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Couplings and attractiveness for general exclusion processes

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Abstract. Attractiveness is a fundamental tool to study interacting particle systems and the basic coupling construction is a usual route to prove this property, as for instance in the simple exclusion process. We consider here general exclusion processes where jump rates from an occupied site to an empty one depend not only on the location of the jump but also possibly on the whole configuration. These processes include in particular exclusion processes with speed change introduced by F. Spitzer in [18]. For such processes we derive necessary and sufficient conditions for attractiveness, through the construction of a coupled process under which, in any coupled transition, discrepancies on the involved sites do not increase, or even decrease. We emphasize the fact that basic coupling is never attractive for this class of processes, except in the case of simple exclusion, and that the coupled processes presented here necessarily differ from it. We study various examples, for which we determine the set of extremal translation invariant and invariant probability measures.

Keywords. Particle systems, attractiveness, couplings, discrepancies, invariant measures, exclusion processes with speed change.

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1 Introduction

Dedicated to Errico Presutti.

Exclusion processes are among the most studied interacting particle systems: despite their very simple form, these Markov processes exhibit characteristic features that make them ideal toy models for many physical or biological phenomena.

In an exclusion process, particles evolve on a countable set of sites S, e.g. $\mathbb{Z}^d$, on which multiple occupancy is forbidden. This exclusion rule is encoded in the structure of the state space which is thus defined as $\Omega = \{0,1\}^S$. For a configuration $\eta \in \Omega$ and for $x \in S$, $\eta(x)$ is the occupation number at site x, that is $\eta(x) = 1$ whenever a particle is present on site x, while $\eta(x) = 0$ when site x is empty. Particles jump from one site to another, empty, site according to a probability transition $p(.,.)$ on S (for $S = \mathbb{Z}^d$, we consider only translation invariant cases).

The most widely studied exclusion model is the simple exclusion process (SEP), in which particles have all the same speed one, that is the transition rate for a particle in a configuration η to jump from its position at site x to an empty site y does not depend on the location of other particles and thus simply reads $\eta(x)(1-\eta(y))p(x,y)$. Endowing Ω with the coordinatewise (partial) order, that is, for $\eta, \xi \in \Omega$,

$$\eta \leq \xi \Leftrightarrow \forall x \in S,\, \eta(x) \leq \xi(x), \tag{1.1}$$

we can define the *monotonicity* property as follows. There exists a coupling such that this partial order is maintained through the (coupled) evolution whenever it holds at initial time; as mentioned in [15, Chapter II, Definition 2.3] this property is called *attractiveness* for particle systems, so that we will generally say attractive rather than monotone.

Attractiveness is a fundamental property of SEP and a key tool to determine the set $(\mathcal{I} \cap \mathcal{S})_e$ of extremal translation invariant and invariant probability measures for the dynamics (see e.g. Chapter VIII of [15]). This set consists in a one parameter family $\{\nu_\rho, \rho \in [0,1]\}$ of Bernoulli product measures, where ρ represents the average particle density per site. It is also crucial in establishing hydrodynamics for asymmetric transition probability $p(.,.)$, see e.g. [17, 12]). In such a problem, attractiveness is embodied through the "basic coupling" construction of two copies $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ of simple exclusion processes, under which particles move together as much as possible. In other words, if at some time s particles of both copies attempt to jump, they will try to go from the same departure site x to the same arrival site y according to $p(x,y)$, as long as those jumps are permitted (that is if $\eta_s(x) = \xi_s(x) = 1$ and $\eta_s(y) = \xi_s(y) = 0$), otherwise only the possible jump will take place. Thanks to basic coupling, when the initial distribution is translation invariant, it is possible to control the evolution of the density of *discrepancies* between $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$, that is, the sites on which the configurations differ. Therefore, for SEP, the basic coupling is a coupling whose marginals will eventually become ordered with probability 1, even if it was not the case initially. Combined with some irreducibility property for the probability transition $p(.,.)$, the control of discrepancies is the essential step to derive $(\mathcal{I}\cap\mathcal{S})_e$ (see [13, 15]).

However, ever since the seminal paper [18] by Frank Spitzer in which simple exclusion process was first defined, other exclusion processes have been considered, named exclusion processes with speed change, in which jump rates may depend on the configuration around the particle departure site. Though such a dependence can be treated within a basic coupling construction for (non conservative) spin flip models, it appeared to be not so simple for conservative ones. In order to determine the set $(\mathcal{I}\cap\mathcal{S})_e$ for such models, more involved attractiveness conditions and related coupling constructions were to be found. Sufficient conditions for attractiveness have been obtained by Tom Liggett in his Saint-Flour lecture notes [14] for the models introduced in [18], as well as a related coupling leading to $(\mathcal{I}\cap\mathcal{S})_e$ whenever these conditions are fulfilled.

Totally asymmetric versions of exclusion processes with speed change are also natural models of traffic (see e.g. [9]). Recently, there has been a renewed interest in exclusion processes, in particular those related to integrable models, such as the facilitated exclusion processes (see e.g. [4, 6, 2]), or the q-Hahn exclusion process (see [5]). These models have been analyzed through other existing techniques such as duality, or through an ad-hoc correspondence with attractive dynamics, such as (generalized) zero-range processes. In the present paper we give various examples of exclusion processes with speed change, and among them traffic models for which the constructed coupling that yields attractiveness has very different features than basic coupling: for instance it requires jumps between the two coupled processes with either different departure sites or different arrival sites.

In this work, we consider a general exclusion process on $\mathbb{Z}^d$ and state necessary and sufficient conditions under which attractiveness holds. Here jump rates depend not only on the position and occupation numbers of the sites at which a jump occurs, but also possibly on the whole configuration, so that the basic coupling construction does not hold beyond SEP (see our examples in Section 4). We proceed in the spirit of our previous papers on particle systems of misanthrope type [8, 7], in which the richer structure of the local state space already imposes non trivial monotonicity conditions even when rates depend on the configuration only through the sites at which a jump occurs. Our monotonicity conditions are inspired by the work of William Massey ([16]). In the present paper, we first give necessary and sufficient conditions for monotonicity of our dynamics, through the construction of a coupling that we call increasing. We then refine this coupling construction to derive what we call a quasi attractive coupling, under which the discrepancies involved in a coupled transition do not increase. Finally, assuming some irreducibility and an additional assumption on the dynamics, thanks to another coupling that we call attractive, we describe, as in Chapter VIII of [15], the set of extremal translation invariant and invariant probability measures of generalized exclusion processes.

The paper is organized as follows. In Section 2 we define the generalized exclusion model, and state our main results: necessary and sufficient conditions for monotonicity (Theorem 2.4), the existence, for a monotone process, of an increasing coupling that is quasi attractive (Theorem 2.7), and the determination of the set $(\mathcal{I}\cap\mathcal{S})_e$ (Theorem 2.9). In Section 3, we prove Theorem 2.4 and give in a series of propositions the construction of the successive generators leading to the proofs of Theorems 2.7 and 2.9. These propositions are proved in Section 5. In Section 4, we illustrate our results with examples, showing first that our construction reduces to basic coupling in the case of simple exclusion and only there. We then consider exclusion processes with speed change, extending the results of [18, 14]. Finally, we turn to traffic models, considering first a generalization of the totally asymmetric 2-step exclusion process studied in [10], and a symmetrized version of the totally asymmetric traffic model from [9]. In all cases, we compute explicitly the attractive coupling rates and give the set of invariant measures $(\mathcal{I}\cap\mathcal{S})_e$.

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2 Model and main results

In this section, we define the class of exclusion models we consider and state our two main results: Theorem 2.4 gives necessary and sufficient conditions for monotonicity, and Theorem 2.7 links monotonicity and attractiveness for this model, through a coupling construction.

We first introduce a general exclusion process $(\eta_t)_{t\geq 0}$ on $S=\mathbb{Z}^d$, together with some notation and general properties. Let $\Omega=\{0,1\}^S$ be its state space and $\mathcal{L}$ its formal generator, acting on any cylinder function f

and for any configuration $\eta \in \Omega$,

$$\mathcal{L}f(\eta) \quad = \quad \sum_{x,y\in S} \eta(x)(1-\eta(y))\Gamma_\eta(x,y)\big[f(\eta^{x,y})-f(\eta)\big], \qquad (2.1)$$

where $\Gamma_\eta(x,y)$ is independent of $\eta(x)$ and $\eta(y)$, and where for any $(x,y) \in S^2$, $\eta^{x,y}$ is the configuration obtained from η by exchanging the occupation numbers in configuration η at sites x and y

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases} \qquad (2.2)$$

The process is thus conservative, and the quantity $\eta(x)+\eta(y)$ is conserved in a jump from site x to site y. We denote by $(T(t), t \geq 0)$ the semi-group of this process.

Remark 2.1. *When the jump rates $\Gamma_\eta(x,y)$ are independent of the configuration η, and reduce to a probability transition $(p(x,y), x, y \in S)$ on S,*

$$\Gamma_\eta(x,y) = p(x,y), \qquad (2.3)$$

one recovers the simple exclusion process.

We assume the following conditions on the jump rates, so that (2.1) is the infinitesimal generator of a well defined Markov process (see [15, Chapter I]):

$$\sup_{v\in S}\sum_{u\in S}\sup_{\eta\in\Omega}\Gamma_\eta(u,v) < +\infty \quad \text{and} \quad \sup_{u\in S}\sum_{v\in S}\sup_{\eta\in\Omega}\Gamma_\eta(u,v) < +\infty. \qquad (2.4)$$

Of course, these generic conditions can be alleviated, depending on the example at hand.

Let us recall the *monotonicity property* for particle systems, quoting [15, Chapter II]. We denote by $\mathcal{M}$ the set of all bounded, non-decreasing, continuous functions f on Ω. The partial order (1.1) induces a stochastic order on the set $\mathcal{P}$ of probability measures on Ω endowed with the weak topology:

$$\forall \nu, \nu' \in \mathcal{P},\ \nu \leq \nu' \Leftrightarrow \big(\forall f \in \mathcal{M}, \nu(f) \leq \nu'(f)\big). \qquad (2.5)$$

Theorem 2.2 ([15, Chapter II, Theorem 2.2]). *For the particle system $(\eta_t)_{t\geq 0}$ the following two statements are equivalent.*

(a) $f \in \mathcal{M}$ implies $T(t)f \in \mathcal{M}$ for all $t \geq 0$.

(b) For $\nu, \nu' \in \mathcal{P}$, $\nu \leq \nu'$ implies $\nu T(t) \leq \nu' T(t)$ for all $t \geq 0$.

Definition 2.3 ([15, Chapter II, Definition 2.3]). The particle system $(\eta_t)_{t\geq 0}$ is monotone if the equivalent statements of Theorem 2.2 are satisfied.

Our first main result is the following set of necessary and sufficient conditions for monotonicity.

Theorem 2.4. *The exclusion process defined by* (2.1) *is* monotone *if and only if for any couple of configurations* $(\xi,\zeta)\in\Omega^2$ *such that* $\xi\leq\zeta$*, the following hold:*

1) For all $y\in S$ *such that* $\zeta(y)=0$*,*

$$\sum_{x\in S}\xi(x)\big[\Gamma_\xi(x,y)-\Gamma_\zeta(x,y)\big]^+\leq\sum_{x\in S}\zeta(x)(1-\xi(x))\Gamma_\zeta(x,y), \tag{2.6}$$

2) for all $x\in S$ *such that* $\xi(x)=1$*,*

$$\sum_{y\in S}(1-\zeta(y))\big[\Gamma_\zeta(x,y)-\Gamma_\xi(x,y)\big]^+\leq\sum_{y\in S}\zeta(y)(1-\xi(y))\Gamma_\xi(x,y). \tag{2.7}$$

Remark 2.5. *Equations* (2.6)–(2.7) *can be interpreted in the following way. First, by conditions* (2.4)*, the sums appearing in* (2.6)–(2.7) *are always finite. The left hand side of* (2.6) *measures the excess rate at which an empty site* y *is filled in the smaller configuration* ξ*, so that coupling jumps in both configurations from the same initial sites* x *to* y *will be clearly not sufficient to preserve partial order if this sum is different from zero. Equation* (2.6) *suggests that partial order could be preserved by coupling such "excess rate" jumps with jumps involved in the right hand side, that is jumps to* y *from sites occupied in configuration* ζ*, but empty in* ξ*. Equation* (2.6) *just states that such rates are sufficient to do so.*

Equation (2.7) *can be interpreted in a similar way: Now the left hand side measures the excess rate at which a filled site* x *is depleted in the larger configuration* ζ*, so that again partial order could not be preserved by coupling jumps in both configurations from site* x *to the same site* y *whenever this sum differs from zero. Again equation* (2.7) *suggests that partial order could be preserved by coupling this second set of "excess rate jumps" with jumps in the smaller configuration* ξ *from the same site* x *to any site* y*, empty in configuration* ξ *but already filled in* ζ*. Again equation* (2.7) *states that the jump rates are just sufficient to do so.*

In Section 3, we prove that these conditions (2.6)–(2.7) are necessary and rely on them to build in Propositions 3.2 and 3.3 a coupling between two copies of the process. A coupling is called *increasing* if it preserves the stochastic order between marginal configurations. In Section 5, we achieve the proof of Proposition 3.3, that is, this coupling is proven to be *increasing* under the hypothesis that inequalities (2.6)–(2.7) hold, showing in turn that these conditions are also sufficient.

Beyond monotonicity, a coupling construction turns out to be essential to characterize the set $(\mathcal{I}\cap\mathcal{S})_e$ of extremal invariant and translation invariant probability measures of $(\eta_t)_{t\geq 0}$. In our setting, the marginals of the coupled process built in Propositions 3.2 and 3.3 are not necessarily ordered, and the evolution of the *discrepancies* between them is the main object to control:

Definition 2.6. In a coupled process $(\xi_t,\zeta_t)_{t\geq 0}$, there is a *discrepancy* at site $z\in S$ at time t if $\xi_t(z)\neq\zeta_t(z)$.

As recalled in the introduction, for the simple exclusion process (SEP) endowed with basic coupling, in any coupled transition the number of discrepancies on the involved sites remains constant whenever the values of the two marginal configurations are ordered, but decreases otherwise. Beyond this case, an increasing coupling does not necessarily impose constraints on the coupled evolution of unordered pairs of configurations, so that the number of involved discrepancies in a transition is not necessarily non-increasing. However here we have the following:

Theorem 2.7. *Suppose that the process defined by* (2.1) *is monotone on* $\Omega=\{0,1\}^S$. *Then there exists an increasing coupled process on* $\Omega\times\Omega$ *such that in any coupled transition, the number of discrepancies does not increase. Such a coupling is called a quasi attractive coupling.*

The proof of Theorem 2.7 relies on the explicit construction of such a *quasi attractive* coupling, which refines the previous *increasing* one. It is described in Proposition 3.9, while proofs of existence and attractiveness are postponed to Section 5.

To conclude with the characterization of the set $(\mathcal{I}\cap\mathcal{S})_e$, we need not only that in the coupling process the number of discrepancies involved in a coupled transition does not increase with time, but also that this number decreases. For this, we need to construct again another coupling process, But it requires an additional assumption of the dynamics.

Definition 2.8. For an exclusion process with generator (2.1), an *open edge* (x,y) (for $x,y\in S$) is an edge such that $\Gamma_\xi(x,y)>0$ for any configuration $\xi\in\Omega$. The set S is then *fully connected* if for all $(x,y)\in S^2$, $x\neq y$, there exists a finite open path in S between x and y, that is a sequence $\{x_0,\cdots,x_n\}$ for some $n>0$ such that (x_{i-1},x_i) is open for $i\in\{1,\cdots,n\}$ with either $x_0=x$ and $x_n=y$, or $x_0=y$ and $x_n=x$.

In Subsection 3.3, we will explain how, whenever for the dynamics S is fully connected, it is possible to construct a coupling such that any pair of discrepancies of opposite sign have a positive probability to disappear in finite time. We call this coupling an *attractive* coupling. When the jump rates are translation invariant, this reduces the derivation of the set $(\mathcal{I} \cap \mathcal{S})_e$ essentially to the classical proof, originally applied to the simple exclusion process (going back to [13]), which leads to the following theorem.

Theorem 2.9. *Let* $(\eta_t)_{t\geq 0}$ *be an exclusion process with generator* (2.1) *and translation invariant jump rates, such that* S *is fully connected in the sense of Definition 2.8. If* $(\eta_t)_{t\geq 0}$ *is attractive then*

1) The set of translation invariant, extremal invariant measures $(\mathcal{I} \cap \mathcal{S})_e$ *is a one parameter family* $\{\mu_\rho, \rho \in \mathcal{R}\}$*, where* $\mathcal{R}$ *is a closed subset of* $[0, 1]$ *containing* $\{0, 1\}$*, and for every* $\rho \in \mathcal{R}$*,* μ_ρ *is a translation invariant probability measure on* Ω *with* $\mu_\rho[\eta(0)] = \rho$*; furthermore, the measures* μ_ρ *are stochastically ordered, that is,* $\mu_\rho \leq \mu_{\rho'}$ *if* $\rho \leq \rho'$*;*

2) if $(\eta_t)_{t\geq 0}$ *possesses a one parameter family* $\{\mu_\rho\}_\rho$ *of product invariant and translation invariant probability measures, we have* $(\mathcal{I} \cap \mathcal{S})_e = \{\mu_\rho\}_\rho$*.*

Our results can be extended in various ways, to more general conservative models, as well as to some mixed non conservative models with both exchanges and configuration independent birth-death events, but this is beyond the scope of the present paper.

3 Proofs of main theorems and coupling constructions

This section is devoted to the construction of the coupling necessary to the proof of Theorem 2.4 (in Subsection 3.1), in three steps. We first prove that inequalities (2.6)–(2.7) are necessary conditions. In order to prove that these conditions are also sufficient, we introduce in Proposition 3.2 the general form $\overline{\mathcal{L}}$ of a Markovian coupling generator associated to $\mathcal{L}$, depending on a set of coupled transition rates $G_{\xi,\zeta}(.)$. Those rates are defined in Proposition 3.3 and we prove in turn that with such a choice, and whenever inequalities (2.6)–(2.7) are fulfilled, the generator $\overline{\mathcal{L}}$ defines an increasing coupling. We continue this section (in Subsection 3.2) with the proof of Theorem 2.7, introducing in Proposition 3.9 the generator $\overline{\mathcal{L}}^D$ of a quasi attractive coupling. Finally we explain in Subsection 3.3 how to prove Theorem 2.9 by refining the construction of an attractive coupling (in Propositions 3.11 and 3.12). Proofs of the above Propositions are given in Section 5.

