_u, \partial_v \rangle$ and Γ^1, Γ^2 are the Christoffel symbols defined by

$$\nabla_{\partial_u}\partial_v = \Gamma^1\partial_u + \Gamma^2\partial_v. \tag{8.2}$$

In the case of complex conjugate coordinates, we denote $F = \langle \partial_z, \partial_{\bar z} \rangle$ where $\langle\ ,\ \rangle$ also stands for the $\mathbb{C}$-bilinear extension of the metric of L^2, and we have that

$$\nabla_{\partial_z}\partial_{\bar z} = \Gamma\partial_z + \bar\Gamma\partial_{\bar z}, \tag{8.3}$$

where ∇ also denotes the $\mathbb{C}$-bilinear extension of the Riemannian connection.

Proposition 8.1. *Let $g\colon L^2 \to \mathbb{S}^m_\epsilon$ be a space-like surface and $h = i \circ g\colon L^2 \to \mathbb{E}^{m+1}$. Then the following assertions are equivalent:*

(i) The coordinates (u, v) are either real conjugate or complex conjugate for g.

(ii) The position vector of h satisfies

$$h_{uv} - \Gamma^1 h_u - \Gamma^2 h_v + Fh = 0 \tag{8.4}$$

in the case of real conjugate coordinates and

$$h_{z\bar{z}} - \Gamma h_z - \bar{\Gamma} h_{\bar{z}} + Fh = 0 \tag{8.5}$$

in the case of complex conjugate coordinates.

Proof. The condition $\alpha^g(\partial_u, \partial_v) = 0$ is equivalent to

$$\alpha^h(\partial_u, \partial_v) + Fh = 0$$

whereas that $\alpha^g(\partial_z, \partial_{\bar{z}}) = 0$ is equivalent to

$$\alpha^h(\partial_z, \partial_{\bar{z}}) + Fh = 0.$$

The preceding two equations can also be written as (8.4) and (8.5), respectively. □

Proposition 8.2. *If the surface $g\colon L^2 \to \mathbb{S}^m_\epsilon$ is hyperbolic (respectively, elliptic), then there exists locally a real conjugate (respectively, complex conjugate) system of coordinates on L^2 for g. Conversely, if there exists a real conjugate (respectively, complex conjugate) system of coordinates on L^2, then $g\colon L^2 \to \mathbb{S}^m_\epsilon$ is hyperbolic (respectively, elliptic).*

Proof. Assume that g is hyperbolic, and let X, Y be a frame of eigenvectors of J associated with the eigenvalues 1 and -1, respectively. Then there exists a local system of coordinates (u, v) in L^2 such that the coordinate vector fields ∂_u and ∂_v are collinear with X and Y, respectively. Hence

$$\alpha^g(\partial_u, \partial_v) = \alpha^g(J\partial_u, \partial_v) = \alpha^g(\partial_u, J\partial_v) = -\alpha^g(\partial_u, \partial_v) = 0.$$

Conversely, if (u, v) are real conjugate coordinates on L^2 for g, let J be the tensor defined by $J\partial_u = \partial_u$ and $J\partial_v = -\partial_v$. Then $J^2 = I$ and (8.1) holds, since this is satisfied for $X, Y \in \{\partial_u, \partial_v\}$. Thus g is hyperbolic with respect to J. The proof for the elliptic case is similar. □

We call a hyperbolic surface $g\colon L^2 \to \mathbb{S}^m_\epsilon$ endowed with a system of real conjugate coordinates as in Proposition 8.2 a *special hyperbolic surface* if the Christoffel symbols Γ^1, Γ^2 given by (8.2) satisfy the condition

$$\Gamma^1_u = \Gamma^2_v. \tag{8.6}$$

Proposition 8.3. *Let $g\colon L^2 \to \mathbb{S}^m_\epsilon$ be a simply connected special hyperbolic surface and, up to a constant factor, let $\mu \in C^\infty(L)$ be the unique positive solution of*

$$d\mu + 2\mu\omega = 0, \tag{8.7}$$

where $\omega = \Gamma^2 du + \Gamma^1 dv$. Then $\varphi \in C^\infty(L)$ is a solution of the equation

$$\varphi_{uv} - \Gamma^1\varphi_u - \Gamma^2\varphi_v + F\varphi = 0 \quad \text{where} \quad F = \langle \partial_u, \partial_v \rangle \tag{8.8}$$

if and only if $\psi = \sqrt{\mu}\varphi$ satisfies

$$\psi_{uv} + M\psi = 0, \tag{8.9}$$

where

$$M = F - \frac{\mu_{uv}}{2\mu} + \frac{\mu_u\mu_v}{4\mu^2} \cdot . \tag{8.10}$$

In particular, the map $k = \sqrt{\mu}\, h \colon L^2 \to \mathbb{E}^{m+1}$ where $h = i \circ g$, satisfies

$$k_{uv} + Mk = 0.$$

Conversely, for a system of coordinates (u, v) on an open subset $U \subset \mathbb{R}^2$ let $\{k_1, \ldots, k_{m+1}\}$ be a set of solutions of the equation (8.9) for $M \in C^\infty(U)$. Assume that the map $k = (k_1, \ldots, k_{m+1}) \colon U \to \mathbb{E}^{m+1}$ satisfies $\mu = \|k\|^2 > 0$ and that the map $h = (1/\sqrt{\mu})\, k \colon U \to \mathbb{E}^{m+1}$ is a space-like immersion if $\epsilon = 1$. Then $g \colon U \to \mathbb{S}^m_\epsilon$ defined by $h = i \circ g$ is a special hyperbolic surface.

Proof. Notice that (8.6) is the integrability condition of (8.7). Since $\mu \in C^\infty(U)$ is a solution of (8.7), it satisfies

$$\Gamma^1 = -\frac{\mu_v}{2\mu} \quad \text{and} \quad \Gamma^2 = -\frac{\mu_u}{2\mu} \cdot$$

Hence (8.8) becomes

$$\varphi_{uv} + \frac{\mu_v}{2\mu}\varphi_u + \frac{\mu_u}{2\mu}\varphi_v + F\varphi = 0$$

which takes the form (8.9) for $\psi = \sqrt{\mu}\,\varphi$ and M given by (8.10).

We now prove the converse. It is easily seen that $h = (1/\sqrt{\mu})\, k \colon U \to \mathbb{L}^{m+1}$ satisfies

$$h_{uv} + \frac{\mu_v}{2\mu}h_u + \frac{\mu_u}{2\mu}h_v + Fh = 0, \tag{8.11}$$

where $F = M + \frac{\mu_{uv}}{2\mu} - \frac{\mu_u\mu_v}{4\mu^2}$. If h is a space-like immersion and $g \colon U \to \mathbb{S}^m_\epsilon$ is the surface defined by $h = i \circ g$, then (8.11) implies that (u, v) are real conjugate coordinates for g and that the Christoffel symbols of the metric induced by g are

$$\Gamma^1 = -\frac{\mu_v}{2\mu} \quad \text{and} \quad \Gamma^2 = -\frac{\mu_u}{2\mu} \cdot$$

It follows that (8.6) is satisfied and that μ is a positive solution of (8.7). $\square$

We call an elliptic surface $g\colon L^2 \to \mathbb{S}^m_\epsilon$ endowed with a system of complex conjugate coordinates as in Proposition 8.2 a *special elliptic surface* if the Christoffel symbol Γ given by (8.3) satisfies the condition

$$\Gamma_z = \bar{\Gamma}_{\bar{z}}, \tag{8.12}$$

that is, if Γ_z is real-valued.

Proposition 8.4. *Let $g\colon L^2 \to \mathbb{S}^m_\epsilon$ be a simply connected special elliptic surface and, up to a constant factor, let $\mu \in C^\infty(L)$ be the unique real-valued positive solution of*

$$\mu_{\bar{z}} + 2\mu\Gamma = 0. \tag{8.13}$$

Then $\varphi \in C^\infty(L)$ is a solution of

$$\varphi_{z\bar{z}} - \Gamma\varphi_z - \bar{\Gamma}\varphi_{\bar{z}} + F\varphi = 0 \ \text{ where } \ F = \langle\partial_z, \partial_{\bar{z}}\rangle = (1/4)(\|\partial_u\|^2 + \|\partial_v\|^2) \tag{8.14}$$

if and only if $\psi = \sqrt{\mu}\varphi$ satisfies

$$\psi_{z\bar{z}} + M\psi = 0, \tag{8.15}$$

where

$$M = F - \frac{\mu_{z\bar{z}}}{2\mu} + \frac{\mu_z\mu_{\bar{z}}}{4\mu^2}\cdot. \tag{8.16}$$

In particular, the map $k = \sqrt{\mu}\, h\colon L^2 \to \mathbb{E}^{m+1}$ where $h = i \circ g$, satisfies

$$k_{z\bar{z}} + Mk = 0.$$

Conversely, for a system of coordinates (u, v) on an open subset $U \subset \mathbb{R}^2$ let $\{k_1, \ldots, k_{m+1}\}$ be a set of solutions of (8.15) where $M \in C^\infty(U)$. Assume that the map $k = (k_1, \ldots, k_{m+1})\colon U \to \mathbb{E}^{m+1}$ satisfies that $\mu = \|k\|^2 > 0$ and that the map $h = (1/\sqrt{\mu})\, k\colon U \to \mathbb{E}^{m+1}$ is a space-like immersion if $\epsilon = 1$. Then $g\colon U \to \mathbb{S}^m_\epsilon$ defined by $h = i \circ g$ is a special elliptic surface.

Proof. Notice that (8.12) is the integrability condition of (8.13). Since $\mu \in C^\infty(L)$ is a real-valued solution of (8.13) then $\Gamma = -(1/2\mu)\mu_{\bar{z}}$. Hence (8.14) becomes

$$\varphi_{z\bar{z}} + \frac{\mu_z}{2\mu}\varphi_{\bar{z}} + \frac{\mu_{\bar{z}}}{2\mu}\varphi_z + F\varphi = 0$$

which takes the form (8.15) for $k = \sqrt{\mu}\,\varphi$ and M given by (8.16).

We prove the converse. It is easily seen that $h = (1/\sqrt{\mu})\, k\colon L^2 \to \mathbb{E}^{m+1}$ satisfies

$$h_{z\bar{z}} + \frac{\mu_z}{2\mu}h_{\bar{z}} + \frac{\mu_{\bar{z}}}{2\mu}h_z + Fh = 0, \tag{8.17}$$

where $F = M + \frac{\mu_{z\bar{z}}}{2\mu} - \frac{\mu_z\mu_{\bar{z}}}{4\mu^2}$. If h is a space-like immersion and $g\colon L^2 \to \mathbb{S}^m_\epsilon$ is the surface defined by $h = i\circ g$, then (8.17) implies that (u,v) are complex conjugate coordinates for g and that the complex Christoffel symbol of the metric induced by g is $\Gamma = -(1/2\mu)\mu_{\bar{z}}$. The equality (8.12) is satisfied and μ is a positive solution of (8.13). □

Proposition 8.5. *For a simply connected surface $g\colon L^2 \to \mathbb{S}^m_\epsilon$ the following assertions are equivalent:*

(i) The surface g is special hyperbolic (respectively, special elliptic).

(ii) The surface g is hyperbolic (respectively, elliptic) with respect to a tensor $J \in \Gamma(End(TL))$ that satisfies $J^2 = I$ and $J \neq I$ (respectively, $J^2 = -I$) and there is a nowhere vanishing function $\mu \in C^\infty(L)$ such that $D = \mu J$ is a Codazzi tensor on L^2, that is,

$$(\nabla_X D)Y = (\nabla_Y D)X$$

for any $X, Y \in \mathfrak{X}(L)$.

Proof. Let g be a hyperbolic surface as in part (ii) and let (u,v) be local real conjugate coordinates on L^2 given by Proposition 8.2. Then the equation

$$(\nabla_{\partial_u} D)\,\partial_v - (\nabla_{\partial_v} D)\,\partial_u = 0 \tag{8.18}$$

is easily seen to be equivalent to (8.7).

