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From Chern classes to Milnor classes

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Abstract. We survey how the Milnor number of complex map-germs with an isolated critical point extends via the theory of Chern classes of singular varieties, to the concept of Milnor classes of varieties with arbitrary singular set in complex manifolds.

Keywords. Chern classes, Milnor classes, Lê classes.

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In memory of
Roberto Callejas-Bedregal†

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Introduction

Given a (real or complex) vector bundle E over a manifold M, a basic point is to know how trivial the bundle is. This is equivalent to asking how many sections E admits that are linearly independent everywhere (over $\mathbb{R}$ or $\mathbb{C}$). Of course this all makes sense also in the holomorphic category.

For instance, it is known that every continuous vector field on S^2 must have singularities and therefore, in particular, the tangent bundle TS^2 is not trivial. From Poincaré-Hopf's theorem one may deduce that every odd-dimensional sphere does admit a vector field with no singularities, *i.e.*, a trivial 1-dimensional subbundle.

Chern classes of complex vector bundles can be defined in various ways and they are obstructions for constructing linearly independent sections. These are powerful invariants with plenty of applications in topology and in differential and algebraic geometry. In fact these are now fundamental concepts for instance in string theory, Chern–Simons theory and Gromov–Witten invariants.

The theory of Chern classes for singular varieties, initiated by M. H. Schwartz [50], D. P. Sullivan [54], R. MacPherson [29], and continued by J. P. Brasselet (see for instance [8]), W. Fulton [20] and others, keeps growing fastly and it is now a rich theory that can be regarded from several points of view and has deep connections with several areas of mathematics. There are various notions extending to the singular case the classical Chern classes of complex manifolds, having each its own properties and interest.

The classes introduced by M. H. Schwartz are an extension for stratified singular varieties of the usual Chern classes regarded as obstructions for constructing linearly independent sections of vector bundles. The classes introduced by MacPherson proved affirmatively a conjecture stated by Deligne with ideas by Grothendieck, somehow motivated by Sullivan's work for the Stiefel-Whitney classes. MacPherson's construction actually assigns a "theory of homology Chern classes" to each constructible function on a compact complex algebraic variety X; these classes are natural. Then Brasselet and Schwartz showed in [11] that Schwartz' and MacPherson's construction for the constructible function $\mathbb{1}_X$ actually coincide up to Alexander duality isomorphism; hence these are named

Schwartz-MacPherson classes, that we denote c_*^{SM}. On the other hand the Fulton classes c_*^{Fu} are defined using the classical Segre classes in algebraic geometry. All of these can be regarded in the singular homology or in the Chow group in the algebraic case (see [4]).

In the 1990s P. Aluffi, studying which schemes can arise as singular schemes of hypersurfaces in complex manifolds, realized that it was important to compare the Schwartz-MacPherson and the Fulton classes. This same issue, comparing the c_*^{SM} and c_*^{Fu} classes, also arose at almost the same time and by different reasons in the work of Parusiński-Pragacz, Yokura and Brasselet-Lehmann-Seade-Suwa. In [40, 41, 42, 43] this appears in relation with the generalized Milnor number and the topology of degeneracy loci of sections of vector bundles. In Yokura's work this appeared in [62, 61] while looking at Chern classes in bivariant theory (cf. Brasselet's work [6]), searching for a Verdier-Riemann-Roch type theorem for the MacPherson classes of singular varieties. On the other hand this comparison of the c_*^{SM} and c_*^{Fu} classes was a natural continuation of Brasselet-Schwartz' theorem showing that the MacPherson and the Schwartz classes coincide (up to Alexander duality). Seade and Suwa proved [52, 55] that in the case of compact local complete intersections with only isolated singularities, the difference $c_*^{SM} - c_*^{Fu}$ is, up to sign, the sum of certain numerical invariants called Milnor numbers, that spring from the local study of the geometry and topology of the critical points holomorphic map-germs. Then, looking for an extension of Parusinski's generalized Milnor number to the case of complete intersections naturally led the authors of [10] to comparing the Schwartz-MacPherson and the Fulton classes.

The Milnor Fibration of holomorphic maps introduced in [36] is a fundamental object for the study of the local topology of complex hypersurfaces. When the map-germ has an isolated critical point one has the associated Milnor number, which is the most important numerical invariant associated to an isolated complex hypersurface singularity. This invariant was extended by Hamm [23] to isolated complete intersection singularities. The literature about the Milnor number is vast and we refer for instance to [27, 51] for recent accounts of the subject. It is a topological invariant, easily computable and it determines the homeomorphism type of the Milnor fiber.

When considering non-isolated complex hypersurface singularities there are two important viewpoints extending the Milnor number: one is local and mostly due to work by Lê D. T., B. Teissier and D. Massey, who introduced in [30, 32] the notions of Lê cycles and Lê numbers. These spring from the theory of polar varieties developed by Lê and Teissier, with roots in ideas by René Thom. There is a Lê cycle (and number) in each complex dimension from 0 to that of the singular set; these encode deep information about the singularity germ and they determine the homeomorphism type of the local Milnor fiber (see for instance [32, 33]). The other viewpoint is global and is due to A. Parusiński who introduced in [39] the notion of a generalized Milnor number: this is an integer associated to each connected component of the singular set of a complex hypersurface in a compact complex manifold. There are several interpretations of that invariant by Parusiński-Pragacz and by other authors.

If X is a complete intersection in a compact complex manifold M, then the difference $c_*^{SM} - c_*^{Fu}$ between the total Schwartz-MacPherson and the Fulton class has support at the singular set X_{sing}. Hence, when X_{sing} consists of isolated points, these classes coincide in all dimensions greater than 0, and as mentioned before, in dimension 0 the difference is the sum of the local Milnor numbers. It is thus natural to call $c_*^{SM} - c_*^{Fu}$ the total Milnor class of X. There is one Milnor class in each dimension, from 0 to that of X_{sing}, and in dimension 0, for hypersurfaces this coincides with Parusinsky's generalized Milnor number.

Milnor classes are interesting invariants, which are defined globally but have support at the singular set. What information these classes encode is not yet understood and little is known about their geometry.

In this work we present and review all these concepts and survey our work and contributions with Roberto Callejas-Bedregal, our dear friend and co-author who passed away in April 2021. So this article is dedicated to his memory. Chapters 5 and 6 concern our articles [14, 15]. We also refer to [16] that was being written by the three of us when Roberto passed away.

We begin this article by presenting some concepts about complex vector bundles over manifolds, highlighting the role of the tangent and cotangent bundle and the local Poincaré-Hopf index theorem, which is the paradigm of Chern classes (see [12] for a thorough account on the subject and [8] for a recent expository article that includes an interesting account on the birth of Chern classes; see [16] for an account on the relations with the Milnor number). The next chapter defines the Chern classes of vector bundles as elements in the cohomology of the base space, using first algebraic topology. Then, following Fulton's book [20], we use algebraic geometry tools to describe these classes as elements in the Chow group.

Chern classes of complex manifolds are associated to their tangent bundle, and in chapter 3 we present different notions extending these concepts to singular varieties. We highlight the Schwartz classes, MacPherson classes and Fulton classes, and following an important observation that we learned from J.-P. Brasselet, we notice that each of these is associated to a way of extending the tangent bundle over the singular set. In chapter 4 we describe Milnor classes and their basic properties. In chapter 5 we plunge into the geometry of these classes, focusing on our articles with Callejas-Bedregal; we present the concept of global Lê cycles and their relationship with local Lê cycles, introduced by D. Massey, and study their relations with Milnor classes, following [14, 15].

In [14] we use work by Schürmann and Tibăr for affine varieties [49], to show that Massey's concept of Lê cycles can be globalized to projective hypersurfaces and, surprisingly, the information encoded in those classes is equivalent to the information encoded in the Milnor classes, since the global Lê classes determine the Milnor classes and conversely.

In [15], somehow inspired by [46], we first get Verdier-Riemann–Roch type formulae for the total classes $c_*^{SM}(X)$ and $c_*^{Fu}(X)$, and use these to prove a surprisingly simple formula for the total Milnor class when X is defined by a finite number of local complete intersection $X_1, \dots, X_r$ in a complex manifold, satisfying certain transversality conditions. As applications, we obtain a Parusinski–Pragacz type formula and an Aluffi type formula for the Milnor class, and a description of the Milnor classes of X in terms of the global Lê classes of the X_i.

Most of the work on Milnor classes in the literature is for hypersurfaces, the complete intersection case being much harder (cf. [10, 34]). The case of varieties which are not complete intersection is far more difficult and it is not even clear what the definition of Milnor classes should be, since there are several possible candidates that coincide for complete intersections.

Chapter 1

Preliminaries on complex vector bundles

Let (M, τ) be a topological space and $\{U_i\}_{i \in J}$ be a family of non-empty open subsets of M, where the U_i are an open covering of the space M. Let also be given a collection of maps $\forall\, i \in J$, $\phi_i : U_i \to W_i \subset \mathbb{R}^n$, where ϕ_i is a homeomorphism. Hence all W_i are open in $\mathbb{R}^n$ and the transition maps

$$\psi_{ji} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \subset \mathbb{R}^n \to \phi_j(U_j \cap U_i) \subset \mathbb{R}^n$$

are also homeomorphisms.

Definition 1.1. A pair (U_i, ϕ_i) is called a coordinate chart of M and a collection $(U_i, \phi_i)_{i \in J}$ is called an atlas of M. Since the transition maps ψ_{ji} are all continuous, one says that M is a topological manifold (of dimension n). A topological manifold M is called a (real) differentiable manifold if the transition maps ψ_{ji} are differentiable. Since $\psi_{ji}^{-1} = \psi_{ij}$, this implies that the transition functions are diffeomorphisms.

Let $(M,\mathcal{U})$ be a real differentiable manifold with atlas $\mathcal{U} = (U_i, \phi_i)_{i \in J}$ of (real) dimension $2n$ and assume that M is connected (as topological space):

$$\psi_i : U_i \to W_i \subset \mathbb{R}^{2n},\ \psi_{ji} : \phi_i(U_i \cap U_j) \subset \mathbb{R}^{2n} \to \phi_j(U_i \cap U_j) \subset \mathbb{R}^{2n},$$

where $W_i \subset \mathbb{R}^{2n}$ is open and ψ_{ji} is differentiable. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard identification:

$$(z_1, \ldots, z_n) \to (Re(z_1), Im(z_1), \ldots, Re(z_n), Im(z_n)).$$

Denote $U_{ji} := \phi_i(U_i \cap U_j)$. Then $\psi_{ji} : U_{ji} \subset \mathbb{C}^n \to \mathbb{C}^n$ becomes a complex map on the open set $U_{ji} \subset \mathbb{C}^n$.

Definition 1.2. The above real differentiable manifold $(M,\mathcal{U})$ is called a complex manifold if the maps ψ_{ji} are biholomorphic $\forall\, i,j \in J$.

Definition 1.3. Let M be a complex manifold of dimension n and $N \subset M$ be closed. N is called a complex submanifold of M if $\forall\, y \in N$, there is a coordinate chart (U, ϕ) of M with $\phi : U \to W \subset \mathbb{C}^n$, W open, $y \in U$ such that $\phi(U \cap N) \cong W \cap (\mathbb{C}^k \times \{0\})$, for some $0 \le k \le n$, where k is the dimension of N.

Using the Implicit Function Theorem and the Constant Rank Theorem, one can show that:

A subset N in a complex manifold M of dimension n is a complex submanifold of dimension k if and only if it can locally be written as the zero set of locally holomorphic functions for which the Jacobian matrix has maximal rank. Hence for all $y \in N$, there is an open neighborhood U of y in M (we may choose U sufficiently small such that (U, ϕ) is a chart at y) and there are holomorphic functions $f_i : U \to \mathbb{C},\ i = 1, \ldots, n-k$ such that

$$U \cap N = \bigcap_{i=1}^{n-k} f_i^{-1}(\{0\})$$

and

$$rank \left(\frac{\partial (f_i \circ \phi^{-1})}{\partial z_j}(z) \right)_{i,j} = n-k, \forall\, z \in U \cap N. \tag{1.1}$$

Let X in $\mathbb{C}^n$ be defined by p holomorphic functions $g_1, \ldots, g_p$. A point $z_0 \in X$ is called a singular point of X if the rank of the Jacobian matrix of the g_i at z_0 is not maximal and X is called regular if it does not contain singular points.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and consider its vanishing set $V(f) = f^{-1}(0)$. Note that f being holomorphic, thus continuous, $V(f)$ is closed in $\mathbb{C}^n$, but until now it is not yet a submanifold of $\mathbb{C}^n$. This occurs if and only if

$$grad\ f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right) \neq 0,\ \forall\, z \in V(f).$$

In fact, since f is globally defined and $V(f)$ is the zero set of the holomorphic function f, we can take $U = \mathbb{C}^n$, thus $V(f) \cap \mathbb{C}^n = f^{-1}(0)$. So by (1.1) it is necessary and sufficient to show that the Jacobian matrix associated to f has maximal rank on the set $V(f)$. But there is only one (globally defined) function, so $n - k = 1$ and

$$rank\ J(f)(z) = 1 \Leftrightarrow J(f)(z) \neq 0 \Leftrightarrow grad\ f(z) \neq 0, \forall \in z \in V(f).$$

Hence if $V(f)$ is a submanifold of $\mathbb{C}^n$, then $V(f)$ is smooth.

Let M and E be complex manifolds and let $\pi : E \to M$ be a surjective differentiable map that is a family of vector spaces (over $\mathbb{C}$), *i.e.*, $\pi^{-1}(m)$ is a vector space over $\mathbb{C}$ for all $m \in M$.

Definition 1.4. A triple (π, E, M) as above is called a complex vector bundle if there is a differentiable atlas for M with open covering $\{U_i\}_{i\in J}$ such that for all $i \in J$, one has that $\pi|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$ is isomorphic (as a family of vector spaces) to the standard trivial family $p_1 : U_i \times \mathbb{C}^r \to U_i$ (first projection), where r is said to be the rank of this vector bundle. If E and M actually are complex manifolds, the above atlas for M is holomorphic and the projection p is holomorphic, then we say that (π, E, M) is a holomorphic vector bundle.

The concept of a vector bundle can also be described in the context of schemes (see [20, B.5.5]).

For short, we say that E is a vector bundle over M if we mean that $\pi : E \to M$ is a vector bundle.

Example 1.5. Let M be a complex manifold of dimension n, $p \in M$ and let (U, ϕ) be a chart at p in M. The tangent space of M at p is defined by $T_pM := d_{\phi(p)}\phi^{-1}(\mathbb{C}^n)$. It is easy to see that this definition does not depend on the chart at p. We know that T_pM is a n-complex dimensional vector space and a basis is given by $\left\{\frac{\partial}{\partial z_1}|_p, \ldots, \frac{\partial}{\partial z_n}|_p\right\}$ (partial derivatives evaluated at p).

The cotangent space of M at p, T_p^*M, is the dual space of the tangent space of M at p.

The tangent bundle $TM := \bigsqcup_{p\in M} T_pM$ with projection-map $\pi : T_pM \to M$, $\pi(v_p) = p$, is a vector bundle of rank n.

In a similar way we have the cotangent bundle T^*M.

Definition 1.6. Let E be a vector bundle over M. A holomorphic map $s : M \to E$ is called a (global) section of E if $\pi \circ s = id_E$.

Example 1.7. Sections of the tangent bundle TM of a complex manifold M are nothing but holomorphic vector fields on M: $\xi : M \to TM$, $\xi(m) = v_m \in T_mM$. Similarly the sections of the cotangent bundle T^*M are the differential 1-forms.