3.1 Proof of Theorem 2.4

Inequalities (2.6)–(2.7) are particular instances (and in turn the worst cases) of a larger set of inequalities (first derived by A.W. Massey [16]) that the coefficients of the infinitesimal generator of a monotone Markov process need to fulfill. We sketch their derivation hereafter and we refer to [16] for a thorough derivation (see also [8] for details). The idea is to derive sensible necessary conditions on the jump rates for a Markov process to be monotone, using the fact that the characteristic function of any increasing (or decreasing) cylinder set $V \subset \Omega$, is a monotone cylinder function on Ω. Let $(\xi_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ two instances of a monotone process with initial conditions ξ_0 and ζ_0 such that $\xi_0 \leq \zeta_0$, then $\mathbf{1}_V(\xi_t) \leq \mathbf{1}_V(\zeta_t)$ (and reverse inequality for a decreasing set). In addition if initial conditions are chosen so that $\xi_0 \notin V$ and $\zeta_0 \notin V$, the same inequality holds for the ratios

$$\frac{1}{t}\left(\mathbf{1}_V(\xi_t) - \mathbf{1}_V(\xi_0)\right) \leq \frac{1}{t}\left(\mathbf{1}_V(\zeta_t) - \mathbf{1}_V(\zeta_0)\right)$$

for all $t > 0$. Taking properly the limit $t \to 0$ gives then inequalities involving the rates of the Markov generator, hereafter named "Massey conditions" and stated below in our case:

Proposition 3.1 (Massey conditions). *If the particle system defined in* (2.1) *is monotone, then for all configurations* $(\xi, \zeta) \in \Omega \times \Omega$ *such that* $\xi \leq \zeta$*,*

1) For all increasing cylinder sets $V \subset \Omega$ *such that* $\zeta \notin V$*,*

$$\sum_{x,y} \xi(x)(1-\xi(y))\Gamma_\xi(x,y)1_V(\xi^{x,y}) \leq \sum_{x,y} \zeta(x)(1-\zeta(y))\Gamma_\zeta(x,y)1_V(\zeta^{x,y}). \quad (3.1)$$

2) For all decreasing cylinder sets $V \subset \Omega$ *such that* $\xi \notin V$*,*

$$\sum_{x,y} \zeta(x)(1-\zeta(y))\Gamma_\zeta(x,y)1_V(\zeta^{x,y}) \leq \sum_{x,y} \xi(x)(1-\xi(y))\Gamma_\xi(x,y)1_V(\xi^{x,y}). \quad (3.2)$$

3.1.1 Proof of necessary conditions

Equations (2.6) follow from (3.1) by taking a particular sequence of cylinder increasing sets and passing to the limit. Equations (2.7) follow in the same way from (3.2). Let ξ, ζ be two configurations such that $\xi \leq \zeta$ and take y such that $\zeta(y) = 0$. For $n > 0$, we construct a configuration η_n as follows

$$\eta_n(x) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } \|x-y\| \leq n \ , \ \xi(x) = 1 \text{ and } \Gamma_\xi(x,y) < \Gamma_\zeta(x,y), \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We define the increasing cylinder set $V_n = \{\rho \in \Omega, \rho \geq \eta_n\}$. Since $\zeta(y) = 0$, configuration ζ (and hence ξ) does not belong to V_n. Equation (3.1) applied to V_n now selects single jumps which allow to enter V_n, hence moving a particle from any site x with $\eta_n(x) = 0$ to site y. We thus get:

$$\sum_{x \in S} \xi(x)(1-\eta_n(x))\Gamma_\xi(x,y) \leq \sum_{x \in S} \zeta(x)(1-\eta_n(x))\Gamma_\zeta(x,y). \tag{3.4}$$

Note that by conditions (2.4), both sums are finite. For all $x \neq y$, we have

$$\begin{aligned} \zeta(x)(1-\eta_n(x)) &= \zeta(x)(1-\eta_n(x))(1-\xi(x)) + \zeta(x)(1-\eta_n(x))\xi(x) \\ &= \zeta(x)(1-\xi(x)) + \xi(x)(1-\eta_n(x)), \end{aligned}$$

where the second line comes from the fact that $\eta_n(x) \leq \xi(x) \leq \zeta(x)$. Inserting this expression in the right hand side of (3.4), we get

$$\sum_{x \in S} \xi(x)(1-\eta_n(x))\big(\Gamma_\xi(x,y) - \Gamma_\zeta(x,y)\big) \leq \sum_{x \in S} \zeta(x)(1-\xi(x))\Gamma_\zeta(x,y),$$

which gives, using definition (3.3) of η_n

$$\begin{aligned} \sum_{x \in S : \|x-y\| \leq n} \xi(x)\big[\Gamma_\xi(x,y) - \Gamma_\zeta(x,y)\big]^+ + \sum_{x \in S : \|x-y\| > n} \xi(x)\big(\Gamma_\xi(x,y) - \Gamma_\zeta(x,y)\big) \\ \leq \sum_{x \in S} \zeta(x)(1-\xi(x))\Gamma_\zeta(x,y) \end{aligned} \tag{3.5}$$

Conditions (2.4) now imply that the second term in the left hand side of (3.5) goes to zero as $n \to \infty$. Taking the limit $n \to \infty$ in (3.5) thus gives

$$\sum_{x \in S} \xi(x)\big[\Gamma_\xi(x,y) - \Gamma_\zeta(x,y)\big]^+ \leq \sum_{x \in S} \zeta(x)(1-\xi(x))\Gamma_\zeta(x,y)$$

which is Equation (2.6).

Equation (2.7) can be derived in a similar way from (3.2). Let again ξ, ζ be two configurations such that $\xi \leq \zeta$ and take now $x \in S$ such that $\xi(x) = 1$. Let $n > 0$ and consider the configuration η_n such that:

$$\eta_n(x) = \begin{cases} 0 & \text{if } y = x, \\ 0 & \text{if } \|x-y\| \leq n \ , \ \zeta(y) = 0 \text{ and } \Gamma_\xi(x,y) > \Gamma_\zeta(x,y), \\ 1 & \text{otherwise.} \end{cases} \tag{3.6}$$

We construct the decreasing cylinder set $V_n = \{\rho \in \Omega, \rho \leq \eta_n\}$. Since $\xi(x) = 1$, the configuration ξ (and thus ζ) does not belong to V_n. Equation (3.2) now selects single jumps which allow to enter the decreasing set, thus removing a particle at x and moving it to any possible site y where $\eta_n(y) = 1$. We thus get

$$\sum_{y\in S} \eta_n(y)(1-\zeta(y))\Gamma_\zeta(x,y) \leq \sum_{y\in S} \eta_n(y)(1-\xi(y))\Gamma_\xi(x,y). \tag{3.7}$$

For all $y \neq x$, we now have

$$\begin{aligned} \eta_n(y)(1-\xi(y)) &= \eta_n(y)(1-\xi(y))(1-\zeta(y)) + \eta_n(y)(1-\xi(y))\zeta(y) \\ &= \eta_n(y)(1-\zeta(y)) + \zeta(y)(1-\xi(y)), \end{aligned}$$

where we have used that $\eta_n(y) \geq \zeta(y) \geq \xi(y)$. Inserting this expression in the right hand side of (3.7) gives

$$\sum_{y\in S} \eta_n(y)(1-\zeta(y))\big(\Gamma_\zeta(x,y)) - \Gamma_\xi(x,y)\big) \leq \sum_{y\in S} \zeta(y)(1-\xi(y))\Gamma_\xi(x,y).$$

Using the definition (3.6) of η_n, we get

$$\begin{aligned} \sum_{\substack{y\in S:\\ \|y-x\|\leq n}} (1-\zeta(y))\big[\Gamma_\zeta(x,y) - \Gamma_\xi(x,y)\big]^+ \;+ \sum_{\substack{y\in S:\\ \|y-x\|> n}} (1-\zeta(y))\big(\Gamma_\zeta(x,y) - \Gamma_\xi(x,y)\big) \\ \leq \sum_{y\in S} \zeta(y)(1-\xi(y))\Gamma_\xi(x,y). \end{aligned} \tag{3.8}$$

In the limit $n \to \infty$, the second term in the left hand side of (3.8) goes to zero and one gets

$$\sum_{y\in S}(1-\zeta(y))\big[\Gamma_\zeta(x,y) - \Gamma_\xi(x,y)\big]^+ \leq \sum_{y\in S} \zeta(y)(1-\xi(y))\Gamma_\xi(x,y),$$

which is Equation (2.7).

3.1.2 Coupling construction

We now use these ideas to construct a coupling process then prove that it is increasing, that is, we proceed with the second and third steps of the proof of Theorem 2.4.

We first define the general form an increasing coupling process should take.

Proposition 3.2. *The operator $\overline{\mathcal{L}}$ defined, for any cylinder function f on $\Omega\times\Omega$ and any pair of configurations $(\xi,\zeta)\in\Omega\times\Omega$, by*

$$
\begin{aligned}
&\overline{\mathcal{L}}f(\xi,\zeta)\\
&=\sum_{x_1,y_1\in S}\xi(x_1)(1-\xi(y_1))\left(\Gamma_\xi(x_1,y_1)-\sum_{x_2,y_2\in S}\zeta(x_2)(1-\zeta(y_2))G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\right)\\
&\qquad\qquad\times\big(f(\xi^{x_1,y_1},\zeta)-f(\xi,\zeta)\big)\\
&+\sum_{x_2,y_2\in S}\zeta(x_2)(1-\zeta(y_2))\left(\Gamma_\zeta(x_2,y_2)-\sum_{x_1,y_1\in S}\xi(x_1)(1-\xi(y_1))G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\right)\\
&\qquad\qquad\times(f(\xi,\zeta^{x_2,y_2})-f(\xi,\zeta))\\
&+\sum_{x_1,y_1\in S}\sum_{x_2,y_2\in S}\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\\
&\qquad\qquad\times\big(f(\xi^{x_1,y_1},\zeta^{x_2,y_2})-f(\xi,\zeta)\big)
\end{aligned}
\tag{3.9}
$$

is the generator of a Markovian coupling between two copies of the Markov process defined by (2.1), provided that for all pairs of configurations $(\xi,\zeta)\in\Omega^2$ the coupling rates $G_{\xi,\zeta}$ are non-negative and the following hold.

1) For all $(x_1,y_1)\in S^2$,

$$\sum_{x_2,y_2\in S}\zeta(x_2)(1-\zeta(y_2))\,G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\le\Gamma_\xi(x_1,y_1),\tag{3.10}$$

2) for all $(x_2,y_2)\in S^2$,

$$\sum_{x_1,y_1\in S}\xi(x_1)(1-\xi(y_1))\,G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\le\Gamma_\zeta(x_2,y_2).\tag{3.11}$$

Proof of Proposition 3.2 is postponed to Section 5. As a shorthand notations for the sums appearing in the left hand side of equations (3.10)–(3.11), we define for all couples of configurations $(\xi,\zeta)\in\Omega\times\Omega$ and all $(x,y)\in S^2$, the quantities

$$\varphi_{\xi,\zeta}(x,y)\quad:=\quad\sum_{x',y'\in S}\zeta(x')(1-\zeta(y'))G_{\xi,\zeta}(x,y;x',y'),\tag{3.12}$$

$$\overline{\varphi}_{\xi,\zeta}(x,y)\quad:=\quad\sum_{x',y'\in S}\xi(x')(1-\xi(y'))G_{\xi,\zeta}(x',y';x,y).\tag{3.13}$$

3.1.3 Definition of coupling rates and proof of increasingness

We now give the set of coupling rates $G_{\xi,\zeta}(x,y;x',y')$ which defines an increasing coupling.

We first introduce some notations. Let ξ and ζ be two configurations in Ω. For all $x\in S$ such that $\xi(x)=\zeta(x)=1$, we define the two sets

$$Y^x_{\xi,\zeta} = \{y\in S:\xi(y)=0,\zeta(y)=1,\Gamma_\xi(x,y)>0\}, \quad (3.14)$$
$$\overline{Y}^x_{\xi,\zeta} = \{y\in S:\xi(y)=\zeta(y)=0,\ \Gamma_\zeta(x,y)>\Gamma_\xi(x,y)\}. \quad (3.15)$$

Whenever they are non empty, we define an arbitrary order on these two sets, possibly depending on ξ, ζ and x, and denote by $y^{x,k}_{\xi,\zeta}$ (respectively $\overline{y}^{x,k}_{\xi,\zeta}$) the k^{th} element in $Y^x_{\xi,\zeta}$ (respectively $\overline{Y}^x_{\xi,\zeta}$).

Similarly, for all $y\in S$ such that $\xi(y)=\zeta(y)=0$, we define

$$X^y_{\xi,\zeta} = \{x\in S:\xi(x)=\zeta(x)=1,\Gamma_\xi(x,y)>\Gamma_\zeta(x,y)\}, \quad (3.16)$$
$$\overline{X}^y_{\xi,\zeta} = \{x\in S:\xi(x)=0,\zeta(x)=1,\Gamma_\zeta(x,y)>0\}. \quad (3.17)$$

We define an arbitrary order on these two sets as well, possibly depending on ξ, ζ and y, and denote by $x^{y,k}_{\xi,\zeta}$ (respectively $\overline{x}^{y,k}_{\xi,\zeta}$) the k^{th} element in $X^y_{\xi,\zeta}$ (respectively $\overline{X}^y_{\xi,\zeta}$).

For definiteness, when one of the above sets is finite or empty, say $|Y^x_{\xi,\zeta}|=C_Y<\infty$, we may extend the ordered sequence of its elements to an infinite one, $(y^{x,n}_{\xi,\zeta})_{n>0}$, by setting arbitrarily $y^{x,n}_{\xi,\zeta}=0$ for all $n>C_Y$.

For all $x\in S$ such that $\xi(x)=\zeta(x)=1$, we define the two series $\left(S^{x,n}_{\xi,\zeta}\right)_{n\geq 0}$ and $\left(\overline{T}^{x,n}_{\xi,\zeta}\right)_{n\geq 0}$ such that $S^{x,0}_{\xi,\zeta}=0$, $\overline{T}^{x,0}_{\xi,\zeta}=0$, and

$$S^{x,n}_{\xi,\zeta} = \sum_{k=1}^{n\wedge|Y^x_{\xi,\zeta}|}\Gamma_\xi(x,y^{x,k}_{\xi,\zeta}) \qquad \forall n>0, \quad (3.18)$$
$$\overline{T}^{x,n}_{\xi,\zeta} = \sum_{k=1}^{n\wedge|\overline{Y}^x_{\xi,\zeta}|}\Big[\Gamma_\zeta(x,\overline{y}^{x,k}_{\xi,\zeta})-\Gamma_\xi(x,\overline{y}^{x,k}_{\xi,\zeta})\Big]^+ \qquad \forall n>0. \quad (3.19)$$

Similarly, for all $y\in S$ such that $\xi(y)=\zeta(y)=0$, we define the two series $\left(T^{y,n}_{\xi,\zeta}\right)_{n\geq 0}$ and $\left(\overline{S}^{y,n}_{\xi,\zeta}\right)_{n\geq 0}$ such that $T^{y,0}_{\xi,\zeta}=\overline{S}^{y,0}_{\xi,\zeta}=0$ and

$$T^{y,n}_{\xi,\zeta} = \sum_{k=1}^{n\wedge|X^y_{\xi,\zeta}|}\Big[\Gamma_\xi(x^{y,k}_{\xi,\zeta},y)-\Gamma_\zeta(x^{y,k}_{\xi,\zeta},y)\Big]^+ \qquad \forall n>0, \quad (3.20)$$
$$\overline{S}^{y,n}_{\xi,\zeta} = \sum_{k=1}^{n\wedge|\overline{X}^y_{\xi,\zeta}|}\Gamma_\zeta(\overline{x}^{y,k}_{\xi,\zeta},y) \qquad \forall n>0. \quad (3.21)$$

Note that by definition, the four series have nonnegative terms and are nondecreasing, and by (2.4), they are also convergent.

Finally, for any two convergent series $(S_n)_{n\geq 0}$ and $(T_n)_{n\geq 0}$, we define the quantity $H_{n,m}(S,T)$ for all $n>0$ and all $m>0$ as

$$H_{m,n}(S_., T_.) = S_m \wedge T_n - S_{m-1} \wedge T_n - S_m \wedge T_{n-1} + S_{m-1} \wedge T_{n-1}. \quad (3.22)$$

Note that $H_{m,n}(S_., T_.) \geq 0$ whenever $S_.$ and $T_.$ are nondecreasing series. Moreover we have

$$H_{m,n}(S_., T_.) = (S_m \wedge T_n - S_{m-1} \vee T_{n-1})^+. \quad (3.23)$$

Indeed to check that the right-hand sides of (3.22) and (3.23) are equal, we consider all the possible cases, that is $S_m \leq T_{n-1}$, $S_{m-1} \leq T_{n-1} \leq S_m \leq T_n$, $S_{m-1} \leq T_{n-1} \leq T_n \leq S_m$, $T_n \leq S_{m-1}$, $S_m \leq T_n$ and $S_{m-1} \leq T_n \leq S_m$.

We can now state Proposition 3.3, which ends the proof of Theorem 2.4.