Conversely, if g is special hyperbolic with real conjugate coordinates (u,v), $J \in \Gamma(\mathrm{End}(TL))$ is given by $J\partial_u = \partial_u$ and $J\partial_v = -\partial_v$, and $\mu \in C^\infty(L)$ satisfies (8.7), then $D = \mu J$ satisfies (8.18) in view of (8.7), and hence is a Codazzi tensor on L^2. The proof for the elliptic case is similar. □

Given a surface $g\colon L^2 \to \mathbb{S}^{m+1}_1$, fix a pseudo-orthonormal basis $e_1, \ldots, e_{m+2}$ of $\mathbb{L}^{m+2}$, which means that

$$\|e_1\| = 0 = \|e_{m+2}\|, \langle e_1, e_{m+2}\rangle = -1/2 \text{ and } \langle e_i, e_j\rangle = \delta_{ij} \text{ if } i \neq 1, m+2, \tag{8.19}$$

and set $g = (g_1, g_2, \ldots, g_{m+2})\colon L^2 \to \mathbb{S}^{m+1}_1 \subset \mathbb{L}^{m+2}$ in terms of this basis. Assume that $g_1 \neq 0$ everywhere, and let the map $h\colon L^2 \to \mathbb{R}^m$ and $r \in C^\infty(L)$ be given by

$$h = r(g_2, \ldots, g_{m+1}) \text{ and } r = 1/g_1. \tag{8.20}$$

Notice that g can be recovered from the pair (h, r) by taking

$$g = r^{-1}(1, h, \|h\|^2 - r^2). \tag{8.21}$$

Proposition 8.6. *We have g is a space-like immersion if and only if h is an immersion and the gradient $\nabla^h r$ of r in the metric induced by h satisfies $\|\nabla^h r\| < 1$.*

Proof. This is Exercise 8.1. □

We call the pair (h, r) formed by a surface $h\colon L^2 \to \mathbb{R}^m$ and a function $r \in C^\infty(L)$ a *special hyperbolic pair* (respectively, *special elliptic pair*) if there exists a special hyperbolic surface (respectively, special elliptic surface) $g\colon L^2 \to \mathbb{S}_1^{m+1}$ such that (h, r) are given by (8.20).

8.2 The classification

The classification of Euclidean hypersurfaces that admit nontrivial conformal infinitesimal bendings is parametric in nature and given in terms of the conformal Gauss parametrization. This parametrization is discussed next limited to the conditions in which it is used here; see [21] for additional details. Then the classification result is given by means of two statements.

Let $f\colon\ M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, be a given oriented hypersurface with Gauss map $N\colon M^n \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Assume that at each point of M^n there is a principal curvature $\lambda > 0$ of multiplicity $n-2$. It follows from Proposition 1.5 that the corresponding eigenspaces form an integrable distribution and that λ is constant along the spherical leaves. Moreover, the so-called *focal map*, namely, the map $f+rN\colon\ M^n \to \mathbb{R}^{n+1}$, $r = 1/\lambda$, induces an isometric immersion $h\colon\ L^2 \to \mathbb{R}^{n+1}$, where the surface L^2 with the induced metric is the quotient space of leaves and $r \in C^\infty(L^2)$ turns out to satisfy that $\|\nabla^h r\| < 1$.

The *conformal Gauss parametrization* goes as follows: The hypersurface f can be locally parametrized along the unit normal bundle $N_1 L$ of h by the map

$$X(\xi) = h - r(h_*\nabla^h r + \sqrt{1 - \|\nabla^h r\|^2}\,\xi).$$

Conversely, given a surface $h\colon\ L^2 \to \mathbb{R}^{n+1}$ and a positive function $r \in C^\infty(L^2)$ whose gradient satisfies $\|\nabla^h r\| < 1$, then on the open subset of regular points the parametrized hypersurface determined as above by the pair (h, r) has, with respect to the Gauss map

$$N = h_*\nabla^h r + \sqrt{1 - \|\nabla^h r\|^2}\,\xi,$$

the principal curvature $\lambda = 1/r$ of multiplicity $n - 2$.

A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is said to be *conformally surface-like* if it is conformally congruent to either a cylinder or a rotation hypersurface over a surface in $\mathbb{R}^3$ or a cylinder over a three-dimensional hypersurface of $\mathbb{R}^4$ that is a cone over a surface in the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. The hypersurface is called *conformally ruled* if M^n carries an integrable $(n-1)$-dimensional distribution such that the restriction of f to each leaf is an umbilical submanifold of $\mathbb{R}^{n+1}$.

The first classification result given next excludes the case of conformally ruled hypersurfaces considered in the sequel.

Theorem 8.7. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, admit a nontrivial conformal infinitesimal variation. Assume that f is neither conformally surface-like nor conformally flat nor conformally ruled on any open subset of M^n. Then, on each connected component of an open dense subset of M^n, the hypersurface can be parametrized in terms of the conformal Gauss parametrization by either a special hyperbolic or a special elliptic pair.*

Conversely, any hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, given in terms of the conformal Gauss parametrization by either a special hyperbolic or special elliptic pair admits a nontrivial conformal infinitesimal variation. Moreover, the conformal infinitesimal bendings associated to any pair of nontrivial conformal infinitesimal variations of f differ by a trivial conformal infinitesimal bending.

For the case of conformally ruled hypersurfaces, we have that the conformal variations being infinitesimal or not does not make a difference.

Theorem 8.8. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a conformally ruled hypersurface that is neither conformally surface-like nor conformally flat on any open subset of M^n. Then f admits on each connected component of an open dense subset of M^n a family of conformal infinitesimal bendings that are in one-to-one correspondence with the set of smooth functions on an interval. Moreover, any such bending is the variational vector field of a conformal variation.*

Let $\mathcal{T}$ be a conformal infinitesimal bending of $f\colon M^n \to \mathbb{R}^{n+1}$ with conformal factor ρ. At any point of M^n we have from Lemma 7.10 that the associated bilinear form $\theta\colon TM \times TM \to \mathbb{R}^4$ defined by

$$\theta(X,Y) = (\langle (A+\mathcal{B})X,Y\rangle, \langle (I+H)X,Y\rangle, \langle (A-\mathcal{B})X,Y\rangle, \langle (I-H)X,Y\rangle) \tag{8.22}$$

is flat with respect to the inner product $\langle\!\langle\,,\,\rangle\!\rangle$ of signature $(2,2)$.

Proposition 8.9. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an isometric immersion free of umbilical points. If $\mathcal{T}$ is a conformal infinitesimal bending of f such that the associated flat bilinear form θ given by* (8.22) *is null at any point of M^n then $\mathcal{T}$ is trivial.*

Proof. That θ is null is equivalent to

$$\langle AX,Y\rangle\mathcal{B}+\langle\mathcal{B}X,Y\rangle A+\langle X,Y\rangle H+\langle HX,Y\rangle I=0$$

for any $X,Y\in\mathfrak{X}(M)$. Fix $x\in M^n$. From our assumptions we have that $A(x)$ is not a multiple of the identity. It follows from the above that $A(x),\mathcal{B}(x)$ and $H(x)$ commute, that is, there exists an orthonormal basis $\{X_i\}_{1\le i\le n}$ of T_xM that diagonalizes simultaneously all of them. If λ_i, b_i and h_i are the respective eigenvalues of $A(x),\mathcal{B}(x)$ and $H(x)$ corresponding to X_i, $1\le i\le n$, then

$$\lambda_i\mathcal{B}+b_iA+H+h_iI=0.$$

If $\lambda_i\neq\lambda_j$, then

$$(\lambda_i-\lambda_j)\mathcal{B}+(b_i-b_j)A+(h_i-h_j)I=0.$$

Hence

$$(\lambda_i-\lambda_j)b_i+(b_i-b_j)\lambda_i+h_i-h_j=0=(\lambda_i-\lambda_j)b_j+(b_i-b_j)\lambda_j+h_i-h_j,$$

and thus

$$(\lambda_i-\lambda_j)(b_i-b_j)=0$$

which gives $b_i=b_j$. If $\lambda_i=\lambda_j$ for $i\neq j$, then

$$(b_i-b_j)A+(h_i-h_j)I=0.$$

But since f has no umbilical points, we necessarily have $b_i=b_j$ and hence $\mathcal{B}=bI$ at any $x\in M^n$. We conclude from Proposition 7.6 that $\mathcal{T}$ is trivial. $\square$

Lemma 8.10. *Let $f\colon M^n\to\mathbb{R}^{n+1}$, $n\geq 5$, be an isometric immersion free of umbilical points and let $\mathcal{T}$ be a nontrivial conformal infinitesimal bending of f. On the connected components of an open and dense subset A, $\mathcal{B}$ and H share a common eigenbundle Δ such that* $\dim\Delta\geq n-2$.

Proof. By Proposition 8.9 the bilinear form θ is not null. Then, by Theorem 1.11 there is an orthogonal decomposition $\mathbb{R}^4=\mathbb{R}^{2,2}=\mathbb{R}^{1,1}\oplus\mathbb{R}^{1,1}$ such that $\theta=\theta_1+\theta_2$ where θ_1 is nonzero but null and θ_2 is flat and $\dim\mathcal{N}(\theta_2)\geq n-2$.

We denote $\Delta=\mathcal{N}(\theta_2)$ and restrict ourselves to connected components of an open and dense subset where $\dim\Delta\geq n-2$ is constant. Since we have that

$$\theta(T,X)=\theta_1(T,X)$$

for any $T\in\Gamma(\Delta)$ and $X\in\mathfrak{X}(M)$, then

$$\langle\langle\theta(T,X),\theta(Y,Z)\rangle\rangle=0$$

for any $T \in \Gamma(\Delta)$ and $X, Y, Z \in \mathfrak{X}(M)$. Equivalently,

$$\langle AT, X\rangle \mathcal{B} + \langle \mathcal{B}T, X\rangle A + \langle T, X\rangle H + \langle HT, X\rangle I = 0 \tag{8.23}$$

for any $T \in \Gamma(\Delta)$ and $X \in \mathfrak{X}(M)$. Taking X orthogonal to T we see that

$$\langle AT, X\rangle \mathcal{B} + \langle \mathcal{B}T, X\rangle A + \langle HT, X\rangle I = 0. \tag{8.24}$$

Fix $x \in M^n$ and assume, by contradiction, that there exists $T \in \Delta(x)$ and $X \in T_xM$ such that $\langle X, T\rangle = 0$ and $\langle \mathcal{B}T, X\rangle \neq 0$. From (8.24) and since f is free of umbilic points, we obtain that A commutes with $\mathcal{B}$, and hence also does H. Let $\{X_i\}_{1\leq i\leq n}$ be an orthonormal basis of T_xM of common eigenvectors of A, $\mathcal{B}$ and H with corresponding eigenvalues λ_i, b_i and h_i. Since $\langle \mathcal{B}T, X\rangle \neq 0$ with $\langle X, T\rangle = 0$, then T is not an eigenvector. Hence, there are two eigenvalues $b_1 \neq b_2$ such that $\langle T, X_1\rangle \neq 0 \neq \langle T, X_2\rangle$. Thus, we have from (8.23) that

$$\lambda_1 \mathcal{B} + b_1 A + H + h_1 I = 0 \;\text{ and }\; \lambda_2 \mathcal{B} + b_2 A + H + h_2 I = 0.$$

Hence

$$(\lambda_1 - \lambda_2)\mathcal{B} + (b_1 - b_2)A + (h_1 - h_2)I = 0, \tag{8.25}$$

from where we obtain that

$$(\lambda_1 - \lambda_2)b_j + (b_1 - b_2)\lambda_j + h_1 - h_2 = 0, \;\; 1 \leq j \leq n.$$

Taking the difference between the cases $j = 1$ and $j = 2$ we have

$$(\lambda_1 - \lambda_2)(b_1 - b_2) = 0,$$

and hence $\lambda_1 = \lambda_2$. It follows from (8.25) that A is a multiple of the identity which is a contradiction.