Definition 1.8. A vector bundle E over a manifold M is trivial if it is isomorphic to a product $M \times \mathbb{C}^k$ where $k \geq 1$.

Notice that this is equivalent to saying that E admits k sections that are lineraly independent everywhere.

Example 1.9. For instance, the spheres S^1 and S^3, and more generally every Lie group G has trivial tangent bundle. To show this just take a basis of the tangent spaces of G at the identity and translate it using the group multiplication. Similarly, every complex Lie group has holomorphically trivial tangent bundle.

As an introduction to the next chapters, we now define the Euler class of a manifold using the Poincaré-Hopf index: this is the paradigm to follow. Consider a real m-dimensional compact smooth oriented manifold M with no boundary, and a vector field v on M regarded as a section of its tangent bundle TM. Assume v has a finite number of isolated singularities (zeros), and let these be $x_1, \dots, x_r$. We use this information to construct from it a canonical cohomology class $\mathrm{Eu(M)} \in H^m(M;\mathbb{Z})$, called the *Euler class* of M, whose Poincaré dual is the homology class of the cycle

$$\sum_{x_i} Ind_{PH}(v, x_i)\{x_i\},$$

where $Ind_{PH}(v, x_i)$ is the Poincaré-Hopf index of v at x_i, and this is the Euler characteristic $\chi(M)$, by Poincaré-Hopf's theorem. We remark that the cohomology class $\mathrm{Eu(M)}$ is independent of v, but the cochain we construct to represent it does depend on the choice of v.

Let (K) be a triangulation of M such that the singularities of v are vertices, (*i.e.*, they are in the 0-skeleton). Now take the barycentric subdivision of (K), denote it $(\widehat{K})$. We use this to construct a cell decomposition dual of (K) that we denote (D_K): to each simplex σ in (K) we associate a cell $d(\sigma)$ which is the union of all simplexes in $(\widehat{K})$ whose closure meets σ exactly at its barycenter $\hat{\sigma}$. For a vertex $x_i \in K^{(0)}$ its dual cell has dimension m and it is the union of all simplexes in $(\widehat{K})$ that have x_i in its closure. Now define an m-cochain as follows: to each m-cell in (D_K) which is dual to a singularity x_i of v, we associate the local Poincaré-Hopf index $Ind_{PH}(v, x_i)$ of v at x_i; to all other m-cells we associate 0, and we extend this to m-chains by linearity. We get a cochain with integer coefficients, which actually is a cocycle, because there is no $m+1$ chain. By definition, its cohomology class is the Euler class of M, $Eu(M)$. The Poincaré duality says that $\chi(M)$ is the evaluation of the class $Eu(M)$ on the fundamental class $[M]$, *i.e.*, $H^m(M;\mathbb{Z}) \stackrel{\cap M}{\to} H_0(M;\mathbb{Z})$, $Eu(M) \cap [M] = \chi(M)$.

Clearly this class is the Poincaré dual of $\chi(M)$, the Euler characteristic regarded as an element in $H_0(M;\mathbb{Z}) \cong \mathbb{Z}$, since $\chi(M)$ equals the total index of v.

Chapter 2

Chern classes of vector bundles

Chern classes of vector bundles play a central role in geometry and topology. In the case of (almost) complex manifolds, by definition their Chern classes are those of its tangent bundle.

There are several alternative ways to define the Chern classes of vector bundles; see for instance [17, 36, 53, 20, 12] for accounts on the subject.

2.1 Algebraic Topology viewpoint

Now we define the Chern classes of a complex vector bundle E over a compact space K that is the geometric realization of a simplicial complex, also denoted K, of real dimension $2m$. We remark that everything we say in this context works similarly if we replace K by a CW-complex with a cell decomposition. We assume the complex dimension of the fibers of E is k. The topological definition is the one given by N. Steenrod [53], using obstruction theory.

Definition 2.1. A *complex r-field* for E, $r \leq k$, on a subcomplex L of K is a set $v^{(r)} = \{v_1, \dots, v_r\}$ of r continuous sections of E defined at all points in L. A singular point of $v^{(r)}$ is a point where the vectors (v_i) fail to be linearly independent. A non-singular r-field is also called an *r-frame.*

The Chern class $c^q(E) \in H^{2q}(K)$, where $q = k-r+1$, is the first possibly non-zero obstruction for constructing an r-frame of E. Let us explain this. Let $W_{r,k}$ be the Stiefel manifold of complex unitary r-frames in $\mathbb{C}^k$. Notice that we will use complex r-frames which are not necessarily unitary, but this does not change the results, because every frame is homotopic to a unitary one. We know (see [53, §25.7.]) that $W_{r,k}$ is $(2k-2r)$-connected and its first non-zero homotopy group is $\pi_{2k-2r+1}(W_{r,k}) \cong \mathbb{Z}$. The bundle of complex r-frames on E, denoted by $W_r(E)$, is the bundle associated with E whose fiber over $x \in K$ is the set of all complex r-frames in the fiber E_x over x (it is diffeomorphic to $W_{r,k}$). In the following, we fix the notation $q = k - r + 1$.

We use the standard stepwise process in obstruction theory to construct this class, similarly to the way we constructed the Euler class of a manifold. Recall that a map $X \to Y$ between topological spaces extends to the cone of X if and only if it is nulhomotopic; and a p-simplex σ is homeomorphic to the cone over $\partial\sigma$.

Let σ be a p-simplex in K. If the section $v^{(r)}$ of $W_r(E)$ is already defined over its boundary $\partial\sigma$, it defines a map :

$$\partial\sigma \simeq \mathbb{S}^{p-1} \xrightarrow{v^{(r)}} W_r(E)|_U \simeq U \times W_{r,k} \xrightarrow{pr_2} W_{r,k},$$

thus an element of $\pi_{p-1}(W_{r,k})$. If $p \leq 2k - 2r + 1$, this homotopy group is zero and therefore the section $v^{(r)}$ can be extended to σ without singularity. This means that we can always construct a section $v^{(r)}$ of $W_r(E)$ without singularity over the $(2q-1)$-skeleton $K^{(2q-1)}$.

If $p = 2(k - r + 1) = 2q$, we meet a possible obstruction. The r-frame on the boundary of each $2q$-simplex σ defines an element, denoted by $\mathrm{Ind}_{\mathrm{PH}}(v^{(r)}, \sigma)$, in the homotopy group $\pi_{2q-1}(W_{r,k}) \cong \mathbb{Z}$. The integer $\mathrm{Ind}_{\mathrm{PH}}(v^{(r)}, \sigma)$ is *the (Poincaré-Hopf) index* of the r-frame $v^{(r)}$ on σ. Similarly to the above case of the Euler class, this defines a cochain

$$\gamma \in C^{2q}(K; \pi_{2q-1}(W_{r,k}))\,,$$

by setting $\gamma(\sigma) = \mathrm{Ind}_{\mathrm{PH}}(v^{(r)}, \sigma)$ for each $2q$-simplex σ and then by extending it linearly. This cochain is actually a cocycle ([53, §41.4.]).

Definition 2.2. The cohomology class of the obtained cocycle is the q-th *Chern class of the bundle* E, $c^q(E) \in H^{2q}(K; \mathbb{Z})$.

Given a complex vector bundle E over K, of fiber dimension k, and its Chern classes defined as above, the total Chern class of E is:

$$c^*(E) = 1 + c^1(E) + \cdots + c^k(E)$$

This is an element in the cohomology ring $H^*(K, \mathbb{Z})$, and actually this is a unit, which therefore has an inverse. This will be used in the sequel.

The class one gets in this way is independent of the various choices involved in its definition.

Example 2.3. Note that if K is a simplicial complex whose geometric realization is a manifold M of complex dimension m and if E is its tangent bundle TM, then the top Chern class $c^m(TM)$ coincides with the Euler class of the underlying real tangent bundle $T_{\mathbb{R}}M$, so Chern classes are a natural generalization of the Euler class. That is, $c^m(TM)$ is the primary obstruction to constructing a never-zero tangent complex vector field on M, where primary means the first possibly non-zero obstruction. Then $c^{m-1}(TM)$ is the primary obstruction to constructing two tangent $\mathbb{C}$-linearly independent vector fields on M and so on; see for instance [36].

We notice also that if the manifold M is compact and has no boundary, then the Poincaré duality isomorphism carries the Chern classes into the homology of M. These are the homology Chern classes, which will also be used in the sequel.

2.2 Algebraic Geometry viewpoint

In this section we follow Fulton's book [20] and define Chern classes of vector bundles over algebraic varieties as operators in the Chow group. In order to do so, we first introduce some necessary background from algebraic geometry, giving references to specific concepts. We refer to Griffiths and Harris book [22] for a different approach.

Let X be a variety (over $\mathbb{C}$) of dimension n, with $\mathcal{O}_X$ the structure sheaf, and denote by $\mathcal{O}_{V,X}$ the local ring of X along a subvariety V. Let $R(X)$ be its field of rational functions and $R(X)^*$ the multiplicative subgroup of its non-zero elements. See [20, B.1.1 and B.1.2.].

For a $(k+1)$-dimensional subvariety W of X and a rational function $r \in R(W)^*$, the divisor of r is the k-cycle on X denoted $[\mathrm{div}(r)]$ and defined by:

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V]\,,$$

where the sum runs over all codimension one subvarieties V of W and ord_V is the order of vanishing of r. If we write $r = a/b$ with $a, b \in \mathcal{O}_{V,X}$, then $\mathrm{ord}_V(r) = \mathrm{ord}_V(a) - \mathrm{ord}_V(b)$, where $\mathrm{ord}_V(a)$ and $\mathrm{ord}_V(b)$ denote the lengths of the $\mathcal{O}_{V,X}$-modules $\frac{\mathcal{O}_{V,X}}{(a)}$ and $\frac{\mathcal{O}_{V,X}}{(b)}$, respectively.

Definition 2.4. A *k-cycle* on X is a finite formal sum $\sum n_i[V_i]$ where the n_i are integers and the V_i are k-dimensional subvarieties of X. The group of k-cycles in X, Z_kX, is the free abelian group generated by the k-dimensional subvarieties of X; to a subvariety V of X corresponds $[V] \in Z_kX$. A *Weil divisor* on X is an $(n-1)$-cycle on X; the Weil divisors form the group $Z_{n-1}X$.

A *k-cycle* α is rationally equivalent to zero, written $\alpha \sim 0$, if it is the divisor of a rational function. That is, if there are a finite number of $(k+1)$-dimensional subvarieties W_i of X, and $r_i \in R(W_i)^*$ such that:

$$\alpha = \sum [\mathrm{div}(r_i)]\,.$$

The cycles rationally equivalent to zero form a subgroup $\mathrm{Rat}_k X$ of $Z_k X$. *The Chow group* $A_k X$ is the group of k-cycles in X modulo rational equivalence:

$$A_k X \; = \; Z_k X / \mathrm{Rat}_k X \, .$$

For each affine open set U of X, let $\mathcal{K}(U)$ be the total quotient ring of the coordinate ring $A(U)$. This determines a presheaf on X, whose associated sheaf is denoted $\mathcal{K}$. Let $\mathcal{K}^*$ denote the (multiplicative) sheaf of invertible elements in $\mathcal{K}$ and $\mathcal{O}^*$ the sheaf of invertible elements in $\mathcal{O} = \mathcal{O}_X$. A Cartier divisor D on X is a section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$. A Cartier divisor is determined by a collection of affine open sets U_i which cover X, and elements $f_i \in \mathcal{K}(U_i)$, such that f_i/f_j is a section of $\mathcal{O}^*$ over $U_i \cap U_j$ (see [20, B.4.1.]).

Consider now a line bundle L over an algebraic variety X. For any k-dimensional subvariety V of X, the restriction of L to V, $L|_V$, is isomorphic to $\mathcal{O}_V(C)$ for some Cartier divisor C on V, determined up to linear equivalence [20, §2.2]. The divisor $[C]$ determines an element in the Chow group $A_{k-1}(V)$, which we denote by $c_1(L) \cap [V]$. That is:

$$c_1(L) \cap [V] = [C] \, .$$

This is extended by linearity to algebraic cycles by $\alpha \mapsto c_1(L) \cap \alpha$, and defines a homomorphism:

$$c_1 \cap -\colon Z_k(X) \to A_{k-1}(X) \, .$$

In fact one has (see [20, 2.5.(a)]) that if α is rationally equivalent to zero on X, then $c_1(L) \cap \alpha = 0$. Hence one has a well-defined homomorphism:

$$c_1 \cap -\colon A_k(X) \to A_{k-1}(X) \, .$$

This defines the *Chern class of the line bundle* L. In fact if V is nonsingular, c_1 is the usual Chern class, defined before by other means, regarded in homology via cap product with the fundamental cycle.

Remark 2.5. The Chern class so defined satisfies various important properties (see [20, Proposition 2.5]), in particular:

1. (Commutativity) If L, L' are line bundles on X, and α is in $Z_k(X)$, then, one has in $A_{k-2}(X)$:

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha).$$

2. (Additivity) If L, L' are line bundles on X, and α is in $Z_k(X)$, then, in $A_{k-1}(X)$ one has:

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha \, ,$$

and
$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha \,,$$
where $L^\vee$ is the dual bundle.

It follows that if $L_1, \ldots, L_n$ are line bundles on X, then arbitrary polynomials in their Chern classes act on A_*X. If P is a homogeneous polynomial of degree d in n variables, then

$$P\big(c_1(L_1), \ldots, c_1(L_n)\big) \cap \alpha$$

is defined inductively in $A_{k-d}(X)$. In particular, and this will be used in the sequel, for a line bundle L on X and $\alpha \in A_k(X)$, $c_1(L)^d \cap \alpha$ is an element in $A_{k-d}(X)$ defined inductively by $c_1(L)^d \cap \alpha = c_1(L) \cap \big(c_1(L)^{d-1} \cap \alpha\big)$.

A morphism $f : X \to Y$ is proper if it is separated, and universally closed, *i.e.*, for all $Y' \to Y$, the induced morphism from $X \times_Y Y'$ takes closed sets to closed sets. (see [20, B.2.4.])

If $f : X \to Y$ is a proper morphism of algebraic varieties, then for any subvariety V of X, its image $f(V)$ is a closed subvariety of Y, and one has an induced embedding of the field of rational functions $R(f(V))$ into $R(V)$. As noticed in [20, Appendix B.2.2], this is a finite field extension if V and $f(V)$ have the same dimension; in this case we denote by $[R(V) : R(f(V)]$ the degree of that field extension. Set:

$$\deg(V/f(V)) \; = \begin{cases} [R(V) : R(f(V)] & \text{if } \dim V = \dim f(V), \\ \quad 0 & \text{if } \dim V > \dim f(V). \end{cases}$$

We then define the push-forward of V by f as:

$$f_*[V] = \deg(V/f(V))[f(V)] \;.$$

This extends linearly to the *push-forward* homomorphism of cycles (see for instance [20, 1.4]):
$$f_* = Z_k X \to Z_k Y \;.$$

Now recall that a homomorphism $A \to B$ of rings is *flat* if every exact sequence of A-modules remains exact after tensoring over A with B. And a morphism $f : X \to Y$ between algebraic varieties is *flat* if for every $p \in X$ the induced map in the local rings

$$f_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p} \,,$$

is flat. Flatness is an open generic condition, and its failure occurs where the map exhibits a type of “discontinuity”. For instance, performing a blow up at a point exhibits a fiber where the dimension “jumps” and we have no flatness there. A flat morphism $f : X \to Y$ always has a relative fiber dimension, say n. In fact if Y is non-singular and X is Cohen-Macaulay, then flatness is equivalent to saying that the fibers have constant dimension.