Proposition 3.3. *Under conditions* (2.6)–(2.7)*, the generator given by* (3.9) *with coupling rates* $G_{\xi,\zeta}$ *below, defines an increasing Markovian coupling.*

$$G_{\xi,\zeta}(x,y;x',y')$$

$$= \begin{cases} \delta(x,x')\,\delta(y,y')\,\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y) \\ \quad +\delta(x,x')\displaystyle\sum_{m,n>0}\delta(y,y^{x,m}_{\xi,\zeta})\,\delta(y',\overline{y}^{x,n}_{\xi,\zeta})\,H_{m,n}(S^{x,\cdot}_{\xi,\zeta},\overline{T}^{x,\cdot}_{\xi,\zeta}) \\ \quad +\delta(y,y')\displaystyle\sum_{m,n>0}\delta(x,x^{y,m}_{\xi,\zeta})\delta(x',\overline{x}^{y,n}_{\xi,\zeta})\,H_{m,n}(T^{y,\cdot}_{\xi,\zeta},\overline{S}^{x,\cdot}_{\xi,\zeta}) \\ \qquad\qquad\qquad\qquad \textit{if } \xi\leq\zeta, \\ \delta(x,x')\,\delta(y,y')\,\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y) \\ \quad +\delta(x,x')\displaystyle\sum_{m,n>0}\delta(y,\overline{y}^{x,m}_{\zeta,\xi})\,\delta(y',y^{x,n}_{\zeta,\xi})\,H_{m,n}(\overline{T}^{x,\cdot}_{\zeta,\xi},S^{x,\cdot}_{\zeta,\xi}) \\ \quad +\delta(y,y')\displaystyle\sum_{m,n>0}\delta(x,\overline{x}^{y,m}_{\zeta,\xi})\,\delta(x',x^{y,n}_{\zeta,\xi})\,H_{m,n}(\overline{S}^{y,\cdot}_{\zeta,\xi},T^{y,\cdot}_{\zeta,\xi}) \\ \qquad\qquad\qquad\qquad \textit{if } \xi>\zeta, \\ 0 \qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases} \quad (3.24)$$

Remark 3.4. *With the above choice, jumps are uncoupled unless* ξ *and* ζ *are ordered. In such a case, the coupling rate* $G_{\xi,\zeta}(x,y;x',y')$ *is possibly non zero only if the two coupled jumps have either the same initial point, the same final point, or both.*

Remark 3.5. *When the two configurations are equal,* $\zeta = \xi$*, both* $\overline{Y}^x_{\xi,\xi} = \emptyset$ *for all* $x \in S$ *and* $X^y_{\xi,\xi} = \emptyset$ *for all* $y \in S$*. The only nonzero coupling rates are thus the diagonal terms* $G_{\xi,\xi}(x,y;x,y) = \Gamma_\xi(x,y)$ *so that marginals remain equal.*

Remark 3.6. *In definition* (3.24)*, the first (resp. second) sum appearing in the right hand side in the case* $\xi \le \zeta$ *is zero except possibly when there is a jump in the first marginal* ξ *from a site* x *to a site* $y \in Y^x_{\xi,\zeta}$ *coupled with a jump in the second marginal* ζ *from the same site* x *to a site* $y' \in \overline{Y}^x_{\xi,\zeta}$ *(respectively a jump in the first marginal from a site in* $X^y_{\xi,\zeta}$ *coupled to a jump in the second marginal from a site in* $\overline{X}^y_{\xi,\zeta}$ *to the same site* y*). Moreover, by the definitions* (3.14)–(3.15) *of* $Y^x_{\xi,\zeta}$ *and* $\overline{Y}^x_{\xi,\zeta}$ *(resp. definitions* (3.16)–(3.17) *of* $X^y_{\xi,\zeta}$ *and* $\overline{X}^y_{\xi,\zeta}$*)* $y \neq y'$ *in the first sum while* $x \neq x'$ *in the second sum (in both cases* $\xi \le \zeta$ *and* $\xi > \zeta$*).*

Remark 3.7. *The ordering in the four ensembles defined in* (3.14)–(3.17) *can be chosen arbitrarily, possibly as a function of the configurations* ξ *and* ζ *and on the (initial or final) common jump site. The best choice (in view of the desired goal) may depend on the particular system at hand, and different choices lead to different increasing couplings. Furthermore, one can prove that all these couplings are extremal in the sense that they cannot be written as a convex combination of other increasing couplings, while any convex combination of these is again an increasing coupling.*

Corollary 3.8. *In the particular case* $S = \mathbb{Z}$ *endowed with the usual order, the coupling rates in Proposition 3.3 have a simpler form: Let* $\xi, \zeta \in \Omega$ *be two configurations. For all* $(x_1, y_1) \in S^2$*,*

$$G_{\xi,\zeta}(x_1,y_1;x_1,y_1) = \Gamma_\xi(x_1,y_1) \wedge \Gamma_\zeta(x_1,y_1). \tag{3.25}$$

For all $(x_1,y_1,x_2,y_2) \in S^4$ *such that* $(x_1,y_1) \neq (x_2,y_2)$*,*

$$G_{\xi,\zeta}(x_1,y_1;x_2,y_2) =$$

$$\begin{cases} \delta_{x_1,x_2}\left[H^i_{\xi,\zeta}(x_1;y_1,y_2)\right]^+ + \delta_{y_1,y_2}\left[H^f_{\xi,\zeta}(x_1,x_2;y_1)\right]^+ & \text{if } \xi \le \zeta, \\ \delta_{x_1,x_2}\left[H^i_{\zeta,\xi}(x_1;y_2,y_1)\right]^+ + \delta_{y_1,y_2}\left[H^f_{\zeta,\xi}(x_2,x_1;y_1)\right]^+ & \text{if } \xi > \zeta, \\ 0 & \text{otherwise.} \end{cases} \tag{3.26}$$

with, for all $(x,y,z) \in S^3$*,*

$$H^i_{\xi,\zeta}(x;y,z) = \Big(\sum_{y'\le y}(1-\xi(y'))\zeta(y')\Gamma_\xi(x,y')\Big)\wedge\Big(\sum_{z'\le z}(1-\zeta(z'))\big[\Gamma_\zeta(x,z')-\Gamma_\xi(x,z')\big]^+\Big)$$
$$-\Big(\sum_{y'<y}(1-\xi(y'))\zeta(y')\Gamma_\xi(x,y')\Big)\vee\Big(\sum_{z'<z}(1-\zeta(z'))\big[\Gamma_\zeta(x,z')-\Gamma_\xi(x,z')\big]^+\Big), \tag{3.27}$$

$$H^f_{\xi,\zeta}(x,y;z) = \Big(\sum_{x'\le x}\xi(x')\big[\Gamma_\xi(x',z)-\Gamma_\zeta(x',z)\big]^+\Big)\wedge\Big(\sum_{y'\le y}\zeta(y')(1-\xi(y'))\Gamma_\zeta(y',z)\Big)$$
$$-\Big(\sum_{x'<x}\xi(x')\big[\Gamma_\xi(x',z)-\Gamma_\zeta(x',z)\big]^+\Big)\vee\Big(\sum_{y'<y}\zeta(y')(1-\xi(y'))\Gamma_\zeta(y',z)\Big). \tag{3.28}$$

3.2 Proof of Theorem 2.7

The above increasing Markovian coupling preserves the ordering between marginals when they are ordered but leaves them otherwise uncoupled. In order to deal with unordered configurations and control their discrepancies, we show in the next proposition how to build an attractive Markov process out of an increasing one.

Proposition 3.9. *Suppose that the process defined by* (2.1) *is monotone on* $\Omega=\{0,1\}^S$. *Let* $\overline{\mathcal{L}}$ *be an associated increasing process defined on* $\Omega\times\Omega$ *as in Proposition 3.2, with the coupling rates defined in Proposition 3.3. The operator* $\overline{\mathcal{L}}^D$ *defined on all cylinder functions on* $\Omega\times\Omega$ *as*

$$\begin{aligned}\overline{\mathcal{L}}^D f(\xi,\zeta) = &\sum_{x_1,y_1\in S}\xi(x_1)(1-\xi(y_1))\Gamma_\xi(x_1,y_1)\big(f(\xi^{x_1,y_1},\zeta)-f(\xi,\zeta)\big)\\ &+\sum_{x_2,y_2\in S}\zeta(x_2)(1-\zeta(y_2))\Gamma_\zeta(x_2,y_2)\big(f(\xi,\zeta^{x_2,y_2})-f(\xi,\zeta)\big)\\ &+\sum_{x_1,y_1\in S}\sum_{x_2,y_2\in S}\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)\\ &\times\big(f(\xi^{x_1,y_1},\zeta^{x_2,y_2})-f(\xi^{x_1,y_1},\zeta)-f(\xi,\zeta^{x_2,y_2})+f(\xi,\zeta)\big),\end{aligned} \tag{3.29}$$

where for all $(\xi,\zeta)\in\Omega\times\Omega$, *all* $(x_1,y_1)\in S^2$ *and all* $(x_2,y_2)\in S^2$,

$$
\begin{aligned}
G^D_{\xi,\zeta}&(x_1,y_1;x_2,y_2)\\
&= \sum_{x,y\in S} (\xi\vee\zeta)(x)(1-(\xi\vee\zeta)(y))\\
&\qquad\qquad \times \frac{1}{N_{\xi,\zeta}(x,y)} G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y)\, G_{\xi\vee\zeta,\zeta}(x,y;x_2,y_2), \qquad (3.30)
\end{aligned}
$$

$$
N_{\xi,\zeta}(x,y) \quad = \quad \begin{cases} \Gamma_{\xi\vee\zeta}(x,y) & \textit{if } \Gamma_{\xi\vee\zeta}(x,y)>0,\\ 1 & \textit{otherwise.}\end{cases} \qquad (3.31)
$$

is a quasi attractive coupling under which, in any coupled transition, the discrepancies in the involved sites do not increase.

Remark 3.10. *When the configurations ξ,ζ are ordered, $\xi\le\zeta$, for all $(x_1,y_1,x_2,y_2)\in S^4$ such that $\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))\neq 0$, we have*

$$
G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2) = G_{\xi,\zeta}(x_1,y_1;x_2,y_2), \qquad (3.32)
$$

so that $\overline{\mathcal{L}}^D f(\xi,\zeta)$ in (3.29) reduces to $\overline{\mathcal{L}} f(\xi,\zeta)$ in (3.9) when marginals are ordered.

3.3 Invariant measures

In Proposition 3.9 above, in any coupled transition, the discrepancies in the involved sites are proven to be non increasing, but the characterization of the set of invariant measures, Theorem 2.9, requires a bit more, namely the proof that there is a positive probability that any pair of discrepancies of opposite sign (that is, the marginals have opposite occupation numbers, $\xi(x)>\zeta(x)$, $\xi(y)<\zeta(y)$ for some x, y in S) disappears in finite time under the coupled process. As for the case of simple exclusion process, this requires additional hypotheses on the process. One may consider processes for which S is fully connected in the sense of Definition 2.8. We then have the following:

Proposition 3.11. *Consider an exclusion process with generator* (2.1) *such that S is fully connected in the sense of Definition 2.8. Whenever the jump rates are such that all inequalities in* (2.6) *and* (2.7) *are strict, there exists an increasing coupling that is attractive, so that for the coupled process extremal, translation invariant, invariant probability measures are supported on the set of coupled configurations $\{(\xi,\zeta):\xi\le\zeta\}\cup\{(\xi,\zeta):\xi>\zeta\}$.*

Next proposition gives an example of such an attractive coupling.

Proposition 3.12. *Consider an exclusion process with generator* (2.1) *such that S is fully connected in the sense of Definition 2.8, and such that for the jump rates, all inequalities in* (2.6) *and* (2.7) *are strict. Then the following set of coupling rates defines a new increasing coupling.*

$$G_{\xi,\zeta}(x,y;x',y') = \begin{cases} \delta(x,x')\,\delta(y,y')\,\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y) \\ \quad +\delta(x,x')\mathbf{1}_{y\in Y^x_{\xi,\zeta}}\,\mathbf{1}_{y'\in\overline{Y}^x_{\xi,\zeta}}\,\dfrac{1}{N^{x,*}_{\xi,\zeta}}\,\Gamma_\xi(x,y)\,\left[\Gamma_\zeta(x,y')-\Gamma_\xi(x,y')\right]^+ \\ \quad +\delta(y,y')\mathbf{1}_{x\in X^y_{\xi,\zeta}}\,\mathbf{1}_{x'\in\overline{X}^y_{\xi,\zeta}}\,\dfrac{1}{\overline{N}^{y,*}_{\xi,\zeta}}\,\left[\Gamma_\xi(x,y)-\Gamma_\zeta(x,y)\right]^+\,\Gamma_\zeta(x',y) \\ \hfill \text{if } \xi\le\zeta, \\ \delta(x,x')\,\delta(y,y')\,\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y) \\ \quad +\delta(x,x')\mathbf{1}_{y\in\overline{Y}^x_{\zeta,\xi}}\,\mathbf{1}_{y'\in Y^x_{\zeta,\xi}}\,\dfrac{1}{N^{x,*}_{\zeta,\xi}}\,\left[\Gamma_\xi(x,y)-\Gamma_\zeta(x,y)\right]^+\,\Gamma_\zeta(x,y') \\ \quad +\delta(y,y')\mathbf{1}_{x\in\overline{X}^y_{\zeta,\xi}}\,\mathbf{1}_{x'\in X^y_{\zeta,\xi}}\,\dfrac{1}{\overline{N}^{y,*}_{\zeta,\xi}}\,\Gamma_\xi(x,y)\,\left[\Gamma_\zeta(x',y)-\Gamma_\xi(x',y)\right]^+ \\ \hfill \text{if } \xi>\zeta, \\ 0 \hfill \text{otherwise,} \end{cases} \tag{3.33}$$

where

$$N^{x,*}_{\xi,\zeta} = \begin{cases} S^{x,*}_{\xi,\zeta} & \text{if } S^{x,*}_{\xi,\zeta}>0, \\ 1 & \text{otherwise,} \end{cases} \tag{3.34}$$

and similar definitions for the other normalization factors, where $S^{x,*}_{\xi,\zeta}$, $\overline{S}^{y,*}_{\xi,\zeta}$, $S^{x,*}_{\zeta,\xi}$ and $\overline{S}^{y,*}_{\zeta,\xi}$ are the limits of the series (3.18)–(3.21), *and are properly defined in Equations* (5.2)–(5.5).

Using a similar construction as in Proposition 3.9, a new quasi attractive coupling $\overline{\mathcal{L}}^D$ can be constructed on top of the above increasing coupling (3.33). *Whenever the conditions of Proposition 3.11 hold, this new coupling is attractive.*

Proposition 3.11 is the crucial step in the determination of the set $(\mathcal{I}\cap\mathcal{S})_e$, and in proving Theorem 2.9. This theorem is analogous to [3, Proposition 3.1] and to [8, Theorem 5.13], to which we refer for a full description of this approach. It has the same (classical skeleton of) proof, although the transition rates in our case depend on more sites than the departure and arrival sites of a jump. The key point of the proof is to establish that for the coupled process, all extremal, translation invariant and invariant probability measures are supported on the set of coupled configurations $\{(\xi,\zeta) : \xi \le \zeta\} \cup \{(\xi,\zeta) : \xi > \zeta\}$, and this is given by Proposition 3.11.

In the next Section, we apply our results to various simple but non trivial examples.

4 Applications

In this section we illustrate our results through various examples, and check for them monotonicity conditions of Theorem 2.4. Whenever these conditions are fulfilled, we construct the coupled generators $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}^D$ by applying Propositions 3.2 and 3.9. In one example we also determine the coupling rates for an attractive coupling given in Proposition 3.12.

In Subsection 4.1, we show that in the case of simple exclusion, our construction reduces to basic coupling. In Subsection 4.2 we consider the exclusion process with speed change introduced by F. Spitzer in [18] and studied by T.M. Liggett in [14]. In this case, we extend the range of previously known attractiveness conditions to necessary and sufficient ones. Finally, in Subsections 4.3 and 4.4 we introduce and study models inspired by traffic flows.

4.1 Simple exclusion

For the simple exclusion process (see Remark 2.1), jump rates are independent on the configuration,

$$\Gamma_\zeta(x,y) - \Gamma_\xi(x,y) = 0 \tag{4.1}$$

for all ξ, ζ in Ω and all x, y in S.

Monotonicity conditions (2.6)–(2.7) reduce to non negativity of jump rates and are thus always satisfied. We show below that the coupling defined in Proposition 3.9 reduces to basic coupling in this case. In fact, using simple exclusion rates (2.3), the jump rates defined through Formula (3.24) become, for all $(x_1, y_1, x_2, y_2) \in S^4$:

$$G_{\xi,\zeta}(x_1,y_1;x_2,y_2) = \begin{cases} \delta_{x_1,x_2}\,\delta_{y_1,y_2}\,p(x_1,y_1) & \text{if } \xi \le \zeta \text{ or } \xi > \zeta, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Therefore, the increasing Markovian coupling $\overline{\mathcal{L}}$ defined through Proposition 3.3 coincides with basic coupling on configurations with ordered marginals. Hence we have

$$\overline{\varphi}_{\xi,\xi\vee\zeta}(x,y) = \xi(x)(1-\xi(y))p(x,y), \tag{4.3}$$
$$\varphi_{\xi\vee\zeta,\zeta}(x,y) = \zeta(x)(1-\zeta(y))p(x,y), \tag{4.4}$$

and

$$N_{\xi,\zeta}(x,y) = \begin{cases} p(x,y) & \text{if } p(x,y) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.5}$$

and using (4.2),

$$\begin{aligned} G^D_{\xi,\zeta}&(x_1,y_1;x_2,y_2) \\ &= \sum_{x,y} (\xi\vee\zeta)(x)(1-(\xi\vee\zeta)(y)) \\ &\qquad\qquad \times \frac{1}{N_{\xi,\zeta}(x,y)} G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y) G_{\xi\vee\zeta,\zeta}(x,y;x_2,y_2) \\ &= \delta_{x_1,x_2}\,\delta_{y_1,y_2}\,(\xi\vee\zeta)(x_1)(1-(\xi\vee\zeta)(y_1))\,p(x_1,y_1). \end{aligned} \tag{4.6}$$

Finally, the generator of the coupling process defined in Proposition 3.9 reads

$$\begin{aligned} &\overline{\mathcal{L}}^D f(\xi,\zeta) = \\ &\quad \sum_{x,y\in S} p(x,y)\xi(x)(1-\xi(y))\zeta(x)(1-\zeta(y))\big(f(\xi^{x,y},\zeta^{x,y}) - f(\xi,\zeta)\big) \\ &\quad + \sum_{x,y\in S} p(x,y)\xi(x)(1-\xi(y))\big(1-\zeta(x)(1-\zeta(y))\big)\big(f(\xi^{x,y},\zeta) - f(\xi,\zeta)\big) \\ &\quad + \sum_{x,y\in S} p(x,y)\zeta(x)(1-\zeta(y))\big(1-\xi(x)(1-\xi(y))\big)\big(f(\xi,\zeta^{x,y}) - f(\xi,\zeta)\big). \end{aligned} \tag{4.7}$$

Hence $\overline{\mathcal{L}}^D$ identifies to the basic coupling generator for SEP. This comes from the fact that non zero coupling rates in (4.2) are diagonal, so that the summation in formula (3.30) reduces here to a single, diagonal, term.

We need to emphasize here that SEP is the only exclusion process for which the basic coupling is monotone. Let us consider some exclusion process with rates $\Gamma_\eta(x,y)$, possibly dependent on the configuration η. We now prove that monotonicity of the associated basic coupling implies that the exclusion process is the simple exclusion, that is the rates are independent on the configurations. The generator of the basic coupling reads

$$
\begin{aligned}
&\tilde{\mathcal{L}}f(\xi,\zeta) = \\
&\quad \sum_{x,y\in S} \xi(x)(1-\xi(y))\zeta(x)(1-\zeta(y))\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y) \\
&\qquad\quad \times\big(f(\xi^{x,y},\zeta^{x,y})-f(\xi,\zeta)\big) \\
&\quad + \sum_{x,y\in S} \xi(x)(1-\xi(y))\big(\Gamma_\xi(x,y)-\zeta(x)(1-\zeta(y))\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y)\big) \\
&\qquad\quad \times\big(f(\xi^{x,y},\zeta)-f(\xi,\zeta)\big) \\
&\quad + \sum_{x,y\in S} \zeta(x)(1-\zeta(y))\big(\Gamma_\zeta(x,y)-\xi(x)(1-\xi(y))\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y)\big) \\
&\qquad\quad \times\big(f(\xi,\zeta^{x,y})-f(\xi,\zeta)\big).
\end{aligned}
\tag{4.8}
$$

Now take two configurations ξ, ζ such that $\xi \leq \zeta$ and $(x,y) \in S^2$ such that $\xi(x) = 1$ and $\zeta(y) = 0$. Monotonicity implies then that the rates of uncoupled jumps from x to y are identically zero since otherwise order would be broken either in x (in case of an uncoupled jump of the ζ-particle) or in y (in case of an uncoupled jump of the ξ-particle). Therefore monotonicity of the basic coupling implies that the jump rates are equal, $\Gamma_\xi(x,y) = \Gamma_\zeta(x,y)$. This can be extended to any pair of configurations (ξ,ζ) whenever $\xi(x) = \zeta(x) = 1$ and $\xi(y) = \zeta(y) = 0$, as follows. Let $\sigma = \xi \wedge \zeta$; clearly $\sigma(x) = 1$, $\sigma(y) = 0$ and both $\sigma \leq \xi$ and $\sigma \leq \zeta$. Thus from the above result, $\Gamma_\sigma(x,y) = \Gamma_\xi(x,y)$, $\Gamma_\sigma(x,y) = \Gamma_\zeta(x,y)$ and thus $\Gamma_\xi(x,y) = \Gamma_\zeta(x,y)$. Jump rates are thus independent on the configuration, $\Gamma_\eta(x,y) = p(x,y)$ and the only exclusion process for which basic coupling is monotone is the simple exclusion process.

