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# Vanishing topology and versality of singularities of mappings: 30 years of the Mond conjecture

**Juan J. Nuño-Ballesteros and Guillermo Peñafort Sanchis**

**Abstract.** This paper is a survey on the Mond conjecture, which is an inequality of type $\mu \geq \tau$ for singularities of mappings. We present all the ingredients necessary to understand the statement of the conjecture, as well as a proof in the case of surfaces, based on the construction of a Jacobian module, whose length controls the image Milnor number. We also include some historical notes with the known cases and some other inequalities of the same nature and indicate directions for future work.

**Keywords.** Image Milnor number, Mond conjecture, singularities of mappings.

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# Contents

# Chapter 1

# Introduction

The Mond conjecture is an inequality of type $\mu \geq \tau$ for singularities of mappings. It was stated by D. Mond in 1991 [50] as follows:

**Mond conjecture:** Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a holomorphic map germ, $S \subset \mathbb{C}^n$ a finite set, with isolated instability and such that $(n, n+1)$ are in the range of nice dimensions of Mather (that is, $n < 15$, see [42]), then

$$\operatorname{codim}_{\mathscr{A}_e}(f) \leq \mu_I(f),$$

with equality if $f$ is weighted homogeneous.

The number $\operatorname{codim}_{\mathscr{A}_e}(f)$, called the extended $\mathscr{A}$-codimension, is the number of parameters of a miniversal unfolding of $f$. Hence, it plays the role of the Tjurina number $\tau(X, 0)$ of a hypersurface $(X, 0)$ with isolated singularity (abbreviated as IHS).

On the other hand, $\mu_I(f)$ is the image Milnor number, analogous to the classical Milnor number $\mu(X, 0)$ in the sense that it encodes the vanishing topology of the singularity. In fact, Mond showed that the image of a stable perturbation of $f$ has the homotopy type of a bouquet of $n$-spheres and he defined $\mu_I(f)$ as the number of such spheres. The existence of a stable perturbation is guaranteed only when $(n, n+1)$ are nice dimensions, so this is a necessary condition.

In the classical case of an IHS, both invariants can be computed algebraically in a simple way:

$$\tau(X, 0) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{n+1}}{J(g) + (g)}, \quad \mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{n+1}}{J(g)},$$

where $\mathscr{O}_{n+1}$ is the local ring of holomorphic functions on $(\mathbb{C}^{n+1}, 0)$, the function $g \in \mathscr{O}_{n+1}$ is a reduced equation of $(X, 0)$ and $J(g)$ is the Jacobian ideal, generated by the partial derivatives of $g$. It is obvious that

$\tau(X,0) \leq \mu(X,0)$. Moreover, when $(X,0)$ is weighted homogeneous, the Euler identity implies $g \in J(g)$ and we get an equality. The inequality $\mu \geq \tau$ holds for isolated complete intersection singularities (ICIS) of positive dimension as well, but the proof is much harder. The inequality was shown by Loojenga and Steenbrick [37]. Before that, Greuel had shown that the equality holds for weighted homogeneous ICIS [25]. Finally, Voosegard showed that the equality of the two numbers characterizes the weighted homogeneity of the ICIS [71].

Per contra, the Mond conjecture is only known to be true for dimensions $n = 1, 2$ [50, 15, 51] and, despite plentiful evidence from many examples and particular cases, the case of dimension $n \geq 3$ is still open. Chapter 2 contains a brief account of the known cases where the conjecture holds , as well as other $\mu \geq \tau$-type inequalities of the same nature. To give a sense of how the two contexts of these inequalities compare one to another, we introduce a basic example that can be examined from both angles.

*Example* 1.1. Consider a cusp

$$(X,0) = V(x^3 - y^2) \subseteq (\mathbb{C}^2, 0).$$

We may think of $(X,0)$ as a germ of hypersurface in $\mathbb{C}^2$ or, alternatively, we may parametrize $(X,0)$ as the image of the map-germ $f\colon (\mathbb{C},0) \to (\mathbb{C}^2,0)$, given by

$$t \mapsto (t^2, t^3).$$

From one point of view, $(X,0)$ is a germ of hypersurface with isolated singularity and, from the other point of view, it is the image of a mapping $f$ with an isolated instability (this is a misleading aspect of this example as, for any $n > 1$, the image of a singular map-germ $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ fails to have isolated singularities).

When we think of $(X,0)$ as a hypersurface, the vanishing homology comes from the Milnor fibre

$$\{x^3 + y^2 = \delta\} \cap B_\epsilon,$$

with $0 < \delta \ll \epsilon \ll 1$, where $B_\epsilon$ stands for a closed ball of radius $\epsilon$ centered at the origin of $\mathbb{C}^2$. This space, represented in the middle of Figure 1.1, has the homotopy type of a bouquet of two spheres of dimension one.

We can figure this out without thinking about this Milnor fibre topologically, because the number of spheres is the dimension of the Milnor algebra

$$\mu(X,0) = \dim_{\mathbb{C}} \frac{\mathscr{O}_2}{\langle 3x^2, 2y \rangle} = 2.$$

Similarly, the Tjurina algebra of $(X,0)$ is

$$\frac{\mathscr{O}_2}{\langle x^3 - y^2, 3x^2, 2y \rangle} \cong \mathrm{Span}_{\mathbb{C}}\{1, x\},$$

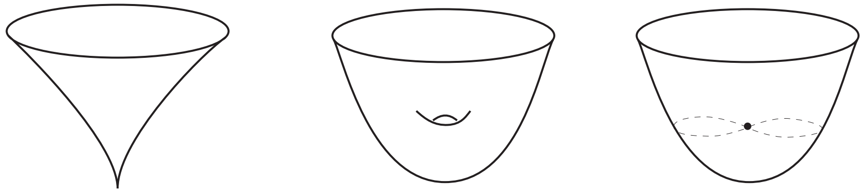

Figure 1.1: A cusp, its Milnor fibre and its disentanglement

from where it follows that the Tjurina number is

$$\tau(X,0) = 2,$$

and that a miniversal deformation is

$$X_{s_1,s_2} = V(x^3 - y^2 + s_1x + s_2).$$

When we think of the cusp $(X,0)$ as the image of the map germ $f(t) = (t^2, t^3)$, we are not allowed to perturb $(X,0)$ directly from its equation, but only by perturbing the parametrization $f$. The vanishing homology comes now from what we call the disentanglement of $(X,0)$, which is the image in $B_\epsilon$ of a small enough (with respect to $\epsilon$) stable perturbation $f_\delta$ of $f$. In our example, such perturbation has the form

$$f_\delta(t) = (t^2, t^3 - \delta t).$$

One sees easily that $f_\delta$ identifies the points $t = \pm\sqrt{\delta}$, and it is one-to-one away from them. Therefore the image of $f_\delta$ is a disk with two points identified, as in the right side of figure 1.1. In particular, the image of $f_\delta$ has the homotopy type of a real sphere of dimension 1, hence

$$\mu_I(f) = 1.$$

Finally, the $\mathscr{A}_e$-codimension of $f$ (see Definition 3.7) is obtained by computing directly

$$T^1_{\mathscr{A}_e}(f) \cong \operatorname{Span}_{\mathbb{C}}\{(0,t)\}.$$

This gives us

$$\operatorname{codim}_{\mathscr{A}_e}(f) = 1$$

and ensures that a miniversal unfolding of $f$ is given by

$$F(t,s) = (f_s, s) = (t^2, t^3 + st, s).$$

We refer to Chapter 3 for the definitions and basic properties about singular mappings used in this example.

The main difficulty in the proof of the Mond conjecture is the fact we do not have an algebra, or more generally, a module, which plays the role of the Jacobian algebra of an IHS and which controls the image Milnor number $\mu_I(f)$. In Chapter 6, following the approach of [18], we present a candidate to Jacobian module, denoted by $M(g)$, associated to each reduced equation $g$ of the image of $f$, with the following property:

$$\dim_{\mathbb{C}} M(g) = \operatorname{codim}_{\mathscr{A}_e}(f) + \dim_{\mathbb{C}} K(g),$$

where $K(g) = (J(g)+(g))/J(g)$. Consequently, the Mond conjecture holds if we are able to prove that $\mu_I(f) = \dim_{\mathbb{C}} M(g)$.