Given any subvariety V of Y, set:

$$f^*[V] = [f^{-1}(V)] .$$

Notice that $f^{-1}(V)$ is a subvariety of X of pure dimension $\dim Y + n$. This extends by linearity to the *pull-back homomorphism* of cycles:

$$f^* : Z_k Y \to Z_{k+n} X .$$

We now have all the ingredients we need to define the Segre classes, and therefore Chern classes, which are their inverses.

Let $E \xrightarrow{p} X$ be a holomorphic vector bundle of rank r over a complex variety X. Let $P = P(E)$ be the projective bundle of lines in E, and let $\mathcal{O}(1) = \mathcal{O}_E(1)$ be the canonical line bundle on $P(E)$. For each i, define a homomorphism in the Chow group of X by:

$$s_i(E) \cap - \colon A_k X \to A_{k-i} X \;, k \geq i \,,$$

by the formula

$$s_i(E) \cap \alpha = p_*\big(c_1(\mathcal{O}(1)^{r+i}) \cap p^*\alpha\big) \,,$$

where p^* is the flat pull back from $A_k X$ to $A_{k+r} P$. Then $(c_1(\mathcal{O}(1)^{r+i}) \cap p^*\alpha)$ is the iterated first Chern class homomorphism from $A_{k+r}P$ to $A_{k-i}P$, and p_* is the push-forward from $A_{k-i}P$ to $A_{k-i}X$.

Here the $s_i(E)$ are regarded as endomorphisms of the Chow group A_*X, with products being compositions that commute, so there is no ambiguity.

The total Segre class of the vector bundle E over X is:

$$s(E) = 1 + s_1(E) + s_2(E) + \dots \,.$$

Consider the formal power series

$$s_t(E) = \sum_{i \geq 0} s_i(E) t^i = 1 + s_1(E)t + s_2(E)t^2 + \dots \,,$$

define the Chern polynomial $c_t(E) = \sum_{i \geq 0} c_i(E) t^i = 1 + c_1(E)t + c_2(E)t^2 + \dots$ to be the inverse power series. Explicitly,

$$c_1(E) = -s_1(E), \quad c_2(E) = s_1(E)^2 - s_2(E), \; \dots \,,$$

$$c_k(E) = -s_1(E)c_{k-1}(E) - s_2(E)c_{k-2}(E) - \dots - s_k(E) \,.$$

Then the total Chern class of E is:

$$c(E) = 1 + c_1(E) + c_2(E) + \dots + c_r(E),$$

where $c_k(E) = 0$, for all $k > r$.

Remark 2.6. Each k-dimensional complex variety V has a cycle class $cl(V) \in H_{2k}(V)$, where H_* denotes homology with locally finite supports (Borel-Moore homology). If V is a subvariety of an n-dimensional complex manifold X, then $H_{2k}(V) \cong H^{2n-2k}(X, X-V)$, by Alexander duality.

For a (cellular) tube $\mathcal{T}$ around V in X, *i.e.*, the union of (closed) cells (D) which are dual of (K)-simplexes situated in V (see [8, Section 5.4.2]), the Alexander isomorphism is the resulting composition:

$$H_{2k}(V) \cong H^{2n-2k}(\mathcal{T}, \partial\mathcal{T}) \cong H^{2n-2k}(\mathcal{T}, \mathcal{T}\backslash V) \cong H^{2n-2k}(X, X\backslash V).$$

Alexander duality is a duality theory initiated by J. W. Alexander in 1915. It applies, in particular, to studying the homology of the complement of a subspace in a manifold. This follows, for instance, by a standard combination of Poincaré duality and excision. We refer to the literature for its definition; in particular the reader may look for it in [5].

The resulting homomorphism from cycles to homology passes to algebraic equivalence. This yields a cycle map

$$cl : A_*(X) \to H_*(X)$$

for complex schemes X, which is covariant for proper morphisms, and compatible with Chern classes of vector bundles.

Example 2.7. Let X be a singular variety in a complex manifold M. Then one has the Nash blow up $\widetilde{X} \xrightarrow{\nu} X$ that we briefly describe in section 3.2, and the corresponding Nash bundle $\widetilde{T} \to \widetilde{X}$. The corresponding total Chern class $c^*(\widetilde{T})$ is an element in the cohomology of $\widetilde{X}$. This is a pseudomanifold that has a fundamental cycle $[\widetilde{X}]$ (see [5]). Then taking the cap product of $c^*(\widetilde{T})$ with $[X]$ we get elements in the homology of X, which can be pushed forward by ν_*, the induced homomorphism, to the homology of X. These are by definition the Mather classes of X, that will appear in the sequel.

Example 2.8. If X is a non-singular submanifold of a complex manifold M defined by a regular section s of a holomorphic bundle E over M, then the restriction $E|_X$ is isomorphic to the normal bundle of X and by the usual properties of Chern classes, the total Chern class of X is:

$$c^*(TX) = c^*(TM|_X) \cdot c^*(E|_X)^{-1} .$$

If we now consider a holomorphic bundle E over M and a regular section s which defines a singular complete intersection X, then we no longer have a tangent bundle TX, but we can still define cohomology classes by:

$$c^*_{\mathrm{vir}}(X) = c^*(TM|_X) \cdot c^*(E|_X)^{-1} .$$

Taking the cap product of these classes with the fundamental cycle $[X]$ we get the Fulton classes mentioned below.

Chapter 3

Chern classes for singular varieties

When looking at singular varieties, there is no longer a tangent bundle. A point for defining Chern classes is what plays the role of the tangent bundle at the singular set. There are several candidates, as for instance (in the sequel we say more about each of these):

- One may consider a singular variety X embedded in a complex manifold M equipped with a Whitney stratification adapted to X and consider stratified vector fields. This leads to the Chern-Schwartz classes.

- One has the Nash bundle $\widetilde{T}$ that somehow extends over the singular set the tangent bundle of the regular part of X. This leads to the Mather classes. And considering the Mather classes with "appropriate weights" given by the local Euler obstruction one arrives to the MacPherson classes. These satisfy the important functoriality properties predicted by a conjecture of Deligne and Grothendieck.

- If X is defined by a regular section of a complex vector bundle E over M, then one has its virtual tangent bundle of X, $TX := TM|_X - E|_X$ and its total Chern class is determined by the Chern classes of $TM|_X$ and $E|_X$. This leads to the Fulton and the Fulton-Johnson classes of X. These classes actually are defined for every algebraic variety in a complex manifold by means of the Segre class.

So there are different notions of Chern classes extending to singular varieties the classical notion for complex manifolds.

3.1 Schwartz classes

The first generalization of Chern classes to singular varieties is due to M.-H. Schwartz [50]. These classes are the primary obstructions for constructing stratified frames on a singular variety X in a complex manifold (cf. [12]). Let us recall this.

We consider a compact complex analytic n-dimensional variety X embedded in a complex m-manifold M endowed with a complex analytic Whitney stratification $\{X_\alpha\}$ adapted to X (see for instance [58, 59, 60]).

Adding the stratum $M \setminus X$ we obtain a Whitney stratification of M. Let us denote by $TM|_X$ the restriction to X of the tangent bundle of M. A *stratified vector field* v on X means a continuous section of $TM|_X$ such that if $x \in X_\alpha$ then $v(x) \in T_x(X_\alpha)$.

Definition 3.1. Let L be a subspace of M which is a union of strata. A *stratified* r-field (or frame) $v^{(r)} = \{v_1, \ldots, v_r\}$ on L is an r-field (or frame) on M, defined at the points in L, consisting of stratified vector fields.

A basic ingredient in the work of M. H. Schwartz is what she called "radial extension". The idea is simple though there are technical difficulties that we shall omit. See [12] for a more detailed exposition of this construction. First we describe the local process, then we say a few words about the global process.

Let v_α be a vector field in a neighborhood of a point $x \in X_\alpha$ with possibly a singularity at x. By the local topological triviality of Whitney stratifications (see [57, 21]), there is a product neighborhood $W \cong \Delta \times U_\alpha$ of x in the ambient space, where U_α is a neighborhood of x in X_α, Δ is a small disc in the ambient manifold, transversal to X_α at x and $V \cap W$ is a product $(\Delta \cap X) \times U_\alpha$. We may assume that x is the only one possible singularity of v_α in U_α. Denoting by $p_1 : W \to \Delta$ and $p_2 : W \to U_\alpha$ the projections on the two factors of the product, we have a decomposition

$$TW = p_1^* T\Delta \oplus p_2^* TU_\alpha.$$

On the one hand, the pull-back $p_2^* v_\alpha$ is a continuous vector field on W, which is "parallel" to v_α. It is stratified, since it is tangent to the fibers of p_1. On the other hand, let Δ be equipped with the induced stratification and let v_Δ be a stratified vector field on Δ, which is singular at x and it is radial in the usual sense, (*i.e.*, pointing outwards in all directions of Δ). Then $p_1^* v_\Delta$ is a stratified vector field on W since it is tangent to the fibers of p_2 and v_Δ is stratified. It is thus radial in each slice $\Delta \times \{q\}$ for q in U_α. The local radial extension of v_α in W is the following:

Definition 3.2. The local radial extension of v_α, denoted by v, is the stratified vector field defined on the neighborhood W as the sum:

$$v = p_1^* v_\Delta + p_2^* v_\alpha.$$

A fundamental property of the local radial extension is that v has no singularity along the boundary of W, it is pointing outward W along its boundary, and if v_α has a singularity at x with index $\mathrm{Ind}_{\mathrm{PH}}(v_\alpha, x; X_\alpha)$, then the local radial extension v of v_α has x as unique singular point in W, and one has

$$\mathrm{Ind}_{\mathrm{PH}}(v, x; W) = \mathrm{Ind}_{\mathrm{PH}}(v_\alpha, x; X_\alpha). \tag{3.1}$$

Recall that if $\mathbb{S}_\varepsilon$ is a small sphere in W around x, then the (local) Poincaré-Hopf index of v at x, here denoted $\mathrm{Ind}_{\mathrm{PH}}(v, x; W)$, is the degree of the Gauss map $v/\|v\|$ from $\mathbb{S}_\varepsilon$ into the unit sphere in $\mathbb{R}^m$.

Definition 3.3. (cf. [25, 52, 19]) *The radial (or Schwartz) index* of v at $\underline{x} \in W$ is:

$$\mathrm{Ind}_{\mathrm{rad}}(v, x; W) = 1 + \mathrm{Ind}_{\mathrm{PH}}(v, x; W)\,.$$

The local radial extension allows to define the global radial extension. For this we filter X by the dimension of the strata as follows:

$$X = \overline{X}_{\mathrm{reg}} = \overline{X}_n \supset \overline{X}_{n-1} \supset \cdots \supset \overline{X}_{\alpha_j} \supset \cdots \supset \overline{X}_{\alpha_2} \supset \overline{X}_{\alpha_1} \supset X_{\alpha_0}$$

where X_{α_j} are the (not necessarily connected) strata and X_{α_0} is the lowest dimensional stratum. The radial extension is defined by induction on the dimension of the strata, starting with X_{α_0}. In the first step one considers a vector field v_{α_0} with isolated singularities on X_{α_0}, which is compact. One performs the local radial extension around X_{α_0} in a tube $\mathcal{T}(X_{\alpha_0})$, union of neighborhoods W as above (see [5] for the construction of these tubes). The vector field v is pointing outward $\mathcal{T}(X_{\alpha_0})$ along its boundary and the singularities of v in $\mathcal{T}(X_{\alpha_0})$ are exactly those of v_{α_0} in X_{α_0}. The vector field v extends to the next element in the above filtration since the X_α are complex manifolds. We iterate this process and we arrive to the following theorem of M. H. Schwartz (see [12] for details):

Theorem 3.4. *([12, Thm 2.3.1]) Let X be a complex analytic variety in a complex manifold M, and let $(X_\alpha)_{\alpha\in A}$ be a complex analytic Whitney stratification of M adapted to X. Then there exist stratified vector fields on a neighborhood of X in M constructed by radial extension as above, and every such vector field v satisfies:*

(1) *Given any stratum (X_α), the total Poincaré-Hopf index of v on the tube $\mathcal{T}(\overline{X_\alpha})$ is $\chi(\overline{X_\alpha})$.*

(2) *v is transverse, outwards pointing, to the boundary of some suitable small regular neighborhood of X in M.*

(3) *The Poincaré-Hopf index of v at each singularity x is the same if we regard v as a vector field on the stratum that contains x or as a vector field in a neighborhood of x in M. Hence the total Schwartz (or radial) index of v on X is $\chi(X)$.*

Now we are ready to define the Schwartz classes of the compact complex analytic singular variety X in a complex manifold M. Let n, m be the complex dimensions of X and M, respectively. We endow M with a Whitney stratification adapted to X and consider a triangulation (K) of M compatible with the stratification, *i.e.*, that each open simplex is located in one and only one stratum. We denote by (D) a cellular decomposition of M dual to (K). If a $2q$-cell of (D) meets X, then it intersects X transversally. To define the Schwartz classes one considers particular stratified r-frames $v^{(r)}$. A key-step is:

Theorem 3.5. *([12, §10.3, p.175]) Let n, m be, respectively, the complex dimensions of X and M, and we equip M with a complex analytic Whitney stratification adapted to X, a triangulation K for which every stratum is a union of open simplexes, and its dual cell decomposition (D) in M. Let U be a compact regular neighborhood of X in M obtained as a union of all cells in (D) which intersect V. Then, for every $r = 1, \ldots, n$, there exist stratified r-fields $v^{(r)}$ on the skeleton $(D)^{(2q)} \cap U$, $q = (m - r + 1)$, such that:*

- *If we write $v^{(r)} = (v^{(r-1)}, v_r)$, where $v^{(r-1)}$ denotes the $(r-1)$-field consisting of the first $(r-1)$ vector fields in $v^{(r)}$, then $v^{(r-1)}$ is non-singular on $(D)^{(2q)}$ and it is constructed by parallel translation of a non-singular r-frame on $(D)^{(2q-1)}$ (using Whitney (a)-property).*
- *The last vector field v_r is constructed by radial extension and the singularities of $v^{(r)}$ are the singularities v_r.*
- *$v^{(r)}$ has only isolated singularities on $(D)^{(2q)}$, and these are all in X;*

Given U and $v^{(r)}$, with $r = 2m-2q+1$, a neighborhood of X in M and a frame as in Theorem 3.5, we have that the only singularities of $v^{(r)}$ are the singularities of v_r. These are all contained in X. One has a Poincaré-Hopf type local index for the frame at each singularity, which is the index of v_r. This can also be regarded as an element in the homotopy group of the Stiefel-Manifold $\pi_{2q-1}(W_{r,m}) \cong \mathbb{Z}$. We then get an integer associated to each $2q$ in U, given by the local index of the frame. By linearity this gives rise to a $2q$-cochain which actually is a cocycle that represents a class in $H^{2q}(U, U \setminus X)$.

Definition 3.6. The *Chern-Schwartz class*, or simply the *Schwartz class*, $c^q_{Sc}(X)$ of X is class in $H^{2q}(U, U\setminus) \cong H^{2q}(M, M \setminus X)$ determined as above (by an r-frame as in Theorem 3.5). We have Schwartz classes from dimensions m (for $r = 1$) to $m - n + 1$ (for $r = n$).

It is known that the classes so obtained depend only on X and not on the stratification, nor on the triangulation.

Notice that the usual Chern classes are defined using arbitrary frames and here we are using stratified frames obtained by radial extension, which is the original way of defining these classes. Yet, we know from [12] that one can use arbitrary stratified frames. The key point is defining an appropriate index, a way of counting the contribution of each singularity of an arbitrary stratified field. Let us recall this.