4.2 Exclusion processes with speed change

We consider here a family of models, introduced by F. Spitzer in his seminal paper [18], and later studied by T.M. Liggett in [14, Part II, Sections 1.1, 4.1]. The jump rates from a site x to a site y are defined as the product of a configuration dependent velocity $c_\xi(x)$ for the particle at site x and a configuration independent jump intensity between sites x and y. This form is particularly interesting in the original context of a lattice gas. The jump rates thus read

$$\Gamma_\eta(x,y) = q(x,y)c_\eta(x), \tag{4.9}$$

where $q : S \times S \to [0,+\infty)$ satisfies for all $x \in S$, $q(x,x) = 0$ and

$$\sup_{x\in S}\sum_{y\in S}[q(x,y)+q(y,x)] < +\infty \tag{4.10}$$

and c satisfies

$$\sup_{x\in S,\eta\in X} c_\eta(x) < +\infty; \qquad \sup_{x\in S}\sum_{y\in S}\sup_{\eta\in X}|c_{\eta^y}(x)-c_\eta(x)| < +\infty, \tag{4.11}$$

where

$$\eta^y(z) = \begin{cases} 1-\eta(y) & \text{if } z=y,\\ \eta(z) & \text{otherwise.}\end{cases} \tag{4.12}$$

In this context, the monotonicity conditions (2.6)–(2.7) of Theorem 2.4 read: *For all pair of configurations* $(\xi,\zeta)\in\Omega^2$ *such that* $\xi\le\zeta$*, one has*

1) *for all* $y\in S$ *such that* $\zeta(y)=0$*,*

$$\sum_{x\in S}\xi(x)q(x,y)\left[c_\xi(x)-c_\zeta(x)\right]^+ \le \sum_{x\in S}\zeta(x)(1-\xi(x))\,q(x,y)\,c_\zeta(x), \tag{4.13}$$

2) *and for all* $x\in S$ *such that* $\xi(x)=1$*,*

$$\Big(\sum_{y\in S}(1-\zeta(y))\,q(x,y)\Big)\left[c_\zeta(x)-c_\xi(x)\right]^+ \le \Big(\sum_{y\in S}\zeta(y)(1-\xi(y))\,q(x,y)\Big)c_\xi(x). \tag{4.14}$$

Note that due to the special form of the jump rates (4.9), jump velocities factorize on both sides of the equations in (4.14), while there is no similar simplification in (4.13).

Within this class of models, T.M. Liggett [14] introduced another set of sufficient conditions for monotonicity, which read in our notations as conditions (4.15)–(4.16) below: *For all pair of configurations* $(\xi,\zeta)\in\Omega^2$ *such that* $\xi\le\zeta$*, one has*

1) *for all* $x\in S$*,*

$$c_\xi(x)\le c_\zeta(x), \tag{4.15}$$

2) *for all* $x\in S$ *such that* $\xi(x)=1$*,*

$$\Big(\sum_{y\in S}(1-\zeta(y))\,q(x,y)\Big)\,c_\zeta(x) \le \Big(\sum_{y\in S}(1-\xi(y))\,q(x,y)\Big)c_\xi(x). \tag{4.16}$$

The purpose of this subsection is to compare both results and show that (4.15)–(4.16) are sufficient, but not necessary, conditions.

First, we have the following

Proposition 4.1. *For exclusion process with speed change and rates defined as in* (4.9)*, conditions* (4.15)–(4.16) *are sufficient conditions for monotonicity.*

Proof. we verify that conditions (4.15)–(4.16) imply necessary and sufficient conditions (4.13)–(4.14).

Suppose that conditions (4.15)–(4.16) hold. On one hand, under the non-decreasing conditions (4.15), the left hand side of inequation (4.13) is identically zero; inequations (4.13) are therefore trivially verified since their right hand side is always non negative. On the other hand, for any two configurations such that $\xi \le \zeta$ and for all $x \in S$ such that $\xi(x) = 1$, using conditions (4.15)–(4.16), the left hand side of condition (4.14) can be bounded as:

$$\begin{aligned}
&\sum_{y\in S}(1-\zeta(y))\,q(x,y)\big[c_\zeta(x)-c_\xi(x)\big]^+\\
&\qquad=\sum_{y\in S}(1-\zeta(y))\,q(x,y)\big(c_\zeta(x)-c_\xi(x)\big)\\
&\qquad=\sum_{y\in S}(1-\zeta(y))\,q(x,y)\,c_\zeta(x)-\sum_{y\in S}(1-\zeta(y))\,q(x,y)\,c_\xi(x)\\
&\qquad\le\sum_{y\in S}(1-\xi(y))\,q(x,y)\,c_\xi(x)-\sum_{y\in S}(1-\zeta(y))\,q(x,y)\,c_\xi(x)\\
&\qquad=\sum_{y\in S}(\zeta(y)-\xi(y))\,q(x,y)\,c_\xi(x)\\
&\qquad=\sum_{y\in S}\zeta(y)(1-\xi(y))\,q(x,y)\,c_\xi(x).
\end{aligned}\tag{4.17}$$

In the above calculation, first line follows from conditions (4.15), third line from conditions (4.16) applied to the first sum, and last line from the identity $\xi(y) = \zeta(y)\,\xi(y)$, which holds whenever $\xi \le \zeta$. Last line is the right hand side of (4.14).Therefore inequalities (4.14) are also verified under conditions (4.15)–(4.16). □

Now, within conditions (4.15)–(4.16), one can define a class of attractive exclusion processes with speed change, as follows:

Proposition 4.2. *Consider jump intensities* $q(\cdot,\cdot)$ *as in* (4.10) *and velocities* c_η *defined as*

$$c_\eta(x)=\varphi\Big(\sum_{y\in S}(1-\eta(y))\,q(x,y)\Big) \text{ for all } x\in S \text{ and all } \eta\in\Omega, \tag{4.18}$$

where $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ *is a decreasing function, differentiable with bounded derivative, such that* $u \mapsto u\,\varphi(u)$ *is increasing. Then the exclusion process with jump rates* $\Gamma_\eta(x,y) = q(x,y)c_\eta(x)$ *is well defined and attractive.*

Proof. We first prove that the velocities c_η verify conditions (4.11).

Since φ is a decreasing function on $\mathbb{R}^+$, we have first

$$\sup_{x\in S,\eta\in X} c_\eta(x) \le \varphi(0) < +\infty;$$

On the other hand, since φ is differentiable on $\mathbb{R}^+$ with bounded derivative, we get, using definition (4.12):

$$\begin{aligned}
&\sup_{x\in S}\sum_{y\in S}\sup_{\eta\in X}\big|c_{\eta^y}(x)-c_\eta(x)\big| \\
&= \sup_{x\in S}\sum_{y\in S}\sup_{\eta\in X}\Big|\varphi\Big(\sum_{z\in S}(1-\eta^y(z))\,q(x,z)\Big)-\varphi\Big(\sum_{z\in S}(1-\eta(z))\,q(x,z)\Big)\Big| \\
&\le \sup_{x\in S}\sum_{y\in S}\sup_{\eta\in X}\Big\{\|\varphi'\|_\infty\Big|\Big(\sum_{z\in S}(1-\eta^y(z))\,q(x,z)\Big)-\Big(\sum_{z\in S}(1-\eta(z))\,q(x,z)\Big)\Big|\Big\} \\
&= \|\varphi'\|_\infty \sup_{x\in S}\sum_{y\in S} q(x,y) \\
&< +\infty,
\end{aligned}$$

where the last inequality follows from equation (4.10) and $\|\varphi'\|_\infty < \infty$. The exclusion process of Proposition 4.1 is thus well defined. We now prove that it fulfills conditions (4.15)–(4.16).

First, for all $\eta \in \Omega$ and all $x \in S$, we define $u_\eta(x)$ as the quantity

$$u_\eta(x) = \sum_{y\in S}(1-\xi(z))q(x,y). \tag{4.19}$$

Thus we have

$$c_\eta(x) = \varphi\big(u_\eta(x)\big). \tag{4.20}$$

We note that for any pair of configurations (ξ,ζ) such that $\xi \le \zeta$ and for any $x \in S$, one has

$$u_\xi(x) \ge u_\zeta(x).$$

Since φ is a decreasing function on $\mathbb{R}^+$, we thus get for all $x \in S$ and all $\xi \le \zeta$,

$$c_\xi(x) = \varphi\big(u_\xi(x)\big) \le \varphi\big(u_\zeta(x)\big) = c_\zeta(x).$$

Conditions (4.15) are verified.

Now, for any pair of configurations (ξ,ζ) such that $\xi \le \zeta$ and for any $x \in S$, one has $u_\xi(x) \ge u_\zeta(x)$, and since $u \mapsto u\varphi(u)$ is an increasing function, we get

$$u_\xi(x)\,\varphi\big(u_\xi(x)\big) \ge u_\zeta(x)\,\varphi\big(u_\zeta(x)\big).$$

Inserting (4.20) and (4.19) in the above equation, we get

$$\big(\sum_{y\in S}(1-\zeta(y))\,q(x,y)\big)\,c_\zeta(x) \le \big(\sum_{y\in S}(1-\xi(y))\,q(x,y)\,\big)c_\xi(x), \qquad (4.21)$$

which is (4.16). □

Though conditions of increasing speeds (4.15) appear to be an additional requirement beyond conditions (4.13)–(4.14), it was nevertheless unclear whether monotone exclusion processes with non increasing speed could exist. In what follows, we answer positively to this question giving an explicit example of a monotone process with decreasing speeds.

In the case of decreasing speeds, the role of equations (4.13) and (4.14) are exchanged with respect to the previous case and equations (4.14) are trivially fulfilled. However, with the rates (4.9), equations (4.13) do not factorize as before but become in this case:

For all $y \in S$ such that $\zeta(y) = 0$,

$$\sum_{x\in S}\xi(x)q(x,y)\,c_\xi(x) \le \sum_{x\in S}\zeta(x)\,q(x,y)\,c_\zeta(x), \qquad (4.22)$$

so that speed functions have to fulfil a set of coupled inequalities indexed by all possible values of y, which are difficult to solve on a general ground. Nevertheless, we have the following

Proposition 4.3. *Let $S = \mathbb{Z}$. For $L \in \mathbb{N}\setminus\{0\}$ fixed, define*

$$\forall (x,y)\in S^2, q(x,y) = \mathbf{1}_{\{1<y-x\le L\}}, \qquad (4.23)$$

$$\forall \eta\in\Omega, \forall x\in S, c_\eta(x) = 2L - \eta(x)\eta(x+1). \qquad (4.24)$$

Then the exclusion process on $\{0,1\}^{\mathbb{Z}}$ with jump rates $\Gamma_\eta(x,y) = q(x,y)c_\eta(x)$ is well defined and attractive.

Proof. Clearly, from definition (4.23), $q(x,x) = 0$ and

$$\sup_{x\in S}\sum_{y\in S}[q(x,y)+q(y,x)] \le 2L < +\infty,$$

so that conditions (4.10) hold. On the other hand $c_\eta(x)$ is a bounded cylindrical function, so

$$\sup_{x\in S,\eta\in X} c_\eta(x) \le 2L < +\infty$$

and

$$\sup_{x\in S}\sum_{y\in S}\sup_{\eta\in X}|c_{\eta^y}(x)-c_\eta(x)| = \sup_{x\in S}\sup_{\eta\in X}|\eta(x)+\eta(x+1)| \le 2 < +\infty.$$

Conditions (4.11) hold and the exclusion process in Proposition 4.3 is well defined.

Furthermore, for all pairs of configurations (ξ,ζ) such that $\xi \le \zeta$ and for all $x \in S$,

$$c_\xi(x) = 2L - \xi(x)\xi(x+1) \ge 2L - \zeta(x)\zeta(x+1) = c_\zeta(x). \qquad (4.25)$$

and the jump speeds are decreasing. Thus the left hand side of equations (4.14) is identically zero while the right hand side is non negative. Equations (4.14) are thus trivially fulfilled.

Now for any configuration $\eta \in \omega$ and any $y \in S$, we have the bound

$$(2L-1)\sum_{x=y-L}^{y-1}\eta(x) \le \sum_{x=y-L}^{y-1}\eta(x)\,q(x,y)\,c_\zeta(x) \le (2L+1)\sum_{x=y-L}^{y-1}\eta(x). \qquad (4.26)$$

Now for $\xi \le \zeta$, either for all $x \in [y-L, y-1]$, $c_\xi(x) = c_\zeta(x)$ and equation (4.22) is fulfilled since $\xi \le \zeta$, or there is $x \in [y-L, y-1]$ such that $\xi(x) = 0$ and $\zeta(x) = 1$. In that case, using the bounds (4.26), we get

$$\sum_{x=y-L}^{y-1}\eta(x)\,q(x,y)\,c_\eta(x) - \sum_{x=y-L}^{y-1}\eta(x)\,q(x,y)\,c_\xi(x) \ge -2(L-1)+2L \ge 2 > 0, \qquad (4.27)$$

and equations (4.22) are verified. Thus, since the speeds are decreasing, equations (4.13) also hold. The exclusion process with decreasing speeds defined in proposition 4.3 is monotone. □

4.3 k-step exclusion process and related models

The k-step exclusion process was introduced in [10] as an auxiliary model to study the long range exclusion process (see also [1, 11]). It generalizes the simple exclusion process, we study this model in dimension 1, when $k = 2$, in Subsection 4.3.1. We then introduce in Subsection 4.3.2 a first variation of the latter model, that we call 2^*-step exclusion process. Finally, in Subsection 4.3.3, we combine both models to build and analyse a traffic model that we call a range 2 traffic model.

4.3.1 The one-dimensional k-step exclusion process

The state space of the k-step exclusion process is $\{0,1\}^{\mathbb{Z}^d}$. Its jumps follow a translation invariant probability transition on $\mathbb{Z}^d$. In words, if a particle on site x tries to jump, it follows for at most k steps a random walk $(X_n^x)_{n\geq 0}$ with $X_0^x = x$ until it finds an empty site y before returning to x; if all the sites encountered during the k steps are occupied, the particle stays on x. The generator of the one-dimensional k-step exclusion process is given by

$$\mathfrak{L}_k f(\eta) = \sum_{j=1}^{k} \sum_{x,y\in\mathbb{Z}} \eta(x)(1-\eta(y))c_j(x,y,\eta)\left[f(\eta^{x,y}) - f(\eta)\right], \qquad (4.28)$$

where $c_j(x,y,\eta) = \mathbf{E}^x\left[\prod_{i=1}^{j-1}\eta(X_i), \sigma_y = j \leq \sigma_x\right]$ with σ_y the first (non zero) arrival time at site y, $\sigma_y = \inf\{n \geq 1 : X_n^x = y\}$.
For the sake of simplicity, we restrict ourselves to the particular case of the totally asymmetric nearest-neighbor 2-step exclusion on $S = \mathbb{Z}$, for which we have

$$\sum_{j=1}^{2} c_j(x,y,\eta) = 1_{\{y=x+1\}} + 1_{\{y=x+2\}}\eta(x+1) =: \Gamma_\eta(x,y). \qquad (4.29)$$

The totally asymmetric nearest-neighbor 2-step exclusion is attractive, and, as for the simple exclusion process, the set $(\mathcal{I}\cap\mathcal{S})_e$ of extremal translation invariant and invariant probability measures for the dynamics consists of a one parameter family $\{\nu_\rho, \rho \in [0,1]\}$ of Bernoulli product measures, where ρ represents the average density per site, see [10]. This process is a particular case of the range 2 traffic model studied in subsection 4.3.3, hence its coupling rates are derived as a particular case of Proposition 4.6 below.

4.3.2 The one-dimensional totally asymmetric 2^*-step exclusion process

On $S = \mathbb{Z}$, we define

$$\Gamma_\eta(x,y) = 1_{\{y=x+1\}} + 1_{\{y=x+2\}}(1-\eta(x+1)). \qquad (4.30)$$

We call *totally asymmetric* 2^**-step exclusion process* the exclusion process with generator (2.1) for the rate $\Gamma_\eta(x,y)$ given in (4.30). The totally asymmetric nearest-neighbor 2*-step exclusion is a particular case of the range 2 traffic model studied in Subsection 4.3.3 below, hence its attractiveness follows from Proposition 4.5, and its coupling rates are derived as a particular case of Proposition 4.6.
This model is also a particular case of a more general 2^*-step exclusion process of transition rate given by

$$\Gamma_\eta(x,y) = p(x,y) + \sum_{z\in\mathbb{Z}} p(x,z)p(z,y)(1-\eta(z)) \tag{4.31}$$

for a translation invariant transition probability $p(.,.)$.

Proposition 4.4. *The Bernoulli product measures* $\{\nu_\rho, \rho \in [0,1]\}$ *are invariant for the 2*-step exclusion process of transition rate* $\Gamma_\eta(x,y)$ *given in* (4.31).