There is also a relative version $M_{\mathrm{rel}}(G)$ for $r$-parameter unfoldings $F$ which specialises to $M(g)$ when we make the parameters equal to zero. Moreover, if $F$ is an unfolding whose bifurcation set $\mathscr{B}(F)$ is contained properly in the parameter space $(\mathbb{C}^r, 0)$, then $\mu_I(F)$ is equal to the Samuel multiplicity $e(\mathfrak{m}_r; M_{\mathrm{rel}}(G))$ with respect to the parameter ideal $\mathfrak{m}_r$. As a consequence, the equality $\mu_I(f) = \dim_{\mathbb{C}} M(g)$ is equivalent to the fact that $M_{\mathrm{rel}}(G)$ is Cohen-Macaulay (see Chapters 7 and 8 for details).

There is another candidate module whose vector space dimension is conjecturally equal to $\mu_I(f)$, namely the module $N\mathscr{K}_{G,e}i$, where $i$ is a map inducing $f$ from a stable unfolding $F$ by transverse fibre product, $G$ is the equation of the image of $F$ and the module $N\mathscr{K}_{G,e}i$ is Damon's normal extended module of the orbit of $i$ with respect to the $\mathscr{K}_G$-action. This is one of the geometric subgroups of Damon, which is in fact the subgroup of the contact group $\mathscr{K}$ of diffeomorphisms which preserve $G$ (see [11]). This approach was used by Damon and Mond in [13] to show an inequality of type $\mu \geq \tau$ for map germs $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, with $n \geq p$. In such case, instead of $\mu_I(f)$ we have to consider the discriminant Milnor number $\mu_\Delta(f)$, which is defined analogously but taking the discriminant $\Delta$ of a stable perturbation of $f$ instead of the image. This is explained with more detail in Chapter 2, Section 2.8.

In Chapter 9 we present a proof of the Mond conjecture for surfaces ($n = 2$), which is different from the ones in [50, 15]. Our proof is based on an argument due to Pellikaan [64], used to prove that certain modules given by a quotient of two ideals are Cohen-Macaulay.

Finally, we conclude the paper with Chapter 10, dedicated to some recent advances on the conjecture, as well as related questions and possible generalisations. These include the question of whether the equality implies that $f$ is weighted homogeneous, up to $\mathscr{A}$-equivalence, the extension of the conjecture to singularities of mappings defined on ICIS, or to singularities of frontals and about a possible upper bound for the quotient $\mu/\tau$.

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# Chapter 2

# Known cases and related problems

This chapter is an overview of the cases where Mond conjecture is known to be true, and on some of the ideas behind the proofs. Some basic knowledge on singularities of mappings is assumed, for which we refer to Chapter s 3, 4 and 5.

Following Mond's account in [50], the conjectured inequality was first observed as an empirical fact in [48] and [38]. The first proof for surfaces was found by de Jong and Pellikaan (unpublished) and the first published proofs were given by de Jong and Van Straten [15] and Mond [50]. We will give the proof for surfaces in Chapter 9, as a consequence of a more general result. The case of dimension one was also known to be true from the beginning, but the proof was published later [51] and it was not as simple as the one we give here.

## 2.1 The one-dimensional case

Quite unsurprisingly, one of the cases where we know the Mond conjecture to be true is that of germs

$$f\colon (\mathbb{C}, S) \to (\mathbb{C}^2, 0).$$

The original proof is found in [51], with an approach similar to that of higher dimensions, described in Chapter 6 (it is worth mentioning that the one-dimensional case requires some modifications with respect to higher dimensional ones. Observe for example that Proposition 6.5 does not apply, since the codimension of the non-inmersive points is smaller than required).

Apart from the mere low dimensionality, this case has a unique advantage, which allows us to give an alternative proof for the Mond conjecture:

the image of an $\mathscr{A}$-finite map germ $f \in \mathscr{O}(1,2)$ is an isolated curve singularity (and conversely, any reduced plane curve can be parameterized, giving rise to a $\mathscr{A}$-finite mapping). Consequently, taking

$$(X,0) = \operatorname{Im} f,$$

we may try to relate the invariants of $f$ to those of $(X,0)$, in the hope that the Mond conjecture will follow from the $\mu \geq \tau$ inequality. More precisely, to prove the Mond conjecture, we only need to show the relation

$$\mu_I(f) - \operatorname{codim}_{\mathscr{A}_e}(f) = \mu(X,0) - \tau(X,0).$$

Luckily enough, this equality follows from two equalities which we introduce next. The first one, harder to prove, is as follows [26, Chapter II, Proposition 2.30]:

**Theorem 2.1.** *Let $f\colon (\mathbb{C},S) \to (\mathbb{C}^2,0)$ be an $\mathcal{A}$-finite map germ and $(X,0) = \operatorname{Im} f$. Then*

$$\tau(X,0) = \operatorname{codim}_{\mathscr{A}_e}(f) + \delta(X,0).$$

The second equality is

$$\mu(X,0) = \mu_I(f) + \delta(X,0)$$

and it is easier to understand: consider a stabilisation $F = (f_t, t)$ of $f$, let $X_t = \operatorname{Im} f_t$ and take a disentaglement of $f$,

$$X_t \cap B_\epsilon,$$

as in Definition 5.2. It follows from conservation of Milnor number [58, Theorem 4.2] (see also [8]) that

$$\mu(X,0) = \beta_1(X_t \cap B_\epsilon) + \sum_{x \in \operatorname{Sing} X_t \cap B_\epsilon} \mu(X_t, x)$$

But $\beta_1(X_t \cap B_\epsilon) = \mu_I(f)$, because $X_t \cap B_\epsilon$ is a disentanglement of $f$. Moreover, $X_t$ has just nodes as singularities, because nodes are the only stable singular multi-germ in these dimensions. Since nodes have Milnor number 1, the sum of the $\mu(X_t, x)$ in $B_\epsilon$ equals the number of nodes, which is precisely $\delta(X,0)$. This shows the desired equality.

## 2.2 Folding maps and a taste of alternating homology

Beyond the case of surfaces, we have some empirical evidence for Mond conjecture in higher dimensions, in the form of an infinite family of singularities in all dimensions and a collection of examples in dimensions $n = 3, 4$

and 5, for which we know the conjecture to be true. In this chapter we comment on the infinite family, known as the *folding map family*, where the proof of the Mond conjecture still boils down to the inequality $\mu \geq \tau$ for hypersurfaces. Explaining the proof for this family is a good excuse to introduce the reader to alternating homology and to point them to the image-computing spectral sequence, a very interesting machinery for the description of the topology of singular mappings.

Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finite map germ and let $k \leq n$ be an integer. The $k$th multiple-point space of $f$ is

$$D^k(f) = \operatorname{cl}(\{(x^{(1)}, \dots, x^{(k)}) \in \mathbb{C}^n \times \overset{k}{\cdots} \times \mathbb{C}^n \mid x^{(i)} \neq x^{(j)}, f(x^{(i)}) = f(x^{(j)})\}),$$

thought as a germ at $S \times \overset{k}{\cdots} \times S$ and where cl stands for the analytic closure operator. A more sophisticated version of this definition gives spaces $D^k(f)$ for all finite maps and map-germs between regular spaces of all dimensions, without the condition $k \leq n$ and regardless of $\mathcal{A}$-finiteness (see, e.g. [59]). We will use the more sophisticated versions, but it is a good idea for the reader to pretend that they are defined just as the ones above.