We have the following definitions 10.1.3 and 10.1.4 from [12]:

Definition 3.7. We say that $v^{(r)}$ is *normally radial* at a_σ if for each stratum X_β having a_σ in its closure and for each sufficiently small tube $\mathcal{T}_\varepsilon(X_\alpha)$ around X_α in M, one has that each component $v_1, \dots, v_r$ of $v^{(r)}$ is transverse (pointing outwards) to the intersection $\overline{X_\beta} \cap \mathcal{T}_\varepsilon(X_\alpha)$. We say that $v^{(r)}$ is actually *radial* at a_σ if it is normally radial and it is also radial in its stratum.

So the framings constructed by radial extension are homotopic to normally radial frames but they may not be radial.

We need to define the local Schwartz index for arbitrary (stratified) frames; this is similar to the definition of the radial index in (3.3). Let $v^{(r)}$ be an r-frame defined on the boundary of a (D)-simplex σ of dimension $2m-2r+2$, whose barycenter is a point $a_\sigma \in X_\alpha \subset X$. We extend $v^{(r)}$ to a stratified frame on all of $\sigma \setminus \{a_\sigma\}$. By construction, the simplex σ meets transversally all the Whitney strata X_β containing X_α in their closure. Let $v^{(r)}_{\mathrm{rad}}$ be a stratified radial frame around a_σ. We define the difference between $v^{(r)}$ and $v^{(r)}_{\mathrm{rad}}$ at a_σ as follows. Consider sufficiently small spheres $\mathbb{S}_\varepsilon$, $\mathbb{S}_{\varepsilon'}$ in M, $\varepsilon > \varepsilon' > 0$, centered at a_σ, and consider the frame $v^{(r)}$ on $\mathbb{S}_\varepsilon \cap \sigma \cap X$ and $v^{(r)}_{\mathrm{rad}}$ on $\mathbb{S}_{\varepsilon'} \cap \sigma \cap X$. We use again the Schwartz's technique of radial extension to get a stratified r-frame $w^{(r)}$ on the intersection of σ with the cylinder

$$C = [(X \cap \mathbb{B}_\varepsilon) \setminus (X \cap \overset{\circ}{\mathbb{B}}_{\varepsilon'})]$$

in X bounded by $K_\varepsilon = \mathbb{S}_\varepsilon \cap X$ and $K_{\varepsilon'} = \mathbb{S}_{\varepsilon'} \cap X$, having finitely many singularities in the interior of C. At each of these singular points its index in the stratum, $\mathrm{Ind}_{\mathrm{PH}}(w^{(r)}, C \cap \sigma)$, equals its index in the ambient space M. The *difference* of $v^{(r)}$ and $v^{(r)}_{\mathrm{rad}}$ is defined as:

$$d(v^{(r)}, v^{(r)}_{\mathrm{rad}}) = \sum \mathrm{Ind}_{\mathrm{PH}}(w^{(r)}, C \cap \sigma)\,,$$

where the sum on the right runs over the singular points of $w^{(r)}$ in C and each singularity is being counted with the local index of $w^{(r)}$ in the corresponding stratum. As in the work of M.-H. Schwartz, we can check that this integer does not depend on the choice of $w^{(r)}$.

Definition 3.8. *The Schwartz* (radial) *index* of a stratified r-field $v^{(r)}$ at $a_\sigma \in X$ is:

$$\mathrm{Ind}_{\mathrm{Sch}}\big(v^{(r)}, a_\sigma; X\big) \;=\; 1 + d(v^{(r)}, v^{(r)}_{\mathrm{rad}})\;.$$

As before, a stratified r-frame $v^{(r)}$, $r \geq 1$, which is non-singular on $(D)^{(2m-2r+1)}$ and has isolated singularities on $(D)^{2m-2r+2}$, defines a cochain in the obvious way, and this cochain is actually a cocycle. One obtains a relative class

$$c^q(U, \partial U; v^{(r)}) \in H^{2q}(U, U \setminus X) \cong H^{2q}(M, M \setminus X)\,, \tag{3.2}$$

where U is a regular neighborhood of X in M. One has [12, Theorem 2.14]:

Theorem 3.9. *Given $X \subset M$ as before, equipped with a Whitney stratification adapted to X and a compatible triangulation (K), let (D) be its dual cellular decomposition and denote $(D)^j$ the union of all cells of dimension j. If $v^{(r)}$ is a stratified r-frame, $r \geq 1$, which is non-singular on $(D)^{(2m-2r+1)}$ and has isolated singularities on $(D)^{2m-2r+2}$, then the Schwartz indices of $v^{(r)}$, defined as in 3.8, determine a class $c^q(X; v^{(r)}) \in H^{2q}(M, M \setminus X)$, $2q = 2m - 2r + 2$, and this cocycle represents the corresponding Schwartz class of X, independently of the choice of the frame $v^{(r)}$ obtained by the Schwartz's radial extension procedure.*

The proof is immediate from the definitions and properties of Schwartz index.

Remark 3.10. In short, this theorem is telling us that the Schwartz class $c^q_{Sc}(X)$ of a singular variety X of dimension n in a complex manifold M is the primary obstruction for constructing a stratified r-frame of $TM|_X$. Unlike the classical case, now the cell decomposition must be dual to a triangulation of X compatible with a Whitney stratification adapted to X.

3.2 MacPherson's classes

In his paper [54] in the famous 1969 Liverpool singularities symposium, D. P. Sullivan discusses the existence of homology Stiefel classes for real analytic varieties. In the last page he explains that Deligne outlined a general conjectural theory of Chern classes for singular varieties based on ideas of Grothendieck and Hironaka's theorem about resolution of singularities. Nowadays this is known as the Deligne-Grothendieck conjecture, and it was proved by MacPherson in [29] by a different way. Let us say a few words about this.

A *constructible set* in a complex analytic variety X is a set obtained from its subvarieties by finitely many of the usual set-theoretic operations: unions, intersections and differences. A $\mathbb{Z}$-valued constructible function on X is a function $\phi : X \to \mathbb{Z}$ for which X has a finite partition into constructible sets so that ϕ is constant on each set. Or equivalently, there exists a complex analytic Whitney stratification of X such that ϕ is constant on each stratum. One has [29, Proposition 1]:

Proposition 3.11. *There is a unique covariant functor* $\mathbf{F}$ *from the category* $\mathcal{V}$ *of compact complex algebraic varieties to the category of abelian groups* $\mathcal{A}b$*, whose value on a variety* X *is the group* $F(X)$ *of constructible functions* $\phi : X \to \mathbb{Z}$ *on* X *such that for every map* $f : X \to Y$ *the function* $f_* : F(X) \to F(Y)$ *satisfies :*

$$f_*(\mathbb{1}_W)(p) := \chi(f^{-1}(p) \cap W)\,,$$

where $\mathbb{1}_W$ *is the characteristic function of subsets* $W \subset X$*, defined by* $\mathbb{1}_W(x) = 1$ *for* $x \in W$ *and* $\mathbb{1}_W(x) = 0$ *for* $x \notin W$*, and* χ *denotes the usual Euler characteristic.*

MacPherson then proves the Deligne-Grothendieck conjecture:

Theorem 3.12. *There exists a natural transformation from the functor* $\mathbf{F}$ *to homology, which for manifolds assigns to the constant function* $\mathbb{1}$ *the Poincaré dual of the total Chern class.*

Explicitly, to any constructible function α on a compact complex algebraic variety X we can assign an element $c_*(\alpha)$ in $H_*(X)$ satisfying:

1. $f_*c_*(\alpha) = c_*f_*(\alpha)$;
2. $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$;
3. If X is non-singular of complex dimension n, then $c_*(\mathbb{1}_X)$=$c^{n-*}(X)\cap [X]$ where $c^{n-*}(X)$ is the total cohomology Chern class of TX and $[X]$ the fundamental class of X.

Definition 3.13. *The total Chern-MacPherson class* $c_*^{MP}(X)$ of any compact complex algebraic variety X is c_* applied to the constant function $\mathbb{1}$ on X. More generally, for a constructible function α on X, the homology class $c_*(\alpha)$ is *the total Chern-MacPherson class of the constructible function.* For simplicity, we shall often call this the MacPherson class of the constructible function α; and call $c_*(\mathbb{1}_X)$ the MacPherson class of X.

MacPherson's proof of Theorem 3.12 uses three important ingredients; one of these is the local Euler obstruction Eu_X of an complex variety (cf. [29]).

Let $(V, 0)$ be a reduced, pure-dimensional complex analytic singularity germ of dimension n in an open set $U \subset \mathbb{C}^m$. Let $G(n, m)$ denote the Grassmanian of complex n-planes in $\mathbb{C}^m$. On the regular part V_{reg} of V there is a map $\sigma : V_{\mathrm{reg}} \to U \times G(n, m)$ defined by $\sigma(x) = (x, T_x(V_{\mathrm{reg}}))$. The *Nash transformation* $\widetilde{V}$ of V is the closure of $\mathrm{Im}(\sigma)$ in $U \times G(n, m)$. It is a complex analytic space endowed with an analytic projection map

$$\nu : \widetilde{V} \longrightarrow V$$

which is a biholomorphism away from $\nu^{-1}(V_{\mathrm{sing}})$.

Now consider the tautological bundle over $G(n,m)$, the bundle where the fiber at point $P \in G(n,m)$, is the set of vectors v in the n-plane P and denote by $\mathcal{T}$ the corresponding product extension bundle over $U \times G(n,m)$. We denote by π the projection map of this bundle and let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{V}$, with projection map π.

We notice that given such a variety X, its Nash transform $\widetilde{X}$ is defined in the obvious way, that springs from the local definition. Similarly one has a bundle $\widetilde{T}$ over $\widetilde{X}$ defined as above.

Definition 3.14. The bundle $\widetilde{T}$ over the Nash transform $\widetilde{X}$ of X is called the Nash bundle of X (both, in the local and global cases).

Given $(V,0)$ as before, an element of $\widetilde{T}$ is written (x,P,v) where $x \in U$, P is an n-plane in $\mathbb{C}^m$ based at x and v is a vector in P. So we have maps:

$$\widetilde{T} \xrightarrow{\pi} \widetilde{V} \xrightarrow{\nu} V\,.$$

Notice that ν is a biholomorphism over the regular part $V_{\mathrm{reg}} := V \backslash V_{\mathrm{sing}}$ and the Nash bundle over $\nu^{-1}(V_{\mathrm{reg}})$ is isomorphic to the tangent bundle.

Let us consider a complex analytic Whitney stratification (V_α) of V (see for instance [58]). Adding the stratum $U \setminus V$ we obtain a Whitney stratification of U. Let us denote by $TU|_V$ the restriction to V of the tangent bundle of U. A *stratified vector field* v on V means a continuous section of $TU|_V$ such that if $x \in V_\alpha \cap V$ then $v(x) \in T_x(V_\alpha)$. By Whitney condition (a) one has the following lemma in [11]:

Lemma 3.15. *Every stratified vector field v on a subset $A \subset V$ has a canonical lifting to a section $\widetilde{v}$ of the Nash bundle $\widetilde{T}$ over $\nu^{-1}(A) \subset \widetilde{V}$.*

Now consider a stratified radial vector field $v(x)$ in a neighborhood of 0 in V, *i.e.*, there is ε_0 such that for every $0 < \varepsilon \le \varepsilon_0$, $v(x)$ is pointing outwards the ball open ball in $\mathbb{C}^m$ centered at 0 of radius ε, $\mathbb{B}_\varepsilon$, over the boundary $\mathbb{S}_\varepsilon := \partial\mathbb{B}_\varepsilon$.

The following interpretation of the Euler obstruction is given in [11]. We refer to [29] for the original definition using 1-forms.

Definition 3.16. Let v be a radial vector field on $V \cap \mathbb{S}_\varepsilon$ and $\widetilde{v}$ the lifting of v on $\nu^{-1}(V \cap \mathbb{S}_\varepsilon)$ to a section of the Nash bundle. The *local Euler obstruction* (or simply the Euler obstruction) $\mathrm{Eu}_V(0)$ is defined to be the obstruction to extending $\widetilde{v}$ as a nowhere zero section of $\widetilde{T}$ over $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$. This is an integer, actually a class in $H^{2n}\big(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \mathbb{S}_\varepsilon)\big) \cong \mathbb{Z}$.

The second ingredient used by MacPherson is the Mather class that we now introduce; the third is the so-called graph construction in the algebraic context. We remark that the analyticity of the graph construction was proved by M. Kwieciński [26] and therefore MacPherson's theorem and proof work in the complex analytic category.

Let X be a complex analytic variety of dimension n, let $\widetilde{X} \xrightarrow{\nu} X$ be its Nash transformation and let $\widetilde{T} \to \widetilde{X}$ be the Nash bundle. Then one has the usual Chern classes of $\widetilde{T}$ defined as above, $c_i(\widetilde{T}) \in H^*(\widetilde{X})$. The variety $\widetilde{X}$ is singular in general, but since it is complex analytic, it is automatically a pseudomanifold (see for instance [5]) and therefore there is a fundamental class $[\widetilde{X}] \in H_{2n}(\widetilde{X})$ and a Poincaré homomorphism $H^*(\widetilde{X}) \to H_{2n-*}(\widetilde{X})$. Composing this with the homomorphism in homology induced by the projection ν, we get classes in the homology of X: these are the Mather classes, introduced in [29] (see Example 2.7):

Definition 3.17. *The Mather classes* of X, $c_j^{Ma}(X)$, are the Chern classes of the Nash bundle of X, carried to the homology of the Nash transform $\widetilde{X}$ by the Poincaré homomorphism, and then pushed forward to the homology of X by the homomorphism induced by the projection. The total Mather class is $c^{Ma}(X) = \nu_*(c(\widetilde{T}) \cap [\widetilde{X}])$. More generally, to any algebraic cycle $\sum n_j V_j$ in X, where the n_j are integers and the V_j are irreducible subvarieties of X, we can associate its *Mather class*:

$$c^{Ma}\Big(\sum n_j V_j\Big) = \sum n_j \iota_{j*} c^{Ma}(V_j) \ ,$$

where ι_j is the inclusion of V_j in X.

MacPherson's next step is writing a formula that expresses $c_*(\alpha)$ as the Mather class of an associated algebraic cycle. For this he proves [29, Lemma 2]:

Lemma 3.18. *There exists an isomorphism T from the group of algebraic cycles in X to the group of constructible functions on X defined by:*

$$T\Big(\sum n_j V_j\Big)(p) = \sum n_j Eu_{V_j}(p) \ ,$$

where $Eu_{V_j}(p)$ is the local Euler obstruction of V_j at the point $p \in X$.

Then MacPherson proves ([29, Theorem 2] and [20, Example 19.1.7]):

Theorem 3.19. $c_* := c^{Ma} T^{-1}$ *satisfies the requirements for c_* in Theorem 3.12.*

Then $c^{Ma} T^{-1}(\mathbb{1}_X)$ is the (total) MacPherson class of X that we denote by $c_*^{MP}(X)$. Notice that one actually has a total MacPherson class $c^{MP}(\alpha)$ for every constructible function α on X, and we know from [29] that one has:

$$c^{Ma}(X) = c^{MP}(Eu_X) \,, \tag{3.3}$$

where Eu_X is the local Euler obstruction of X, which is a constructible function.

Recall that in the previous section we defined the Schwartz classes of a singular analytic variety X of dimension n embedded in a complex manifold M of dimension m. Brasselet and Schwartz proved in [11] that these classes coincide with MacPherson's classes $c_*(\mathbb{1}_X)$ via Alexander duality isomorphism $H^{m-*}(M, M \setminus X) \to H_*(X)$.