Proof. We proceed as in the proof of [15, Theorem VIII.2.1], by checking that $\int Lf_A d\nu_\rho = 0$, where A is a finite set of sites and f_A is the cylinder function defined by

$$f_A(\eta) = \prod_{x\in A} \eta(x). \tag{4.32}$$

We have, denoting by $\mathcal{L}_{\mathrm{SEP}}$ the generator of the simple exclusion process and by $\mathcal{L}_{2*\mathrm{s}}$ the second part of the generator of the 2*-step exclusion process,

$$\begin{aligned} \mathcal{L}f_A(\eta) &= \mathcal{L}_{\mathrm{SEP}} f_A(\eta) + \mathcal{L}_{2*\mathrm{s}} f_A(\eta) \\ \mathcal{L}_{2*\mathrm{s}} f_A(\eta) &= \sum_{x,y\in S, x\neq y} \sum_{z\in S, z\neq x,y} p(x,z)p(z,y)\eta(x)(1-\eta(y))(1-\eta(z)) \\ &\qquad\qquad \times \big[f_A(\eta^{x,y}) - f_A(\eta)\big]. \end{aligned}$$

Since

$$\int f_A(\eta)\eta(x)(1-\eta(z))(1-\eta(y))d\nu_\rho(\eta) = \begin{cases} 0, & \text{if } y\in A \text{ or } z\in A, \\ (1-\rho)^2\rho^{|A\cup\{x\}|}, & \text{if } y\notin A, z\notin A, \end{cases} \tag{4.33}$$

and

$$\int f_A(\eta^{x,y})\eta(x)(1-\eta(z))(1-\eta(y))d\nu_\rho(\eta) = \begin{cases} 0, & \text{if } x\in A \text{ or } z\in A, \\ (1-\rho)^2\rho^{|A\cup\{x\}\setminus\{y\}|}, & \text{if } x\notin A, z\notin A, \end{cases} \tag{4.34}$$

we have

$$\begin{aligned} \int \mathcal{L}_{2*\mathrm{s}} f_A(\eta) d\nu_\rho(\eta) &= \sum_{x,y:x\neq y, x\notin A} \sum_{z:z\neq x,y, z\notin A} p(x,z)p(z,y)(1-\rho)^2\rho^{|A\cup\{x\}\setminus\{y\}|} \\ &\quad - \sum_{x,y:x\neq y, y\notin A} \sum_{z:z\neq x,y, z\notin A} p(x,z)p(z,y)(1-\rho)^2\rho^{|A\cup\{x\}|}. \end{aligned} \tag{4.35}$$

Taking $x \notin A, y \notin A$ in the first sums of the two terms on the right hand side gives 0, hence we are left with $y \in A$ for the first term, and $x \in A$ for the second term. Exchanging the indexes x and y in the second term gives

$$\begin{aligned}
&\int \mathcal{L}_{2*s} f_A(\eta) d\nu_\rho(\eta) \\
&= (1-\rho)^2 \rho^{|A|} \sum_{x,y:x\neq y, x\notin A, y\in A} \ \sum_{z:z\neq x,y,z\notin A} \big[p(x,z)p(z,y) - p(y,z)p(z,x)\big] \\
&= 0
\end{aligned}$$

because A is finite and $p(.,.)$ is bi-stochastic. □

4.3.3 A range 2 traffic model

On $S = \mathbb{Z}$, for $\alpha, \beta \in [0,1]$, we define

$$\Gamma_\eta(x,y) = 1_{\{y=x+1\}} + 1_{\{y=x+2\}}[\alpha\eta(x+1) + \beta(1-\eta(x+1))]. \tag{4.36}$$

We call *range 2 traffic model* the exclusion process with generator (2.1) for the rate $\Gamma_\eta(x,y)$ given in (4.36). This rate is a convex combination of the respective rates for one-dimensional totally asymmetric simple exclusion, 2-step exclusion and 2^*-step exclusion. The traffic interpretation is that a car can either go one step ahead, or 2 steps ahead by overtaking another car or by accelerating.

Proposition 4.5. *The range 2 traffic model is attractive if and only if $|\beta - \alpha| \leq 1$. The case $\beta = \alpha = 0$ corresponds to simple exclusion, the case $\beta = 0, \alpha \neq 0$ to 2-step exclusion, and the case $\alpha = 0, \beta \neq 0$ to 2^*-step exclusion.*

Proof. We have to check inequalities (2.6)–(2.7). Let $(\xi, \zeta) \in \Omega^2$ be such that $\xi \leq \zeta$.

We begin with (2.6). Let $y \in \mathbb{Z}$ be such that $\zeta(y) = 0$, hence $\xi(y) = 0$. Then we write

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}} \zeta(x)(1-\xi(x))\Gamma_\zeta(x,y) \\
&\qquad = \zeta(x-1)(1-\xi(x-1)) + \zeta(x-2)(1-\xi(x-2))(\alpha\zeta(x-1) \\
&\qquad\qquad + \beta(1-\zeta(x-1))) \qquad (4.37)\\
&\sum_{x\in\mathbb{Z}} \xi(x)\big[\Gamma_\xi(x,y) - \Gamma_\zeta(x,y)\big]^+ \\
&\qquad = \xi(x-2)\big[(\alpha\xi(x-1) + \beta(1-\xi(x-1))) - (\alpha\zeta(x-1) \\
&\qquad\qquad + \beta(1-\zeta(x-1)))\big]^+. \qquad (4.38)
\end{aligned}$$

First, if $\xi(x-1)=1$ then $\zeta(x-1)=1$, hence (4.38) is null; secondly, if $\zeta(x-1)=0$ then $\xi(x-1)=0$, hence (4.38) is null; in both cases, (2.6) is satisfied. Finally, if $\xi(x-1)=0$ and $\zeta(x-1)=1$, then (4.37) is equal to $1+\alpha\zeta(x-2)(1-\xi(x-2))$ while (4.38) is equal to $\xi(x-2)(\beta-\alpha)^+$: either $\xi(x-2)=0$ and (2.6) is satisfied, or $\xi(x-2)=1$ and $(\beta-\alpha)^+\le 1$ is required for (2.6) to be satisfied.

We now check (2.7). Let $x\in\mathbb{Z}$ be such that $\xi(x)=1$, hence $\zeta(x)=1$. Then we write

$$\begin{aligned}\sum_{y\in\mathbb{Z}}&\zeta(y)(1-\xi(y))\Gamma_\xi(x,y)\\ &=\zeta(x+1)(1-\xi(x+1))+\zeta(x+2)(1-\xi(x+2))(\alpha\xi(x+1)\\ &\qquad\qquad+\beta(1-\xi(x+1))),\end{aligned}\tag{4.39}$$

$$\begin{aligned}\sum_{y\in\mathbb{Z}}&(1-\zeta(y))\left[\Gamma_\zeta(x,y)-\Gamma_\xi(x,y)\right]^+\\ &=(1-\zeta(x+2))\Big[(\alpha\zeta(x+1)+\beta(1-\zeta(x+1)))\\ &\qquad\qquad-(\alpha\xi(x+1)+\beta(1-\xi(x+1)))\Big]^+.\end{aligned}\tag{4.40}$$

First, if $\xi(x+1)=1$ then $\zeta(x+1)=1$, hence (4.40) is null; secondly, if $\zeta(x+1)=0$ then $\xi(x+1)=0$, hence (4.40) is null; in both cases, (2.7) is satisfied. Finally, if $\xi(x+1)=0$ and $\zeta(x+1)=1$, then (4.39) is equal to $1+\beta\zeta(x+2)(1-\xi(x+2))$ while (4.40) is equal to $(1-\zeta(x+2))(\alpha-\beta)^+$: either $\zeta(x+2)=1$ and (2.7) is satisfied, or $\zeta(x+2)=0$ and $(\alpha-\beta)^+\le 1$ is required for (2.7) to be satisfied. □

Invariant measures. Since it is the case for simple exclusion, 2-step exclusion and 2*-step exclusion processes (see Proposition 4.4) the Bernoulli product measures $\{\nu_\rho,\rho\in[0,1]\}$ are invariant for the range 2 traffic model. In this model S is fully connected if α,β are positive, in which case the Bernoulli product measures are the extremal translation invariant and invariant probability measures for the dynamics, by Theorem 2.9.

Therefore, for the range 2 traffic model, applying Propositions 3.2, 3.3, and formulas (3.25)–(3.28) from Corollary 3.8, we obtain first the following formulas for the coupling rates $G_{\xi,\zeta}(x_1,y_1;x_2,y_2)$, taking into account that in formula (3.9), they are multiplied by the prefactor $\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))$, so that $\xi(x_1)=1-\xi(y_1)=\zeta(x_2)=1-\zeta(y_2)=1$:

$$
\begin{aligned}
G_{\xi,\zeta}(x,x+1;x,x+1) =& 1,\\
G_{\xi,\zeta}(x,x+2;x,x+2) =& [\alpha\xi(x+1)+\beta(1-\xi(x+1))]\\
&\wedge [\alpha\zeta(x+1)+\beta(1-\zeta(x+1))],
\end{aligned}
$$

$$
G_{\xi,\zeta}(x,x+1;x,x+2) = \begin{cases} [\alpha-\beta]^{+}\zeta(x+1) & \text{when } \xi \le \zeta,\\ 0 & \text{when } \xi > \zeta, \end{cases}
$$

$$
G_{\xi,\zeta}(x,x+2;x+1,x+2) = \begin{cases} [\beta-\alpha]^{+}(1-\xi(x+1)) & \text{when } \xi \le \zeta,\\ 0 & \text{when } \xi > \zeta, \end{cases}
$$

$$
G_{\xi,\zeta}(x+1,x+2;x,x+2) = \begin{cases} 0 & \text{when } \xi \le \zeta,\\ (\beta-\alpha)^{+}(1-\zeta(x+1)) & \text{when } \xi > \zeta, \end{cases}
$$

$$
G_{\xi,\zeta}(x,x+2;x,x+1) = \begin{cases} 0 & \text{when } \xi \le \zeta,\\ [\alpha-\beta]^{+}\xi(x+1) & \text{when } \xi > \zeta. \end{cases}
$$

Some more computations to get the formulas in Proposition 3.9 yield:

Proposition 4.6. *The coupled generator of the range 2 traffic model writes*

$$
\overline{\mathcal{L}}^{D} f(\xi,\zeta) = \overline{\mathcal{L}}_1^{D} f(\xi,\zeta) + \overline{\mathcal{L}}_2^{D} f(\xi,\zeta) + \overline{\mathcal{L}}_{3,1}^{D} f(\xi,\zeta) + \overline{\mathcal{L}}_{3,2}^{D} f(\xi,\zeta) \tag{4.41}
$$

where $\overline{\mathcal{L}}_1^{D}$ deals with coupled jumps with the same departure and arrival sites, $\overline{\mathcal{L}}_2^{D}$ with coupled jumps with a different site either for departure or for arrival, and $\overline{\mathcal{L}}_{3,1}^{D}$, $\overline{\mathcal{L}}_{3,2}^{D}$ deal with uncoupled jumps. They are given by

$$
\begin{aligned}
\overline{\mathcal{L}}_1^{D} f(\xi,\zeta) =& \sum_{x\in S} \xi(x)(1-\xi(x+1))\zeta(x)(1-\zeta(x+1))\\
&\qquad \times \big(f(\xi^{x,x+1},\zeta^{x,x+1}) - f(\xi,\zeta)\big)\\
&+ \sum_{x\in S} \xi(x)\zeta(x)(1-(\xi\vee\zeta)(x+2))\\
&\qquad \times \big[\alpha\xi(x+1)\zeta(x+1) + \beta(1-(\xi\vee\zeta)(x+1))\\
&+ (\alpha\wedge\beta)\{\xi(x+1)(1-\zeta(x+1)) + \zeta(x+1)(1-\xi(x+1))\}\big]\\
&\qquad \times \big(f(\xi^{x,x+2},\zeta^{x,x+2}) - f(\xi,\zeta)\big),
\end{aligned} \tag{4.42}
$$

$$\begin{aligned}
\overline{\mathcal{L}}_2^D f(\xi,\zeta) = &\sum_{x\in S} \xi(x)(1-\xi(x+1))\zeta(x+1)(1-(\xi\vee\zeta)(x+2)) \\
&\quad \times (\beta-\alpha)^+ \big(f(\xi^{x,x+2},\zeta^{x+1,x+2}) - f(\xi,\zeta)\big) \\
&+ \sum_{x\in S} \xi(x+1)(1-\zeta(x+1))\zeta(x)(1-(\xi\vee\zeta)(x+2)) \\
&\quad \times (\beta-\alpha)^+ \big(f(\xi^{x+1,x+2},\zeta^{x,x+2}) - f(\xi,\zeta)\big) \\
&+ \sum_{x\in S} \xi(x)\zeta(x)\xi(x+1)(1-\zeta(x+1))(1-(\xi\vee\zeta)(x+2)) \\
&\quad \times (\alpha-\beta)^+ \big(f(\xi^{x,x+2},\zeta^{x,x+1}) - f(\xi,\zeta)\big) \\
&+ \sum_{x\in S} \xi(x)\zeta(x)(1-\xi(x+1))\zeta(x+1)(1-(\xi\vee\zeta)(x+2)) \\
&\quad \times (\alpha-\beta)^+ \big(f(\xi^{x,x+1},\zeta^{x,x+2}) - f(\xi,\zeta)\big), \qquad (4.43)
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{L}}_{3,1}^D f(\xi,\zeta) = &\sum_{x\in S} \xi(x)(1-\xi(x+1)) \\
&\times \big[1-\zeta(x)(1-\zeta(x+1)) - (1-\zeta(x))\zeta(x-1)(1-\zeta(x+1))(\beta-\alpha)^+ \\
&\qquad -\zeta(x)\zeta(x+1)(1-(\xi\vee\zeta)(x+2))(\alpha-\beta)^+\big] \\
&\times \big(f(\xi^{x,x+1},\zeta) - f(\xi,\zeta)\big) \\
&+ \sum_{x\in S} \xi(x)(1-\xi(x+2)) \\
&\times \big[\alpha\xi(x+1) + \beta(1-\xi(x+1)) - \zeta(x)(1-\zeta(x+2)) \\
&\times \big\{\alpha\xi(x+1)\zeta(x+1) + \beta(1-(\xi\vee\zeta)(x+1)) \\
&\quad + (\alpha\wedge\beta)\{\xi(x+1)(1-\zeta(x+1)) + \zeta(x+1)(1-\xi(x+1))\}\big\} \\
&-\zeta(x)\xi(x+1)(1-\zeta(x+1))(1-\zeta(x+2))(\alpha-\beta)^+\big] \\
&\times \big(f(\xi^{x,x+2},\zeta) - f(\xi,\zeta)\big), \qquad (4.44)
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{L}}_{3,2}^D f(\xi,\zeta) = &\sum_{x\in S} \zeta(x)(1-\zeta(x+1)) \\
&\quad \times \big[1-\xi(x)(1-\xi(x+1)) \\
&\qquad - \xi(x)\xi(x+1)(1-(\xi\vee\zeta)(x+2))(\alpha-\beta)^+ \\
&\qquad -(1-\xi(x))\xi(x-1)(1-\xi(x+1))(\beta-\alpha)^+\big] \\
&\quad \times \big(f(\xi,\zeta^{x,x+1}) - f(\xi,\zeta)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{x\in S}\zeta(x)(1-\zeta(x+2))\\
&\quad\times\Big[\alpha\zeta(x+1)+\beta(1-\zeta(x+1))-\xi(x)(1-\xi(x+2))\\
&\qquad\times\big\{\alpha\xi(x+1)\zeta(x+1)+\beta(1-(\xi\vee\zeta)(x+1))\\
&\qquad\quad+(\alpha\wedge\beta)\{\zeta(x+1)(1-\xi(x+1))\\
&\qquad\qquad+\xi(x+1)(1-\zeta(x+1))\}\big\}\\
&\quad-\xi(x)(1-\xi(x+1))\zeta(x+1)(1-\xi(x+2))(\alpha-\beta)^+\Big]\\
&\quad\times\big(f(\xi,\zeta^{x,x+2})-f(\xi,\zeta)\big). \qquad (4.45)
\end{aligned}
$$

Remark 4.7. *Taking $\alpha=\beta=0$ gives the basic coupling generator for TASEP, while taking $\alpha=0,\beta=1$ gives a coupled generator for 2^*-step exclusion, and taking $\alpha=1,\beta=0$ gives a coupled generator for 2-step exclusion. The latter is different from the one used in [10].*

Alternative coupling rates. We now compute the rates given in Proposition 3.11. We obtain the following formulas for the coupling rates $G_{\xi,\zeta}(x_1,y_1;x_2,y_2)$, taking into account that they are multiplied by the prefactor $\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))$, so that $\xi(x_1)=1-\xi(y_1)=\zeta(x_2)=1-\zeta(y_2)=1$:

$$
\begin{aligned}
&G_{\xi,\zeta}(x,x+1;x,x+1)=1,\\
&G_{\xi,\zeta}(x,x+2;x,x+2)=[\alpha\xi(x+1)+\beta(1-\xi(x+1))]\\
&\qquad\qquad\wedge[\alpha\zeta(x+1)+\beta(1-\zeta(x+1))],\\
&G_{\xi,\zeta}(x,x+1;x,x+2)=\zeta(x+1)(1-\xi(x+2))\frac{[\alpha-\beta]^+}{1+\beta}\quad\text{if }\xi\le\zeta,\\
&G_{\xi,\zeta}(x,x+2;x+1,x+2)=\zeta(x)(1-\xi(x+1))\frac{[\beta-\alpha]^+}{1+\alpha}\quad\text{if }\xi\le\zeta,\\
&G_{\xi,\zeta}(x+1,x+2;x,x+2)=\xi(x)(1-\zeta(x+1))\frac{[\beta-\alpha]^+}{1+\alpha}\quad\text{if }\xi>\zeta,\\
&G_{\xi,\zeta}(x,x+2;x,x+1)=\xi(x+1)(1-\zeta(x+2))\frac{[\alpha-\beta]^+}{1+\beta}\quad\text{if }\xi>\zeta.
\end{aligned}
$$

4.4 From a non-attractive traffic model to an attractive dynamics

We begin with an exclusion process with the transition rates introduced in [9] in the context of a cellular automaton dynamics. There, $S=\mathbb{Z}$, and the transitions are nearest neighbor and totally asymmetric. For all $x\in S$, $\eta\in X$ such that $\eta(x)=1$ and $\eta(x+1)=0$

$$\Gamma_\eta(x,x+1) = \begin{cases} \alpha & \text{if } \eta(x-1)=1,\ \eta(x+2)=0,\ \text{[accelerating]},\\ \beta & \text{if } \eta(x+2)=1,\ \eta(x-1)=0,\ \text{[braking]},\\ \gamma & \text{if } \eta(x-1)=\eta(x+2)=1,\ \text{[congested]},\\ \delta & \text{if } \eta(x-1)=\eta(x+2)=0,\ \text{[driving]}. \end{cases} \tag{4.46}$$

where the parameters $\alpha,\beta,\gamma,\delta$ are positive. This model is not attractive, unless it reduces to simple exclusion, that is $\alpha=\beta=\gamma=\delta$. Indeed, for any other choice, conditions (2.6)–(2.7) from Theorem 2.4 are not satisfied. Here, it is possible to turn the dynamics into an attractive one, just by considering a symmetrized version, in which the non zero rates are the previous, rightwards, ones, (4.46), together with the following symmetric, leftwards rates:

$$\Gamma_\eta(x+1,x) = \begin{cases} \alpha & \text{if } \eta(x+2)=1,\ \eta(x-1)=0,\\ \beta & \text{if } \eta(x-1)=1,\ \eta(x+2)=0,\\ \gamma & \text{if } \eta(x+2)=\eta(x-1)=1,\\ \delta & \text{if } \eta(x+2)=\eta(x-1)=0. \end{cases} \tag{4.47}$$

Then applying conditions in Theorem 2.4 leads to the following result.