The key aspect of the spaces $D^k(f)$ is that they keep track of how points in the source of $f$ are glued together to form the image of $f$. From this point of view, it seems reasonable that the topology of $\operatorname{Im} f$ must be encoded by the topology of the collection of spaces $D^k(f)$. The mathematical object that carries this information is the so called image-computing spectral sequence, introduced in [24] (see [9] for an updated version). The applications of the spectral sequence reach further than the study of the image Milnor number, but we cannot cover them here. In fact, we are not even going to introduce the spectral sequence itself, for which refer the reader to the original sources cited above. Instead, we introduce only the objects involved in a result, consequence of the spectral sequence, which expresses the image Milnor number from the homology of $D^k(f)$ in the corank one case:

The symmetric group $S_k$ acts on $D^k(f)$ by permutation of the entries $x^{(j)}$, that is, a permutation $\sigma \in S_k$ acting on $(x^{(1)}, \dots, x^{(k)})$ gives the point $(x^{(\sigma(1))}, \dots, x^{(\sigma(k))})$. Since the action is continuous, it passes to homology and we define the $j$th alternating homology of $D^k(f)$ (with rational coefficients) as the following subgroup of $H_j(D^k(f); \mathbb{Q})$:

$$H_j^{\mathrm{Alt}}(D^k(f); \mathbb{Q}) = \{c \in H_j(D^k(f); \mathbb{Q}) \mid \sigma c = \operatorname{sign}(\sigma) c\}.$$

These objects are the entries of the first page of the spectral sequence. As it turns out, if $f \in \mathcal{O}(n, n+1)$ is $\mathcal{A}$-finite and has corank one, then, for each $k \leq n+1$, $D^k(f)$ is an isolated complete intersection singularity of dimension $n-k$, and if $f_t$ is a stable perturbation of $f$ (see Definition 3.6), then $D^k(f_t)$ is a Milnor fibre of $D^k(f)$. This implies the collapse of the spectral sequence at the first page, from where the following result follows:

**Theorem 2.2.** *Let $f \in \mathcal{O}(n, n+1)$ be $\mathcal{A}$-finite and of corank one and let $f_t$ be a stable perturbation of $f$. Then*

$$\mu_I(f) = \sum_{j+k=n+1} \dim_{\mathbb{Q}} H_j^{\mathrm{Alt}}(D^k(f_t); \mathbb{Q}).$$

*(for the definition of stable perturbation, see Definition 3.6).*

To go from this result to a proof of the Mond conjecture, one still needs to have good control of the terms at the right hand side of the equality, and to relate them to $\operatorname{codim}_{\mathcal{A}_e}(f)$. As we will see, this is possible for folding mappings, thanks to their very simple geometry.

A folding map $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a map germ of the form

$$f(x) = (x_1, \ldots, x_{n-1}, x_n^2, H(x)),$$

with $H \in \mathfrak{m}_n^2$. It is not difficult to see that a mono-germ $f$ is $\mathscr{A}$-equivalent to the germ of a folding map if and only if $m(f) = 2$, where $m(f)$ is the multiplicity of $f$, defined as

$$m(f) = \dim_{\mathbb{C}} \frac{\mathscr{O}_n}{(f_1, \ldots, f_{n+1})},$$

Clearly, every folding map has multiplicity two. For the converse implication, assume $m(f) = 2$. Then, necessarily we have an isomorphism of $\mathbb{C}$-algebras

$$\frac{\mathscr{O}_n}{(f_1, \ldots, f_{n+1})} \cong \frac{\mathbb{C}\{t\}}{(t^2)}.$$

This means that $f$ can be written, up to $\mathscr{A}$-equivalence, as

$$f(x) = (x_1, \ldots, x_{n-1}, K(x), H(x)),$$

for some $K, H \in \mathfrak{m}_n^2$ such that $K$ is regular of order 2 in the variable $x_n$. The map germ given by the $n$ first coordinates

$$\tilde{f}(x) = (x_1, \ldots, x_{n-1}, K(x))$$

is now $\mathscr{A}$-equivalent to the Whitney fold $x \mapsto (x_1, \ldots, x_{n-1}, x_n^2)$. Such $\mathscr{A}$-equivalence induces in a natural way another one which puts $f$ in the desired form.

**Theorem 2.3.** *Folding maps satisfy the Mond conjecture.*

This was shown by Kevin Houston in [29] (Houston uses a different notation, and proves a more general result about folding maps $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, for all dimensions $p > n$). We do not go into the details of the proof, but we give a sketch that highlights its dependence on the alternating homology.

A folding map may be thought as obtained from the graph of $H(x)$, by identifying every pair of points of the form $(Y_1, \dots, Y_{n-1}, \pm Y_n, Y_{n+1})$. To be more precise, let the group $\mathbb{Z}/2$ act on $\mathbb{C}^{n+1}$ by changing the sign of the $n$th coordinate. The quotient map of this group action can be realized as the (trivially unfolded) *Whitney fold* $\omega\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, given by

$$(Y_1, \dots, Y_n, Y_{n+1}) \mapsto (Y_1, \dots, Y_n^2, Y_{n+1}).$$

Now a folding map is just the composition $\omega \circ h$, where $h(x) = (x, H(x))$ is the graph embedding of $H$.

The fact that folding maps factorize through an embedding and such a simple quotient mapping is the reason behind their previously alluded very simple geometry. It forces, for instance, the absence of triple and higher multiplicity points. Moreover, the double point space $D$ of $f$ (This space is defined in Definition 6.3 but, as a set, it is the projection of $D^2(f) \subseteq (\mathbb{C}^n, 0) \times (\mathbb{C}^n, 0)$ on $(\mathbb{C}^n, 0)$) can be computed just as

$$D = V(\lambda),$$

for the holomorphic function

$$\lambda = \frac{H(x_1, \dots, x_{n-1}, x_n) - H(x_1, \dots, x_{n-1}, -x_n)}{x_n}.$$

It turns out that $f$ is $\mathscr{A}$-finite if and only if $D$ is an isolated hypersurface singularity, and in that case the space $D' = D \cap \{x_n = 0\}$ is an isolated hypersurface singularity as well. From the absence of higher multiplicity points and the particularities of the double points of folds, Houston shows that Theorem 2.2 translates into the formula

$$\mu_I(f) = \frac{1}{2}(\mu(D) + \mu(D')).$$

Furthermore, by direct calculation he shows that for folding mappings one has

$$\operatorname{codim}_{\mathcal{A}_e}(f) \leq \frac{1}{2}(\tau(D) + \tau(D')),$$

with equality in the weighted homogeneous case (this is not exactly how Houston puts it, but it follows from his Theorem 2.8 and Lemma 2.9 and his considerations in the proof of Theorem 2.11). This reduces the Mond conjecture for folding maps to the usual $\mu \geq \tau$ inequality.

## 2.3 Higer dimensional examples

Apart from the folding maps from the previous chapter , there are some more degenerate germs in certain dimensions for which we know the Mond conjecture to be true. These germs are the ones found in Houston and Kirk's classification [32], and the collection of $\mathscr{A}$-finite weighted-homogeneous germs exhibited by Sharland (née Altıntaş) [2, 68]. These are interesting not only as new evidence for the conjecture, but also because they force us to find new ways to check the conjecture, not involving a reduction to the $\mu \geq \tau$ inequality. One has to keep in mind that, while the Mond conjecture for curves and folding maps have been proven via the $\mu \geq \tau$ inequality, there is no obvious way to relate the two inequalities for higher dimensional more degenerate mappings.

Houston and Kirk classified all simple $\mathscr{A}$-finite mono-germs $\mathscr{O}(3,4)$ of corank 1, and checked the conjecture for them. They proceeded just by computing $\mu_I$ and $\operatorname{codim}_{\mathscr{A}_e}(f)$ for every germ on the classification, then comparing the numbers. Here, the corank one hypothesis is key because it forces the multiple point spaces $D^k(f)$ to be complete intersections with isolated singularities (see [38] for details), and the Milnor numbers of these complete intersections can be used to compute $\mu_I$. While the problem of computing $\operatorname{codim}_{\mathscr{A}_e}(f)$ is less sensitive to the corank, computing $\mu_I$ for map germs of corank greater than one is usually much more challenging.