In fact the theorem in [11] makes this statement precise and gives an explicit cycle representing the MacPherson class. Let us recall this.

We endow M with a Whitney stratification adapted to X and consider a triangulation (K) of M compatible with the stratification. We denote by (D) a cellular decomposition of M dual to (K). Recall that if a $2q$-cell d_α of (D) meets X, then it is dual to a $2(m-q)$-simplex σ_α of (K) contained in X. We recall too that to define the Schwartz classes one considers particular stratified r-frames $v^{(r)}$. These have no singularity on the $(2q-1)$-skeleton of (D), where $q = m - r + 1$, and (at most) isolated singularities on the $2q$-cells d_α. At each such cell, the frame $v^{(r)}$ has a Poincaré-Hopf type index at the corresponding singularity $\widehat{\sigma}_\alpha$ which is the barycenter of σ_α, also barycenter of d_α, that we may denote by $I(v^{(r)}, \widehat{\sigma}_\alpha)$; of course this index is 0 if there is no singularity of $v^{(r)}$ in that cell. Then we have the following theorem of Brasselet and Schwartz:

Theorem 3.20. *The Alexander duality isomorphism* $H^{m-*}(M, M \setminus X) \to H_*(X)$ *carries the Schwartz class* $c^*_{Sc}(X) \in H^*(M, M \setminus X)$ *to the Chern-MacPherson class* $c^{MP}_*(X) \in H_*(X)$. *In fact, the MacPherson class* $c^{MP}_{r-1}(X)$ *is represented in* $H_{2(r-1)}(X)$ *by the cycle:*

$$\sum_{\sigma_\alpha \subset X} I(v^{(r)}, \widehat{\sigma}_\alpha) \cdot \sigma_\alpha \ ,$$

where the sum runs over all the simplexes σ_α *of dimension* $2(r-1)$ *which are contained in* X, *and* $I(v^{(r)}, \widehat{\sigma}_\alpha)$ *is the (Poincaré-Hopf) index in the dual cell of each such simplex* σ_α *of a stratified vector field* $v^{(r)}$ *constructed by radial extension.*

Hence, from now on we denote the classes so obtained in homology by $c^{SM}_*(X)$ and call them the *Chern-Schwartz-MacPherson classes of* X, or simply *Schwartz-MacPherson classes.*

Remark 3.21. Brylinski, Dubson and Kashiwara [13] showed that the MacPherson classes of a singular variety can be studied by means of D-modules. In fact the micro-local viewpoint, through the theory of Lagrangian cycles, has proved to be very important and fruitful to study these characteristic classes (see for instance [45, 47]).

Remark 3.22. In this section we defined the Mather and MacPherson's classes of singular varieties X as elements in the homology of X. We remark however that the construction of Chern classes of vector bundles as the inverse of the Segre classes, shows that if X is algebraic, then the Mather and the MacPherson classes actually live in the Chow group of X.

3.3 Fulton classes

Consider first the case where the singular variety X is a complete intersection in some complex manifold M, defined by the zero set of a regular section of some holomorphic complex vector bundle E over M.

In this case, there is a morphism $i : X \to M$ that is local complete intersection, as in [20, B.7.6.]. If $M = \mathbb{C}P^n$, X is a complete intersection if the number of generators of the ideal of X is exactly the codimension of X.

Then E plays the role of a normal bundle $N_X M$ (see [20, B.7.1.]) and one has the virtual tangent bundle:

$$\tau X := TM|_X - E|_X$$

an element in the Grothendieck group of complex vector bundles on X. The total Chern class of τX is well-defined and this is:

$$c^*(\tau X) := c^*(TM|_X) \cdot c^*(E|_X)^{-1} .$$

The equivalence between these and the usual Chern classes is discussed in [20, Ch. 19].

Cap product with the fundamental class $[X]$ carries these Chern cohomology classes into the homology of X: in this particular case, these are the Fulton classes of X.

This definition works in general, for varieties that may not be complete intersection, using the Segre class, which for bundles is the inverse of the Chern class.

The Segre classes extend to the more general setting of (algebraic) cones over an algebraic variety (or scheme). This includes several familiar examples, including all vector bundles. And it also includes many other important families. One of these is the normal cone $C = C_X Y$ of a closed subvariety X in a variety Y. Let us say a few words about this.

As a motivation, recall first that in algebraic geometry one studies algebraic sets, *i.e.*, subsets of K^n, where K is an algebraically closed field, that here we take to be the complex numbers $K = \mathbb{C}$. The algebraic sets are by definition the common zeros of a set of polynomials in n variables. If X is such an algebraic set, one considers the commutative ring R of all polynomial functions $X \to \mathbb{C}$. Since $K = \mathbb{C}$ is algebraically closed, the maximal ideals of R correspond to the points of X, and the prime ideals of R correspond to the irreducible subvarieties of X.

Let us now forget this information for a moment and consider an arbitrary commutative ring R, and define its spectrum, denoted $\mathrm{Spec}(R)$, to be the set of all prime ideals. For any ideal I of R, define V_I to be the set of all prime ideals that contain I, and we equip $\mathrm{Spec}(R)$ with the Zariski topology by defining the closed sets to be

$$\big\{V_I \,|\, I \text{ is an ideal of } R\big\} .$$

Coming back to the previous example where R is the ring of polynomial functions $X \to \mathbb{C}$, the spectrum of R consists of the points of X together with elements corresponding to all subvarieties of X. The points of X are closed in the spectrum, while the elements corresponding to subvarieties of positive dimension have a closure consisting of all their points and subvarieties.

Therefore the topological space $\mathrm{Spec}(R)$ somehow is a refinement of the algebraic space X with its Zariski topology. By studying spectra of rings instead of algebraic sets, one can generalize concepts of algebraic geometry to non-algebraically closed fields and beyond, eventually arriving to the concept of schemes, due to A. Grothendieck.

There is a relative version of this concept (actually a functor) called the *relative or global spectrum.* If X is an algebraic variety and we are given a quasi-coherent sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras, there is a scheme $\mathrm{Spec}_X(\mathcal{A})$ and a morphism $f : \mathrm{Spec}_X(\mathcal{A}) \to X$ satisfying certain important properties (see [24, §5]). This allows us, among other things, to define key concepts for this presentation: *The normal cone and the Segre class of a subvariety X in a variety Y.*

Definition 3.23. The normal cone to X in Y, $C = C_X Y$ is defined by:

$$C = \mathrm{Spec}\Big(\sum_{n=0}^{\infty} \mathfrak{I}^n/\mathfrak{I}^{n+1}\Big)$$

where $\mathfrak{I}$ is the ideal sheaf defining X in Y (see [20, B.6.1.]).

When X and Y are non-singular, this corresponds to the usual normal bundle. More generally, if the embedding of X in Y is regular, the normal cone is the vector bundle on X corresponding to the dual of the sheaf $\mathfrak{I}/\mathfrak{I}^2$, and it is also called the normal bundle of X (see [20, B.7.1.]).

Definition 3.24. The *(total) Segre class* of X in Y, denoted $s(X,Y)$, is:

$$s(X,Y) = \sum_{i \geq 0} p_* \left(c_1(\mathcal{O}(1))^i \cap [P(C_X Y)]\right) \in A_* X,$$

where $P(C_X Y)$ is the projectivized normal cone, p the projection from $P(C_X Y)$ to X and $\mathcal{O}(1)$ is the dual tautological bundle of $C_X Y \oplus 1$ (see [20, B.6.3.]).

In case X is regularly embedded in Y, then the normal cone is a vector bundle and [20, Proposition 4.1] implies that the Segre class $s(X,Y)$ is the cap product of the total inverse Chern class of the normal bundle with $[X]$. By Poincaré homomorphism, that is:

$$s(X,Y) = c(N_X Y)^{-1} \cap [X]\,. \tag{3.4}$$

The following result [20, Corollary 4.2.2] gives a beautiful and useful characterization of the Segre class. This could be taken as a definition of the Segre class of X in Y with no need of introducing the previous concepts:

Proposition 3.25. *Let X be a subvariety of a compact variety Y, and let $\widetilde{Y}$ be the blow-up of Y along X. Let $\widetilde{X} \subset \widetilde{Y}$ be the exceptional divisor and $\eta : \widetilde{X} \to X$ the projection. Then the total Segre class of X in Y is:*

$$s(X,Y) = \sum_{i \geq 0} \eta_*\big(c_1(\mathcal{O}(1))^i \cap [\widetilde{X}]\big)\,.$$

We remark that all terms in this formula make sense in the complex analytic category, so we can take this as the definition of the Segre class in that setting.

Observe that if X is a complex submanifold (*i.e.*, non-singular) of a complex manifold M, then one has a C^∞ splitting of the tangent bundle of M restricted to X:

$$TM|_X = TX \oplus N_X M$$

where the latter is the normal bundle. By general properties of Chern classes (see for instance [36]) this implies:

$$c^*(TM|_X) = c^*(TX) \cdot c^*(N_X M)$$

regarded in the cohomology of X. Notice too that $TM|_X, TX$ and $N_X M$ are all complex vector bundles and in the Grothendieck group $K(X)$ of vector bundles on X we have:

$$[TX] = TM|_X - N_X M\,.$$

Now following Fulton [20, 4.2.6], let X be an algebraic variety embedded in a compact algebraic manifold M, and consider the class:

$$c_*^{Fu}(X) := c^*(TM|_X) \cap s(X,M) \in A_*(X)\,.$$

This class is independent of the choice of embedding, and if X is a local complete intersection in M, then one has the *virtual tangent bundle* of X:

$$T_{\mathrm{vir}} X := TM|_X - N_X M$$

a well-defined element in the corresponding Grothendieck group $K(X)$, and one has:

$$c_*^{Fu}(X) = c^*(TM|_X)\; c(N_X M)^{-1} \cap [X] = c^*(T_{\mathrm{vir}} X) \cap [X] \in A_*(X)\,.$$

Definition 3.26. Let X be an n-dimensional complex algebraic variety embedded in a compact algebraic manifold M. Then the class:

$$c_*^{Fu}(X) := c^*(TM|_X) \cap s(X, M) \in A_*(X)\,,$$

is called the total *Fulton class* of X.

By the above comments, if X is a local complete intersection in M, this is the cap product of the Chern class of the virtual tangent bundle with $[X]$. By definition:

$$c_*^{Fu}(X) = 1 + c_1^{Fu}(X) + \ldots + c_n^{Fu}(X)\,,$$

with $c_i^{Fu}(X) \in A_i(X)$. The various $c_i^{Fu}(X)$ are called *the Fulton classes* of X.

If X and M are complex analytic, not necessarily algebraic, the above definitions hold in the homology of X.

Remark 3.27. If X is regularly embedded in M the Fulton class coincides with another class called the Fulton-Johnson class denoted by $c_*^{FJ}(X)$, which uses the conormal sheaf $\mathcal{N}_X M$ of X in M, *i.e.*, if $\mathfrak{I}$ is as in Definition 3.23, $\mathcal{N}_X M = \mathfrak{I}/\mathfrak{I}^2$.

Chapter 4

Milnor classes

So far we have discussed the Schwartz-MacPherson and the Fulton classes of singular varieties. It is natural to ask how these are related, and that is the topic we explore in this chapter.

Let X be an n-dimensional complex variety embedded in a compact manifold M.

Definition 4.1. The total Milnor class of X is, up to sign, the difference between the total Schwartz-MacPherson and Fulton classes:

$$\mathcal{M}(X) := (-1)^n \left(c^{Fu}(X) - c^{SM}(X)\right). \tag{4.1}$$

This is the sum of the corresponding *Milnor classes* $\mathcal{M}_r(X)$ in all (even) dimensions. Milnor classes are defined globally on X, yet one has (see [43, 10, 55, 3]) that these classes have support in the singular set X_{sing} and therefore they vanish in dimensions higher than that of X_{sing}.

Milnor classes appeared first implicitly in [1, 2] and [41]. The actual name of Milnor classes was coined by various authors at about the same time (see [9, 10, 62, 43]).

The genesis of the name is related to an important invariant associated with germs of holomorphic functions.

Consider a holomorphic function $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ with a critical point at $\underline{0}$. Let $\mathbb{B}_\varepsilon$ be an open ball in $\mathbb{C}^{n+1}$ centered at $\underline{0}$ of radius $\varepsilon > 0$ sufficiently small and let $N(\varepsilon, \delta) = f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$ for $0 < \delta \ll \varepsilon$, where $\partial\mathbb{D}_\delta$ is the boundary of the disc in $\mathbb{C}$ of radius $\delta > 0$ and centered at 0. Then,

$$f : N(\varepsilon, \delta) \longrightarrow \partial\mathbb{D}_\delta \cong \mathbb{S}^1$$

is a locally trivial fibration. The fiber $F_t = f^{-1}(t) \cap \mathbb{B}_\varepsilon$ with $t \in \partial\mathbb{D}_\delta$ is called *the Milnor fiber of* f and denoted by F_f.

One knows from [35] that F_f has the homotopy type of a CW-complex of middle dimension n. Furthermore, if f has an isolated critical point at $\underline{0}$

then F_f actually has the homotopy type of a bouquet of spheres of middle dimension n, $F_f \simeq \bigvee_\mu S^n$. One has:

Definition 4.2. If the map f has an isolated critical point, say at $\underline{0}$, then the number μ above is *the Milnor number* of f at $\underline{0}$.

Example 4.3. Let f be the complex polynomial:

$$f(z_0, ..., z_n) \;=\; z_0^2 + \cdots z_n^2 \,.$$

Then f has a unique critical point at $\underline{0}$ and it is an exercise to show that the Milnor fiber is diffeomorphic to the total space of the unit tangent bundle of the n-sphere S^n. Hence the Milnor number is 1.

In general one has (see [35, Theorem 7.2] that if f has an isolated critical point, then its Milnor number equals the multiplicity:

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1,0}}{\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right)} \,.$$

Thus, for instance, if f is the Pham-Brieskorn polynomial $z_0^{a_0} + \cdots z_n^{a_0}$, $a_i \geq 2$, then (see [35, Thm. 9.1]):

$$\mu(f) = (a_0 - 1) \cdot (a_1 - 1) \cdot \ldots \cdot (a_n - 1).$$

It is well-known (see, e. g., [12, 3.2.1]) that if f has an isolated critical point at $\underline{0}$, so that $V = f^{-1}(0)$ is a hypersurface with an isolated singularity, and v is a vector field on V with an isolated singularity at 0, then the Milnor number of f is, up to sign, the difference between the radial and the GSV indices of v, which are two invariants extending for singular varieties the classical Poincaré-Hopf index. This is at the core of the following theorem [52, Theorem 2.4] and its corollary below.

Theorem 4.4. *Let X be the zero locus of a regular section s of a holomorphic complex vector bundle E of rank $k \geq 1$ over a compact complex manifold M of dimension $n+k$; assume the singular set of X consists of isolated points, say $x_1, \ldots, x_r$. Then the Fulton and the Schwartz-MacPherson classes in $H_0(X)$ differ by the sum of the local Milnor numbers:*

$$c_0^{Fu}(X) \;=\; c_0^{SM}(X) + (-1)^{n-1} \sum_{i=1}^{r} \mu_i \,.$$

The proof of Theorem 4.4 is via Chern-Weil theory, using the virtual index of vector fields, which is a localization of the top Fulton class (cf [56]). It was first proved by Suwa in [55] that the Milnor classes are localized at the singular set, and therefore the theorem above yields:

Corollary 4.5. *With the hypotheses of Theorem 4.4, the total Milnor class of X is the sum of the local Milnor numbers:*

$$\mathcal{M}(X) = \sum_{i=1}^{r} \mu_i \,.$$

So Milnor classes are a generalization of the classical Milnor number to compact varieties X with arbitrary singular set.