Therefore $\langle \mathcal{B}T, X\rangle = 0$ for any $T \in \Delta(x)$ and $X \in T_xM$ with $\langle X, T\rangle = 0$. This implies that Δ is an eigenspace of $\mathcal{B}$. If $\langle AT, X\rangle \neq 0$, for some $T \in \Delta(x)$ and $X \in T_xM$ with $\langle T, X\rangle = 0$, then we obtain from (8.24) that B is a multiple of the identity and this is contradiction. Hence Δ is also an eigenspace of A, and consequently of H. $\square$

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface that carries a principal curvature of constant multiplicity $n-2$ with corresponding eigenbundle Δ. In what follows, we write $\nabla^h_T X = (\nabla_T X)_{\Delta^\perp}$ for $T \in \Gamma(\Delta)$ and $X \in \Gamma(\Delta^\perp)$. If f is not conformally surface-like on any open subset of M^n, we say that f is *hyperbolic* (respectively, *parabolic* or *elliptic*) if there exists $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ satisfying the following conditions:

(i) $J^2 = I$ and $J \neq I$ (respectively, $J^2 = 0$ with $J \neq 0$, or $J^2 = -I$),

(ii) $\nabla^h_T J = 0$ for all $T \in \Gamma(\Delta)$,

(iii) $C_T \in \operatorname{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$.

Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be an oriented hypersurface with a principal curvature λ of constant multiplicity $n-2$. By composing with an appropriate inversion, if necessary, and given that f is orientable, we can always assume that $\lambda > 0$ at any point of M^n. Recall that an *inversion* $\varphi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with respect to the hypersphere of radius $r > 0$ centered at $x_0 \in \mathbb{R}^{n+1}$ is the conformal map given by

$$\varphi(x) = x_0 + r^2 \frac{x - x_0}{\|x - x_0\|^2}.$$

Let A be the second fundamental form associated to the Gauss map N of f and let $\Delta(x) \subset T_xM$ be the eigenspace corresponding to $\lambda(x)$ at $x \in M^n$. Fix an embedding Ψ as in (7.20) and let $S\colon M^n \to \mathbb{L}^{n+3}$ be the map given by

$$S(x) = \lambda(x)\Psi(f(x)) + \Psi_* N(x). \tag{8.26}$$

Then $S(x) \in \mathbb{S}_1^{n+2} \subset \mathbb{L}^{n+3}$ and

$$S_* X = X(\lambda)\Psi(f(x)) - \Psi_* f_*(A - \lambda I)X \tag{8.27}$$

for any $X \in \mathfrak{X}(M)$. From (8.27) it follows that S is constant along the leaves of Δ. Let L^2 be the quotient space of leaves of Δ and let $\pi\colon M^n \to L^2$ be the canonical projection. Thus S induces an immersion $s\colon L^2 \to \mathbb{S}_1^{n+2} \subset \mathbb{L}^{n+3}$ such that $S = s \circ \pi$. Moreover, the metric $\langle\,,\,\rangle'$ on L^2 induced by s satisfies

$$\langle \bar{X}, \bar{Y} \rangle' = \langle (A - \lambda I)X, (A - \lambda I)Y \rangle, \tag{8.28}$$

where $X, Y \in \mathfrak{X}(M)$ are the horizontal lifts of $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$.

A tensor $D \in \Gamma(\operatorname{End}(\Delta^\perp))$ is said to be *projectable* with respect to $\pi\colon M^n \to L^2$ if it is the horizontal lift of a tensor $\bar{D}$ on L^2, that is, if

$$\pi_* D X = \bar{D}\pi_* X = \bar{D}\bar{X} \circ \pi \quad \text{if} \quad \pi_* X = \bar{X} \circ \pi.$$

The following result is Corollary 11.7 of [21].

Lemma 8.11. *A tensor $D \in \Gamma(End(\Delta^\perp))$ is projectable if and only if*

$$\nabla^h_T D = [D, C_T]$$

for all $T \in \Gamma(\Delta)$.

Proposition 8.12. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be an oriented hypersurface and let $\mathcal{T}$ be a nontrivial conformal infinitesimal bending of f. Assume that the principal curvature $\lambda > 0$ of A determined by Δ in Lemma 8.10 has constant multiplicity $n - 2$. Then, on each connected component* *of an open dense subset of M^n either f is conformally surface-like or f is hyperbolic, parabolic or elliptic with respect to $J \in \Gamma(End(\Delta^\perp))$ and there exists $\mu \in \mathbb{C}^\infty(M)$ nowhere vanishing and constant along the leaves of Δ such that $D = \mu J \in \Gamma(End(\Delta^\perp))$ satisfies:*

(i) $(A - \lambda I)D$ *is symmetric,*

(ii) $\nabla^h_T D = 0,$

(iii) $(\nabla_X (A - \lambda I)D)Y - (\nabla_Y (A - \lambda I)D)X = X \wedge Y(D^t \nabla\lambda),$

(iv) $$\langle(\nabla_Y D)X - (\nabla_X D)Y, \nabla\lambda\rangle + \operatorname{Hess}\lambda(DX, Y) - \operatorname{Hess}\lambda(X, DY) \\ = \lambda(\langle AX, (A - \lambda I)DY\rangle - \langle(A - \lambda I)DX, AY\rangle),$$

(v) $(A - \lambda I)DX \wedge (A - \lambda I)Y - (A - \lambda I)DY \wedge (A - \lambda I)X = 0$

for any $T \in \Gamma(\Delta)$ and $X, Y \in \Gamma(\Delta^\perp)$.

Conversely, assume that f as above is either hyperbolic, parabolic or elliptic with respect to $J \in \Gamma(End(\Delta^\perp))$ and that there is $0 \neq D = \mu J \in \Gamma(End(\Delta^\perp))$ that satisfies conditions (i) through (v). If M^n is simply connected there exists a nontrivial conformal infinitesimal bending $\mathcal{T}$ of f determined by D that is unique up to trivial conformal infinitesimal bendings.

Proof. We have from Lemma 8.10 that Δ is a common eigenbundle for A, $\mathcal{B}$ and H. Thus $\mathcal{B}|_\Delta = bI$ and $H|_\Delta = hI$ where $b, h \in C^\infty(M)$. We obtain from (8.23) that
$$bA + \lambda\mathcal{B} + H + hI = 0.$$
In particular $\lambda b + h = 0$, and thus locally
$$bA + \lambda(\mathcal{B} - bI) + H = 0. \tag{8.29}$$

From (7.19) we have
$$T(b - \lambda\rho) = T(b) - \lambda T(\rho) = 0 \tag{8.30}$$
for any $T \in \Gamma(\Delta)$. Notice that (7.19) is equivalent to
$$(\nabla_X(\mathcal{B} - bI))Y - (\nabla_Y(\mathcal{B} - bI))X + (X \wedge Y)(A\nabla\rho - \nabla b) = 0. \tag{8.31}$$
Then it follows from (8.30) and (8.31) that
$$(\nabla^h_T(\mathcal{B} - bI))X = (\mathcal{B} - bI)C_T X \tag{8.32}$$
for any $X \in \Gamma(\Delta^\perp)$ and $T \in \Gamma(\Delta)$.

We regard $A - \lambda I$ and $\mathcal{B} - bI$ as tensors on $\Delta^\perp$. We obtain from (8.32) and the Codazzi equation $\nabla^h_T A = (A - \lambda I)C_T$ that
$$(\mathcal{B} - bI)C_T = C^t_T(\mathcal{B} - bI) \text{ and } (A - \lambda I)C_T = C^t_T(A - \lambda I).$$

We have that $D \in \Gamma(\mathrm{End}(\Delta^\perp))$ defined by

$$D = (A - \lambda I)^{-1}(\mathcal{B} - bI)$$

satisfies $D \neq 0$ since $\mathcal{T}$ is nontrivial. Hence

$$\begin{aligned}(A - \lambda I)DC_T = (\mathcal{B} - bI)C_T &= C_T^t(\mathcal{B} - bI) = C_T^t(A - \lambda I)D\\ &= (A - \lambda I)C_T D,\end{aligned}$$

and therefore

$$[D, C_T] = 0. \tag{8.33}$$

We also have

$$(A - \lambda I)C_T D = (\nabla_T^h A)D$$

and

$$\begin{aligned}(A - \lambda I)DC_T = (\mathcal{B} - bI)C_T &= \nabla_T^h(\mathcal{B} - bI) = \nabla_T^h((A - \lambda I)D)\\ &= \nabla_T^h(AD) - \lambda\nabla_T^h D.\end{aligned}$$

Thus

$$(A - \lambda I)\nabla_T^h D = (A - \lambda I)[D, C_T],$$

and hence

$$\nabla_T^h D = 0 \tag{8.34}$$

for any $T \in \Gamma(\Delta)$.

It follows from (8.33), (8.34) and Lemma 8.11 that D is projectable with respect to $\pi\colon M^n \to L^2$, that is, the horizontal lift of a tensor $\bar{D}$ on L^2. Since $H = \lambda(bI - \mathcal{B}) - bA$ from (8.29), we have from (7.18) that

$$(\mathcal{B} - bI)X \wedge (A - \lambda I)Y - (\mathcal{B} - bI)Y \wedge (A - \lambda I)X = 0 \tag{8.35}$$

for any $X, Y \in \mathfrak{X}(M)$. From (8.35) and the definition of D we obtain that

$$\langle(((A-\lambda I)DX\wedge(A-\lambda I)Y-(A-\lambda I)DY\wedge(A-\lambda I)X)(A-\lambda I)Z, (A-\lambda I)W\rangle = 0$$

for any $X, Y, Z, W \in \Gamma(\Delta^\perp)$. This implies that

$$\langle(\bar{D}\bar{X} \wedge \bar{Y} - \bar{D}\bar{Y} \wedge \bar{X})\bar{Z}, \bar{W}\rangle' = 0$$

for any $\bar{X}, \bar{Y}, \bar{Z}, \bar{W} \in \mathfrak{X}(L)$. In other words, we have

$$\bar{D}\bar{X} \wedge \bar{Y} - \bar{D}\bar{Y} \wedge \bar{X} = 0$$

with respect to the metric $\langle\,,\,\rangle'$. Thus $\mathrm{tr}\bar{D} = 0$.

We have that $\bar{D}$ has either two smooth distinct real eigenvalues, a single real eigenvalue of multiplicity two or a pair of smooth complex conjugate eigenvalues. Thus there is $\bar{\mu} \in C^\infty(L)$ such that $\bar{D} = \bar{\mu}\bar{J}$, $\bar{J} \neq I$, where the tensor $\bar{J} \in \Gamma(\mathrm{End}(TL))$ satisfies $\bar{J}^2 = \epsilon I$, for $\epsilon = 1, 0$ or -1. Hence $D = \mu J$ where J is the lifting of $\bar{J}$ and $\bar{\mu} = \mu \circ \pi$. In particular $\mathrm{tr} D = 0$.

If $\mathrm{span}\{C_T : T \in \Delta\} \subset \mathrm{span}\{I\}$ we have from Corollary 9.33 in [21] that f is conformally surface-like. Hence, we assume $\mathrm{span}\{C_T : T \in \Delta\} \not\subset \mathrm{span}\{I\}$ and obtain from (8.33) that $C_T \in \mathrm{span}\{I, J\}$ for any $T \in \Gamma(\Delta)$.

Since J is projectable, being the lifting of $\bar{J}$, we have from Lemma 8.11 that

$$\nabla^h_T J = [J, C_T]$$

for all $T \in \Delta$. Then (8.33) gives that $\nabla^h_T J = 0$ and hence the hypersurface f is either hyperbolic, parabolic or elliptic.