Sharland has been able to produce a collection of $\mathscr{A}$-finite map germs of coranks 2 and 3, in dimensions $3 \leq n \leq 5$, and to check the conjecture for them. Just to give a taste of how these map germs look like, some of the germs in $\mathscr{O}(3,4)$ for which the conjecture has been proven are

$$\hat{B}_{2\ell+1}\colon (x,y,z) \mapsto (x, y^2+xz, z^2+xy, y^{2\ell+1}+y^{2\ell}z+yz^{2\ell}-z^{2\ell+1})$$

for $2 \leq \ell \leq 5$, and

$$\hat{f}_\ell\colon (x,y,z) \mapsto (x, y^2+x^\ell z, z^2-x^\ell y, y^3+y^2z+yz^2-z^3),$$

for all $\ell \geq 1$. In $\mathscr{O}(4,5)$, for all $\ell \geq 1$ we have $\hat{J}_\ell$, mapping $(x,y,z,w)$ to

$$(x, w, y^3+xz+x^4y+w^{2\ell}y^2, yz+w^{5\ell}y, z^2+y^5).$$

In $\mathscr{O}(5,6)$, for all $\ell \geq 1$, we have $\hat{N}_\ell$, mapping $(x,y,z,w,v)$ to

$$(x, v, w, y^3+xz+x^2y+wy, yz+v^\ell z, z^2+y^5+w^2y+v^{4\ell}y+v^{3\ell}y^2).$$

These germs are obtained as special unfoldings of lower dimensional germs of corank 2, which we call *generalized augmentations* (Sharland calls them augmentations, but her construction is more general than the augmentations of Section 2.7). The $\mathscr{A}_e$-codimension of augmentations can be computed, thanks to an isomorphism

$$T^1_{\mathscr{A}_e}(f) \cong N\mathscr{K}_{V,e}g,$$

where $V$ and $g$ are related to the augmentation construction (see [2] for details). Sharland used this method to compute (see [2, Tables 1 and 2 and Section 3])

$$\operatorname{codim}_{\mathscr{A}_e}(\hat{B}_3) = 33, \quad \operatorname{codim}_{\mathscr{A}_e}(\hat{B}_5) = 252, \quad \operatorname{codim}_{\mathscr{A}_e}(\hat{B}_7) = 837,$$
$$\operatorname{codim}_{\mathscr{A}_e}(\hat{B}_9) = 1968, \quad \operatorname{codim}_{\mathscr{A}_e}(\hat{B}_{11}) = 3825,$$
$$\operatorname{codim}_{\mathscr{A}_e}(\hat{f}_\ell) = 45\ell - 12, \quad \operatorname{codim}_{\mathscr{A}_e}(\hat{J}_\ell) = 2144\ell - 186,$$
$$\operatorname{codim}_{\mathscr{A}_e}(\hat{N}_\ell) = 1759\ell - 350.$$

Moreover, one can show that an augmentation satisfies the Mond conjecture by showing that certain relative module $N\mathscr{K}_{H,e/\mathbb{C}}G$ is Cohen-Macaulay over the parameter space (this is similar to the criterion that we introduce in Chapter 7 and to the one used by Damon and Mond in [13]). In fact, the Cohen-Macaulay condition implies that $N\mathscr{K}_{H,e/\mathbb{C}}G$ is a free module of rank equal to $\mu_I(f)$. Such rank is also equal to the length of the tensor product of $N\mathscr{K}_{H,e/\mathbb{C}}G$ with the parameter ring, which turns out to be $N\mathscr{K}_{V,e}g$. This is how the conjecture is established for these examples. However, we notice that the proof of the Cohen-Macaulayness of $N\mathscr{K}_{H,e/\mathbb{C}}G$ in [2, page 10] seems incomplete, or, at the very least, hard to follow.

It is interesting to note that this computes their $\mu_I$ values, but rather indirectly: given that the examples are weighted homogeneous and satisfy the conjecture, their $\mu_I$ must be equal to the already computed $\mathscr{A}_e$-codimension.

But there are even more degenerate map germs, like Sharland's corank 3 map germ

$$f_3\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0),$$

given by

$$(x, y, z) \mapsto (y^2 + xz, x^5 + yz + xy^2, x^6 + y^3 + z^2, x^{13} + x^{10}z + xz^4 + y^5 z).$$

This germ cannot be an augmentation, since it has rank zero. The way the conjecture was checked for this and other corank 3 examples in $\mathscr{O}(3, 4)$ was simply to compute $\mu_I$ and the $\mathscr{A}_e$-codimension separately and compare the numbers [68]. The $\mathscr{A}_e$-codimension was computed with Singular [73] by means of the formula

$$\operatorname{codim}_{\mathscr{A}_e}(f) = \dim_{\mathbb{C}} \frac{(f^*)^{-1}(J(g) \cdot \mathscr{O}_n)}{J(g) + (g)}.$$

that we introduce in Proposition 6.6 (the formula was independently found by Sharland [68] and Bobadilla and the authors [18]) and it is

$$\operatorname{codim}_{\mathscr{A}_e}(f) = 127295.$$

The image Milnor number computation is obtained by means of a formula in terms of weights and degrees, found in Theorem 2.5 in the next section.

In the same paper [68], Sharland's gives two more corank 3 examples, whose extended $\mathscr{A}$-codimensions are 18967 and 41244. These numbers are so enormous that the coincidence of codimension and image Milnor number makes these examples convincing evidence for the conjecture.

## 2.4 The $\mu_I$ formulas for weighted-homogeneous mono-germs

To study the Mond conjecture, it is convenient to be able to compute the image Milnor number of map germs. The problem is that, while the Milnor number of hypersurfaces is computed as the dimension of the Milnor algebra, no such result exists for $\mu_I$, and computing the image Milnor number directly from its topological definition can be very hard.

For small dimensions, we know well the singularities that the disentanglement is allowed to have, and we can try to use this knowledge to determine the topology of the disentanglement. For example, the disentanglement of a map germ $f\colon (\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ can only have double points, triple points and crosscaps. Thinking about how the Euler characteristic of the disentanglement relates to these singularities, one concludes that the following holds:

$$\mu_I = \frac{1}{2}\big(\mu(D^2) - 4T + C - 1\big),$$

where $T$ and $C$ are the number of triple points and crosscaps exhibited by the disentanglement, and $D^2$ is the closure of the set of pairs of point $(x,x')\in\mathbb{C}^2\times\mathbb{C}^2$ with $x\neq x'$, such that $f(x)=f(x')$ [50].

The right hand side of the equality is more computable because, as it turns out, $\mu(D^2), T$ and $C$ can be computed as dimensions of suitable $\mathbb{C}$-vector spaces. For weighted homogeneous germs the situation is even better, because, as Mond realized in [49], one can use the grading $w_1, w_2, d_1, d_2, d_3$ to determine the Hilbert series of the involved vector spaces. By this procedure, he obtained the following result:

**Theorem 2.4.** *Let $f\colon (\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ be an $\mathscr{A}$-finite weighted homogeneous mono-germ, with grading $(w_1,w_2,d_1,d_2,d_3)$. Then,*

$$\mu_I(f) = \frac{1}{6w_1w_2}\Big((s_0-\epsilon)(s_0+\epsilon-3(w_1+w_2)) - (d_2d_3+d_1d_3+d_1d_2) + (w_1+w_2)\epsilon + w_1w_2\Big),$$

*with $\epsilon = d_1+d_2+d_3-w_1-w_2$ and $s_0 = \dfrac{d_1d_2d_3}{w_1w_2}$.*

Trying to reproduce the same idea in higher dimensions becomes much harder, because the geometry is much more complicated and we do not know how to compute some of invariants as dimensions of $\mathbb{C}$-vector spaces. In fact, a priori it is unclear that these formulas must exist.

With a more sophisticated approach based on the theory of Thom polynomials, Ohmoto [61] has shown that $\mu_I$ formulas indeed exist for weighted-homogeneous mono-germs $f \in \mathscr{O}(n, n+1)$, for $n \leq 5$. The formula for $n = 3$ was found by Ohmoto, and the cases $n = 4, 5$ by Pallarés and the second author [63]. These formulas are very cumbersome when written in terms of the weights and degrees, but they become more tractable if expressed adequately:

First, given weights $w = (w_1, \dots, w_n)$ and degrees $d = (d_1, \dots, d_{n+1})$, consider the auxiliary expressions

$$\sigma_k = \sum_{1 \leq j_1 < \dots < j_k \leq n} w_{j_1} \cdot \ldots \cdot w_{j_k},$$

for $k = 1, \dots, n$, and

$$\delta_k = \sum_{1 \leq i_1 < \dots < i_k \leq n+1} d_{i_1} \cdot \ldots \cdot d_{i_k},$$

for $k = 1, \dots, n+1$. Then, the $\mu_I$ formulas will be expressed in terms of

$$s_0 = \frac{\delta_{n+1}}{\sigma_n} \quad \text{and} \quad c_k = \sum_{0 \leq i \leq k} (-1)^{k-i} \delta_i \sum_{|\alpha| = k-i} w^\alpha,$$

with the usual multi-index notation for $\alpha$. With these at hand, the expression of $\mu_I$ for $n = 2$ in Theorem 2.4 can be rewritten as

$$\mu_I(f) = \frac{1}{\sigma_2} \big( \frac{1}{2!}(-s_0 + c_1)\sigma_1 + \frac{1}{3!}(s_0^2 - c_1^2 - c_2) \big).$$