This led Parusiński to extending the notion of Milnor number to non-isolated hypersurface singularities. We refer to [39] for details on the original definition.

We now recall another way to view this invariant, given in [40], which is responsible for an important characterization of the Milnor classes of hypersurfaces.

We first call to mind the classical Gauss-Bonnet theorem. This says that if M is a compact m-dimensional complex manifold with tangent bundle TM, then its topological Euler characteristic can be expressed as:

$$\chi(M) = \int_M \Omega$$

where Ω is an $m-form$ representing the top cohomology Chern class.

As pointed out in [40, Section 5], if $\mathcal{L}$ is a holomorphic line vector bundle over M and s is a section transverse to the zero section, so its zero set Z is a non-singular hypersurface in M and its normal bundle is isomorphic to $\mathcal{L}|_Z$, then the Gauss-Bonnet theorem yields:

$$\chi(Z) = \int_M c_1(\mathcal{L}) \cdot c(M) \cdot c(\mathcal{L})^{-1} \,,$$

where $c(\,)$ denotes the total cohomology Chern class. If we now drop the hypothesis of s being transversal to the zero section, then its divisor Z is a hypersurface in M with singular set the points of non-transversality with the zero section. In this setting, Parusiński's generalized Milnor number can be regarded as being (up to sign) the correction term coming from the singular set in the above formula:

$$\mu(Z) := (-1)^n \Big(\chi(Z) - \int_M c_1(\mathcal{L}) \cdot c(M) \cdot c(\mathcal{L})^{-1}\Big) \,.$$

If Z has only isolated singularities then the formula above implies that $\mu(Z)$ is the sum of the usual Milnor numbers at the singularities of Z.

In [42] the authors give a formula for the invariant $\mu(Z)$ in the vein of [41], describing the generalized Milnor number in terms of a local invariants of the singularities of Z and Chern-Schwartz-MacPherson's classes. For this, consider a complex analytic Whitney stratification $\mathcal{S} = \{S\}$ of Z

with connected strata, such that Z_{sing} is union of strata; let γ_S be the function defined on each stratum S as follows. For each $x \in S$, let F_x be a *local Milnor fibre*, and let $\chi(F_x)$ be its Euler characteristic. Then

$$\mu(x; Z) := (-1)^n \left(\chi(F_x) - 1\right),$$

is the *local Milnor number of Z at x*. This number is constant on each Whitney stratum, so we denote it by μ_S. Then γ_S is defined inductively (starting with the strata S of largest dimension) by:

$$\gamma_S = \mu_S - \sum_{S' \neq S,\ \overline{S'} \supset S} \gamma_{S'}. \tag{4.2}$$

Then [42, Theorem 4] says:

Theorem 4.6.

$$\mu(Z) = \sum_{S \in \mathcal{S}} \gamma_S \int_{\overline{S}} \left(c(\mathcal{L}_{|\overline{S}})^{-1} \cap c^{SM}(\overline{S})\right).$$

Yokura conjectured (unpublished; cf. [62]) that Theorem 4.6 could be extended to a theorem concerning all Milnor classes. This was proved by Parusiński and Pragacz in [43]:

Theorem 4.7. *If M is an n-dimensional compact complex manifold, and Z is a hypersurface in M, then its total Milnor class can be expressed as:*

$$\mathcal{M}(Z) := \sum_{S \in \mathcal{S}} \gamma_S \left(c(\mathcal{L}_{|Z})^{-1} \cap (i_{\overline{S},Z})_* c^{SM}(\overline{S})\right),$$

where $i_{\overline{S},Z} : \overline{S} \hookrightarrow Z$ is the inclusion.

Chapter 5

Lê classes

Lê cycles are analytic cycles encoding deep information about singularity germs $f : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ and allow describing the topology and diffeomorphism type of the local Milnor fibres. These were introduced by D. Massey in [32].

In the affine context, we have the description by Schürmann and Tibăr in [49] about the Schwartz-MacPherson classes of a complex algebraic proper subset $X \subset \mathbb{C}^N$ using algebraic cycles. Motivated by this description the definition of affine Lê cycles appears and they are a global extension of Massey's local Lê cycles. These are generalized to the compact projective setting via projective Lê cycles. The explanation of the relationship between the affine and projective Lê cycles can be seen in [37].

Then, we consider the class of Lê cycles globalized to projective hypersurfaces and we show that the information encoded in those classes is equivalent to the information encoded in the Milnor classes, since the global Lê classes determine the Milnor classes and conversely.

5.1 Local Lê cycles

Let us recall first the definition of Lê cycles and Lê numbers of germs of complex analytic functions introduced by D. Massey in [30] (see also [32]).

Let U be an open subset of $\mathbb{C}^{n+1}$ containing the origin, $h : (U, 0) \to (\mathbb{C}, 0)$ the germ of an analytic function, $z = (z_0, \cdots, z_n)$ a linear choice of coordinates in $\mathbb{C}^{n+1}$ and $\Sigma(h) = V\left(\frac{\partial h}{\partial z_0}, \ldots, \frac{\partial h}{\partial z_n}\right)$ the critical set of h. To define the Lê cycles we need to define the relative polar cycles first, which are associated to the relative polar varieties:

Definition 5.1. For each k with $0 \leq k \leq n$, the *k-th local polar variety* $\Gamma^k_{h,z}$ is the analytic space $V\left(\frac{\partial h}{\partial z_k}, \ldots, \frac{\partial h}{\partial z_n}\right)/\Sigma(h)$.

Hence the analytic structure of $\Gamma^k_{h,z}$ does not depend on the structure of $\Sigma(h)$ as a scheme, but only as an analytic set. At the level of ideals, $\Gamma^k_{h,z}$ consists of those components of $V\left(\frac{\partial h}{\partial z_k}, \ldots, \frac{\partial h}{\partial z_n}\right)$ which are not contained in the set $\Sigma(h)$. Massey denotes by $\left[\Gamma^k_{h,z}\right]$ the cycle associated with the space $\Gamma^k_{h,z}$ (see [32, p. 9]).

Definition 5.2. For each $0 \leq k \leq n$, the *k-th local Lê cycle* $\Lambda^k_{h,z}$ of h with respect to the coordinate system z is the cycle:

$$\Lambda^k_{h,z} := \left[\Gamma^{k+1}_{h,z} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right] - \left[\Gamma^k_{h,z}\right].$$

If a point $p = (p_0, \cdots, p_n) \in U$ is an isolated point of the intersection of $\Lambda^k_{h,z}$ with the cycle of $V(z_0 - p_0, \cdots, z_{k-1} - p_{k-1})$, then the *$k$-th Lê number* $\lambda^k_{h,z}(p)$ is the intersection number at p :

$$\lambda^k_{h,z}(p) := \left(\Lambda^k_{h,z} \cdot V(z_0 - p_0, \ldots, z_{k-1} - p_{k-1})\right)_p.$$

It is proved in [31, Theorem 7.5] (see also [32, Theorem 10.18]) that for a generic choice of linear coordinates, all the Lê numbers of h at p are defined and they are independent of the coordinates choice. Hence, these are called the generic Lê numbers of h at p and they are denoted simply by $\lambda^k_h(p)$.

Furthermore, the generic Lê numbers of h are constant along the strata of any Whitney stratification of $V(h)$ (see [32, Thm 10.19]).

An important feature of the generic Lê numbers is that they allow to describe a handle decomposition of the Milnor fiber $F_{h,p}$ of h at p. In fact, Massey proved in [32, Theorem 3.3; Theorem 10.3] the following:

Theorem 5.3. *Let U be an open subset of $\mathbb{C}^{n+1}$, let $h : (U, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function, let s denote $\dim_0 \Sigma(h)$, and let $z = (z_0, \cdots, z_n)$ be a generic choice of linear coordinates in $\mathbb{C}^{n+1}$. Then the local Lê cycles are a collection of analytic cycle germs $\Lambda^i_{h,z}$ in $\Sigma(h)$ at the origin such that each $\Lambda^i_{h,z}$ is purely i-dimensional and properly intersects $V(z_0, \ldots, z_{i-1})$ at the origin, and for all $p \in \Sigma(h)$ near 0 we have that*

1. *If $s \leq n-2$, then $F_{h,p}$ is obtained up to diffeomorphism from a real $2n$-ball by successively attaching $\lambda^{n-k}_{h,z}(p)$ k-handles, where $n - s \leq k \leq n$;*

2. *If $s = n-1$, then $F_{h,p}$ is obtained up to diffeomorphism from a real $2n$-manifold with a homotopy-type of a bouquet $\lambda^{n-1}_{h,z}(p)$ circles by successively attaching $\lambda^{n-k}_{h,z}(p)$ k-handles, where $2 \leq k \leq n$.*

3. *The reduced Euler characteristic of the Milnor fiber of* h *at* p *is given by*

$$\tilde{\chi}\left(F_{h,p}\right) := \chi\left(F_{h,p}\right) - 1 = \sum_{i=0}^{n} (-1)^{n-i} \lambda^{i}_{h,z}(p).$$

Massey gives an alternative characterization of the local Lê cycles of a hypersurface singularity, which leads to a generalization of the Lê numbers that can be applied to any constructible complex of sheaves. From this more general viewpoint, the case of the Lê numbers of a function h is just the case where the underlying constructible complex of sheaves is the sheaf of vanishing cycles along h. Let us explain this. We assume some basic knowledge on derived categories, hypercohomology and sheaves of vanishing cycles as described in [18].

If X is a complex analytic space then $\mathcal{D}^b_c(X)$ denotes the derived category of bounded, constructible complexes of sheaves of $\mathbb{C}$-vector spaces on X. We denote the objects of $\mathcal{D}^b_c(X)$ by a notation of the form $F^\bullet$. The shifted complex $F^\bullet[l]$ is defined by $(F^\bullet[l])^k = F^{l+k}$ and its differential is $d^k_{[l]} = (-1)^l d^{k+l}$. The constant sheaf $\mathbb{C}_X$ on X induces an object $\mathbb{C}^\bullet_X \in \mathcal{D}^b_c(X)$ by letting $\mathbb{C}^0_X = \mathbb{C}_X$ and $\mathbb{C}^k_X = 0$ for $k \neq 0$.

If $h : X \to \mathbb{C}$ is an analytic map and $F^\bullet \in \mathcal{D}^b_c(X)$ then we denote the sheaf of vanishing cycles of $F^\bullet$ with respect to h by $\phi_h F^\bullet$.

For $F^\bullet \in \mathcal{D}^b_c(X)$ and $p \in X$, we denote by $\mathcal{H}^*\left(F^\bullet\right)_p$ the stalk cohomology of $F^\bullet$ at p, and by $\chi\left(F^\bullet\right)_p$ its Euler characteristic. That is

$$\chi\left(F^\bullet\right)_p = \sum_k (-1)^k \dim_{\mathbb{C}} \mathcal{H}^k\left(F^\bullet\right)_p.$$

We also denote by $\chi\left(X, F^\bullet\right)$ the Euler characteristic of X with coefficients in $F^\bullet$, *i.e.*,

$$\chi\left(X, F^\bullet\right) = \sum_k (-1)^k \dim_{\mathbb{C}} \mathbb{H}^k\left(X, F^\bullet\right),$$

where $\mathbb{H}^*\left(X, F^\bullet\right)$ denotes the hypercohomology groups of X with coefficients in $F^\bullet$.

When $F^\bullet \in \mathcal{D}^b_c(X)$ is $\mathcal{S}$-constructible, where $\mathcal{S}$ is a Whitney stratification of X, we denote it by $F^\bullet \in \mathcal{D}^b_{\mathcal{S}}(X)$. We would like to point out the following result which appears in [18, Theorem 4.1.22]:

$$\chi\left(X, F^\bullet\right) = \sum_{S \in \mathcal{S}} \chi\left(F^\bullet_S\right) \chi(S),$$

where $\chi\left(F^\bullet_S\right) = \chi\left(F^\bullet\right)_p$ for an arbitrary point $p \in S$.

Let M be a complex manifold. For a complex analytic subspace X of M, we denote its conormal space by $T^*_X M$. That is

$$T^*_X M := \text{closure}\left\{(p, \theta) \in T^*M \mid p \in X_{\text{reg}} \text{ and } \theta|_{T_p X_{\text{reg}}} \equiv 0\right\},$$

where T^*M is the cotangent bundle of M and X_{reg} is the regular part of X. The following definition is standard in the literature:

Definition 5.4. Let X be an analytic subspace of a complex manifold M, $\{S_\alpha\}$ a Whitney stratification of M adapted to X and $p \in S_\alpha$ a point in X. Consider $g : (M,p) \to (\mathbb{C},0)$ a germ of holomorphic function such that d_pg is a non-degenerate covector at p with respect to the fixed stratification, that is, $d_pg \in T^*_{S_\alpha}M$ and $d_pg \notin T^*_{S'}M$, for all stratum $S' \neq S_\alpha$. And let N be a germ of a closed complex submanifold of M which is transversal to S_α, with $N \cap S_\alpha = \{p\}$. Define the *complex link* l_{S_α} of S_α by:

$$l_{S_\alpha} := X \cap N \cap B_\delta(p) \cap \{g = w\}$$

for $0 < |w| \ll \delta \ll 1$ and $B_\delta(p)$ is the ball is with center p and radius δ.

The *normal Morse datum* of S_α is defined by:

$$NMD\,(S_\alpha) := \left(X \cap N \cap B_\delta(p), l_{S_\alpha}\right),$$

and the *normal Morse index* $\eta\,(S_\alpha, F^\bullet)$ of the stratum is:

$$\eta\,(S_\alpha, F^\bullet) := \chi\,(NMD\,(S_\alpha)\,, F^\bullet)\,,$$

where the right-hand-side means the Euler characteristic of the relative hypercohomology.