We have from (8.29) that

$$X(b)AY + X(\lambda)\mathcal{B}Y - X(\lambda b)Y + b(\nabla_X A)Y + \lambda(\nabla_X \mathcal{B})Y + (\nabla_X H)Y = 0. \tag{8.36}$$

On the other hand, the Gauss equation yields

$$(\nabla_X H)Y - (\nabla_Y H)X = R(X,Y)\nabla\rho = \langle AY, \nabla\rho\rangle AX - \langle AX, \nabla\rho\rangle AY. \tag{8.37}$$

Then (7.19), (8.36), (8.37) and the Codazzi equation imply that

$$\begin{aligned} X(b)AY + X(\lambda)\mathcal{B}Y - X(\lambda b)Y - Y(b)AX - Y(\lambda)\mathcal{B}X + Y(\lambda b)X \\ - \lambda(X \wedge Y)A\nabla\rho + \langle AY, \nabla\rho\rangle AX - \langle AX, \nabla\rho\rangle AY = 0 \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Then

$$\begin{aligned} \langle X, \nabla b - A\nabla\rho\rangle(A - \lambda I)Y - \langle Y, \nabla b - A\nabla\rho\rangle(A - \lambda I)X \\ + \langle X, \nabla\lambda\rangle(\mathcal{B} - bI)Y - \langle Y, \nabla\lambda\rangle(\mathcal{B} - bI)X = 0 \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. For $X, Y \in \Gamma(\Delta^\perp)$ we have

$$\langle X, \nabla b - A\nabla\rho\rangle Y - \langle Y, \nabla b - A\nabla\rho\rangle X + \langle X, \nabla\lambda\rangle DY - \langle Y, \nabla\lambda\rangle DX = 0.$$

Taking X and Y orthonormal, we obtain

$$\langle Y, \nabla b - A\nabla\rho\rangle - \langle X, \nabla\lambda\rangle\langle DY, X\rangle + \langle Y, \nabla\lambda\rangle\langle DX, X\rangle = 0$$

and

$$\langle X, \nabla b - A\nabla\rho\rangle + \langle X, \nabla\lambda\rangle\langle DY, Y\rangle - \langle Y, \nabla\lambda\rangle\langle DX, Y\rangle = 0.$$

Using that $\mathrm{tr} D = 0$ this gives

$$D^t \nabla\lambda = \nabla b - A\nabla\rho, \tag{8.38}$$

where D^t denotes the transpose of D.

So far we have that (i) holds from the definition of D, (ii) is (8.34), (iii) follows from (8.31) and (8.38), and (v) is (8.35). Thus, it remains to prove that (iv) holds. To do this, fix a pseudo-orthonormal basis $e_1 \dots, e_{n+3}$ of $\mathbb{L}^{n+3}$ as in (8.19), and set $v = e_1$ and $w = -2e_{n+3}$. Let $\Psi\colon \mathbb{R}^{n+1} \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $S\colon M^n \to \mathbb{L}^{n+3}$ be given by (7.20) and (8.26) respectively. We see next that the immersion $s\colon L^2 \to \mathbb{S}_1^{n+2} \subset \mathbb{L}^{n+3}$ induced by S satisfies $s = g$, where g is given by (8.21), $h\colon L^2 \to \mathbb{R}^{n+1}$ is induced by $f + rN$ and $r = \lambda^{-1}$. In fact, we have that $\Psi(y) = (1, y, \|y\|^2)$. Then

$$
\begin{aligned}
S(x) &= \lambda(1, f(x), \|f(x)\|^2) + (0, N(x), 2\langle f(x), N(x)\rangle) \\
&= \lambda(1, f(x) + rN, \|f(x)\|^2 + 2r\langle f(x), N(x)\rangle).
\end{aligned}
$$

Since $h \circ \pi = f + rN$, it follows that

$$
s = r^{-1}(1, h, \|h\|^2 - r^2) = g.
$$

Let $X, Y \in \Gamma(\Delta^\perp)$ be the horizontal lifts of $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$. We have

$$
\begin{aligned}
\tilde{\nabla}'_X S_* DY &= \tilde{\nabla}'_{\pi_* X} g_* \pi_* DY = \tilde{\nabla}'_{\bar{X}} g_* \bar{D}\bar{Y} \\
&= g_* \nabla'_{\bar{X}} \bar{D}\bar{Y} + \alpha^g(\bar{X}, \bar{D}\bar{Y}) - \langle \bar{X}, \bar{D}\bar{Y}\rangle' g \circ \pi,
\end{aligned}
$$

where $\tilde{\nabla}'$ and ∇' are the connections in $\mathbb{L}^{n+3}$ and L^2, respectively. From (8.27) we obtain that

$$
\begin{aligned}
&\tilde{\nabla}'_X \Psi_* f_*(A - \lambda I)DY = X\langle DY, \nabla\lambda\rangle \Psi \circ f + \langle DY, \nabla\lambda\rangle \Psi_* f_* X \\
&- g_* \nabla'_{\bar{X}} \bar{D}\bar{Y} - \alpha^g(\bar{X}, \bar{D}\bar{Y}) + \langle (A - \lambda I)X, (A - \lambda I)DY\rangle(\lambda\Psi \circ f + \Psi_* N).
\end{aligned}
$$

On the other hand, using (7.21) and (8.27) it follows that

$$
\begin{aligned}
&\tilde{\nabla}'_X \Psi_* f_*(A - \lambda I)DY \\
&\quad= \Psi_* \bar{\nabla}_X f_*(A - \lambda I)DY + \alpha^\Psi(f_* X, f_*(A - \lambda I)DY) \\
&\quad= \Psi_* f_* \nabla_X (A - \lambda I)DY + \langle AX, (A - \lambda I)DY\rangle \Psi_* N - \langle X, (A - \lambda I)DY\rangle w \\
&\quad= \Psi_* f_*(\nabla_X (A - \lambda I)D)Y + \Psi_* f_*(A - \lambda I)D\nabla_X Y \\
&\qquad + \langle AX, (A - \lambda I)DY\rangle \Psi_* N - \langle X, (A - \lambda I)DY\rangle w \\
&\quad= \Psi_* f_*(\nabla_X (A - \lambda I)D)Y + \langle D\nabla_X Y, \nabla\lambda\rangle \Psi \circ f - g_* \bar{D}\pi_* \nabla_X Y \\
&\qquad + \langle AX, (A - \lambda I)DY\rangle \Psi_* N - \langle X, (A - \lambda I)DY\rangle w.
\end{aligned}
$$

We obtain from the last two equations that

$$
\begin{aligned}
&\langle DY, \nabla\lambda\rangle \psi_* f_* X - g_*(\nabla'_{\bar{X}} \bar{D})\bar{Y} - g_* \bar{D}\nabla'_{\bar{X}} \bar{Y} \\
&- \alpha^g(\bar{X}, \bar{D}\bar{Y}) - \lambda\langle X, (A - \lambda I)DY\rangle \Psi_* N \\
&+ (\langle (\nabla_X D)Y, \nabla\lambda\rangle + \langle DY, \nabla_X \nabla\lambda\rangle + \lambda\langle (A - \lambda I)X, (A - \lambda I)DY\rangle)\Psi \circ f \\
&\qquad\qquad = \Psi_* f_*(\nabla_X (A - \lambda I)D)Y - g_* \bar{D}\pi_* \nabla_X Y - \langle X, (A - \lambda I)DY\rangle w.
\end{aligned}
$$

Hence, we have from $\pi_*[X,Y] = [\bar{X},\bar{Y}]$ that

$$\begin{aligned} g_*((\nabla'_{\bar{Y}}\bar{D})\bar{X} - (\nabla'_{\bar{X}}\bar{D})\bar{Y}) + \alpha^g(\bar{Y},\bar{D}\bar{X}) - \alpha^g(\bar{X},\bar{D}\bar{Y}) \\ = \Psi_* f_*\Omega(X,Y) - \lambda\psi(X,Y)\Psi_* N + \varphi(X,Y)\Psi\circ f + \psi(X,Y)w, \end{aligned}$$

where

$$\Omega(X,Y) = (\nabla_X(A-\lambda I)D)Y - (\nabla_Y(A-\lambda I)D)X - X\wedge Y(D^t\nabla\lambda), \tag{8.39}$$

$$\psi(X,Y) = \langle Y,(A-\lambda I)DX\rangle - \langle X,(A-\lambda I)DY\rangle, \tag{8.40}$$

$$\begin{aligned} \varphi(X,Y) = \langle(\nabla_Y D)X - (\nabla_X D)Y,\nabla\lambda\rangle + \mathit{Hess}\,\lambda(DX,Y) - \mathit{Hess}\,\lambda(X,DY) \\ - \lambda(\langle(A-\lambda I)X,(A-\lambda I)DY\rangle - \langle(A-\lambda I)DX,(A-\lambda I)Y\rangle). \end{aligned} \tag{8.41}$$

It follows from (8.31) and (8.38) that Ω vanishes. The symmetry of $\mathcal{B}$ yields $\psi = 0$. Hence

$$g_*((\nabla'_{\bar{Y}}\bar{D})\bar{X} - (\nabla'_{\bar{X}}\bar{D})\bar{Y}) + \alpha^g(\bar{Y},\bar{D}\bar{X}) - \alpha^g(\bar{X},\bar{D}\bar{Y}) = \varphi(X,Y)\Psi\circ f.$$

Since the term on the left-hand side is constant along the leaves of Δ then φ has to vanish, and this proves (iv).

We prove the converse. Let $D = \mu J \in \Gamma(\mathrm{End}(\Delta^\perp))$ verify the conditions (i) through (v). In the sequel, we extend D to an element of $\mathrm{End}(TM)$ defining $DT = 0$ for any $T\in\Gamma(\Delta)$. Then (v) holds for any $X,Y\in\mathfrak{X}(M)$.

Set $F = \Psi\circ f\colon M^n \to \mathbb{V}^{n+2}\subset\mathbb{L}^{n+3}$. Let $\beta\colon TM\times TM\to N_FM$ be the symmetric tensor defined by

$$\beta(X,Y) = \langle(A-\lambda I)DX,Y\rangle(\Psi_*N + \lambda F), \tag{8.42}$$

where N is a Gauss map of f. Then let $B_\eta\in\Gamma(\mathrm{End}(TM))$ be given by

$$\langle B_\eta X,Y\rangle = \langle\beta(X,Y),\eta\rangle$$

for any $\eta\in\Gamma(N_FM)$. For simplicity, we write $N = \Psi_*N$. We have $B_N = (A-\lambda I)D$ and $B_w = \lambda B_N$. Since

$$\alpha^F(X,Y) = \langle AX,Y\rangle N - \langle X,Y\rangle w, \tag{8.43}$$

we obtain from (v) and $A|_\Delta = \lambda I$ that

$$A^F_{\beta(Y,Z)}X + B_{\alpha^F(Y,Z)}X - A^F_{\beta(X,Z)}Y - B_{\alpha^F(X,Z)}Y = 0 \tag{8.44}$$

for any $X,Y,Z\in\mathfrak{X}(M)$, where A^F_η is the shape operator with respect to $\eta\in\Gamma(N_FM)$.

We define $\mathcal{E}\colon TM \times N_FM \to N_FM$ by

$$\mathcal{E}(X,N) = \langle DX, \nabla\lambda\rangle F, \;\; \mathcal{E}(X,w) = -\langle DX, \nabla\lambda\rangle N \;\text{ and }\; \mathcal{E}(X,F) = 0 \tag{8.45}$$

for any $X \in \mathfrak{X}(M)$. Observe that $\mathcal{E}$ satisfies the condition

$$\langle \mathcal{E}(X,\eta), \xi\rangle = -\langle \mathcal{E}(X,\xi), \eta\rangle \tag{8.46}$$

for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_FM)$.

It follows from (iii) that

$$(\nabla_X B_N)Y - (\nabla_Y B_N)X = \langle DY, \nabla\lambda\rangle X - \langle DX, \nabla\lambda\rangle Y \tag{8.47}$$

for any $X, Y \in \Gamma(\Delta^\perp)$. Using (ii) and that $[D, C_T] = 0$, we obtain

$$\begin{aligned}(\nabla_X B_N)T - (\nabla_T B_N)X &= B_N C_T X - (\nabla_T(A-\lambda I))DX - (A-\lambda I)(\nabla_T D)X\\ &= (A-\lambda I)C_T DX - (\nabla_T(A-\lambda I))DX\end{aligned}$$

for any $T \in \Gamma(\Delta)$. Now using the Codazzi equation, we have

$$\begin{aligned}&(\nabla_X B_N)T - (\nabla_T B_N)X\\ &\quad= (A-\lambda I)C_T DX - (\nabla_{DX} A)T\\ &\quad= (A-\lambda I)C_T DX - \langle DX, \nabla\lambda\rangle T - (A-\lambda I)C_T DX\\ &\quad= -\langle DX, \nabla\lambda\rangle T.\end{aligned} \tag{8.48}$$

Since Δ is integrable, we obtain

$$(\nabla_T B_N)S - (\nabla_S B_N)T = 0 \tag{8.49}$$

for any $T, S \in \Gamma(\Delta)$. It follows from (8.47), (8.48) and (8.49) that

$$(\nabla_X B_N)Y - (\nabla_Y B_N)X = A^F_{\mathcal{E}(X,N)}Y - A^F_{\mathcal{E}(Y,N)}X \tag{8.50}$$

for any $X, Y \in \mathfrak{X}(M)$.