After showing that the formulas exist, still remains the problem of finding them. For dimensions 3 and higher, this was not done by relating $\mu_I$ to easier invariants, but by a purely interpolative method that finds the coefficients in the formula by studying a big enough number of map germs. The formula found by Ohmoto [61] is the following one:

**Theorem 2.5.** *Let $f\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ be an $\mathscr{A}$-finite weighted homogeneous germ. Then,*

$$\mu_I(f) = -\frac{1}{\sigma_3} \big( \frac{1}{2!}(-s_0 + c_1)\sigma_2 + \frac{1}{3!}(s_0^2 - c_1^2 - c_2)\sigma_1 + \\ + \frac{1}{4!}(-s_0^3 - 2s_0^2 c_1 + s_0 c_1^2 + 16 s_0 c_2 + 2c_1^3 - 10 c_1 c_2) \big).$$

Observe the similarity between the expression for $n = 2$ and the first terms in the formula for $n = 3$. This phenomenon would be much less apparent if we would write the formula directly in terms of $w$ and $d$, indicating that $s_0$ and $c_k$ carry some meaningful information. Of course, this has to do with the theory of Thom polynomials, which justifies the existence of this formulas and their form as rational functions, whose numerator is obtained from the n-th degree truncation of the so called Segre-MacPherson Thom polynomial $tp^{SM}(\alpha_{image})$ series (see [61, Section 6.5] for the original results and [63, Section 3] for a summary). For our purposes, $s_0$ and the $c_k$ are just convenient expressions. The remaining formulas for the image Milnor number [63] are as follows:

**Theorem 2.6.** *Let $f\colon (\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ be an $\mathscr{A}$-finite weighted homogeneous germ. If $n=4$, then*

$$\begin{aligned}\mu_I(f) = \frac{1}{\sigma_4}\Big(&\frac{1}{2!}(-s_0+c_1)\sigma_3+\frac{1}{3!}(s_0^2-c_1^2-c_2)\sigma_2\\ &+\frac{1}{4!}(-s_0^3-2s_0^2c_1+s_0c_1^2+16s_0c_2+2c_1^3-10c_1c_2)\sigma_1\\ &+\frac{1}{5!}(s_0^4+5s_0^3c_1+5s_0^2c_1^2-50s_0^2c_2-5s_0c_1^3-20s_0c_1c_2\\ &+60s_0c_3-6c_1^4+34c_1^2c_2-64c_1c_3+108c_2^2+4c_4)\Big).\end{aligned}$$

*If $n=5$, then*

$$\begin{aligned}\mu_I(f) = -\frac{1}{\sigma_5}\Big(&\frac{1}{2!}(-s_0+c_1)\sigma_4+\frac{1}{3!}(s_0^2-c_1^2-c_2)\sigma_3+\\ &+\frac{1}{4!}(-s_0^3-2s_0^2c_1+s_0c_1^2+16s_0c_2+2c_1^3-10c_1c_2)\sigma_2\\ &+\frac{1}{5!}(s_0^4+5s_0^3c_1+5s_0^2c_1^2-50s_0^2c_2-5s_0c_1^3-20s_0c_1c_2\\ &+60s_0c_3-6c_1^4+34c_1^2c_2-64c_1c_3+108c_2^2+4c_4)\sigma_1\\ &+\frac{1}{6!}(-s_0^5-9s_0^4c_1-25s_0^3c_1^2+110s_0^3c_2-15s_0^2c_1^3+270s_0^2c_1c_2\\ &-240s_0^2c_3+26s_0c_1^4+16s_0c_1^2c_2+24s_0c_1c_3-1138s_0c_2^2+336s_0c_4\\ &+24c_1^5-156c_1^3c_2+276c_1^2c_3+108c_1c_2^2-396c_1c_4+600c_2c_3)\Big).\end{aligned}$$

## 2.5 Reduction to families of unbounded multiplicity

In the previous chapter s we have discussed a good number of examples for which the Mond conjecture is known to hold. One can think that this does not bring us any closer to proving the conjecture but, as we are going to explain next, this is not entirely true and, actually, the conjecture must be true if it holds for a good enough collection of examples.

Fixed a dimension $n$, and a subset $S \subseteq \mathbb{C}^n$, consider a collection

$$\{f_M\}_{M\in\mathbb{N}}$$

of weighted homogeneous $\mathscr{A}$-finite germs $f_M \in \mathscr{O}_S(n, n+1)$, such that the coordinate functions of any branch of $f_M$ have no terms of degree smaller than $M$. We call such a collection a *family of unbounded multiplicity.* The reason why these are interesting is the following:

**Theorem 2.7.** *If the Mond conjecture holds for a family of unbounded multiplicity, then it holds for every $\mathscr{A}$-finite mono-germ $f \in \mathscr{O}(n, n+1)$.*

This was shown in [18] and, in fact, what is shown is that if $f_{M+1}$ satisfies the Mond conjecture, then the conjecture holds for all $M$-determined germs (with same dimensions and number of branches as the ones in the family). Moreover, the notion of family of unbounded multiplicity can be adapted, to consider a family of map germs of some corank, and then checking the conjecture on such a family would prove it only for germs of that corank.

The problem is that we do not know of any family of unbounded multiplicity in dimensions where the Mond conjecture is open. Producing such a family is easy for $f\colon (\mathbb{C}, S) \to (\mathbb{C}^2, 0)$ but, beyond that, even if we omit the requirement that the $f_M$ must be weighted homogeneous, the only known family of $\mathscr{A}$-finite map germs where the multiplicities are unbounded is the family

$$(x, y) \mapsto (x^a, y^b, (x + y)^c),$$

with $a, b, c$ pairwise coprime, found in [65]. The obstacle to producing such families is checking the finite determinacy. At least in the nice dimensions, it seems reasonable that a generic choice of polynomial coordinate functions, with degrees greater than $M$, would yield an $\mathscr{A}$-finite map germ but, even if that is the case, showing it to be $\mathscr{A}$-finite can be quite challenging when $M$ is big.

Observe that, if a family of unbounded multiplicity is found for dimensions $n \leq 5$, the image Milnor number of mono-germs $f_M$ could be computed immediately with the formulas of Section 2.4. Hence, proving the Mond conjecture for mono-germs in such dimension $n$ would amount to computing the invariant $\operatorname{codim}_{\mathscr{A}_e}(f_M)$, for all $M \in \mathbb{N}$.

## 2.6 Weak versions of the Mond conjecture

If $f \in \mathscr{O}_S(n, n + 1)$ is stable multi-germ, then $\mu_I(f)$ must vanish. This is simply because any stabilisation $F$ of $f$ (or any unfolding of $f$ for that matter) must be trivial. Therefore, the image of a small enough representative of $F$ will be a trivial fibration $\mathcal{X} \to D$ and have contractible fibres, showing the disentanglement of $f$ to be contractible.

This sort of simple argument does not work in the opposite direction, and it is not clear a priori that a multi-germ with $\mu_I(f) = 0$ must be stable. The fact that the implication goes both ways was shown by Giménez Conejero and the first author [23] (see [21] for a different proof in the corank one case), as follows: Let $F$ be a 1-parameter stabilisation (see Definition 3.6) of an $\mathscr{A}$-finite germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ and take

$$(\mathcal{X}, 0) = \operatorname{Im} F = V(G),$$

for certain germ of function $G \in \mathscr{O}_{\mathbb{C}^{n+2}}$ so that $V(G)$ is reduced. To this data one can associate an ideal $FT(\pi, G)$ of $\mathscr{O}_{n+2}$, whose zero locus is the germ at the origin of following set of points:

- $(0, 0) \in \mathbb{C}^{n+1} \times \mathbb{C}$, if $f$ is unstable.
- $(y, t) \in \mathbb{C}^{n+1} \times \mathbb{C}$, if $G(y, t) \neq 0$ and $y$ is a critical point of the function $g_t \in \mathscr{O}_{n+1}$, given by $g_t(y) = G(y, t)$.