By the result of M. Goresky and R. MacPherson in [21, Theorem 2.3] we get that the number $\eta\,(S_\alpha, F^\bullet)$ does not depend on the choices of $p \in S_\alpha, g$ and N. Notice that by [18, Remark 2.4.5(ii)], it follows that

$$\eta\,(S_\alpha, F^\bullet) = \chi\,(X \cap N \cap B_\delta(p), F^\bullet) - \chi\,(l_{S_\alpha}, F^\bullet)\,.$$

Lemma 5.5. *Let $F^\bullet \in \mathcal{D}^b_{\mathcal{S}}(X)$ with $\mathcal{S} = \{S_\alpha\}$ a Whitney stratification of X. Let $p \in S_\alpha$ and $g : (M,p) \to (\mathbb{C},0)$ be a holomorphic function germ such that d_pg is a non-degenerate covector at $p \in S_\alpha$ with respect to the fixed stratification. Set $d = \dim X, d_\alpha = \dim S_\alpha$ and $m_\alpha := (-1)^{d-d_\alpha-1}\chi\left(\phi_{g_{|_N}} F^\bullet_{|_N}\right)_p$, where $\phi_{g_{|_N}} F^\bullet_{|_N}$ is the sheaf of vanishing cycles of $F^\bullet_{|_N}$ with respect to $g_{|_N}, p \in S_\alpha$ and N is a germ of a closed complex submanifold which is transversal to S_α with $N \cap S_\alpha = \{p\}$. Then*

$$m_\alpha = (-1)^{d-d_\alpha}\eta\,(S_\alpha, F^\bullet)\,.$$

Proof. By [18, Equation (4.1), p. 106] we have that

$\mathcal{H}^i\left(\phi_g F^\bullet\right)_p \simeq \mathbb{H}^{i+1}\left(B_\epsilon(p) \cap X, B_\epsilon(p) \cap X \cap g^{-1}(\varsigma), F^\bullet\right)$, for $0 < |\varsigma| \ll \epsilon \ll 1$. Hence

$$\chi\left(\phi_{g_{|_N}} F^\bullet_{|_N}\right)_p = -\chi\left(B_\epsilon(p) \cap X \cap N, B_\epsilon(p) \cap X \cap N \cap g^{-1}(\varsigma), F^\bullet\right),$$

and therefore $m_\alpha = (-1)^{d-d_\alpha}\eta\,(S_\alpha, F^\bullet)$. $\square$

Remark 5.6. Everything we have defined so far for a constructible complex of sheaves is defined by J. Schürmann and M. Tibăr in [49] for constructible functions, and the two are equivalent constructions. In fact, given $F^\bullet \in \mathcal{D}_c^b(X)$, we have naturally associated the constructible function on X given by

$$\beta(p) = \chi\left(F^\bullet\right)_p.$$

Moreover, the converse also holds (see [48]), *i.e.*, given any constructible function β on X there is $F^\bullet \in \mathcal{D}_c^b(X)$ such that

$$\beta(p) = \chi\left(F^\bullet\right)_p.$$

In particular, for any constructible function β on X we have that

$$\eta\left(S_\alpha, \beta\right) = \chi\left(X \cap N \cap B_\delta(x), \beta\right) - \chi\left(l_{S_\alpha}, \beta\right). \tag{5.1}$$

Let X be an analytic germ of an d-dimensional space which is embedded in some affine space, $M := \mathbb{C}^{n+1}$, so that the origin is a point of X. Consider a bounded, constructible sheaf $F^\bullet$ on X or M.

For a generic choice of linear coordinates $z = (z_0, \ldots, z_n)$ for $\mathbb{C}^{n+1}$, Massey in [31, Proposition 0.1] proves that there exists analytic cycles $\Lambda^i_{F^\bullet,z}$ in X which are purely i-dimensional, such that $\Lambda^i_{F^\bullet,z}$ and $V\left(z_0 - p_0, \ldots, z_{i-1} - p_{i-1}\right)$ intersect properly at each point $p = (p_0, \ldots, p_n) \in X$ near the origin, and such that

$$\chi\left(F^\bullet\right)_p = \sum_{i=0}^{d} (-1)^{d-i}\left(\Lambda^i_{F^\bullet,z} \cdot V\left(z_0 - p_0, \ldots, z_{i-1} - p_{i-1}\right)\right)_p.$$

Moreover, whenever such analytic cycles $\Lambda^i_{F^\bullet,z}$ exist, they are unique. Massey also sets $\lambda^i_{F^\bullet,z}(p) = \left(\Lambda^i_{F^\bullet,z} \cdot V\left(z_0 - p_0, \ldots, z_{i-1} - p_{i-1}\right)\right)_p$ and calls it the i-th characteristic polar multiplicity $F^\bullet$. When $\beta(p) = \chi\left(F^\bullet\right)_p$ we also denote $\Lambda^i_{F^\bullet,z}$ by $\Lambda^i_{\beta,z}$.

In [32, Corollary 10.15] was proved that for a generic choice of linear coordinates $z = (z_0, \ldots, z_n)$, if we let L^i be the i-dimensional linear subspace $V\left(z_0, \ldots, z_{n-i}\right)$ then,

$$\Lambda^k_{F^\bullet,z} = \sum_\alpha m_\alpha P_k\left(\overline{S_\alpha}\right) = \sum_\alpha (-1)^{d-d_\alpha}\eta\left(S_\alpha, F^\bullet\right) P_k\left(\overline{S_\alpha}\right). \tag{5.2}$$

where $P_k\left(\overline{S_\alpha}\right)$ is the absolute affine k-dimensional polar variety, with respect to the flag given by the L^i above, as defined by Lê and Teissier in [28]. We are going to define these affine polar varieties later on.

Remark 5.7. By [32, Remark 10.5, Remark 10.7] it follows that if we have $h : (U, 0) \to (\mathbb{C}, 0)$ with U an open neighborhood of the origin in $\mathbb{C}^{n+1}$, if $X = \Sigma(h)$ is the critical set of h, with $d = \dim_0 X$ and if we let

$$P^\bullet = \left(\phi_h \mathbb{C}^\bullet_U\right)_{|_{\Sigma(h)}}[n-d],$$

then for generic linear coordinates z, for all i and for all $p \in X$ near the origin, we have $\Lambda^i_{P^\bullet,z} = \Lambda^i_{h,z}$ and $\lambda^i_{P^\bullet,z} = \lambda^i_{h,z}(p)$. Also

$$m_\alpha = (-1)^{d-d_\alpha} \eta\left(S_\alpha, P^\bullet\right) = (-1)^{d-d_\alpha} \eta\left(S_\alpha, w\right),$$

where w is the constructible function defined by $w(p) = \chi\left(P^\bullet\right)_p = \chi\left(F_{h,p}\right) - 1$ with $F_{h,p}$ being the Milnor fiber of h at p. Hence, by equation (5.2) we have that

$$\Lambda^i_{h,z} = \sum_\alpha (-1)^{d-d_\alpha} \eta\left(S_\alpha, w\right) P_i\left(\overline{S_\alpha}\right).$$

This is the description of the local Lê cycles in terms of local polar varieties we need in order to define the global Lê cycles for compact projective varieties.

5.2 Affine Lê cycles

In the affine context, Schürmann and Tibăr in [49] describe the Schwartz-MacPherson classes of a complex algebraic proper subset $X \subset \mathbb{C}^N$ using algebraic cycles, which were called MacPherson cycles. In this construction a key role is played by the affine polar varieties, which we now describe (see [28]).

Definition 5.8. For each $0 \leq i \leq N$, let L_i be a linear subvariety of $\mathbb{C}^N$ of codimension i. If X is of pure dimension $d < N$, the *k-th affine polar variety* of X, with $0 \leq k \leq d$, is the following algebraic set

$$P_k\left(X, L_{k+1}\right) := \overline{\{x \in X_{reg} \mid \dim\left(T_x X_{reg} \cap L_{k+1}\right) \geq d-k\}}.$$

For L_{k+1} sufficiently general, the polar variety $P_k\left(X, L_{k+1}\right)$ has pure dimension k. We have $P_d(X) := X$ and we set $P_k(X) := \emptyset$ for $k > d$.

We fix an Whitney stratification $\{S_\alpha\}$ of X with connected strata. In this context X does not need to be pure dimensional and we only assume $d = \dim X < N$. Let β be a constructible function on X with respect to this Whitney stratification.

Schürmann and Tibăr make the following definition.

Definition 5.9. The *k-th MacPherson cycle* of β $(0 \leq k \leq d)$ is:

$$MP_k\left(\beta, L_{k+1}\right) := \sum_\alpha (-1)^{d_\alpha} \eta\left(S_\alpha, \beta\right) P_k\left(\overline{S_\alpha}, L_{k+1}\right),$$

where $d_\alpha = \dim S_\alpha$ and $P_k\left(\overline{S_\alpha}, L_{k+1}\right)$ is the k-th global affine polar variety of the algebraic closure $\overline{S_\alpha} \subset \mathbb{C}^N$ of the stratum S_α.

The main result of [49] is that, for generic L_{k+1}, the cycle $MP_k(\beta, L_{k+1})$ represents the k-th dual Schwartz-MacPherson class $\check{c}_k^{SM}(\beta)$ in the Chow group $A_k(X)$, where $\check{c}_k^{SM}(\beta) = (-1)^k c_k^{SM}(\beta)$. That is, Schürmann and Tibăr describe the Schwartz-MacPherson classes via affine polar varieties:

$$\begin{aligned} c_k^{SM}(\beta) &= (-1)^k \left[MP_k(\beta)\right] \\ &= (-1)^k \sum_\alpha (-1)^{d_\alpha} \eta\left(S_\alpha, \beta\right) \left[P_k\left(\overline{S_\alpha}\right)\right]. \end{aligned} \tag{5.3}$$

Definition 5.10. We define the *k-th affine Lê cycle* of β by

$$\Lambda_k^{\mathbb{A}}\left(\beta, L_{k+1}\right) := \sum_\alpha (-1)^{d-d_\alpha} \eta\left(S_\alpha, \beta\right) P_k\left(\overline{S_\alpha}, L_{k+1}\right).$$

Notice that

$$\Lambda_k^{\mathbb{A}}\left(\beta, L_{k+1}\right) = (-1)^d MP_k\left(\beta, L_{k+1}\right).$$

Hence, by equation (5.3) we have that

$$\begin{aligned} c_k^{SM}(\beta) &= (-1)^{k+d} \left[\Lambda_k^{\mathbb{A}}(\beta)\right] \\ &= (-1)^{k+d} \sum_\alpha (-1)^{d-d_\alpha} \eta\left(S_\alpha, \beta\right) \left[P_k\left(\overline{S_\alpha}\right)\right]. \end{aligned} \tag{5.4}$$

An interesting feature of these affine Lê cycles of X is that they are a global extension of the Lê cycles defined by Massey:

Proposition 5.11. *Let X be a closed subvariety of $\mathbb{C}^N$ and let β be a constructible function on X with respect to a Whitney stratification $\{S_\alpha\}$ of X. Let $x \in X$ and let $U \subseteq \mathbb{C}^N$ be an open neighborhood of x. Let $\{x\} = L_N \subset L_{N-1} \subset \cdots \subset L_1 \subset L_0 = \mathbb{C}^N$ be a generic flag of linear subvarieties of $\mathbb{C}^N$ with L_i being of codimension i and such that $L_i \cap U = V\left(z_0, \ldots, z_{i-1}\right)$ where $z = \left(z_0, \ldots, z_{N-1}\right)$ are generic linear coordinates around x. Let $\iota : U \cap X \longrightarrow \mathbb{C}^N$ be the inclusion. Then, the flat pull-back of the affine Lê cycles satisfies the following property*

$$\iota^* \Lambda_k^{\mathbb{A}}\left(\beta, L_{k+1}\right) = \Lambda^k_{\iota^*(\beta), z}.$$

Proof. In fact,

$$\begin{aligned} \iota^* \Lambda_k^{\mathbb{A}}\left(\beta, L_{k+1}\right) &= \iota^* \left(\sum_\alpha (-1)^{d-d_\alpha} \eta\left(S_\alpha, \beta\right) P_k\left(\overline{S_\alpha}, L_{k+1}\right) \right) \\ &= \sum_\alpha (-1)^{d-d_\alpha} \eta\left(S_\alpha, \beta\right) \iota^*\left(P_k\left(\overline{S_\alpha}, L_{k+1}\right)\right) \\ &= \sum_\alpha (-1)^{d-d_\alpha} \eta\left(S_\alpha \cap U, \iota^*(\beta)\right) P_k\left(\overline{S_\alpha \cap U}\right) \\ &= \Lambda^k_{\iota^*(\beta), z}. \end{aligned}$$

□

5.3 Projective Lê cycles

Let X be a complex analytic space in $\mathbb{C}P^N$ of pure dimension d. For each $0 \leq k \leq N$, let L_k be a linear subspace of $\mathbb{C}P^N$ of codimension k.

Definition 5.12. The *k-th projective polar variety* of X, with respect to L_{k+2}, is defined by

$$\mathbb{P}_k\left(X, L_{k+2}\right) := \overline{\{x \in X_{reg} \mid \dim\left(T_x X_{reg} \cap L_{k+2}\right) \geq d-k-1\}},$$

where $T_x X_{\text{reg}}$ is the projective tangent space of X at a regular point x.

We observe that for L_{k+2} sufficiently general, the dimension of $\mathbb{P}_k\left(X, L_{k+2}\right)$ is equal to k. Thus, we are indexing the polar varieties by their dimension and not by their codimension, as it is usually done. Also observe that the class $[\mathbb{P}_k\left(X, L_{k+2}\right)]$ of $\mathbb{P}_k\left(X, L_{k+2}\right)$ modulo rational equivalence in the Chow group $A_k(X)$ does not depend on L_{k+2} provided this is sufficiently general. This class is denoted by $[\mathbb{P}_k(X)]$ and it is called the *k-th projective polar class* of X.

Remark 5.13. For any subvariety Z of $\mathbb{C}P^N$ we denote by $\text{Cone}(Z)$ the cone in $\mathbb{C}^{N+1}$ induced by Z. Analogously, for any conical subvariety through the origin V of $\mathbb{C}^{N+1}$ we denote by $\mathbb{P}(V)$ the induced projective variety in $\mathbb{C}P^N$. Let X be a subvariety of $\mathbb{C}P^N$ and let L_{k+2} be a linear subvariety of $\mathbb{C}P^N$ of codimension $k+2$. In this case, $\text{Cone}\left(L_{k+2}\right)$ is a linear subspace of codimension $k+2$ in $\mathbb{C}^{N+1}$ and $P_{k+1}\left(\text{Cone}(X), \text{Cone}\left(L_{k+2}\right)\right)$ the $(k+1)$-th affine polar variety of $\text{Cone}(X)$ with respect $\text{Cone}\left(L_{k+2}\right)$, that is a conical subvariety of $\mathbb{C}^{N+1}$ of dimension $k+1$. The relationship between the projective and the affine polar varieties is given by

$$\mathbb{P}_k\left(X, L_{k+2}\right) = \mathbb{P}\left(P_{k+1}\left(\text{Cone}(X), \text{Cone}\left(L_{k+2}\right)\right)\right)$$

Definition 5.14. For any given $F^\bullet \in \mathcal{D}^b_{\mathcal{S}}(X)$, where $\mathcal{S} = \{S_\alpha\}$ is a Whitney stratification of X, define the *k-th projective Lê cycle*, with respect to L_{k+2}, by

$$\Lambda^{\mathbb{P}}_k\left(F^\bullet, L_{k+2}\right) := \sum_\alpha (-1)^{d-d_\alpha} \eta\left(S_\alpha, F^\bullet\right) \mathbb{P}_k\left(\bar{S}_\alpha, L_{k+2}\right),$$

where $d_\alpha = \dim S_\alpha$.

Hence, the class of this cycle in the Chow group $A_k(X)$ does not depend on L_{k+2} provided this is sufficiently general. This class is denoted by $\left[\Lambda^{\mathbb{P}}_k\left(F^\bullet\right)\right]$.

If β is the constructible function associated to $F^\bullet$ as in Remark 5.6 we also denote this cycle $\Lambda^{\mathbb{P}}_k\left(F^\bullet, L_{k+2}\right)$ by $\Lambda^{\mathbb{P}}_k\left(\beta, L_{k+2}\right)$ and the class $\left[\Lambda^{\mathbb{P}}_k\left(F^\bullet\right)\right]$ by $\left[\Lambda^{\mathbb{P}}_k(\beta)\right]$. That is,

$$\Lambda^{\mathbb{P}}_k\left(\beta, L_{k+2}\right) := \sum_\alpha (-1)^{d-d_\alpha} \eta\left(S_\alpha, \beta\right) \mathbb{P}_k\left(\bar{S}_\alpha, L_{k+2}\right).$$

The next result is going to relate the affine and projective Lê cycles (see [37, Theorem 4.4]).