We have from (8.47) that

$$\begin{aligned}(\nabla_X B_w)Y - (\nabla_Y B_w)X = &\langle X, \nabla\lambda\rangle B_N Y - \langle Y, \nabla\lambda\rangle B_N X\\ &+ \lambda\langle DY, \nabla\lambda\rangle X - \lambda\langle DX, \nabla\lambda\rangle Y\end{aligned}$$

for any $X, Y \in \Gamma(\Delta^\perp)$. Let $\sigma \in \Gamma(\Delta^\perp)$ be given by $\nabla\lambda = (A-\lambda I)\sigma$. Using (v) we obtain

$$\begin{aligned}(\nabla_X B_w)Y - (\nabla_Y B_w)X &= \langle B_N Y, \sigma\rangle(A-\lambda I)X - \langle B_N X, \sigma\rangle(A-\lambda I)Y\\ &\quad+ \lambda\langle DY, \nabla\lambda\rangle X - \lambda\langle DX, \nabla\lambda\rangle Y\\ &= \langle DY, \nabla\lambda\rangle(A-\lambda I)X - \langle DX, \nabla\lambda\rangle Y + \lambda\langle DY, \nabla\lambda\rangle X - \lambda\langle DX, \nabla\lambda\rangle Y\\ &= \langle DY, \nabla\lambda\rangle AX - \langle DX, \nabla\lambda\rangle AY.\end{aligned} \tag{8.51}$$

Using (8.48) it follows that

$$\begin{aligned}(\nabla_X B_w)T-(\nabla_T B_w)X &= \lambda((\nabla_X B_N)T-(\nabla_T B_N)X)\\ &= -\langle DX,\nabla\lambda\rangle AT\end{aligned}\tag{8.52}$$

for any $T\in\Gamma(\Delta)$. As before, we have that

$$(\nabla_T B_w)S-(\nabla_S B_w)T=0\tag{8.53}$$

for any $S,T\in\Gamma(\Delta)$. We conclude from (8.51), (8.52) and (8.53) that

$$(\nabla_X B_w)Y-(\nabla_Y B_w)X=A^F_{\mathcal{E}(X,w)}Y-A^F_{\mathcal{E}(Y,w)}X\tag{8.54}$$

for any $X,Y\in\mathfrak{X}(M)$.

We have that $B_F=0=\mathcal{E}(X,F)$, and hence it holds trivially that

$$(\nabla_X B_F)Y-(\nabla_Y B_F)X=A^F_{\mathcal{E}(X,F)}Y-A^F_{\mathcal{E}(Y,F)}X\tag{8.55}$$

for any $X,Y\in\mathfrak{X}(M)$.

Next we focus on the covariant derivative of $\mathcal{E}$. Let $\nabla'^{\perp}$ denote the normal connection on N_FM. We have

$$\begin{aligned}(\nabla'^{\perp}_X\mathcal{E})(Y,N) &= \nabla'^{\perp}_X\mathcal{E}(Y,N)-\mathcal{E}(\nabla_XY,N)\\ &= X\langle DY,\nabla\lambda\rangle F-\langle D\nabla_XY,\nabla\lambda\rangle F\\ &= (\langle(\nabla_XD)Y,\nabla\lambda\rangle+Hess\,\lambda(DY,X))F\end{aligned}$$

for any $X,Y\in\mathfrak{X}(M)$. Then

$$\begin{aligned}(\nabla'^{\perp}_X\mathcal{E})(Y,N)-(\nabla'^{\perp}_Y\mathcal{E})(X,N) &= (\langle(\nabla_XD)Y-(\nabla_YD)X,\nabla\lambda\rangle\\ &\quad+Hess\,\lambda(DY,X)-Hess\,\lambda(DX,Y))F\end{aligned}$$

for all $X,Y\in\mathfrak{X}(M)$. From (iv) we have

$$(\nabla'^{\perp}_X\mathcal{E})(Y,N)-(\nabla'^{\perp}_Y\mathcal{E})(X,N)=\lambda(\langle B_NX,AY\rangle-\langle AX,B_NY\rangle)F\tag{8.56}$$

for all $X,Y\in\Gamma(\Delta^{\perp})$. Using (ii) and that $[D,C_T]=0$, we obtain

$$\begin{aligned}&(\nabla'^{\perp}_X\mathcal{E})(T,N)-(\nabla'^{\perp}_T\mathcal{E})(X,N)\\ &\qquad= \mathcal{E}([T,X],N)-\nabla'^{\perp}_T\mathcal{E}(X,N)\\ &\qquad= (\langle DC_TX-(\nabla_TD)X,\nabla\lambda\rangle-Hess\,\lambda(DX,T))F\\ &\qquad= (\langle C_TDX,\nabla\lambda\rangle-Hess\,\lambda(DX,T))F\\ &\qquad= (\langle T,\nabla_{DX}\nabla\lambda\rangle-Hess\,\lambda(DX,T))F\\ &\qquad= 0\end{aligned}\tag{8.57}$$

for any $X\in\Gamma(\Delta^{\perp})$ and $T\in\Gamma(\Delta)$. We also have

$$(\nabla'^{\perp}_T\mathcal{E})(S,N)-(\nabla'^{\perp}_S\mathcal{E})(T,N)=0\tag{8.58}$$

for any $S, T \in \Gamma(\Delta)$. On the other hand, from (8.42) and (8.43) we obtain

$$\begin{aligned}\beta(X, AY) - \beta(AX, Y) + \alpha^F(X, B_N Y) - \alpha^F(B_N X, Y)&\\ = \lambda(\langle B_N X, AY\rangle - \langle AX, B_N Y\rangle)F&\end{aligned} \tag{8.59}$$

for all $X, Y \in \mathfrak{X}(M)$. From (8.56) through (8.59) we conclude that

$$\begin{aligned}&(\nabla'^{\perp}_X \mathcal{E})(Y, N) - (\nabla'^{\perp}_Y \mathcal{E})(X, N)\\ &\quad= \beta(X, AY) - \beta(AX, Y) + \alpha^F(X, B_N Y) - \alpha^F(B_N X, Y)\end{aligned} \tag{8.60}$$

for any $X, Y \in \mathfrak{X}(M)$. Similarly as above, we obtain

$$\begin{aligned}(\nabla'^{\perp}_X \mathcal{E})(Y, w) - (\nabla'^{\perp}_Y \mathcal{E})(X, w) &= \langle(\nabla_Y D)X - (\nabla_X D)Y, \nabla\lambda\rangle N\\ &\quad+ (\mathit{Hess}\,\lambda(DX, Y) - \mathit{Hess}\,\lambda(DY, X))N\end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$. From (iv) it follows that

$$(\nabla'^{\perp}_X \mathcal{E})(Y, w) - (\nabla'^{\perp}_Y \mathcal{E})(X, w) = \lambda(\langle AX, (A - \lambda I)DY\rangle - \langle(A - \lambda I)DX, AY\rangle)N$$

for $X, Y \in \Gamma(\Delta^{\perp})$. As before, we have from (ii) and $[D, C_T] = 0$ that

$$\begin{aligned}(\nabla'^{\perp}_X \mathcal{E})(T, w) - (\nabla'^{\perp}_T \mathcal{E})(X, w) &= (-\langle C_T DX, \nabla\lambda\rangle + \mathit{Hess}\,\lambda(DX, T))N\\ &= (-\langle T, \nabla_{DX}\lambda\rangle + \mathit{Hess}\,\lambda(DX, T))N\\ &= 0\end{aligned}$$

and

$$(\nabla'^{\perp}_T \mathcal{E})(S, w) - (\nabla'^{\perp}_S \mathcal{E})(T, w) = 0$$

for any $T, S \in \Gamma(\Delta)$. In addition, we have

$$\begin{aligned}\beta(X, A^F_w Y) - \beta(A^F_w X, Y) + \alpha^F(X, B_w Y) - \alpha^F(B_w X, Y)&\\ = \lambda(\langle AX, B_N Y\rangle - \langle B_N X, AY\rangle)N&\end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$. Thus

$$\begin{aligned}&(\nabla'^{\perp}_X \mathcal{E})(Y, w) - (\nabla'^{\perp}_Y \mathcal{E})(X, w)\\ &= \beta(X, A^F_w Y) - \beta(A^F_w X, Y) + \alpha^F(X, B_w Y) - \alpha^F(B_w X, Y)\end{aligned} \tag{8.61}$$

for all $X, Y \in \mathfrak{X}(M)$. Finally, we have

$$\beta(X, A^F_F Y) - \beta(A^F_F X, Y) + \alpha^F(X, B_F Y) - \alpha^F(B_F X, Y) = 0$$

for all $X, Y \in \mathfrak{X}(M)$. Since $\mathcal{E}(X, F) = 0$, then

$$\begin{aligned}&(\nabla'^{\perp}_X \mathcal{E})(Y, F) - (\nabla'^{\perp}_Y \mathcal{E})(X, F)\\ &= \beta(X, A^F_F Y) - \beta(A^F_F X, Y) + \alpha^F(X, B_F Y) - \alpha^F(B_F X, Y)\end{aligned} \tag{8.62}$$

for all $X, Y \in \mathfrak{X}(M)$ holds trivially.

We have that β is symmetric and that the tensor $\mathcal{E}$ satisfies condition (8.46). Moreover, the pair $(\mathcal{E}, \beta)$ also satisfies (8.44), (8.50), (8.54), (8.55), (8.60), (8.61) and (8.62). In this situation Theorem (2.8) applies. We conclude that there is an infinitesimal bending $\tilde{\mathcal{T}} \in \Gamma(F^*(T\mathbb{L}^{n+3}))$ of F whose associated pair $(\tilde{\beta}, \tilde{\mathcal{E}})$ satisfies

$$\tilde{\beta} = \beta + C\alpha^F \quad \text{and} \quad \tilde{\mathcal{E}} = \mathcal{E} - \nabla^\perp C, \tag{8.63}$$

where $C \in \Gamma(\mathrm{End}(N_F M))$ is skew-symmetric. Moreover, we have that $\tilde{\mathcal{T}}$ is unique up to trivial isometric infinitesimal bendings.

Write $\tilde{\mathcal{T}}$ as

$$\tilde{\mathcal{T}} = \Psi_* \mathcal{T} + \langle \tilde{\mathcal{T}}, w \rangle F + \langle \tilde{\mathcal{T}}, F \rangle w.$$

Being $\tilde{\mathcal{T}}$ an isometric infinitesimal bending of F, we have

$$\langle \tilde{\nabla}'_X \tilde{\mathcal{T}}, F_* Y \rangle + \langle \tilde{\nabla}'_Y \tilde{\mathcal{T}}, F_* X \rangle = 0$$

for all $X, Y \in \mathfrak{X}(M)$. Then

$$\langle \tilde{\nabla}_X \mathcal{T}, f_* Y \rangle + \langle \tilde{\nabla}_Y \mathcal{T}, f_* X \rangle + 2\langle \tilde{\mathcal{T}}, w \rangle \langle X, Y \rangle = 0$$

for all $X, Y \in \mathfrak{X}(M)$. Hence, setting $\rho = -\langle \tilde{\mathcal{T}}, w \rangle$ we have that $\mathcal{T}$ is a conformal infinitesimal bending of f with conformal factor ρ. Notice that

$$(\tilde{\nabla}'_X \tilde{\mathcal{T}})_{\Psi_* T\mathbb{R}^{n+1}} = \Psi_* \tilde{\nabla}_X \mathcal{T} - \rho F_* X$$

for any $X \in \mathfrak{X}(M)$. It follows from (8.63) that the symmetric tensor $\mathcal{B} \in \Gamma(\mathrm{End}(TM))$ associated to $\mathcal{T}$ has the form $\mathcal{B} = B_N + cI$, where $c = -\langle Cw, N \rangle$. Since $B_N|_{\Delta^\perp} \neq 0$ we conclude that $\mathcal{T}$ is not trivial.