Proposition 4.8. *The symmetrized dynamics with rates* (4.46)–(4.47) *is attractive if and only if* $\alpha,\beta,\gamma,\delta$ *satisfy the following conditions*

$$\beta\le\gamma\wedge\delta\le\gamma\vee\delta\le\alpha,\qquad \alpha\le\beta+\gamma\wedge\delta,\qquad \delta\le 2\beta. \tag{4.48}$$

Note that the facilitated exclusion process ([2, 4, 6]) has rates (4.46) with $\alpha=\gamma=1$, $\beta=\delta=0$. Hence it is not attractive, and its symmetrized version (with the corresponding rates in (4.47)) is not attractive either. Indeed the study of this model required other tools.

Invariant measures. For the symmetrized dynamics, S is fully connected, since the parameters $\alpha,\beta,\gamma,\delta$ are positive. We can thus apply Theorem 2.9.

Coupling rates. Applying Propositions 3.2, 3.3, with formulas (3.25)–(3.28), we obtain first the following formulas for the coupling rates $G_{\xi,\zeta}(x_1,y_1;x_2,y_2)$, taking into account that they are multiplied by the prefactor $\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))$, so that $\xi(x_1)=1-\xi(y_1)=\zeta(x_2)=1-\zeta(y_2)=1$. Note that since the rates (4.46) and (4.47) are symmetric, it is enough to compute the coupling rates in the positive direction to get the ones in the negative direction by symmetry. To simplify the computations, we assume that $\gamma\le\delta$.

$$G_{\xi,\zeta}(x,x+1;x,x+1) = \Gamma_\xi(x,x+1)\wedge\Gamma_\zeta(x,x+1),$$
$$G_{\xi,\zeta}(x,x-1;x,x-1) = \Gamma_\xi(x,x-1)\wedge\Gamma_\zeta(x,x-1).$$

When $\xi \le \zeta$, we have

$$
\begin{aligned}
&G_{\xi,\zeta}(x,x+1;x,x-1) = \zeta(x+1)\big[(1-\zeta(x-2))(\alpha-\delta)+\xi(x-2)(\gamma-\beta)\big],\\
&G_{\xi,\zeta}(x,x-1;x,x+1) = \zeta(x-1)\big[(1-\zeta(x+2))(\alpha-\delta)+\xi(x+2)(\gamma-\beta)\big],\\
&G_{\xi,\zeta}(x,x+1;x+2,x+1) = (1-\xi(x+2))\big[(1-\zeta(x-1))(\delta-\beta)\\
&\qquad +(1-\xi(x-1))\zeta(x-1)(\delta-\gamma)+\xi(x-1)(\alpha-\gamma)\big],\\
&G_{\xi,\zeta}(x,x-1;x-2,x-1) = (1-\xi(x-2))\big[(1-\zeta(x+1))(\delta-\beta)\\
&\qquad +(1-\xi(x+1))\zeta(x+1)(\delta-\gamma)+\xi(x+1)(\alpha-\gamma)\big].
\end{aligned}
$$

When $\xi > \zeta$, we have

$$
\begin{aligned}
&G_{\xi,\zeta}(x,x+1;x,x-1) = \xi(x-1)\big[(1-\xi(x+2))(\alpha-\delta)+\zeta(x+2)(\gamma-\beta)\big],\\
&G_{\xi,\zeta}(x,x-1;x,x+1) = \xi(x+1)\big[(1-\xi(x-2))(\alpha-\delta)+\zeta(x-2)(\gamma-\beta)\big],\\
&G_{\xi,\zeta}(x,x+1;x+2,x+1) = (1-\zeta(x))\big[(1-\xi(x+3))(\delta-\beta)\\
&\qquad\qquad +\zeta(x+3)(\alpha-\gamma)+\xi(x+3)(1-\zeta(x+3))(\delta-\gamma)\big],\\
&G_{\xi,\zeta}(x,x-1;x-2,x-1) = (1-\zeta(x))\big[(1-\xi(x-3))(\delta-\beta)\\
&\qquad\qquad +\zeta(x-3)(\alpha-\gamma)+\xi(x-3)(1-\zeta(x-3))(\delta-\gamma)\big].
\end{aligned}
$$

Finally, applying Proposition 3.9 with formulas (3.30)–(3.31), we obtain the following formulas for the coupling rates $G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)$, taking into account that they are multiplied by the prefactor $\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))$, so that $\xi(x_1)=1-\xi(y_1)=\zeta(x_2)=1-\zeta(y_2)=1$. Again, since the rates (4.46) and (4.47) are symmetric, it is enough to compute the coupling rates in the positive direction.

$$
\begin{aligned}
G^D_{\xi,\zeta}(x,x+1;x,x+1) &= \Gamma_\xi(x,x+1)\wedge\Gamma_\zeta(x,x+1),\\
G^D_{\xi,\zeta}(x,x-1;x,x-1) &= \Gamma_\xi(x,x-1)\wedge\Gamma_\zeta(x,x-1),\\
G^D_{\xi,\zeta}(x,x+1;x,x-1) &= (1-\zeta(x+1))\xi(x-1)\\
&\qquad \times\big[(1-(\xi\vee\zeta)(x+2))(\alpha-\delta)+\zeta(x+2)(\gamma-\beta)\big]\\
&\quad +(1-\xi(x-1))\zeta(x+1)\\
&\qquad \times\big[(1-(\xi\vee\zeta)(x-1))(\alpha-\delta)+\xi(x-2)(\gamma-\beta)\big],\\
G^D_{\xi,\zeta}(x,x-1;x,x+1) &= (1-\zeta(x-1))\xi(x+1)\\
&\qquad \times\big[(1-(\xi\vee\zeta)(x-2))(\alpha-\delta)+\zeta(x-2)(\gamma-\beta)\big]\\
&\quad +(1-\xi(x+1))\zeta(x-1)\\
&\quad \times\big[(1-(\xi\vee\zeta)(x+2))(\alpha-\delta)+(\xi\vee\zeta)(x+2)(\gamma-\beta)\big].
\end{aligned}
$$

$$\begin{aligned}G^D_{\xi,\zeta}(x,x+1;x+2,x+1) &= (1-\zeta(x))\big[(1-(\xi\vee\zeta)(x+3))(\delta-\beta)\\ &\quad +\zeta(x+3)(\alpha-\gamma)+(1-\zeta(x+3))\xi(x+3)(\delta-\gamma)\big]\\ &\qquad \times\big[\xi(x+2)\zeta(x-1)(1-\xi(x-1))\left(\frac{\beta}{\gamma}-1\right)+1\big]\\ &\quad +(1-\xi(x+2))\big[(1-(\xi\vee\zeta)(x-1))(\delta-\beta)\\ &\qquad +\xi(x-1)(\alpha-\gamma)+(1-\xi(x-1))\zeta(x-1)(\delta-\gamma)\big]\\ &\qquad \times\big[\xi(x+3)\zeta(x)(1-\zeta(x+3))\left(\frac{\beta}{\gamma}-1\right)+1\big].\end{aligned}$$

$$\begin{aligned}G^D_{\xi,\zeta}(x,x-1;x-2,x-1) &= (1-\zeta(x))\big[(1-(\xi\vee\zeta)(x-3))(\delta-\beta)\\ &\quad +(1-\zeta(x-3))\xi(x-3)(\delta-\gamma)+\zeta(x-3)(\alpha-\gamma)\big]\\ &\qquad \times\big[\xi(x-2)\zeta(x+1)(1-\xi(x+1))\left(\frac{\beta}{\gamma}-1\right)+1\big]\\ &\quad +(1-\xi(x-2))\big[(1-(\xi\vee\zeta)(x+1))(\delta-\beta)\\ &\qquad +\xi(x+1)(\alpha-\gamma)+(1-\xi(x+1))\zeta(x+1)(\delta-\gamma)\big]\\ &\qquad \times\big[\xi(x-3)\zeta(x)(1-\zeta(x-3))\left(\frac{\beta}{\gamma}-1\right)+1\big].\end{aligned}$$

$$\begin{aligned}&G^D_{\xi,\zeta}(x,x+1;x+2,x+3)\\ &\qquad = (1-\zeta(x+1))(1-\xi(x+2))\xi(x+3)\zeta(x)\frac{(\gamma-\beta)}{\gamma}\\ &\qquad\quad \times\big[(1-(\xi\vee\zeta)(x-1))(\delta-\beta)\\ &\qquad\qquad +(1-\xi(x-1))\zeta(x-1)(\delta-\gamma)+\xi(x-1)(\alpha-\gamma)\big].\\ &G^D_{\xi,\zeta}(x,x-1;x-2,x-3)\\ &\qquad = (1-\zeta(x-1))(1-\xi(x-2))\xi(x-3)\zeta(x)\frac{(\gamma-\beta)}{\gamma}\\ &\qquad\quad \times\big[(1-(\xi\vee\zeta)(x+1))(\delta-\beta)\\ &\qquad\qquad +(1-\xi(x+1))\zeta(x+1)(\delta-\gamma)+\xi(x+1)(\alpha-\gamma)\big].\end{aligned}$$

5 Technical proofs

Proof of Proposition 3.2. Taking into account notations (3.13)–(3.12), we rewrite the generator $\overline{\mathcal{L}}$ (Equation (3.9)) as

$$\begin{aligned}\overline{\mathcal{L}}f(\xi,\zeta) &= \sum_{x_1,y_1\in S} \xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1)-\varphi_{\xi,\zeta}(x_1,y_1)\big)\\ &\qquad \times\big(f(\xi^{x_1,y_1},\zeta)-f(\xi,\zeta)\big)\\ &+ \sum_{x_2,y_2\in S} \zeta(x_2)(1-\zeta(y_2))\big(\Gamma_\zeta(x_2,y_2)-\overline{\varphi}_{\xi,\zeta}(x_2,y_2)\big)\\ &\qquad \times\big(f(\xi,\zeta^{x_2,y_2})-f(\xi,\zeta)\big)\\ &+ \sum_{x_1,y_1\in S}\sum_{x_2,y_2\in S} \xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))\\ &\qquad \times G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\big(f(\xi^{x_1,y_1},\zeta^{x_2,y_2})-f(\xi,\zeta)\big) \qquad (5.1)\end{aligned}$$

In the above expression, the first two terms on the r.h.s. refer to uncoupled transitions, respectively $(\xi,\zeta)\to(\xi^{x_1,y_1},\zeta)$ and $(\xi,\zeta)\to(\xi,\zeta^{x_2,y_2})$, while the third line refers to coupled transitions $(\xi,\zeta)\to(\xi^{x_1,y_1},\zeta^{x_2,y_2})$.

Inequalities (3.11)–(3.10) and non-negativity of $G_{\xi,\zeta}$ insure that the rates of all uncoupled and coupled transitions above are non-negative.

Moreover, if $f(\xi,\zeta)=g(\xi)$ depends only on ξ (resp. $f(\xi,\zeta)=h(\zeta)$ depends only on ζ), we have $\overline{\mathcal{L}}g(\xi)=\mathcal{L}g(\xi)$ (resp. $\overline{\mathcal{L}}h(\zeta)=\mathcal{L}h(\zeta)$). Therefore $\overline{\mathcal{L}}$ defines indeed a coupling of two copies of a generalized exclusion process. □

Proof of Proposition 3.3. • We first consider the limits of the series defined in (3.18)–(3.21). By construction, these series are nonnegative, nondecresing and by (2.4) they are also bounded from above. They are thus (absolutely) convergent and we denote their limits as

$$S^{x,*}_{\xi,\zeta} = \lim_{n\to\infty} S^{x,n}_{\xi,\zeta}, \qquad (5.2)$$

$$\overline{T}^{x,*}_{\xi,\zeta} = \lim_{n\to\infty} \overline{T}^{x,n}_{\xi,\zeta}, \qquad (5.3)$$

$$T^{y,*}_{\xi,\zeta} = \lim_{n\to\infty} T^{y,n}_{\xi,\zeta}, \qquad (5.4)$$

$$\overline{S}^{y,*}_{\xi,\zeta} = \lim_{n\to\infty} \overline{S}^{y,n}_{\xi,\zeta}. \qquad (5.5)$$

In these notations, equations (2.7) and (2.6) read respectively:

For any configurations ξ,ζ in Ω such that $\xi\le\zeta$,

$$\text{For all } x\in S \text{ such that } \xi(x)=1, \qquad \overline{T}^{x,*}_{\xi,\zeta}\le S^{x,*}_{\xi,\zeta}, \qquad (5.6)$$

$$\text{For all } y\in S \text{ such that } \zeta(y)=0, \qquad T^{y,*}_{\xi,\zeta}\le \overline{S}^{y,*}_{\xi,\zeta}. \qquad (5.7)$$

• We now prove that for any two nondecreasing, convergent series $(S_n)_{n\geq 0}$ and $(T_n)_{n\geq 0}$, the quantity defined in (3.22) $H_{m,n}(S_\cdot, T_\cdot)$ is nonnegative for all $m, n > 0$. We have

$$\begin{aligned} H_{m,n}(S_\cdot, T_\cdot) &= S_m \wedge T_n - S_{m-1} \wedge T_n - S_m \wedge T_{n-1} + S_{m-1} \wedge T_{n-1} \\ &= (S_m \wedge T_n - S_{m-1} \wedge (S_m \wedge T_n)) \\ &\qquad - (S_m \wedge T_{n-1} - S_{m-1} \wedge (S_m \wedge \mathbb{T}_{n-1})) \\ &= [S_m \wedge T_n - S_{m-1}]^+ - [S_m \wedge T_{n-1} - S_{m-1}]^+ \\ &\geq 0. \end{aligned} \tag{5.8}$$

In the equations above, we used $S_m \geq S_{m-1}$ to get the second line, the third line is an identity and, finally, positivity comes from the fact that $T_n \geq T_{m-1}$ and the function $t \to [S_m \wedge t - S_{m-1}]^+$ is not decreasing.

In addition, we get that the sums $\sum_{m>0} H_{m,n}(S_\cdot, T_\cdot)$ and $\sum_{m>0} H_{n,m}(S_\cdot, T_\cdot)$ are absolutely convergent for all $n > 0$ whenever the two series converge. In particular, setting $S_* = \lim_{m\to\infty} S_m$ and $T_* = \lim_{n\to\infty} T_n$, one gets

$$\sum_{m>0} H_{m,n}(S_\cdot, T_\cdot) = S_* \wedge T_n - S_* \wedge T_{n-1} \qquad \text{for all } n > 0, \tag{5.9}$$

$$\sum_{n>0} H_{m,n}(S_\cdot, T_\cdot) = S_m \wedge T_* - S_{m-1} \wedge T_* \qquad \text{for all } m > 0. \tag{5.10}$$

We are now ready to turn to the proof of Proposition 3.3.

• We first prove that the coupling rates (3.24) satisfy conditions (3.10)–(3.11) of Proposition 3.2.

First, for all non ordered pairs of configurations $(\xi, \zeta) \in \Omega \times \Omega$, all coupling rates $G_{\xi,\zeta}$ defined by (3.24) are zero, so that the left hand sides of equations (3.10)–(3.11) are identically zero and both equations (3.10)–(3.11) trivially hold.

We now consider the case $(\xi, \zeta) \in \Omega \times \Omega$ with $\xi \leq \zeta$. For all $(x, y) \in S^2$, the left hand side of equation (3.10) reads:

$$\begin{aligned} &\varphi_{\xi,\zeta}(x, y) \\ &\quad = \sum_{x',y'\in S} \zeta(x')\,(1-\zeta(y'))\, G_{\xi,\zeta}(x, y; x', y') \\ &\quad = \zeta(x)\,(1-\zeta(y))\,\Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) \\ &\quad +\zeta(x) \sum_{m,n>0} \delta(y, y^{x,m}_{\xi,\zeta}) \sum_{y'\in S} \delta(y', \overline{y}^{x,n}_{\xi,\zeta})\, H_{m,n}(S^{x,\cdot}_{\xi,\zeta}, \overline{T}^{x,\cdot}_{\xi,\zeta}) \\ &\quad + (1-\zeta(y)) \sum_{m,n>0} \delta(x, x^{y,m}_{\xi,\zeta}) \sum_{x'\in S} \delta(x', \overline{x}^{y,n}_{\xi,\zeta})\, H_{m,n}(T^{y,\cdot}_{\xi,\zeta}, \overline{S}^{x,\cdot}_{\xi,\zeta}) \end{aligned}$$

$$
\begin{aligned}
&= \zeta(x)\,(1-\zeta(y))\,\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y)\\
&+\zeta(x)\zeta(y)\sum_{m>0}\delta(y,y^{x,m}_{\xi,\zeta})\,\left(S^{x,m}_{\xi,\zeta}\wedge\overline{T}^{x,*}_{\xi,\zeta}-S^{x,m-1}_{\xi,\zeta}\wedge\overline{T}^{x,*}_{\xi,\zeta}\right)\\
&+\zeta(x)\,(1-\zeta(y))\sum_{m>0}\delta(x,x^{y,m}_{\xi,\zeta})\left(T^{y,m}_{\xi,\zeta}\wedge\overline{S}^{y,*}_{\xi,\zeta}-T^{y,m-1}_{\xi,\zeta}\wedge\overline{S}^{y,*}_{\xi,\zeta}\right)\\
&= \zeta(x)\,(1-\zeta(y))\,\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y)\\
&+\zeta(x)\zeta(y)\sum_{m>0}\delta(y,y^{x,m}_{\xi,\zeta})\,\left(S^{x,m}_{\xi,\zeta}\wedge\overline{T}^{x,*}_{\xi,\zeta}-S^{x,m-1}_{\xi,\zeta}\wedge\overline{T}^{x,*}_{\xi,\zeta}\right)\\
&+\zeta(x)\,(1-\zeta(y))\sum_{m>0}\delta(x,x^{y,m}_{\xi,\zeta})\,[\Gamma_\xi(x,y)-\Gamma_\zeta(x,y)]^+\\
&= \zeta(x)\,(1-\zeta(y))\,\Gamma_\xi(x,y)\\
&+\zeta(x)\zeta(y)\sum_{m>0}\delta(y,y^{x,m}_{\xi,\zeta})\,\left(S^{x,m}_{\xi,\zeta}\wedge\overline{T}^{x,*}_{\xi,\zeta}-S^{x,m-1}_{\xi,\zeta}\wedge\overline{T}^{x,*}_{\xi,\zeta}\right). \qquad (5.11)
\end{aligned}
$$

In the second expression, the summation over y' in the second term and the summation over x' in the third term just give 1 and we use the expression (5.10) to compute the summation over $n>0$. The fourth equality is a consequence of relation (5.7), which gives

$$
T^{y,m}_{\xi,\zeta}\wedge\overline{S}^{y,*}_{\xi,\zeta}-T^{y,m-1}_{\xi,\zeta}\wedge\overline{S}^{y,*}_{\xi,\zeta}=T^{y,m}_{\xi,\zeta}-T^{y,m-1}_{\xi,\zeta}=\left[\Gamma_\xi(x^{y,m}_{\xi,\zeta},y)-\Gamma_\zeta(x^{y,m}_{\xi,\zeta},y)\right]^+. \qquad (5.12)
$$

Now using the estimate

$$
S^{x,m}_{\xi,\zeta}\wedge\overline{T}^{x,*}_{\xi,\zeta}-S^{x,m-1}_{\xi,\zeta}\wedge\overline{T}^{x,*}_{\xi,\zeta}\le S^{x,m}_{\xi,\zeta}-S^{x,m-1}_{\xi,\zeta}=\Gamma_\xi(x,y^{x,m}_{\xi,\zeta})
$$

we get the inequality

$$
\varphi_{\xi,\zeta}(x,y)\le\zeta(x)\,(1-\zeta(y))\,\Gamma_\xi(x,y)+\zeta(x)\zeta(y)\Gamma_\xi(x,y)\le\Gamma_\xi(x,y). \qquad (5.13)
$$

Thus inequality (3.10) holds for $\xi\le\zeta$.