On one hand, it is showed in [23, Theorem 2.10] that $\mu_I(f)$ is equal to the Samuel multiplicity of $\mathscr{O}_{n+2}/FT(\pi, G)$ considered as a module over the parameter space. In more geometrical words, $\mu_I(f)$ is equal to the local intersection number of the complex space $V(FT(\pi, G))$ and the hyperplane $t = 0$. On the other hand, by [23, Proposition 2.14], $\dim V(FT(\pi, G)) = 1$ if not empty, which forces that the local intersection number must be at least 1. This shows the following result:

**Theorem 2.8.** *Let $f \in \mathscr{O}(n, n+1)$ be an $\mathscr{A}$-finite multi-germ. Then, $\mu_I(f) = 0$ if and only if $f$ is stable.*

We may think of this as a weak version of Mond conjecture, stating that the inequality $\operatorname{codim}_{\mathcal{A}_e}(f) \leq \mu_I(f)$ holds for those multi-germs having $\mu_I(f) = 0$.

As a corollary of this result, we obtain an extension of a result Cooper, Mond and Wik Atique [10], originally stated for the corank one case only:

**Proposition 2.9.** *Every multigerm $f \in \mathscr{O}(n, n+1)$ with $\operatorname{codim}_{\mathscr{A}_e}(f) = 1$ satisfies the Mond conjecture.*

*Proof.* Since $\operatorname{codim}_{\mathscr{A}_e}(f) = 1$, then $f$ is unstable, which forces $\mu_I \neq 0$, that is $\mu_I \geq 1$. Moreover, by Proposition 6.5 and Corollary 8.2,

$$\mu_I \leq \dim_{\mathbb{C}} M(g) = \operatorname{codim}_{\mathscr{A}_e}(f) + \dim_{\mathbb{C}} K(g) = 1 + \dim_{\mathbb{C}} K(g),$$

where $g$ is a reduced equation of the image of $f$, $M(g)$ is the Jacobian module (see Definition 6.1) and $K(g) = (J(g) + (g))/J(g)$. If $f$ is weighted homogeneous, then we can choose $g$ also weighted homogeneous, so $K(g) = 0$ and $\mu_I = 1$. □

## 2.7 Relation to augmentations

Consider an $\mathscr{A}$-finite mono-germ $f \in \mathcal{O}(n, n+1)$ and assume that it admits a one-parameter stable unfolding, that is, a germ $F \in \mathcal{O}(n+1, n+2)$, which is stable and has the form

$$F(x,t) = (f_t(x), t),$$

with $f_0 = f$ (this is not to be confused with a stabilisation of $f$, since here we ask for $F$ to be stable, rather than $f_\epsilon$). Now take a germ $g \in \mathcal{O}_r$, such that the hypersurface $V(g) \subset (\mathbb{C}^r, 0)$ has an isolated singularity. The *augmentation of $f$ by $F$ and $g$* is the map germ $A_{F,g}(f) \in \mathcal{O}(n+r, n+1+r)$ given by

$$(x, s) \mapsto (f_{g(s)}(x), s).$$

Any map germ $A$ which is $\mathscr{A}$-equivalent to one of the form $A_{F,g}(f)$ is said to be an *augmentation* of $f$. One can show that augmentations are $\mathscr{A}$-finite [29]. The following series of equalities and inequalities relate the invariants of the augmentation $A$ to the invariants of $f$, $g$ and $F$. The first one is an equality for $\mu_I(A)$. It is due to Houston in the corank one case [30, Corollary 6.4]) and was extended to the general case by Giménez Conejero and the first author in [23, Corollary 2.17].

**Theorem 2.10.** *Let $A = A_{F,g}(f)$ be an augmentation of $f$. Then,*

$$\mu_I(A) = \mu(g) \cdot \mu_I(f).$$

The second one is a lower bound for $\operatorname{codim}_{\mathscr{A}_e}(A)$ and was also obtained by Houston [31]:

**Theorem 2.11.** *Let $A = A_{F,g}(f)$ be an augmentation of $f$. Then,*

$$\tau(g) \operatorname{codim}_{\mathscr{A}_e}(f) \leq \operatorname{codim}_{\mathscr{A}_e}(A),$$

*with equality if $g$ is weighted homogeneous.*

Finally, the upper bound for $\operatorname{codim}_{\mathscr{A}_e}(A)$ has been recently computed by Breva and Oset [5, Theorem 3.3]:

**Theorem 2.12.** *Let $A = A_{F,g}(f)$ be an augmentation of $f$. Then,*

$$\operatorname{codim}_{\mathscr{A}_e}(A) \leq \mu(g) \operatorname{codim}_{\mathscr{A}_e}(f)$$

*with equality if $g$ is weighted homogeneous.*

Combining these results, Breva and Oset obtain the following consequence (see [5, Theorem 3.7]):

**Corollary 2.13.** *Assume that $f$ satisfies the Mond conjecture, and let $A = A_{F,g}(f)$ be an augmentation of $f$. Then,*

$$\mu_I(A) \geq \operatorname{codim}_{\mathscr{A}_e}(A),$$

*and the equality holds if $A$ is $\mathscr{A}$-equivalent to an augmentation $A_{\tilde{F},\tilde{g}}(f)$, with $\tilde{F}$ and $\tilde{g}$ weighted-homogeneous.*

Observe that this does not solve completely the "equality for weighted homogeneous" part of the Mond conjecture for the augmentation $A$ because, a priori, $A$ could be weighted homogeneous without any $f$ and $g$ so that $A = A_{f,g}$ being weighted homogeneous. We however do not know any particular example where this is the case.

## 2.8 The discriminant Milnor number

We finish this overview of the state of the art of the Mond conjecture with a known case which, even though it is not strictly speaking included in our statement of the Mond conjecture, it is very much related.

Given a mapping $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, one defines the critical set as the locus where the differential of $f$ is not surjective, and the discriminant $\Delta(f)$ as the image of the critical set (see the lines before Definition 3.3). One can prove that, for germs with isolated instabilities, the critical locus is a reduced Cohen-Macaulay space of dimension $p-1$, and from this it follows that, for $p \leq n+1$, the discriminant is a hypersurface (notice that in the particular case of $p = n+1$, the discriminant is just the image of $f$). Then, it follows that the discriminant of a stable perturbation is a wedge of spheres (the proof is similar to that of Theorem 5.4). The number of spheres is called the *discriminant Milnor number*, written $\mu_\Delta(f)$. This discriminant Milnor number can then be seen as a generalization, for dimensions $p \leq n+1$ of the image Milnor number. This calls for the obvious question: Does the Mond conjecture hold in this context? As it turns out, the new cases with $p \leq n$ are much more tractable that the one where $p = n+1$ and an affirmative answer was given by Damon and Mond [13]:

**Theorem 2.14.** *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a germ with isolated instability, and assume that $(n,p)$ is in the range of nice dimensions and $n \geq p$. Then*

$$\operatorname{codim}_{\mathscr{A}_e}(f) \leq \mu_\Delta(f),$$

*and the equality holds if and only if $f$ is weighted homogeneous.*

The result was originally stated for mono-germs in [13], but the same proof works for multi-germs [53, Theorem 8.10]. The proof is strongly based on the fact that the discriminant of stable mapping for dimensions $n \geq p$ is a free divisor, see [13] for details.

# Chapter 3

# Singularities of mappings

In this chapter, we provide a overview of the fundamental definitions and properties related to singularities of mappings, essential for formulating the Mond conjecture. Main references for this chapter are the recent book by Mond and the second author [53] and their survey [54].

Some of the results discussed apply to both smooth (i.e., $C^\infty$) mappings between smooth manifolds and holomorphic mappings between complex manifolds. However, since our primary point of interest is the Mond conjecture, which is meaningful exclusively in the complex context, we will discuss only the complex case.

Let $S \subset \mathbb{C}^n$ be a finite set. We use the notation $\mathscr{O}_S(n,p)$ to represent the set of holomorphic map germs $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$. When the set $S$ is evident from the context or when no confusion arises, we simply use $\mathscr{O}(n,p)$.

Our first definition pertains to $\mathscr{A}$-equivalence, also known as right-left equivalence of germs. Two germs are considered equivalent if they can be transformed into each other through changes of coordinates in the source and target.