Any other conformal infinitesimal bending $\mathcal{T}'$ arising in this manner has the associated tensor $\mathcal{B}' = B_N + c'I$. Then Corollary 7.6 gives that $\mathcal{T} - \mathcal{T}'$ is trivial, and this concludes the proof. $\square$

Proposition 8.13. *Any parabolic hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, that admits a nontrivial conformal infinitesimal variation is conformally ruled.*

Conversely, let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a simply connected conformally ruled hypersurface free of points with a principal curvature of multiplicity at least $n-1$ and that is not conformally surface-like on any open subset of M^n. Then f is parabolic and admits a family of conformal infinitesimal bendings that are in one-to-one correspondence with the set of smooth functions on an interval.

Proof. From the proof of Proposition 8.12, we have in this case that $D = \mu J$ where $J^2 = 0$. Let $Y \in \Gamma(\Delta^\perp)$ be of unit-length such that

$JY = 0$ and let $X \in \Gamma(\Delta^\perp)$ be orthogonal to Y satisfying $JX = Y$. Note that $\nabla^h_T J = 0$ for any $T \in \Gamma(\Delta)$ is equivalent to

$$\nabla^h_T Y = 0 = \nabla^h_T X \tag{8.64}$$

for all $T \in \Gamma(\Delta)$. Hence, replacing J by $\|X\|J$, one can assume that X is of unit-length.

For the sequel, we extend J to TM as being zero on Δ. Recall that

$$\mathcal{B} - bI = (A - \lambda I)D = \mu(A - \lambda I)J$$

is symmetric. Then

$$\langle (A - \lambda I)Y, Y\rangle = \langle (A - \lambda I)JX, Y\rangle = 0. \tag{8.65}$$

Hence $(A - \lambda I)Y = \nu X$ where $\nu = \langle AX, Y\rangle \neq 0$ by assumption. Then

$$(\nabla_X \mu(A-\lambda I)J)Y - (\nabla_Y \mu(A-\lambda I)J)X = -\mu(A-\lambda I)J\nabla_X Y - \nabla_Y(\mu\nu X).$$

On the other hand, we obtain from (iii) that

$$(\nabla_X \mu(A - \lambda I)J)Y - (\nabla_Y \mu(A - \lambda I)J)X = -\mu Y(\lambda)Y. \tag{8.66}$$

Hence

$$\mu(A - \lambda I)J\nabla_X Y + \nabla_Y(\mu\nu X) = \mu Y(\lambda)Y.$$

Taking the inner product with X and Y, respectively, gives

$$Y(\mu\nu) = \mu\nu\langle \nabla_X X, Y\rangle \tag{8.67}$$

and

$$Y(\lambda) = -\nu\langle \nabla_Y Y, X\rangle. \tag{8.68}$$

Since $C_T \in \operatorname{span}\{I, J\}$, we obtain

$$\langle \nabla_Y T, X\rangle = -\langle C_T Y, X\rangle = 0 \tag{8.69}$$

for any $T \in \Gamma(\Delta)$. Let $T \in \Gamma(\Delta)$ have unit length. The inner product of the Codazzi equation

$$(\nabla_T A)Y - (\nabla_Y A)T = 0$$

with T easily gives

$$Y(\lambda) = -\nu\langle \nabla_T T, X\rangle. \tag{8.70}$$

It follows from (8.64), (8.68), (8.69) and (8.70) that the subspaces $\Delta \oplus \operatorname{span}\{Y\}$ form an umbilical distribution. Moreover, we have from (8.65) that f restricted to any leaf of $\Delta \oplus \operatorname{span}\{Y\}$ is umbilical in $\mathbb{R}^{n+1}$. Thus f is conformally ruled.

We now prove the converse. Let L be an $(n-1)$-dimensional umbilical distribution of M^n such that the restriction of f to any leaf is also umbilical. Therefore, there is $\lambda \in C^\infty(M)$ such that $L \subset \ker((A-\lambda I)_L)$, that is, $(A-\lambda I)(L) \subset L^\perp$. By assumption, we have that $\Delta = \ker(A-\lambda I)$ satisfies $\dim \Delta = n-2$.

Let X, Y be an orthonormal frame of $\Delta^\perp$ with X orthogonal to L. Hence

$$\langle (A-\lambda I)Y, Y\rangle = 0. \tag{8.71}$$

We have that $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ defined by $JX = Y$ and $JY = 0$ verifies $J^2 = 0$. It follows from (8.71) that $(A-\lambda I)J$ is symmetric. Now, since L is umbilical, we have

$$\nabla^h_T Y = 0, \tag{8.72}$$

and this is equivalent to $\nabla^h_T J = 0$ for any $T \in \Gamma(\Delta)$. To see that $C(\Gamma(\Delta)) \subset \mathrm{span}\{I, J\}$ it suffices to prove that $C_T \circ J = J \circ C_T$ for any $T \in \Gamma(\Delta)$. This is equivalent to

$$\langle \nabla_Y T, X\rangle = 0 \ \text{ and } \ \langle \nabla_X X, T\rangle = \langle \nabla_Y Y, T\rangle \tag{8.73}$$

for all $T \in \Gamma(\Delta)$. The first equation holds since L is umbilical. From (8.71) we have

$$(A-\lambda I)Y = \nu X, \tag{8.74}$$

where $\nu = \langle AX, Y\rangle \neq 0$. From the Codazzi equation we obtain

$$\nabla^h_T A = (A-\lambda I)C_T,$$

and hence the right-hand side is symmetric. We have

$$\langle (A-\lambda I)C_T X, Y\rangle = \nu\langle \nabla_X X, T\rangle \ \text{ and } \ \langle (A-\lambda I)C_T Y, X\rangle = \nu\langle \nabla_Y Y, T\rangle.$$

This implies

$$\langle \nabla_X X, T\rangle = \langle \nabla_Y Y, T\rangle$$

for any $T \in \Gamma(\Delta)$. Thus f is parabolic with respect to J.

To show that f admits a nontrivial conformal infinitesimal bending it suffices to prove the existence of a smooth function μ such that the tensor $D = \mu J \in \Gamma(\mathrm{End}(\Delta^\perp))$ satisfies all conditions in Proposition 8.12. We already know that $(A-\lambda I)J$ is symmetric, hence condition (i) is satisfied for any function μ. We assume that μ is constant along the leaves of Δ, and now condition (ii) follows from (8.72). From the definition of D it is easy to see that also (v) holds.

Condition (iii) is just (8.66). Since $\Delta = \ker(A-\lambda I)$ and (8.74) holds, then the inner product of the Codazzi equation

$$(\nabla_T A)Y - (\nabla_Y A)T = 0$$

with $T \in \Gamma(\Delta)$ of unit-length gives that (8.70) holds for any such T. Since $L = \Delta \oplus \mathrm{span}\{Y\}$ is an umbilical distribution we obtain that (8.68) holds. But (8.68) is just the Y-component of (8.66). The X-component of (8.66) is (8.67), which can be stated as

$$Y(\log \mu\nu) = \langle \nabla_X X, Y\rangle.$$

After choosing an arbitrary function as initial condition along one maximal integral curve of X, there exists a unique function μ such that $T(\mu) = 0$ for all $T \in \Gamma(\Delta)$ and $\mu\nu$ is a solution of the preceding equation. Therefore, there are as many tensors D satisfying (iii) as smooth functions on an open interval.

We have that

$$\langle(\nabla_Y \mu J)X - (\nabla_X \mu J)Y, \nabla\lambda\rangle = (Y(\mu) - \mu\langle\nabla_X X, Y\rangle)Y(\lambda) + \mu\langle\nabla_Y Y, X\rangle X(\lambda).$$

Choose any D satisfying condition (iii). Then (8.67) and (8.68) yield

$$\langle(\nabla_Y \mu J)X - (\nabla_X \mu J)Y, \nabla\lambda\rangle = -\frac{\mu}{\nu}Y(\lambda)(Y(\nu) + X(\lambda)).$$

We have using (8.68) that

$$\begin{aligned} \mathit{Hess}\,\lambda(\mu JX, Y) - \mathit{Hess}\,\lambda(X, \mu JY) &= \mu(YY(\lambda) - \langle\nabla_Y Y, X\rangle X(\lambda)) \\ &= \mu(YY(\lambda) + \frac{1}{\nu}Y(\lambda)X(\lambda)) \end{aligned}$$

and using (8.74) that

$$\lambda(\langle(A - \lambda I)\mu JX, AY\rangle - \langle AX, (A - \lambda I)\mu JY\rangle) = \lambda\mu\nu^2.$$

The last three equations give that condition (iv) is equivalent to

$$YY(Y) - \frac{1}{\nu}Y(\lambda)Y(\nu) = -\lambda\nu^2,$$

that can also be written as

$$Y((1/\nu)Y(\lambda)) = -\lambda\nu. \tag{8.75}$$

To conclude, we show that (8.75) is just the Gauss equation

$$\begin{aligned} \langle R(Y,T)T, X\rangle &= \langle AT, T\rangle\langle AY, X\rangle - \langle AY, T\rangle\langle AT, X\rangle \\ &= \lambda\nu. \end{aligned}$$

In fact, using (8.70) and (8.73) we have

$$\begin{aligned} \langle\nabla_Y\nabla_T T, X\rangle &= Y\langle\nabla_T T, X\rangle + \langle\nabla_T T, Y\rangle\langle\nabla_Y Y, X\rangle \\ &= -Y((1/\nu)Y(\lambda)) + \langle\nabla_T T, Y\rangle\langle\nabla_Y Y, X\rangle. \end{aligned}$$

Also, since L is an umbilical distribution, we have

$$\langle\nabla_T\nabla_Y T, X\rangle = -\langle\nabla_Y T, \nabla_T X\rangle = 0.$$

Using (8.72) we obtain

$$\begin{aligned}\langle\nabla_{[Y,T]}T, X\rangle &= -\langle\nabla_{\nabla_T Y}T, X\rangle\\ &= \langle\nabla_T T, Y\rangle\langle\nabla_T T, X\rangle.\end{aligned}$$

The last three equations yield

$$\langle R(Y,T)T, X\rangle = -Y((1/\nu)Y(\lambda)).$$

Now the proof follows from Proposition 8.12. $\square$

Proof of Theorem 8.7: If f admits a nontrivial conformal infinitesimal bending and is not conformally surface-like nor conformally flat nor conformally ruled, we obtain from Proposition 8.12 and Proposition 8.13 that f is either hyperbolic or elliptic. From the proof of Proposition 8.12 we have that $D = \mu J$ is the lifting of a tensor $\bar{D} = \bar{\mu}\bar{J}$ on L^2. Also from that proof, we that

$$\begin{aligned}&g_*((\nabla'_{\bar{Y}}\bar{D})\bar{X} - (\nabla'_{\bar{X}}\bar{D})\bar{Y}) + \alpha^g(\bar{Y}, \bar{D}\bar{X}) - \alpha^g(\bar{X}, \bar{D}\bar{Y})\\ &= \Psi_* f_*\Omega(X,Y) - \lambda\psi(X,Y)\Psi_* N + \varphi(X,Y)\Psi\circ f + \psi(X,Y)w,\end{aligned} \tag{8.76}$$

where $X, Y \in \Gamma(\Delta^\perp)$ are the liftings of $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$ and Ω, ψ and φ are given by (8.39), (8.40) and (8.41) respectively. Recall that D satisfies the conditions (i) through (v). Therefore, we have

$$(\nabla'_{\bar{X}}\bar{D})\bar{Y} = (\nabla'_{\bar{Y}}\bar{D})\bar{X} \tag{8.77}$$

and, since $\bar{D} = \bar{\mu}\bar{J}$, that

$$\alpha^g(\bar{X}, \bar{J}\bar{Y}) = \alpha^g(\bar{J}\bar{X}, \bar{Y}).$$

Finally, that g is a special hyperbolic or elliptic surface follows from Proposition 8.5 and the integrability condition of $\bar{\mu}$ in (8.77).