Theorem 5.15. *Let $X \subseteq \mathbb{C}P^N$ be a d-dimensional projective variety endowed with a Whitney stratification $\mathcal{S} = \{S_\alpha\}$ with connected strata. Let L_{k+2} be a linear subvariety of $\mathbb{C}P^N$ of codimension $k+2$. Let $\pi : \mathbb{C}^{N+1}\backslash\{0\} \longrightarrow \mathbb{C}P^N$ be the natural projection. Let β be a constructible function on X, with respect to this stratification. Then*

1. *$\mathcal{S}' := \left\{\pi^{-1}\left(S_\alpha\right)\right\} \cup \{\{0\}\}$ is a Whitney stratification of $\mathrm{Cone}(X)$.*

2. *β induces a constructible function $\tilde{\beta}$ on $\mathrm{Cone}(X)$ with respect to the Whitney stratification $\mathcal{S}'$.*

3. *$\Lambda_k^{\mathbb{P}}\left(\beta, L_{k+2}\right) = \mathbb{P}\left(\Lambda_{k+1}^{\mathbb{A}}\left(\tilde{\beta}, \mathrm{Cone}\left(L_{k+2}\right)\right)\right)$.*

Summarizing we get:

Theorem 5.16. *The affine Lê cycles restricted to every point in X give the local Lê cycles, and the projective Lê cycles are the projectivization of the affine Lê cycles of the affine cone defined by a projective variety.*

5.4 Lê classes and Milnor classes of hypersurfaces

Lê cycles are originally associated to map-germs $\mathbb{C}^{n+1} \to \mathbb{C}$ and determine the diffeomorphism type of the Milnor fiber. These were extended above to invariants of projective manifolds. On the other hand Milnor classes are by definition the difference between two extensions of the classical Chern classes to the case of singular varieties.

It was proved in [14] that these two concepts are remarkably linked together in a deep way. In fact the main result in [14] says that the information encoded in the Milnor classes is essentially equivalent to the information encoded in the Lê cycles.

Let M be a smooth complex submanifold of $\mathbb{C}P^N$ of dimension $n+1$, let Z be the hypersurface in M defined by the set of zeroes of a reduced holomorphic section s of a line bundle L on M.

Definition 5.17. The *k-th Lê class* of Z is $\Lambda_k(Z) := [\Lambda_k^{\mathbb{P}}(\omega, L_{k+2})]$, where $\omega(x) = \chi(F_x) - 1$ (Euler characteristic of the Milnor fiber of Z at x) and L_{k+2} is a generic linear subspace of $\mathbb{C}P^N$ of codimension $k+2$.

One has:

Theorem 5.18. *Assume M, L and Z are as above. Set $h := c_1(\mathcal{O}_{\mathbb{P}^N}(1)|_Z)$ and denote by $\mathcal{M}_k(Z)$ the k-th Milnor class of Z. Then, for each $k = 0, \dots, r = \dim(Z_{sing})$, there are cycles, obtained with respect to the choice of a linear subspace of $\mathbb{C}P^N$, which give rise to well defined classes $\Lambda_k(Z)$ of Z in the Chow group and integral homology group of Z, that we call the global Lê classes of Z, and these are related to the Milnor classes $\mathcal{M}_k(Z)$ by the formulas:*

$$\mathcal{M}_k(Z) = \sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(L|_Z)^j h^{i-k-j} \cap \Lambda_i(Z)$$

and conversely:

$$\Lambda_k(Z) = \sum_{j \geq 0} (-1)^{k+j} \binom{k+j+1}{k+1} h^j \cap \left(\mathcal{M}_{k+j}(Z) + c_1(L|_Z) \mathcal{M}_{k+j+1}(Z) \right).$$

One gets the corollary below, which extends and strengthens [10, Corollary 5.13] in the hypersurface case:

Corollary 5.19. *Assume M, L and Z are as above and equip M with a Whitney stratification $\{Z_\beta\}$ adapted to Z. Let d be the dimension of the singular set Z_{sing}. Then we have the following equalities of cycles in the Chow group of Z:*

$$\mathcal{M}_d(Z) \;=\; \sum_{S_\beta \subset Z_{\mathrm{sing}}} \lambda^d_{S_\beta} \, [\overline{S}_\beta] \;=\; (-1)^d \Lambda_d(Z) \,,$$

where the sums run over the strata of dimension d which are contained in Z_{sing} and $\lambda^d_{S_\beta}$ is the d-th Lê number of S_β.

The trail for getting to Theorem 5.18 can be roughly described as follows. The first step is recalling the main theorem of A. Parusinski and P. Pragacz in [43], Theorem 4.7 above. This expresses the total Milnor class as a function of the Schwartz-MacPherson classes of the closure of the strata of a Whitney stratification:

$$\mathcal{M}(Z) := \sum_{S_\alpha \in \mathcal{S}} \gamma_{S_\alpha} \left(c(L_{|_Z})^{-1} \cap (i_{S_\alpha, Z})_* c^{SM}(\overline{S}_\alpha) \right). \tag{5.5}$$

Then one has the aforementioned MacPherson cycles [49], associated to any constructible function on a complex algebraic proper subset $X \subset \mathbb{C}^N$ that represent the (dual) Schwartz-MacPherson classes in the Borel-Moore homology group, and also in the Chow group. We already described above the analogous result in the projective case. In this construction a key role is played by the projective polar varieties.

Next one uses R. Piene characterization in [44] of the Mather classes via polar varieties to give a formula for the Schwartz-MacPherson classes in terms of polar varieties and the normal Morse indices (see Definition 5.4). Finally we use the above described characterization of the global Lê cycles for constructible sheaves via polar varieties. This also answers a conjecture posed by J.-P. Brasselet in [7], claiming that Milnor classes can be expressed in terms of polar varieties.

Chapter 6

Milnor classes of complete intersections

From now on, let M be an n-dimensional compact complex manifold. Set $M^{(r)} := M \times \ldots \times M$, r times. We let E be a holomorphic vector bundle over $M^{(r)}$ of rank d. Consider $\Delta : M \to M^{(r)}$ the diagonal morphism, which is a regular embedding of codimension $nr - n$. Let t be a regular holomorphic section of E. The set of zeros of t is a closed subvariety $Z(t)$ of $M^{(r)}$ of dimension $nr - d$. Consider $Z(\Delta^*(t))$ the set of zeros of the pull back section of t by Δ of dimension $n - d$.

Following [20, Chapter 6] we have that Δ induces the refined Gysin homomorphism

$$\Delta^! : H_{2k}(Z(t)) \to H_{2(k-nr+n)}(Z(\Delta^*(t))).$$

The refined intersection product is defined by:

$$\alpha_1 \cdot \ldots \cdot \alpha_r := \Delta^!(\alpha_1 \times \ldots \times \alpha_r)\,.$$

For the usual homology this is defined by duality between homology and cohomology:

$$\Delta^! = \Delta^* : H_{2k}(Z(t);\mathbb{Z}) \simeq H^{2(nr-k)}(Z(t);\mathbb{Z}) \to$$

$$H^{2(nr-k)}(Z(\Delta^*(t));\mathbb{Z}) \simeq H_{2(k-nr+n)}(Z(\Delta^*(t));\mathbb{Z}).$$

In [15, Proposition 1.15, Theorem 1.12, Corollary 1.13] we obtain a Verdier-Riemann-Roch type theorem for the Fulton-Johnson, Schwartz-MacPherson and Milnor classes:

Proposition 6.1. *Assume that $Z(t)$ admits a Whitney stratification $\{\mathcal{T}_\gamma\}$ transversal to $\Delta(M)$ such that the strata $\mathcal{T}_\gamma \cap \Delta(M)$ are connected. The* *refined Gysin morphism satisfies:*

$$\Delta^! \left(c^{FJ}(Z(t)) \right) = c\left((TM|_{Z(\Delta^* t)})^{\oplus r-1} \right) \cap c^{FJ}(Z(\Delta^* t))$$

and

$$\Delta^! \left(c^{SM}(Z(t)) \right) = c\left((TM|_{Z(\Delta^* t)})^{\oplus r-1} \right) \cap c^{SM}(Z(\Delta^* t)) \,.$$

Therefore,

$$\Delta^! \mathcal{M}(Z(t)) = (-1)^{nr-n} c\left((TM|_{Z(\Delta^* t)})^{\oplus r-1} \right) \cap \mathcal{M}(Z(\Delta^* t)) \,.$$

Now, let $\{E_i\}$ be a finite collection of holomorphic vector bundles over M of rank d_i, $1 \leq i \leq r$. For each of these bundles, let s_i be a regular holomorphic section and X_i the $(n - d_i)$-dimensional local complete intersections defined by the zeroes of s_i. We assume that we can equip the product $X_1 \times \ldots \times X_r$ with a Whitney stratification such that the diagonal embedding Δ is transversal to all strata. This transversality condition is necessary for using above proposition.

Let $p_i : M^{(r)} \to M$ be the i^{th}-projection, then we have the holomorphic exterior product section

$$s = s_1 \oplus \ldots \oplus s_r : M^{(r)} \to p_1^* E_1 \oplus \ldots \oplus p_r^* E_r ,$$

given by $s(x_1, \ldots, x_r) = (s_1(x_1), \ldots, s_r(x_r))$. Then $Z(s) = X_1 \times \ldots \times X_r$ and $Z(\Delta^*(s)) = X_1 \cap \ldots \cap X_r$ is a local complete intesection of dimension $n - d_1 - \cdot \ldots \cdot - d_r$.

Theorem 6.2. *([16, Prpositions 2.1 and 2.4]) Set $X = Z(\Delta^*(s))$. Then:*

(i) $c^{SM}(X) = c\left((TM|_X)^{\oplus r-1} \right)^{-1} \cap \left(c^{SM}(X_1) \cdot \ldots \cdot c^{SM}(X_r) \right)$;

(ii) $c^{FJ}(X) = c\left((TM|_X)^{\oplus r-1} \right)^{-1} \cap \left(c^{FJ}(X_1) \cdot \ldots \cdot c^{FJ}(X_r) \right)$; *and therefore*

(iii) $\mathcal{M}(X) = (-1)^{\dim X} c\left((TM|_X)^{\oplus r-1} \right)^{-1} \cap \left(c^{FJ}(X_1) \cdot \ldots \cdot c^{FJ}(X_r) - c^{SM}(X_1) \cdot \ldots \cdot c^{SM}(X_r) \right)$.

Example 6.3. Let Z_1 and Z_2 be the hypersurfaces of $\mathbb{P}^4$ defined by

$$H(x_0, \ldots, x_4) = x_0 x_1 \quad \text{and} \quad G(x_0, \ldots, x_4) = x_3 \,.$$

The line bundle of Z_1 is $\mathcal{O}(2H)$, where $H = c_1(\mathcal{O}(1))$, so the class of the virtual tangent bundle of Z_1 is:

$$(1 + H)^5 2H/(1 + 2H) = 2H + 6H^2 + 8H^3 + 4H^4 ,$$

while the Schwartz-MacPherson class is, by the inclusion-exclusion formula in [1]:

$$2c(T\mathbb{P}^3)-c(T\mathbb{P}^2)=2((1+H)^4H)-(1+H)^3H^2=2H+7H^2+9H^3+5H^4$$

Therefore the Milnor class of Z_1 is $H^2+H^3+H^4$. On the other hand, since Z_2 is smooth, the Schwartz-MacPherson class and the Fulton-Johnson class of Z_2 are $(1+H)^4H=H+4H^2+6H^3+4H^4$. Therefore, by Theorem 6.2, the Milnor class of $Z_1\cap Z_2$ is given by

$$\mathcal{M}(Z_1\cap Z_2)=-c(T\mathbb{P}^4)^{-1}\cap c^{SM}(Z_2)\mathcal{M}(Z_1)=-H^3.$$

If we restrict the discussion to the case where the bundles E_i in question are all line bundles L_i, then we obtain some applications:

i) A Parusiński-Pragacz type formula for local complete intersections. This expresses the Milnor classes using only Schwartz-MacPherson classes, and it answers positively the expected description given by Ohmoto and Yokura in [38] for the total Milnor class of a local complete intersection. We notice that a different generalization of the Parusiński-Pragacz formula for complete intersections has been given recently in [34].

Corollary 6.4. *([16, Corollary 3.5])*

$$\mathcal{M}(X)=(-1)^{nr-n}c\left(\left(TM|_X\right)^{\oplus r-1}\right)^{-1}\cap$$

$$\left(\sum \alpha^{\epsilon_1,\ldots,\epsilon_r}_{S_1,\ldots,S_r}\frac{c(L_1)^{\epsilon_1}\cdot\ldots\cdot c(L_r)^{\epsilon_r}}{c(L_1\oplus\ldots\oplus L_r)}\cap c^{SM}(\overline{S_1})\cdot\ldots\cdot c^{SM}(\overline{S_r})\right),$$

where the sum runs over all possible choices of the strata provided that $(S_1,\ldots,S_r)\neq((X_1)_{reg},\ldots,(X_r)_{reg})$, γ_i is the inductive function obtained by Milnor fibre of X_i (see (4.2)), $\alpha^{\epsilon_1,\ldots,\epsilon_r}_{S_1,\ldots,S_r}=(-1)^{(n-1)(\epsilon_1+\ldots+\epsilon_r)}\gamma^{1-\epsilon_1}_{S_1}\cdot\ldots\cdot\gamma^{1-\epsilon_r}_{S_r}$ and $\epsilon_i=\begin{cases}1, & if\ S_i\subseteq(X_i)_{reg}\\ 0, & if\ \dim(S_i)<n-1\end{cases}$.

ii) A description of the total Milnor class of the local complete intersection X in the vein of Aluffi's formula in [2] for hypersurfaces, using Aluffi's μ-classes.

For each X_i, the Aluffi's μ-class of the singular locus is defined by the formula

$$\mu_{L_i}(\mathrm{Sing}(X_i))=c(T^*M\otimes L_i)\cap s(\mathrm{Sing}(X_i),M).$$

Given a cycle $\alpha\in H_{2*}(X_i,\mathbb{Z})$ and $\alpha=\sum_{j\geq 0}\alpha^j$, where α^j is the codimension j component of α, then Aluffi introduced the following cycles

$$\alpha^\vee:=\sum_{j\geq 0}(-1)^j\alpha^j\quad\text{and}\quad\alpha\otimes L_i:=\sum_{j\geq 0}\frac{\alpha^j}{c(L_i)^j}\ .$$

Then Aluffi proved in [2] that the total Milnor class $\mathcal{M}(X_i)$ can be described as follows:

$$\mathcal{M}(X_i) = (-1)^{n-1} c(L_i)^{n-1} \cap (\mu_{L_i}(\mathrm{Sing}(X_i))^\vee \otimes L_i). \tag{6.1}$$

Corollary 6.5. *([16, Corollary 3.3])*

$$\mathcal{M}(X) = (-1)^{n-1} c\left((TM|_X)^{\oplus r-1}\right)^{-1} \cap$$

$$\left(\sum_{i=1}^{r} (-1)^{r-1} a_{1,i} \cdot \ldots \cdot a_{r-1,i} \cdot c(L_i)^{n-1} \cap (\mu_{L_i}(\mathrm{Sing}(X_i)\,)^\vee \otimes L_i)\right),$$

where $a_{j,i} = \begin{cases} c^{SM}(X_{j+1}) & \text{if} \quad i \leq j \\ c^{FJ}(X_j) & \text{if} \quad i > j \end{cases}$.

iii) As seen in the previous section there is a concept of global Lê classes of a singular hypersurface Z in a smooth complex submanifold M of $\mathbb{C}P^N$, and a formula relating these with the Milnor classes of Z. It is also possible to get a description of the Milnor classes of the local complete intersections $X = X_1 \cap \ldots \cap X_r$ via the Lê classes of each hypersurface X_i (cf. [16, Remark 3.6]).

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