**Definition 3.1.** Two germs $f, g \in \mathscr{O}_S(n,p)$ are called *$\mathscr{A}$-equivalent* if we have a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^n, S) & \xrightarrow{f} & (\mathbb{C}^p, 0) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ (\mathbb{C}^n, S) & \xrightarrow{g} & (\mathbb{C}^p, 0) \end{array}$$

where the columns are biholomorphisms.

Next, we revisit the concept of stability. In simple terms, a germ is deemed stable when any perturbation, up to $\mathscr{A}$-equivalence, does not alter it.

**Definition 3.2.** A *d-parameter unfolding* of $f \in \mathscr{O}_S(n,p)$ is another holomorphic map germ $F \in \mathscr{O}_{S\times\{0\}}(n+d, p+d)$ of the form $F(x,u) = (f_u(x), u)$ and such that $f_0 = f$.

Two unfoldings $F, G$ of $f$ are called *equivalent* if we have a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^d, S\times\{0\}) & \xrightarrow{F} & (\mathbb{C}^p \times \mathbb{C}^d, 0) \\ \downarrow \Phi & & \downarrow \Psi \\ (\mathbb{C}^n \times \mathbb{C}^d, S\times\{0\}) & \xrightarrow{G} & (\mathbb{C}^p \times \mathbb{C}^d, 0) \end{array}$$

where the columns are biholomorphisms which are also unfoldings of the identity.

The map germ $f$ is called *stable* if any unfolding is equivalent to the constant unfolding $f \times \mathrm{id}\colon (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$.

It is evident that the property of a map germ being stable remains unchanged under $\mathscr{A}$-equivalence. Thus, stability can be naturally extended to holomorphic map germs $f\colon (N,S) \to (P,y)$ between complex manifolds $N$ and $P$, simply by taking coordinates.

When dealing with a holomorphic map $f\colon N \to P$ between complex manifolds N and P, we define the *critical set* as the subset $C(f) \subseteq N$ containing points $x$ where the differential $df_x$ is not surjective. The image of this critical set, denoted by $\Delta(f) = f(C(f))$, is referred to as the *discriminant*.

**Definition 3.3.** We say that $f\colon N \to P$ has *finite singularity type* if the restriction $f\colon C(f) \to P$ is finite (i.e., closed and finite-to-one). We say that $f$ is *locally stable* if, in addition, for any $y \in \Delta(f)$, the *multi-germ of $f$ at $y$*, denoted by

$$f_y\colon (N,S) \to (P,y),$$

is stable, where $S = C(f) \cap f^{-1}(y)$.

The condition that the mapping has finite singularity type implies that the discriminant is a closed analytic subset of $P$, by the Remmert finite mapping theorem (see e.g. [14]).

**Definition 3.4.** We say that $f \in \mathscr{O}_S(n,p)$ has *isolated instability* if there exists a representative $f\colon U \to V$ such that

1. $f^{-1}(0) \cap C(f) \subseteq S$ and $f$ has finite singularity type.
2. the restriction $f\colon U \setminus f^{-1}(0) \to V \setminus \{0\}$ is locally stable.

Given $f \in \mathscr{O}_S(n,p)$ and an $r$-parameter unfolding $F(x,u) = (f_u(x), u)$, we can always choose a representative of the form

$$F\colon U \to V \times T, \tag{3.1}$$

where $U, V$ and $T$ are open subsets such that $S \times \{0\} \subset U \subseteq \mathbb{C}^n \times \mathbb{C}^r$, $0 \in V \subseteq \mathbb{C}^p$ and $0 \in T \subseteq \mathbb{C}^r$. For each $u \in T$ we have a mapping $f_u \colon U_u \to V$, where $U_u = \{x \in \mathbb{C}^n \mid (x, u) \in U\}$.

**Definition 3.5.** The *bifurcation set* $\mathscr{B}(F)$ is the set germ in $(\mathbb{C}^r, 0)$ of parameters $u \in T$, such that $f_u : U_u \to V$ is not locally stable.

**Lemma 3.1.** *[53, Lemma 5.3] If $f \in \mathscr{O}_S(n, p)$ has isolated instability, then $\mathscr{B}(F)$ is a closed analytic germ in $(\mathbb{C}^r, 0)$, for any $r$-parameter unfolding $F$.*

**Definition 3.6.** A *stabilisation* of $f$ is a 1-parameter unfolding $F$ whose bifurcation set is $\mathscr{B}(F) = \{0\}$ in $(\mathbb{C}, 0)$. In other words, there exists a representative as in (3.1), such that for all $t \in T \setminus \{0\}$, $f_t \colon U_t \to V$ is locally stable. Such $f_t$ is called a *stable perturbation* of $f$.

The following theorem is well known, see for instance [53].

**Theorem 3.2.** *Let $f \in \mathscr{O}_S(n, p)$ with isolated instability and assume that either $(n, p)$ are nice dimensions or $f$ has kernel rank one. Then $f$ admits a stabilisation $F$.*

The nice dimensions were introduced by Mather [42] in the context of global smooth (i.e., $C^\infty$) mappings between smooth manifolds. They characterize those pairs $(n, p)$ such that the stable proper mappings are dense in $C^\infty(N, P)$ with the Whitney $C^\infty$-topology, for any pair of smooth manifolds $N$ and $P$ of dimensions $n$ and $p$, respectively.

Mather also proved that, for any pair $(n, p)$, the topologically stable proper mappings are always dense in $C^\infty(N, P)$ (see [43]). This fact is based on his construction of the canonical stratification of the jet space. This can be adapted also to the complex local case in our setting to show that any map germ with isolated instability admits a "topological stabilisation". However, we will not consider such a construction in this paper.

We recall now the notion of codimension of $f \in \mathscr{O}_S(n, p)$. This gives an algebraic tool to check whether a germ is stable or has isolated instability. In order to do that we need to introduce the following notation:

- $\mathscr{O}_n$ is the ring of holomorphic function germs $(\mathbb{C}^n, S) \to \mathbb{C}$,
- $\mathscr{O}_p$ is the ring of holomorphic function germs $(\mathbb{C}^p, 0) \to \mathbb{C}$,
- $\theta_n$ is the $\mathscr{O}_n$-module of vector fields on $(\mathbb{C}^n, S)$,
- $\theta_p$ is the $\mathscr{O}_p$-module of vector fields on $(\mathbb{C}^p, 0)$,
- $\theta(f)$ is the $\mathscr{O}_n$-module of vector fields along $f$,
- $tf \colon \theta_n \to \theta(f)$, the morphism $\xi \mapsto df \circ \xi$,
- $\omega f \colon \theta_p \to \theta(f)$, the morphism $\eta \mapsto \eta \circ f$.

**Definition 3.7.** The *$\mathscr{A}_e$-normal space* of $f \in \mathscr{O}_S(n,p)$ is defined as

$$T^1_{\mathscr{A}_e}(f) = \frac{\theta(f)}{tf(\theta_n) + \omega f(\theta_p)}$$

and the *$\mathscr{A}_e$-codimension* is

$$\operatorname{codim}_{\mathscr{A}_e}(f) = \dim_{\mathbb{C}} T^1_{\mathscr{A}_e}(f).$$

The germ $f$ is called *infinitesimally stable* when $\operatorname{codim}_{\mathscr{A}_e}(f) = 0$ and *$\mathscr{A}$-finite* when $\operatorname{codim}_{\mathscr{A}_e}(f) < \infty$.

It is not difficult to check that stability implies infinitesimal stability. Mather showed that the converse is also true, although the proof requires of deeper arguments like the Weierstrass preparation theorem (see [41] or [53]):

**Theorem 3.3 (Mather infinitesimal stability theorem).** *A germ $f \in \mathscr{O}_S(n,p)$ is stable if and only if it is infinitesimally stable.*

The second result which relates the $\mathscr{A}_e$-codimension with the stability is due to Mather and Gaffney and it says that isolated instability is equivalent to $\mathscr{A}$-finiteness. This equivalence is valid only in the complex case, since it uses the notion of coherent sheaves of modules. A detailed proof can be found in [72] or [53], we give a short version in Chapter 4.

**Theorem 3.4 (Mather-Gaffney geometric criterion).** *A germ $f \in \mathscr{O}_S(n,p)$ has isolated instability if and only if it is $\mathscr{A}$-finite.*

An even more geometric interpretation of the $\mathscr{A}_e$-codimension can be obtained if we look at versal unfoldings. Roughly speaking, an unfolding is versal when it contains all possible perturbations of the germ, up to $\mathscr{A}$-equivalence.