Conversely, take $f\colon M^n \to \mathbb{R}^{n+1}$ to be parametrized by the conformal Gauss parametrization in terms of a special hyperbolic or a special elliptic pair. Then f has a nowhere vanishing principal curvature $\lambda(x)$ at $x \in M^n$ of constant multiplicity $n-2$ and corresponding eigenspace $\Delta(x)$. Take $v = e_1$, $w = -2e_{n+3}$ and let $\Psi\colon \mathbb{R}^{n+1} \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ be the embedding given by (7.20). Then $S\colon M^n \to \mathbb{S}_1^{n+2}$ given by (8.26) induces a map $s\colon L^2 \to \mathbb{S}_1^{n+2}$ on the (local) space of leaves L^2 of Δ. Moreover, by the choice of v and w we have that $s = g$.

We obtain from Proposition 8.5 that, at least locally, there is a nowhere vanishing function $\bar{\mu} \in C^\infty(L^2)$ such that $\bar{D} = \bar{\mu}\bar{J}$ is a Codazzi tensor. Let $X, Y \in \Gamma(\Delta^\perp)$ be the liftings of $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$. If $D = \mu J$ is the lifting of $\bar{D}$ we have as before that (8.76) holds. Given that g is special hyperbolic or special elliptic, it follows that $\Omega = \psi = \varphi = 0$. In other words, we obtain that conditions (i), (iii) and (iv) are satisfied.

We recall that

$$\langle(\bar{D}\bar{X} \wedge \bar{Y} - \bar{D}\bar{X} \wedge \bar{Y})\bar{Z}, \bar{W}\rangle' = 0$$

for any $\bar{X}, \bar{Y}, \bar{Z}, \bar{W} \in \mathfrak{X}(L)$. It follows from (8.28) that

$$\langle(((A-\lambda I)DX\wedge(A-\lambda I)Y-(A-\lambda I)DY\wedge(A-\lambda I)X)(A-\lambda I)Z, (A-\lambda I)W\rangle=0,$$

where $X, Y, Z, W \in \Gamma(\Delta^\perp)$ are the liftings of $\bar{X}, \bar{Y}, \bar{Z}$ and $\bar{W}$. Then

$$(A-\lambda I)DX \wedge (A-\lambda I)Y - (A-\lambda I)DY \wedge (A-\lambda I)X = 0$$

for any $X, Y \in \Gamma(\Delta^\perp)$, and hence (v) holds. Given that D is projectable we have from Lemma 8.11 that $\nabla^h_T D = [D, C_T] = 0$ for all $T \in \Gamma(\Delta)$. Hence (ii) also holds. Now the proof follows from Proposition 8.12. □

Proposition 8.14. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a simply connected conformally ruled hypersurface free of points with a principal curvature of multiplicity at least $n-1$ and that is not conformally surface-like on any open subset of M^n. Then any conformal infinitesimal bending of f is the variational vector field of a conformal variation.*

Proof. We have seen that the conformal infinitesimal bendings of f are in one-to-one correspondence with the tensors D given in the proof of Proposition 8.13. Take such a D and let $F\colon M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ be the immersion $F = \Psi \circ f$, where Ψ was given in (7.20). Let $\beta\colon TM \times TM \to N_F M$ and $\mathcal{E}\colon TM \times N_F M \to N_F M$ be given by (8.42) and (8.45), respectively. The pair $(\beta, \mathcal{E})$ is associated to an infinitesimal bending of F, say $\tilde{\mathcal{T}}$, which determines a conformal infinitesimal bending $\mathcal{T}$ of f. Let $\alpha^t\colon TM \times TM \to N_F M$, $t \in (-\epsilon, \epsilon)$, be the symmetric tensor defined by

$$\alpha^t(X, Y) = \alpha^F(X, Y) + t\beta(X, Y)$$

for any $X, Y \in \mathfrak{X}(M)$. Since $\mathcal{E}$ satisfies (8.46) then

$$\bar{\nabla}^t_X \eta = \nabla'^\perp_X \eta + t\mathcal{E}(X, \eta)$$

is a connection on $N_F M$ that is compatible with the induced metric, where $X \in \mathfrak{X}(M)$, $\eta \in \Gamma(N_F M)$ and $\nabla'^\perp$ denotes the normal connection of F.

It follows from (8.44), (8.50), (8.54), (8.55), (8.60), (8.61) and (8.62) together with the Gauss, Codazzi and Ricci equations for F that α^t and

$\bar{\nabla}^t$ verify the Gauss, Codazzi and Ricci equations (see Exercise 8.6). Hence, there is a family of isometric immersions $F_t\colon M^n \to \mathbb{L}^{n+3}$ with $F_0 = F$ together with vector bundle isometries $\Phi_t\colon N_F M \to N_{F_t} M$ satisfying

$$\alpha^{F_t} = \Phi_t \alpha^t \ \text{ and } \ \nabla^{t\perp}\Phi_t = \Phi_t(\bar{\nabla}^t),$$

where α^{F_t} and $\nabla^{t\perp}$ are the second fundamental form and normal connection of F_t. Hence, we have

$$A^t_{\Phi_t F} X = -X \ \text{ and } \ \nabla^{t\perp}_X \Phi_t F = \Phi_t(\bar{\nabla}^t_X F) = 0,$$

where A^t_η denotes the shape operator of F_t in the direction of $\eta \in \Gamma(N_{F_t} M)$. Then $F_t - \Phi_t F = v_t$ is a constant vector field along F_t for any t. Since

$$\langle F_t - v_t, F_t - v_t \rangle = 0,$$

we obtain that $F_t - v_t$ determines an isometric variation of $F_0 = F$ in $\mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$. Hence, we assume that $F_t(x) \in \mathbb{V}^{n+2}$ for all $x \in M^n$. The variational vector field $\tilde{\mathcal{T}}' = \partial/\partial_t|_{t=0} F_t$ is clearly an infinitesimal bending of F and the tensor β' associated to $\tilde{\mathcal{T}}'$ satisfies

$$\beta' = (\partial/\partial_t|_{t=0}\alpha^{F_t})_{N_F M}$$

(see Exercise 2.1).

Next we define $\Phi' \in \Gamma(\mathrm{End}(N_F M))$ as follows. Given $\eta \in \Gamma(N_F M)$ we set $\eta_t = \Phi_t \eta \in \Gamma(N_{F_t} M)$. Then, regarding η_t as an element of $\mathbb{L}^{n+3}$, we define

$$\Phi'\eta = (\partial/\partial_t|_{t=0}\eta_t)_{N_F M}.$$

Given that Φ_t is an isometry for any t, it follows that Φ' is skew symmetric. Since $\alpha^{F_t} = \Phi_t(\alpha^F + t\beta)$ we have from the above that

$$\beta' = \beta + \Phi'\alpha^F.$$

Let $\Pi\colon \mathbb{V}^{n+2} \setminus \mathbb{R}w \to \mathbb{E}^{m+1} = \Psi(\mathbb{R}^{n+1})$ be the map $\Pi(u) = (1/\langle u, w\rangle)u$. Then each F_t induces an immersion $f_t\colon M^n \to \mathbb{R}^{n+1}$ such that $\Psi \circ f_t = \Pi \circ F_t$ for any t. Observe that the metric induced by f_t satisfies

$$\langle f_{t*}X, f_t{*}Y\rangle(x) = \langle (\Pi\circ F_t)_* X, (\Pi \circ F_t)_* Y\rangle(x) = \langle F_t(x), w\rangle^{-2}\langle X, Y\rangle(x)$$

at any $x \in M^n$. Hence, the variation f_t determines a conformal variation of f in $\mathbb{R}^{n+1}$. The variational vector field $\mathcal{T}'$ is a conformal infinitesimal bending of f with associated tensor $\mathcal{B}' = B_N - \langle \Phi' w, N\rangle I$, where $B_N = (A - \lambda I)D$. Hence $\mathcal{T} - \mathcal{T}'$ is trivial, and this concludes the proof. $\square$

Proof of Theorem 8.8: The proof follows from Propositions 8.13 and 8.14. $\square$

Remark 8.15. To obtain in terms of the conformal Gauss parametrization that a nontrivial conformal infinitesimal bending is, in fact, the variational vector field of a conformal variation one has to require the special hyperbolic or special elliptic surface to satisfy a strong additional condition, namely, that $\Gamma^1_u = \Gamma^2_v = 2\Gamma^1\Gamma^2$ in the former case and that $\Gamma_z = 2\Gamma\bar{\Gamma}$ in the latter case, see [20] or [21] for details.

8.3 From conformal to isometric

In this section, it is shown how the classification of the Euclidean hypersurfaces that admit nontrivial infinitesimal variations can be obtained from the classification result given in the preceding section.

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface of constant rank two, this is, a hypersurface with constant index of relative nullity $\nu = n-2$. Then let $\pi\colon M^n \to L^2$ denote the projection onto the quotient space of relative nullity leaves of f. The Gauss map N of f induces an immersion $g\colon L^2 \to \mathbb{S}^n$ and the *support function* $\gamma \in C^\infty(M)$ of f, defined as $\gamma(x) = \langle f(x), N(x)\rangle$, induces a function $\bar{\gamma} \in C^\infty(L)$ such that $N = g\circ\pi$ and $\gamma = \bar{\gamma}\circ\pi$. Set $h = i\circ g$. Then, at least locally, we can recover f from the pair $\{g,\gamma\}$ by means of the *Gauss parametrization* described next.

Let $g\colon L^2 \to \mathbb{S}^n$ be an isometric immersion and let $\gamma \in C^\infty(L)$. Set $h = i\circ g$ and let $\Lambda = N_gL$ denote the normal bundle of g. On the open subset of regular points the map $\psi\colon \Lambda \to \mathbb{R}^{n+1}$ given by

$$\psi(y,w) = \gamma(y)h(y) + h_*\nabla\gamma(y) + i_*w,$$

parametrizes a hypersurface of constant rank two, where $\nabla\gamma$ is the gradient of γ on L^2. Conversely, any hypersurface with constant rank two can be locally parametrized in this way by means of the pair $(g,\bar{\gamma})$ determined by the Gauss map and the support function. We refer to Section 7.3 in [21] for details.

We say that a pair (g,γ) formed by a surface $g\colon L^2 \to \mathbb{S}^n$ and $\gamma \in C^\infty(L)$ is an *special hyperbolic pair* (respectively, *special elliptic pair*), if g is an special hyperbolic surface (respectively, special elliptic surface) and γ satisfies

$$(Hess\,(\gamma) + \gamma I)\circ J = J^t \circ (Hess\,(\gamma) + \gamma I),$$

where $Hess\,(\gamma)$ also denotes the endomorphism of TL determined by the Hessian of γ.

A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is said to be *surface-like* if it is a cylinder over either a surface in $\mathbb{R}^3$ or over a cone of a surface in $\mathbb{S}^3(r) \subset \mathbb{R}^4$. The following result holds for dimensions $n = 3,4$ as shown in [23]. The limitation to $n \geq 5$ is due to the use of Theorem 8.7.

Theorem 8.16. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a hypersurface of constant rank 2 that admits a nontrivial infinitesimal variation. Assume that f is neither surface-like nor ruled on any open subset of M^n. Then, on each connected component of an open dense subset of M^n, the hypersurface can be given in terms of the Gauss parametrization by a special hyperbolic or a special elliptic pair.*

Conversely, any hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, given in terms of the Gauss parametrization by a special hyperbolic or special elliptic pair admits a nontrivial infinitesimal variation. Moreover, the infinitesimal bendings associated to any pair of nontrivial infinitesimal variations of f differ by a trivial infinitesimal bending.