We prove (3.11) for $\xi\le\zeta$ in a similar way, as follows. For all $(x,y)\in S^2$, the left hand side of equation (3.11) reads:

$$
\begin{aligned}
&\overline{\varphi}_{\xi,\zeta}(x,y)\\
&\quad=\sum_{x',y'\in S}\xi(x')\,(1-\xi(y'))\,G_{\xi,\zeta}(x',y';x,y)\\
&\quad=\xi(x)\,(1-\xi(y))\,\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y)\\
&\quad+\xi(x)\sum_{m,n>0}\sum_{y'\in S}\delta(y',y^{x,m}_{\xi,\zeta})\,\delta(y,\overline{y}^{x,n}_{\xi,\zeta})\,H_{m,n}(S^{x,\cdot}_{\xi,\zeta},\overline{T}^{x,\cdot}_{\xi,\zeta})\\
&\quad+(1-\xi(y))\sum_{m,n>0}\sum_{x'\in S}\delta(x',x^{y,m}_{\xi,\zeta})\delta(x,\overline{x}^{y,n}_{\xi,\zeta})\,H_{m,n}(T^{y,\cdot}_{\xi,\zeta},\overline{S}^{x,\cdot}_{\xi,\zeta})
\end{aligned}
$$

$$
\begin{aligned}
&= \xi(x)\,(1-\xi(y))\,\Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) \\
&+\xi(x)\,(1-\xi(y)) \sum_{n>0} \delta(y, \overline{y}^{x,n}_{\xi,\zeta}) \, \left(S^{x,*}_{\xi,\zeta} \wedge \overline{T}^{x,n}_{\xi,\zeta} - S^{x,*}_{\xi,\zeta} \wedge \overline{T}^{x,n-1}_{\xi,\zeta} \right) \\
&+ (1-\xi(x))\,(1-\xi(y)) \sum_{n>0} \delta(x, \overline{x}^{y,n}_{\xi,\zeta}) \left(T^{y,*}_{\xi,\zeta} \wedge \overline{S}^{y,n}_{\xi,\zeta} - T^{y,*}_{\xi,\zeta} \wedge \overline{S}^{y,n-1}_{\xi,\zeta} \right) \\
&= \xi(x)\,(1-\xi(y))\,\Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) \\
&+\xi(x)\,(1-\xi(y)) \sum_{n>0} \delta(y, \overline{y}^{x,n}_{\xi,\zeta}) \, [\Gamma_\zeta(x,y) \wedge \Gamma_\xi(x,y)]^+ \\
&+ (1-\xi(x))\,(1-\xi(y)) \sum_{n>0} \delta(x, \overline{x}^{y,n}_{\xi,\zeta}) \left(T^{y,*}_{\xi,\zeta} \wedge \overline{S}^{y,n}_{\xi,\zeta} - T^{y,*}_{\xi,\zeta} \wedge \overline{S}^{y,n-1}_{\xi,\zeta} \right) \\
&= \xi(x)\,(1-\xi(y))\,\Gamma_\zeta(x,y) \\
&+ (1-\xi(x))\,(1-\xi(y)) \sum_{n>0} \delta(x, \overline{x}^{y,n}_{\xi,\zeta}) \left(T^{y,*}_{\xi,\zeta} \wedge \overline{S}^{y,n}_{\xi,\zeta} - T^{y,*}_{\xi,\zeta} \wedge \overline{S}^{y,n-1}_{\xi,\zeta} \right).
\end{aligned}
\tag{5.14}
$$

In the second expression, the summation over y' in the second term and the summation over x' in the third term just give 1 and we use the expression (5.9) to compute the summation over $n > 0$. To get the fourth expression, we used the relation (5.7) to obtain

$$
S^{x,*}_{\xi,\zeta} \wedge \overline{T}^{x,n}_{\xi,\zeta} - S^{x,*}_{\xi,\zeta} \wedge \overline{T}^{x,n-1}_{\xi,\zeta} = \overline{T}^{x,n}_{\xi,\zeta} - \overline{T}^{x,n-1}_{\xi,\zeta} = \left[\Gamma_\zeta(x, \overline{y}^{x,n}_{\xi,\zeta}) - \Gamma_\xi(x, \overline{y}^{x,n}_{\xi,\zeta}) \right]^+ .
$$

Now we have the estimate

$$
T^{y,*}_{\xi,\zeta} \wedge \overline{S}^{y,n}_{\xi,\zeta} - T^{y,*}_{\xi,\zeta} \wedge \overline{S}^{y,n-1}_{\xi,\zeta} \le \overline{S}^{y,n}_{\xi,\zeta} - \overline{S}^{y,n-1}_{\xi,\zeta} = \Gamma_\zeta(\overline{x}^{y,n}_{\xi,\zeta}, y),
$$

which gives

$$
\begin{aligned}
\overline{\varphi}_{\xi,\zeta}(x,y) &\le \xi(x)\,(1-\xi(y))\,\Gamma_\zeta(x,y) + (1-\xi(x))\,(1-\xi(y))\,\Gamma_\zeta(x,y) \\
&\le \Gamma_\zeta(x,y).
\end{aligned}
\tag{5.15}
$$

Equation (3.11) is proven for $\xi \le \zeta$.

A similar derivation holds in the case $\zeta < \xi$. Thus the coupling rates defined in Proposition 3.3 satisfy the conditions (3.10)–(3.11) of Proposition 3.2.

• We now prove that this coupling is increasing.

We suppose that $\xi \le \zeta$. We first consider coupled transitions. From equation (3.24), we find that a coupled transition $(\xi, \zeta) \to (\xi^{x,y}, \zeta^{x',y'})$ has possibly a non zero coupling rate $G_{\xi,\zeta}(x, y; x', y')$ in three possible cases:

- $x = x'$ and $y = y'$:
  thus

$$\begin{array}{rcll} \xi^{x,y}(x) & = & \zeta^{x',y'}(x) = 0, & \\ \xi^{x,y}(y) & = & \zeta^{x',y'}(y) = 1, & \\ \xi^{x,y}(z) & = & \xi(z) \leq \zeta(z) = \zeta^{x',y'}(z) & \text{for all } z \neq x, y, \end{array}$$

- $x = x'$, $y \in Y^x_{\xi,\zeta}$ and $y' \in \overline{Y}^x_{\xi,\zeta}$
  thus $y \neq y'$, $\zeta(y) = 1$ and

$$\begin{array}{rcll} \xi^{x,y}(x) & = & \zeta^{x',y'}(x) = 0, & \\ \xi^{x,y}(y) & \leq & 1 = \zeta(y) = \zeta^{x',y'}(y), & \\ \xi^{x,y}(y') & \leq & 1 = \zeta^{x',y'}(y'), & \\ \xi^{x,y}(z) & = & \xi(z) \leq \zeta(z) = \zeta^{x',y'}(z) & \text{for all } z \neq x, y, y', \end{array}$$

- $y = y'$, $x \in X^y_{\xi,\zeta}$ and $x' \in \overline{X}^y_{\xi,\zeta}$
  thus $x \neq x'$, $\xi(x') = 0$ and

$$\begin{array}{rcll} \xi^{x,y}(x) & = & 0 \leq \zeta^{x',y'}(x), & \\ \xi^{x,y}(x') & = & \xi(x') = 0 \leq \zeta^{x',y'}(x'), & \\ \xi^{x,y}(y) & = & \zeta^{x',y'}(y') = 1, & \\ \xi^{x,y}(z) & = & \xi(z) \leq \zeta(z) = \zeta^{x',y'}(z) & \text{for all } z \neq x, x', y. \end{array}$$

In all three cases, we find that $\xi^{x,y} \leq \zeta^{x',y'}$. Hence partial order is preserved in coupled transitions for all $\xi \leq \zeta$.

We now turn to uncoupled transitions, $(\xi, \zeta) \to (\xi^{x,y}, \zeta)$ and $(\xi, \zeta) \to (\xi, \zeta^{x,y})$, with rates $(\Gamma_\xi(x, y) - \varphi_{\xi,\zeta}(x, y))$ and $\big(\Gamma_\zeta(x, y) - \overline{\varphi}_{\xi,\zeta}(x, y)\big)$ respectively. In both cases, partial order could be broken if and only if $\xi(y) = \zeta(y) = 1$, $\xi(x) = \zeta(x) = 0$ respectively and the associated transition rate is nonzero. In the first case, $\zeta(y) = 1$ implies that $y \notin Y^x_{\xi,\zeta}$, which allows us to precise the estimate (5.13) and get the value of $\varphi_{\xi,\zeta}(x, y)$, as follows. Note that in the expression (3.24) for $G_{\xi,\zeta}(x, y; x', y')$ when $\xi \leq \zeta$, since $y \notin Y^x_{\xi,\zeta}$, we are in the case $y = y'$ so that $G_{\xi,\zeta}(x, y; x', y')$ is given by the third line in (3.24) (recall (5.12)):

$$\begin{aligned} \varphi_{\xi,\zeta}(x, y) &= \Gamma_\xi(x, y) \wedge \Gamma_\zeta(x, y) \\ &\qquad + \sum_{m,n>0} \delta(x, x^{y,m}_{\xi,\zeta}) \sum_{x' \in S} \delta(x', \overline{x}^{y,n}_{\xi,\zeta})\, H_{m,n}(T^{y,\cdot}_{\xi,\zeta}, \overline{S}^{x,\cdot}_{\xi,\zeta}) \\ &= \Gamma_\xi(x, y) \wedge \Gamma_\zeta(x, y) \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{m>0} \delta(x, x^{y,m}_{\xi,\zeta}) \left(T^{y,m}_{\xi,\zeta} \wedge \overline{S}^{y,*}_{\xi,\zeta} - T^{y,m-1}_{\xi,\zeta} \wedge \overline{S}^{y,*}_{\xi,\zeta} \right) \\
&= \Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) + \sum_{m>0} \delta(x, x^{y,m}_{\xi,\zeta}) \left[\Gamma_\xi(x,y) - \Gamma_\zeta(x,y) \right]^+ \\
&= \Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) + \mathbf{1}_{x \in X^y_{\xi,\zeta}} \left[\Gamma_\xi(x,y) - \Gamma_\zeta(x,y) \right]^+ \\
&= \Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) + \left[\Gamma_\xi(x,y) - \Gamma_\zeta(x,y) \right]^+ \\
&= \Gamma_\xi(x,y). \qquad (5.16)
\end{aligned}
$$

Thus uncoupled transitions in the first marginal that do not preserve partial order in the case $\xi \le \zeta$ have zero transition rates.

In uncoupled transitions for the second marginal, $(\xi, \zeta) \to (\xi, \zeta^{x,y})$ in which partial order could be broken, $\xi(x) = 1$ implies $x \notin \overline{X}^y_{\xi,\zeta}$ and, following the same line as in (5.15), one gets now the value of $\overline{\varphi}_{\xi,\zeta}(x,y)$. Again, in the expression (3.24) for $G_{\xi,\zeta}(x,y;x',y')$ when $\xi \le \zeta$, since $x \notin \overline{X}^y_{\xi,\zeta}$, we are in the case $x = x'$ so that $G_{\xi,\zeta}(x,y;x',y')$ is given by the second line in (3.24) (cf. (5.6) and the reasoning above to go from (5.7) to (5.12)):

$$
\begin{aligned}
\overline{\varphi}_{\xi,\zeta}(x,y) &= \sum_{x',y' \in S} \xi(x') \left(1 - \xi(y')\right) G_{\xi,\zeta}(x',y';x,y) \\
&= \Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) \\
&\qquad + \sum_{m,n>0} \sum_{y' \in S} \delta(y', y^{x,m}_{\xi,\zeta})\, \delta(y, \overline{y}^{x,n}_{\xi,\zeta})\, H_{m,n}(S^{x,\cdot}_{\xi,\zeta}, \overline{T}^{x,\cdot}_{\xi,\zeta}) \\
&= \Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) \\
&\qquad + \sum_{n>0} \delta(y, \overline{y}^{x,n}_{\xi,\zeta}) \left(S^{x,*}_{\xi,\zeta} \wedge \overline{T}^{x,n}_{\xi,\zeta} - S^{x,*}_{\xi,\zeta} \wedge \overline{T}^{x,n-1}_{\xi,\zeta} \right) \\
&= \Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) + \sum_{n>0} \delta(y, \overline{y}^{x,n}_{\xi,\zeta}) \left[\Gamma_\zeta(x,y) - \Gamma_\xi(x,y) \right]^+ \\
&= \Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) + \mathbf{1}_{y \in \overline{Y}^x_{\xi,\zeta}} \left[\Gamma_\zeta(x,y) - \Gamma_\xi(x,y) \right]^+ \\
&= \Gamma_\xi(x,y) \wedge \Gamma_\zeta(x,y) + \left[\Gamma_\zeta(x,y) - \Gamma_\xi(x,y) \right]^+ \\
&= \Gamma_\zeta(x,y). \qquad (5.17)
\end{aligned}
$$

Uncoupled transitions in the second marginal that do not preserve partial order in the case $\xi \le \zeta$ have thus also zero transition rates.

In conclusion, in the generator of the coupling process (3.9) with rates (3.24), for all pairs of configurations $(\xi, \zeta) \in \Omega \times \Omega$ such that $\xi \le \zeta$, all possible transitions, coupled or uncoupled which have a non zero transition rate do preserve the partial order. In the case $\xi > \zeta$, the same result can be obtained along similar lines, and we thus omit its proof. The coupling defined in Proposition 3.3 is thus increasing. □

Proof of Corollary 3.8. Let $\xi, \zeta \in \Omega$ be two configurations. and let $(x_1, y_1, x_2, y_2) \in S^4$. If $(x_1, y_1) = (x_2, y_2)$, the rate (3.25) is identical to the one given in the first line in (3.24). We now assume that $(x_1, y_1) \neq (x_2, y_2)$, and we want to recover the formulas in (3.24) from the ones in (3.26). For $\xi \leq \zeta$ and $x_1 = x_2$, we first compute using (3.27), (3.14)–(3.21) and (3.23)

$$\begin{aligned}
&\big[H^i_{\xi,\zeta}(x_1; y_1, y_2)\big]^+ \\
&\quad = \Big[\Big(\sum_{\substack{y' \leq y_1 \\ y' \in Y^{x_1}_{\xi,\zeta}}} \Gamma_\xi(x_1, y')\Big) \wedge \Big(\sum_{\substack{z' \leq y_2 \\ z' \in \overline{Y}^{x_1}_{\xi,\zeta}}} \big[\Gamma_\zeta(x_1, z') - \Gamma_\xi(x_1, z')\big]^+\Big) \\
&\qquad -\Big(\sum_{\substack{y' < y_1 \\ y' \in Y^{x_1}_{\xi,\zeta}}} \Gamma_\xi(x_1, y')\Big) \vee \Big(\sum_{\substack{z' < y_2 \\ z' \in \overline{Y}^{x_1}_{\xi,\zeta}}} \big[\Gamma_\zeta(x_1, z') - \Gamma_\xi(x_1, z')\big]^+\Big)\Big]^+ \\
&\quad = \sum_{m,n>0} \delta(y_1, y^{x_1,m}_{\xi,\zeta})\, \delta(y_2, \overline{y}^{x_1,n}_{\xi,\zeta})\, H_{m,n}(S^{x_1,\cdot}_{\xi,\zeta}, \overline{T}^{x_1,\cdot}_{\xi,\zeta}).
\end{aligned} \tag{5.18}$$

Then, for $\xi \leq \zeta$ and $y_1 = y_2$, we now compute using (3.28), (3.14)–(3.21) and (3.23)

$$\begin{aligned}
&\big[H^f_{\xi,\zeta}(x_1, x_2; y_1)\big]^+ \\
&\quad = \Big[\Big(\sum_{\substack{x' \leq x_1 \\ x' \in X^{y_1}_{\xi,\zeta}}} \big[\Gamma_\xi(x', y_1) - \Gamma_\zeta(x', y_1)\big]^+\Big) \wedge \Big(\sum_{\substack{y' \leq x_2 \\ y' \in \overline{X}^{y_1}_{\xi,\zeta}}} \Gamma_\zeta(y', y_1)\Big) \\
&\qquad -\Big(\sum_{\substack{x' < x_1 \\ x' \in X^{y_1}_{\xi,\zeta}}} \big[\Gamma_\xi(x', y_1) - \Gamma_\zeta(x', y_1)\big]^+\Big) \vee \Big(\sum_{\substack{y' < x_2 \\ y' \in \overline{X}^{y_1}_{\xi,\zeta}}} \Gamma_\zeta(y', y_1)\Big)\Big]^+ \\
&\quad = \sum_{m,n>0} \delta(x_1, x^{y_1,m}_{\xi,\zeta})\, \delta(x_2, \overline{x}^{y_1,n}_{\xi,\zeta})\, H_{m,n}(T^{y_1,\cdot}_{\xi,\zeta}, \overline{S}^{y_1,\cdot}_{\xi,\zeta}).
\end{aligned} \tag{5.19}$$

We proceed similarly for the other terms. □

Proof of Proposition 3.9. • We first prove that the operator defined by (3.29) is a valid coupling, that is the coefficient associated to each transition is nonnegative. We rewrite the generator $\overline{\mathcal{L}}^D$ as,

$$\begin{aligned}
\overline{\mathcal{L}}^D f(\xi, \zeta) = &\sum_{x_1, y_1 \in S} \xi(x_1)(1 - \xi(y_1)) \\
&\times \big(\Gamma_\xi(x_1, y_1) - \sum_{x_2, y_2 \in S} \zeta(x_2)(1 - \zeta(y_2)) G^D_{\xi,\zeta}(x_1, y_1; x_2, y_2)\big) \\
&\times \big(f(\xi^{x_1,y_1}, \zeta) - f(\xi, \zeta)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{x_2,y_2\in S} \zeta(x_2)(1-\zeta(y_2)) \\
&\qquad \times\big(\Gamma_\zeta(x_2,y_2) - \sum_{x_1,y_1\in S} \xi(x_1)(1-\xi(y_1))G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)\big) \\
&\qquad\quad \times\big(f(\xi,\zeta^{x_2,y_2}) - f(\xi,\zeta)\big) \\
&+ \sum_{x_1,y_1\in S}\sum_{x_2,y_2\in S} \xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2) \\
&\qquad\quad \times\big(f(\xi^{x_1,y_1},\zeta^{x_2,y_2}) - f(\xi,\zeta)\big).
\end{aligned}
$$

In the above expression, the first (respectively second) line refers to uncoupled transitions $(\xi,\zeta)\to(\xi^{x_1,y_1},\zeta)$ (respectively $(\xi,\zeta)\to(\xi,\zeta^{x_2,y_2})$), while the third line refers to coupled transitions $(\xi,\zeta)\to(\xi^{x_1,y_1},\zeta^{x_2,y_2})$.