Proof. Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface of constant rank 2 that admits a nontrivial infinitesimal variation, and let $\mathcal{T}$ be the corresponding nontrivial infinitesimal bending. We may assume that $f(M)$ does not contain the origin $0 \in \mathbb{R}^{n+1}$. Let $\varphi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1}$ be the inversion with respect to the unit sphere $\mathbb{S}^n$. Recall that N is constant along the leaves of Δ, and let $\eta\colon L^2 \to \mathbb{S}^n$ be the map induced on the space of leaves of Δ. We can assume that the support function satisfies $\gamma > 0$, at least locally. Consider the 2-parameter family of tangent affine hyperplanes given by $P(x) = f_*T_xM$ at $x \in M^n$. Notice that $\gamma(x)$ coincides with the distance from the origin 0 to $P(x)$. Applying the inversion φ we obtain a 2-parameter family of spheres, all of which have the origin as a common point.

Let $h\colon L^2 \to \mathbb{R}^{n+1}$ denote the map describing the centers and $r \in C^\infty(L)$ the radius of the 2-parameter family of hyperspheres $\varphi(P(x))$. Since all the spheres contain the origin we have that $r^{-1}h$ has unit length and satisfies $\eta = r^{-1}h$. In addition, we have that

$$2r\bar{\gamma} = 1, \tag{8.78}$$

where $\bar{\gamma} \in C^\infty(L)$ is induced by the support function γ. The map $\tilde{f} = \varphi \circ f$ is an immersion with a principal curvature λ of multiplicity $n-2$ having Δ as its corresponding eigenspace. Then r and h coincide with the maps induced on L^2 by λ^{-1} and the focal map of $\tilde{f}$, respectively. The vector field $\tilde{\mathcal{T}} = \varphi_*\mathcal{T}$ is a nontrivial conformal infinitesimal bending of $\tilde{f}$. From now on, we assume that f is neither surface-like nor ruled. Since f is not surface-like, by Propositions 7.4 and 7.6 of [21] the splitting tensor of Δ with respect to the metric induced by f satisfies $\operatorname{span}\{C_T : T \in \Delta\} \not\subset \operatorname{span}\{I\}$. The metrics induced by f and $\tilde{f}$ are conformal, then we have from the relation between their Levi-Civita connections that the splitting tensor of Δ with respect to the metric induced by $\tilde{f}$ also satisfies that $\operatorname{span}\{C_T : T \in \Delta\} \not\subset \operatorname{span}\{I\}$, and hence $\tilde{f}$ is not conformally surface-like. If $\tilde{f}$ is conformally ruled, then f is also conformally ruled, that is, there is an $(n-1)$-dimensional integrable distribution R whose leaves are mapped by f into umbilical submanifolds of $\mathbb{R}^{n+1}$. Since f has rank 2 we have that $\Delta \cap R$ is not trivial. Therefore f is ruled and that is a contradiction. Thus $\tilde{f}$ is neither conformally surface-like nor conformally ruled.

On connected components of an open dense subset of M^n, in terms of the conformal Gauss parametrization we have from Theorem 8.7 that $\tilde{f}$ is parametrized by the special hyperbolic or special elliptic pair determined by h and r.

Fix a pseudo-orthonormal basis $e_1 \ldots, e_{n+3}$ of $\mathbb{L}^{n+3}$ as in (8.19) with $v = e_1$ and $w = -2e_{n+3}$. Let $\Psi\colon \mathbb{R}^{n+1} \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $S\colon M^n \to \mathbb{L}^{n+3}$ be given by (7.20) and (8.26) respectively. Then, as in the proof of Proposition 8.12, the immersion $s\colon L^2 \to \mathbb{S}_1^{n+2} \subset \mathbb{L}^{n+3}$ induced by S satisfies $s = g$, where g is given by (8.21), that is, $g = \lambda(1, h, 0)$ where $\rho = 1/r$. Notice that both g and $\eta = \rho h$ induce the same metric on L^2. Since g is a special hyperbolic (respectively, special elliptic) surface, the corresponding position vector in $\mathbb{L}^{n+3}$ satisfies (8.4) (respectively, (8.5)) with respect to a system of coordinates (u, v). In particular, the position vector of $\eta \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ also satisfies (8.4) (respectively, (8.5)) with respect to the same coordinate system. Hence (u, v) is a system of real (respectively, complex) conjugate coordinates for η, and thus we have a hyperbolic (resp. elliptic) surface. Moreover, from Proposition 8.5 it follows that η determines a special hyperbolic (resp. elliptic) surface.

Since $g(L^2) \subset \mathbb{S}_1^{n+2}$ and $\langle g, w\rangle = \rho$, then

$$\mathit{Hess}\,(\rho)(X, Y) = \langle \alpha^g(X, Y), w\rangle - \rho\langle X, Y\rangle$$

for any $X, Y \in \mathfrak{X}(L)$. Since g is hyperbolic (elliptic) with respect to $J \in \Gamma(\mathrm{End}(TL))$ satisfying $J \neq I$ and $J^2 = I$ ($J^2 = -I$), we have that

$$(\mathit{Hess}\,(\rho) + \rho I) \circ J = J^t \circ (\mathit{Hess}\,(\rho) + \rho I),$$

where $\mathit{Hess}\,(\rho)$ also denotes the endomorphism of TL determined by the Hessian of ρ. Then, we obtain from (8.78) that $\bar{\gamma}$ satisfies

$$(\mathit{Hess}\,(\bar{\gamma}) + \bar{\gamma} I) \circ J = J^t \circ (\mathit{Hess}\,(\bar{\gamma}) + \bar{\gamma} I).$$

Thus f is parametrized in terms of the Gauss parametrization by a especial hyperbolic or special elliptic pair $(\eta, \bar{\gamma})$.

We outline the proof of the converse. Let $f\colon M^n \to \mathbb{R}^{n+1}$ be parametrized by means of the Gauss parametrization in terms of a special hyperbolic or special elliptic pair (g, γ). Then f has constant index of relative nullity $\nu = n - 2$. As in the proof of Theorem 8.7, we obtain from Proposition 8.5 that, at least locally, there is a nowhere vanishing function $\bar{\mu} \in C^\infty(L^2)$ such that $\bar{D} = \bar{\mu}\bar{J}$ is a Codazzi tensor. Let $D = \mu J$ be the lifting of $\bar{D}$ and define the tensor $B \in \Gamma(\mathrm{End}(TM))$ by

$$B|_{\Delta^\perp} = AD \;\text{ and }\; B|_\Delta = 0.$$

Then $B \neq 0$ is a symmetric Codazzi tensor that satisfies (2.21). Hence it follows from Theorem 2.11 that B determines a unique infinitesimal bending of f. $\square$

In the case of ruled hypersurfaces, in analogy with Propositions 8.13 and 8.14, we have the following. A simply connected ruled hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, free of flat points and that is not surface-like on any open subset, admits a family of nontrivial conformal infinitesimal bendings that is in one-to-one correspondence with the set of smooth functions on an interval. Moreover, any of its infinitesimal bendings is the variational vector field of an isometric variation. Again, the proofs of these facts can be seen in [21].

8.4 Exercises

Exercise 8.1. Prove the statement given by Proposition 8.6.

Exercise 8.2. Let V and W be vector spaces of dimensions 2 and $p \geq 2$, respectively, and let $\alpha\colon V \times V \to W$ be a symmetric bilinear form. Assume that there exist a basis X, Y of V and $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 + c^2 \neq 0$ and

$$a\alpha(X,X) + 2c\alpha(X,Y) + b\alpha(Y,Y) = 0.$$

Show that $ab - c^2$ being positive (respectively, negative) is independent of the basis X, Y, and that it is equivalent to the existence of an endomorphism $J \neq I$ of V such that $J^2 = \epsilon I$ with $\epsilon = -1$ (respectively, $\epsilon = 1$) and

$$\alpha(JX, Y) = \alpha(X, JY)$$

for all $X, Y \in V$.

Hint: See hint of Exercise 11.2 in [21].

Exercise 8.3. Let $f\colon M^n \to N^m$ be an isometric immersion and let $g \in C^\infty(N)$. Show that the gradient and Hessian of g and $h = g \circ f$ are related by

$$f_*\operatorname{grad} h = (\operatorname{grad} g)^T$$

and

$$\mathit{Hess}\, h(X,Y) = \mathit{Hess}\, g(f_*X, f_*Y) + \langle \operatorname{grad} g, \alpha(X,Y)\rangle$$

for all $x \in M^n$ and $X, Y \in T_xM$. Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion and let $h^v \in C^\infty(M)$ be the height function

$$h^v(x) = \langle f(x), v\rangle$$

with respect to the hyperplane normal to $v \in \mathbb{R}^m$. Show that

$$\mathit{Hess}\, h^v(x)(X,Y) = \langle \alpha^f(X,Y), v\rangle$$

for all $x \in M^n$ and $X, Y \in T_xM$.

Hint: See Proposition 1.2 and Corollary 1.3 in [21].

Exercise 8.4. Given a surface $g\colon L^2 \to \mathbb{S}^n_\epsilon$ set $h = i \circ g\colon L^2 \to \mathbb{E}^{n+1}$. Show that the condition (8.1) holds if and only for any $v \in \mathbb{E}^{n+1}$ the height function $h^v = \langle h, v\rangle$ satisfies

$$(Hess\, h^v + h^v I) \circ J = J^t \circ (Hess\, h^v + h^v I),$$

where $Hess\, h^v$ denotes the endomorphism of TL associated with the Hessian of h^v with respect to the induced metric.

Exercise 8.5. Give proofs of Propositions 8.2 and 8.5 in the elliptic case.

Exercise 8.6. Verify that the symmetric tensor α^t and the connection $\bar{\nabla}^t$ defined in the proof of Proposition 8.14 satisfy the Gauss, Codazzi and Ricci equations.

Hint: Use the Gauss, Codazzi and Ricci equations together with the fact that β and $\mathcal{E}$ satisfy (2.12), (2.13) and (2.14). Also use that in this case D has kernel in $\Delta^\perp$.

Exercise 8.7. Let L^2 be a Riemannian surface carrying a tensor $J \in \Gamma(\mathrm{End}(TL))$ satisfying $J \neq I$ and $J^2 = I$. Decompose $TL = T'L \oplus T''L$ where $T'L$ and $T''L$ are the eigenbundles corresponding to the eigenvalues 1 and -1 of J, respectively. For $X \in \mathfrak{X}(L)$ write $X = X' + X''$ according to that decomposition. Let $g\colon L^2 \to \mathbb{S}^m_\epsilon$, $m \geq 4$, be an isometric immersion with second fundamental form $\alpha^g\colon TL \times TL \to N_gL$. The (p,q)-components of α^g for $p+q=2$ are given by

$$\alpha^{(2,0)}(X,Y) = \alpha^g(X',Y'),\;\; \alpha^{(0,2)}(X,Y) = \alpha^g(X'',Y'')$$

and

$$\alpha^{(1,1)}(X,Y) = \alpha^g(X',Y'') + \alpha^g(X'',Y'),$$

for any $X, Y \in \mathfrak{X}(L)$.

(i) Show that $2X' = X + JX$ and $2X'' = X - JX$.

(ii) Prove that g is hyperbolic if and only if $\alpha^{(1,1)} = 0$.

Exercise 8.8. Let L^2 be a Riemannian surface carrying a tensor $J \in \Gamma(\mathrm{End}(TL))$ satisfying $J^2 = -I$ (almost complex structure). Let $TM \otimes \mathbb{C}$ be the complexified tangent bundle of L and decompose $TL \otimes \mathbb{C} = T'L \oplus T''L$ where $T'L$ and $T''L$ are the eigenbundles corresponding to the eigenvalues i and $-i$ of J, respectively. For $X \in \Gamma(TM \otimes \mathbb{C})$ write $X = X' + X''$ according to that decomposition. Let $g\colon L^2 \to \mathbb{S}^m_\epsilon$, $m \geq 4$, be an isometric immersion with second fundamental form $\alpha^g\colon TL \times TL \to N_gL$ and let the (p,q)-components of α for $p+q=2$ given in Exercise 8.7.

(i) Show that $2X' = X - iJX$ and $2X'' = X + iJX$.

(ii) Prove that g is elliptic if and only if $\alpha^{(1,1)} = 0$.

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