We first prove that the coefficient associated to an uncoupled transition $(\xi,\zeta)\to(\xi^{x_1,y_1},\zeta)$ is non-negative. It reads

$$
\begin{aligned}
&\xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) \\
&\qquad\qquad - \sum_{x_2,y_2\in S} \zeta(x_2)(1-\zeta(y_2))G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)\big) \\
&= \xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) - \sum_{x_2,y_2\in S} \zeta(x_2)(1-\zeta(y_2)) \\
&\quad \times \sum_{x,y\in S} (\xi\vee\zeta)(x)(1-(\xi\vee\zeta)(y)) \\
&\qquad \times \frac{1}{N_{\xi,\zeta}(x,y)} G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y)G_{\xi\vee\zeta,\zeta}(x,y;x_2,y_2)\big) \\
&= \xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) - \sum_{x,y\in S} (\xi\vee\zeta)(x)(1-(\xi\vee\zeta)(y)) \\
&\qquad \times \frac{1}{N_{\xi,\zeta}(x,y)} \varphi_{\xi\vee\zeta,\zeta}(x,y)\ G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y)\big) \\
&\geq \xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) \\
&\qquad\qquad - \sum_{x,y\in S} (\xi\vee\zeta)(x)(1-(\xi\vee\zeta)(y))G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y)\big) \\
&\geq 0.
\end{aligned}
$$

In this derivation, we used (3.30) to get the first equality, then exchanged the summations and used (3.12) to get the second one; first inequality comes from $\dfrac{1}{N_{\xi,\zeta}(x,y)}\varphi_{\xi\vee\zeta,\zeta}(x,y) \leq 1$ and nonnegativity of the coupling rates $G_{\xi,\xi\vee\zeta}$; the last one follows from inequality (3.10). Non negativity of the coefficients associated to uncoupled transitions $(\xi,\zeta)\to(\xi,\zeta^{x_2,y_2})$ follows along similar lines and inequality (3.11). Non-negativity of $G_{\xi,\zeta}$

insures that the rates $G^D_{\xi,\zeta}$ of coupled transitions $(\xi,\zeta)\to(\xi^{x_1,y_1},\zeta^{x_2,y_2})$ are also non negative.

• We now prove that the new coupling is increasing.

Suppose that $\xi\le\zeta$. We have $\xi\vee\zeta=\zeta$; equations (3.13)–(3.12) and Remark 3.5 give

$$\begin{aligned}\overline{\varphi}_{\xi,\xi\vee\zeta}(x,y) &= \overline{\varphi}_{\xi,\zeta}(x,y)=\sum_{x',y'}\xi(x')(1-\xi(y'))G_{\xi,\zeta}(x',y';x,y),\\ \varphi_{\xi\vee\zeta,\zeta}(x,y) &= \varphi_{\zeta,\zeta}(x,y)=\sum_{x',y'}\zeta(x')(1-\zeta(y'))G_{\zeta,\zeta}(x,y;x',y')\\ &= \zeta(x)(1-\zeta(y))\Gamma_\zeta(x,y).\end{aligned}$$

Inequality (3.11) implies here that $\overline{\varphi}_{\xi,\zeta}(x,y)\le\Gamma_\zeta(x,y)$, and we get from equation (3.31)

$$\zeta(x)(1-\zeta(y))\,\frac{1}{N_{\xi,\zeta}(x,y)}\,\Gamma_\zeta(x,y)=\begin{cases}1 & \text{if } \zeta(x)(1-\zeta(y))\Gamma_\zeta(x,y)\neq 0,\\ 0 & \text{otherwise.}\end{cases}\tag{5.20}$$

This enables us to prove Remark 3.10. For all $(x_1,y_1,x_2,y_2)\in S^4$ such that $\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))\neq 0$, the coupling rates $G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)$ thus read

$$\begin{aligned}&G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)\\ &\quad=\sum_{x,y\in S}\zeta(x)(1-\zeta(y))\,\frac{1}{N_{\xi,\zeta}(x,y)}\,G_{\xi,\zeta}(x_1,y_1;x,y)G_{\zeta,\zeta}(x,y;x_2,y_2)\\ &\quad=\zeta(x_2)(1-\zeta(y_2))\frac{1}{N_{\xi,\zeta}(x_2,y_2)}\Gamma_\zeta(x_2,y_2)G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\\ &\quad=G_{\xi,\zeta}(x_1,y_1;x_2,y_2).\end{aligned}\tag{5.21}$$

First equality is Equation (3.30) in the present case; second equality follows from Remark 3.5; the last one follows from (5.20), $G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\ge 0$ and $G_{\xi,\zeta}(x_1,y_1;x_2,y_2)\le\Gamma_\zeta(x_2,y_2)$ (this last inequality comes from the fact that in $\underline{\mathcal{L}}$, the increasing coupling generator defined in Proposition 3.3, the rates of uncoupled transitions are non-negative, cf. (5.15) and the fact that $\xi(x_1)(1-\xi(y_1))\neq 0$).

Inserting Equation (5.21) in (3.29), we get

$$\begin{aligned}&\overline{\mathcal{L}}^D f(\xi,\zeta)\\ &\quad=\sum_{x_1,y_1\in S}\xi(x_1)(1-\xi(y_1))\Gamma_\xi(x_1,y_1)\big(f(\xi^{x_1,y_1},\zeta)-f(\xi,\zeta)\big)\\ &\qquad+\sum_{x_2,y_2\in S}\zeta(x_2)(1-\zeta(y_2))\Gamma_\zeta(x_2,y_2)\big(f(\xi,\zeta^{x_2,y_2})-f(\xi,\zeta)\big)\end{aligned}$$

$$+\sum_{x_1,y_1\in S}\sum_{x_2,y_2\in S}\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))G_{\xi,\zeta}(x_1,y_1;x_2,y_2)$$
$$\times\big(f(\xi^{x_1,y_1},\zeta^{x_2,y_2})-f(\xi^{x_1,y_1},\zeta)-f(\xi,\zeta^{x_2,y_2})+f(\xi,\zeta)\big)$$
$$=\overline{\mathcal{L}}f(\xi,\zeta).$$

A similar identity holds for $\xi>\zeta$. Since both generators identify on $\{\xi\le\zeta\}\cup\{\xi>\zeta\}$, the coupling with generator $\overline{\mathcal{L}}^D$ is also increasing.
• We now prove that discrepancies cannot increase under $\overline{\mathcal{L}}^D$.

For any finite domain $D\subset S$, the number of discrepancies in D between two configurations ξ, ζ in Ω is defined as

$$\sum_{x\in D}|\xi(x)-\zeta(x)|$$

Each transition in (3.29) with positive transition rate involves a change on a finite number of sites. For any such transition, say $(\xi,\zeta)\longrightarrow(\xi',\zeta')$, and for any finite domain D which contains all sites involved in the transition

$$D\supset\big\{x\in S,\xi'(x)\neq\xi(x)\text{ or }\zeta'(x)\neq\zeta(x)\big\}\tag{5.22}$$

the variation of discrepancies is

$$\begin{aligned}\Delta_D(\xi,\zeta;\xi',\zeta') &= \sum_{x\in D}|\xi'(x)-\zeta'(x)|-\sum_{x\in D}|\xi(x)-\zeta(x)|\\ &= \sum_{x\in D}\big(2\xi'(x)\vee\zeta'(x)-\xi'(x)-\zeta'(x)\big)\\ &\qquad -\sum_{x\in D}\big(2\xi(x)\vee\zeta(x)-\xi(x)-\zeta(x)\big)\\ &= 2\sum_{x\in D}\big(\xi'(x)\vee\zeta'(x)-\xi(x)\vee\zeta(x)\big).\end{aligned}\tag{5.23}$$

The last equality holds since the process is conservative.
• We consider first a coupled transition $(\xi,\zeta)\longrightarrow(\xi^{x_1,y_1},\zeta^{x_2,y_2})$ for some (x_1,y_1), (x_2,y_2) in S^2 with positive transition rate in $\overline{\mathcal{L}}^D$,

$$\xi(x_1)(1-\xi(y_1))\zeta(x_2)(1-\zeta(y_2))G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)>0.\tag{5.24}$$

Turning to the definition (3.30), $G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)>0$ implies that there exists $(x_0,y_0)\in S^2$ such that both $G_{\xi,\xi\vee\zeta}(x_1,y_1;x_0,y_0)>0$ and $G_{\xi\vee\zeta,\zeta}(x_0,y_0;x_2,y_2)>0$. Thus the transitions $(\xi,\xi\vee\zeta)\longrightarrow(\xi^{x_1,y_1},(\xi\vee\zeta)^{x_0,y_0})$ and $(\xi\vee\zeta,\zeta)\longrightarrow((\xi\vee\zeta)^{x_0,y_0},\zeta^{x_2,y_2})$ have positive transition rate in $\overline{\mathcal{L}}$. Since it is the generator of an increasing coupling, $\xi\le(\xi\vee\zeta)$ and

$\zeta \le (\xi \vee \zeta)$ implies that $\xi^{x_1,y_1} \le (\xi \vee \zeta)^{x_0,y_0}$ and $\zeta^{x_2,y_2} \le (\xi \vee \zeta)^{x_0,y_0}$ and thus

$$\xi^{x_1,y_1} \vee \zeta^{x_2,y_2} \le (\xi \vee \zeta)^{x_0,y_0}.$$

Now, for any domain D as in (5.22),

$$\begin{aligned}
&\Delta_D(\xi,\zeta;\xi^{x_1,y_1},\zeta^{x_2,y_2}) \\
&= \Delta_{D\cup\{x_0,y_0\}}(\xi,\zeta;\xi^{x_1,y_1},\zeta^{x_2,y_2}) \\
&= 2 \sum_{x\in D\cup\{x_0,y_0\}} \big(\xi^{x_1,y_1}(x) \vee \zeta^{x_2,y_2}(x) - \xi(x)\vee\zeta(x)\big) \\
&\le 2 \sum_{x\in D\cup\{x_0,y_0\}} \big((\xi\vee\zeta)^{x_0,y_0}(x) - (\xi\vee\zeta)(x)\big) \\
&= 0,
\end{aligned}$$

where the last equality follows from particle conservation. Thus the number of discrepancies does not increase in any coupled transition in $\overline{\mathcal{L}}^D$.

• We now turn to uncoupled transitions in $\overline{\mathcal{L}}^D$. Let us consider a transition in the first marginal, say $(\xi,\zeta) \longrightarrow (\xi^{x_1,y_1},\zeta)$ for some (x_1,y_1) in S^2. For any finite domain D such that $\{x_1,y_1\} \subset D$, the variation in the number of discrepancies reads

$$\begin{aligned}
\Delta_D(\xi,\zeta;\xi^{x_1,y_1},\zeta) &= 2\big(\xi^{x_1,y_1}(x_1)\vee\zeta(x_1) - \xi(x_1)\vee\zeta(x_1)\big) \\
&\quad +2\big(\xi^{x_1,y_1}(y_1)\vee\zeta(y_1) - \xi(y_1)\vee\zeta(y_1)\big) \\
&= 2\big(\zeta(x_1)-\zeta(y_1)\big). \qquad (5.25)
\end{aligned}$$

Thus the variation of discrepancies is non positive except in the case where both $\zeta(x_1) = 1$ and $\zeta(y_1) = 0$. We now prove that such a transition has rate 0 in $\overline{\mathcal{L}}^D$:

First, since $(\xi\vee\zeta)(y_1) = 0$, $y_1 \notin Y^{x_1}_{\xi,\xi\vee\zeta}$ and by (3.24), for any $(x,y) \in S^2$ such that $y \ne y_1$, $G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y) = 0$. Furthermore, $\varphi_{\xi\vee\zeta,\zeta}(x,y_1) = \overline{\varphi}_{\zeta,\xi\vee\zeta,}(x,y_1)$ and since $\zeta(y_1) = 0$, equation (5.17) holds and one has

$$\varphi_{\xi\vee\zeta,\zeta}(x,y_1) = \Gamma_{\xi\vee\zeta}(x,y_1). \qquad (5.26)$$

Now the rate for the transition $(\xi,\zeta) \longrightarrow (\xi^{x_1,y_1},\zeta)$ in $\overline{\mathcal{L}}^D$ reads

$$\begin{aligned}
&\xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) \\
&\qquad - \sum_{x_2,y_2\in S} \zeta(x_2)(1-\zeta(y_2))G^D_{\xi,\zeta}(x_1,y_1;x_2,y_2)\big) \\
&\quad = \xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) - \sum_{x_2,y_2\in S} \zeta(x_2)(1-\zeta(y_2))
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{x,y\in S} (\xi\vee\zeta)(x)(1-(\xi\vee\zeta)(y)) \\
&\qquad \times \frac{1}{N_{\xi,\zeta}(x,y)}\, G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y) G_{\xi\vee\zeta,\zeta}(x,y;x_2,y_2)\big) \\
&= \xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) - \sum_{x,y\in S} (\xi\vee\zeta)(x)(1-(\xi\vee\zeta)(y)) \\
&\qquad \times \frac{1}{N_{\xi,\zeta}(x,y)}\, \varphi_{\xi\vee\zeta,\zeta}(x,y)\; G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y)\big) \\
&= \xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) \\
&\quad - \sum_{x,y\in S} (\xi\vee\zeta)(x)(1-(\xi\vee\zeta)(y)) G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y)\big) \\
&= \xi(x_1)(1-\xi(y_1))\big(\Gamma_\xi(x_1,y_1) - \varphi_{\xi,\xi\vee\zeta}(x_1,y_1)\big) \\
&= 0.
\end{aligned}
$$

The third equality comes from the fact that $\frac{1}{N_{\xi,\zeta}(x,y)}\,\varphi_{\xi\vee\zeta,\zeta}(x,y) = 1$ if $y = y_1$ and $\Gamma_{\xi\vee\zeta}(x,y_1) > 0$, and $G_{\xi,\xi\vee\zeta}(x_1,y_1;x,y) = 0$ otherwise; the last equality comes from $(\xi\vee\zeta)(x_1) = 1$, $\xi \le \xi\vee\zeta$ and equation (5.16).

Thus the number of discrepancies does not increase in any uncoupled, first marginal transition in $\overline{\mathcal{L}}^D$.

• Finally, we consider an uncoupled, second marginal transition $(\xi,\zeta) \longrightarrow (\xi,\zeta^{x_2,y_2})$ for some (x_2,y_2) in S^2. Again, one proves that either the number of discrepancies does not increase, or has zero transition rate. The derivation is essentially identical to the previous one so we skip it.

Collecting all cases, we have shown that in any transition in $\overline{\mathcal{L}}^D$ with nonzero transition rate, the number of discrepancies does not increase. The result is proven. □

Proof of Proposition 3.11 and Proposition 3.12. We consider the coupling with rates given in Proposition 3.12, and we show that it satisfies the requirements for Proposition 3.11. For two configurations ξ and ζ such that $\xi \le \zeta$, one can compute easily the sum of correlated jump rates associated to a jump in a given marginal. One finds, respectively

$$
\begin{aligned}
\varphi_{\xi,\zeta}(x,y) &= \sum_{x',y'\in S} \zeta(x')\,(1-\zeta(y'))\, G_{\xi,\zeta}(x,y;x',y') \\
&= \zeta(x)\,(1-\zeta(y))\,\Gamma_\xi(x,y) + \zeta(x)\,\zeta(y)\,\frac{\overline{T}^{x,*}_{\xi,\zeta}}{S^{x,*}_{\xi,\zeta}}\,\Gamma_\xi(x,y) \qquad (5.27)
\end{aligned}
$$

and

$$\begin{aligned}\overline{\varphi}_{\xi,\zeta}(x,y) &= \sum_{x',y'\in S} \xi(x')\,(1-\xi(y'))\,G_{\xi,\zeta}(x',y';x,y) \\ &= \xi(x)\,(1-\xi(y))\,\Gamma_\zeta(x,y) \\ &\qquad + (1-\xi(x))\,(1-\xi(y))\,\frac{T^{y,*}_{\xi,\zeta}}{\overline{S}^{y,*}_{\xi,\zeta}}\,\Gamma_\zeta(x,y). \end{aligned} \tag{5.28}$$

Clearly coupled jump rates $G_{\xi,\zeta}(x,y;x',y')$ and uncoupled jump rates $\Gamma_\xi(x,y)-\varphi_{\xi,\zeta}(x,y)$, $\Gamma_\zeta(x,y)-\overline{\varphi}_{\xi,\zeta}(x,y)$ are all nonnegative for $\xi\le\zeta$, and similarly for $\xi>\zeta$. Following the same lines as in the proof of Proposition 3.3, one finds that the above rates define an increasing Markovian coupling. Using these new rates, one can define as in Proposition 3.9 a new coupling $\overline{\mathcal{L}}^D$ such that the discrepancies on the involved sites do not increase. Now suppose that for a given pair of non ordered configurations ξ and ζ, there is a discrepancy at site x, say $\xi(x)=1$ and $\zeta(x)=0$. Now the discrepancy can move alongside with the particle in the first marginal to any empty site y such that the edge (x,y) is open at rate $\Gamma_\xi(x,y)>0$, or to any fully occupied site y such that the edge (y,x) is open, alongside with a particle from the second marginal in the opposite direction with rate $\left(1-\frac{\overline{T}^{x,*}_{\xi,\zeta}}{S^{x,*}_{\xi,\zeta}}\right)\Gamma_\zeta(y,x)>0$. In this case, pairs of discrepancies of opposite sign connected through an open path have positive probability to disappear. □

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