**Definition 3.8.** Let $f \in \mathscr{O}_S(n,p)$ and let $F(x,u) = (f_u(x),u)$ be an $r$-parameter unfolding. Given $h\colon (\mathbb{C}^s,0) \to (\mathbb{C}^r,0)$ holomorphic, the *induced* unfolding is defined as the $s$-parameter unfolding $G(x,v) = (f_{h(v)}(x),v)$.

The unfolding $F$ is called *versal* if every unfolding is equivalent to one induced from $F$. A versal unfolding with minimal number of parameters (among versal unfoldings) is called *miniversal.*

Given an $r$-parameter unfolding $F(x,u) = (f_u(x),u)$ of $f \in \mathscr{O}_S(n,p)$, for each $i = 1,\dots,r$ we put

$$\dot{F}_i = \left.\frac{\partial f_u}{\partial u_i}\right|_{u=0}.$$

The following infinitesimal criterion of versality is due to Martinet (see [40] or [53] for a proof):

**Theorem 3.5 (Versality theorem).** *The unfolding $F$ is versal if and only if the residue classes of $\dot{F}_1, \ldots, \dot{F}_r$ in $T^1_{\mathscr{A}_e}(f)$ generate it as a $\mathbb{C}$-vector space. In particular, any $\mathscr{A}$-finite germ always admits a versal unfolding and* $\operatorname{codim}_{\mathscr{A}_e}(f)$ *is the number of parameters of a miniversal unfolding.*

The versality theorem shows the analogy between the $\mathscr{A}_e$-codimension of $f$ and the Tjurina number $\tau(X,0)$ of a hypersurface with isolated singularity $(X,0)$. We recall that $\tau(X,0)$ is defined as

$$\tau(X,0) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{n+1}}{J(g)+(g)},$$

where $g \in \mathscr{O}_{n+1}$ is a reduced equation of $(X,0)$ and $J(g)$ is the Jacobian ideal, generated by the partial derivatives of $g$. It follows that $\tau(X,0)$ is the number of parameters of a miniversal deformation of $(X,0)$ (see for instance [36]).

# Chapter 4

# Say it with sheaves

For any vector bundle $E$ on a analytic set $X$, the sheaf of sections of $E$ is a locally free $\mathcal{O}_X$-module, which we denote by $\mathcal{O}(E)$. A most basic example is the *sheaf of vector fields* on a manifold $N$, which is the sheaf of sections of the tangent bundle $TN$. We write it

$$\theta_N = \mathcal{O}(TN).$$

Similarly, for any holomorphic mapping $f\colon N \to P$, the *sheaf of vector fields along* $f$ is

$$\theta(f) = \mathcal{O}(f^*TP),$$

where $f^*TP = TP \times_P N$ is the pullback of the tangent bundle on $P$. The fibres of this vector bundle are

$$(f^*TP)_x = T_{f(x)}P \times \{x\},$$

they have the $\mathbb{C}$-vector field structure of $T_{f(x)}P$, but there is one fibre for each $x \in N$. The sheaves $\theta_N$ and $\theta(f)$ are locally free $\mathcal{O}_N$-modules of ranks $\dim N$ and $\dim P$, respectively.

The differential $df\colon TN \to TP$ induces a morphism of $\mathcal{O}_N$-modules

$$\theta_N \xrightarrow{tf} \theta(f),$$

by taking a section $s\colon U \to TN$ to the section $U \to TP \times_P N$ given by $x \mapsto (df_x(s(x)), x)$. We write the image of this morphism as $tf(\theta_N)$ and let

$$\mathscr{T}^1_{\mathscr{R}_e} f = \frac{\theta(f)}{tf(\theta_N)}.$$

Being the cokernel of a morphism between locally free $\mathcal{O}_N$-modules, $\mathscr{T}^1_{\mathscr{R}_e} f$ is a coherent $\mathcal{O}_N$-module. The support of this sheaf is the set of points

$x \in N$ where the differential $df_x$ fails to be surjective (see the proof of Proposition 4.8 in [53]), that is,

$$\operatorname{supp} \mathscr{T}^1_{\mathscr{R}_e} f = C(f).$$

In particular, whenever $f$ is of finite singularity type, the restriction of $f$ to the support of $\mathscr{T}^1_{\mathscr{R}_e} f$ is finite and then $f_*(\mathscr{T}^1_{\mathscr{R}_e} f)$ is a coherent $\mathcal{O}_P$-module.

Similarly, there is a morphism of $\mathcal{O}_P$-modules

$$\theta_P \to f_*\theta(f),$$

mapping a section $s\colon V \to TP$ to the section $f^{-1}(V) \to f^*TP$ given by $x \mapsto (s \circ f(x), x)$. Taking the composition with the morphism $f_*\theta(f) \to f_*(\mathscr{T}^1_{\mathscr{R}_e} f)$, we obtain a morphism

$$\theta_P \xrightarrow{\omega f} f_*(\mathscr{T}^1_{\mathscr{R}_e} f).$$

The cokernel of this morphism is the $\mathcal{O}_P$-module

$$\mathscr{T}^1_{\mathscr{A}_e} f = \frac{f_* \mathscr{T}^1_{\mathscr{R}_e} f}{\omega f(\theta_P)},$$

which is coherent, provided that $f$ is of finite type.

Still assuming $f$ to be of finite type, we may associate to each $y \in P$ the multi-germ

$$f_y = (f, S), \quad \text{with} \quad S = f^{-1}(y) \cap C(f).$$

Then, the stalks of $\mathscr{T}^1_{\mathscr{A}_e} f$ are precisely

$$(\mathscr{T}^1_{\mathscr{A}_e} f)_y = T^1_{\mathscr{A}_e}(f_y).$$

As we shall see, the sheaf $\mathscr{T}^1_{\mathscr{A}_e} f$ is the right tool to think about the geometry of instabilities.

**Definition 4.1.** For any mapping $f\colon N \to P$ of finite singularity type, the *instability locus* of $f$ is

$$\operatorname{Inst}(f) = \{y \in P \mid f_y \text{ is unstable}\}.$$

Putting together the description of the stalks of $\mathscr{T}^1_{\mathscr{A}_e} f$, the Infinitesimal Stability Theorem 3.3 and the coherence of $\mathscr{T}^1_{\mathscr{A}_e} f$, we obtain the following result:

**Proposition 4.1.** *For any mapping $f\colon N \to P$ of finite singularity type,*

$$\operatorname{Inst}(f) = \operatorname{supp} \mathscr{T}^1_{\mathscr{A}_e} f.$$

*In particular, the instability locus is an analytic subset of $P$.*

*Proof of Mather-Gaffney geometric criterion (Theorem 3.4).* If $f$ is not of finite singularity type, then it does not have isolated instability, by definition. Morever, it also fails to be $\mathscr{A}$-finite (see [53, Lemma 4.3]). As a consequence, we may assume $f \in \mathscr{O}_S(n,p)$ to be of finite singularity type.

Now the germ $f$ has isolated instability if and only if it admits a representative $\tilde{f}$, such that $\operatorname{Inst}(\tilde{f}) = \{0\}$. By Proposition 4.1, we have that

$$\operatorname{Inst}(\tilde{f}) = \operatorname{supp} \mathscr{T}^1_{\mathscr{A}_e} \tilde{f}.$$

Since $\mathscr{T}^1_{\mathscr{A}_e} \tilde{f}$ is coherent and $(\mathscr{T}^1_{\mathscr{A}_e} \tilde{f})_0 = T^1_{\mathscr{A}_e} f$, the condition that $\operatorname{Inst}(\tilde{f}) = \{0\}$ is equivalent to the condition

$$\sqrt{\operatorname{Ann} T^1_{\mathscr{A}_e} f} = \mathfrak{m}_{n+1},$$

where $\mathfrak{m}_{n+1}$ is the maximal ideal of $\mathscr{O}_{n+1}$. This last condition is equivalent to

$$\dim_{\mathbb{C}} T^1_{\mathscr{A}_e} f < \infty. \qquad \square$$

To end this chapter, we explain how the same ideas can be used to define a relative version of the coherent sheaf $\mathscr{T}^1_{\mathscr{A}_e}(f)$, whose support is the bifurcation set $\mathscr{B}$. This is not strictly necessary for the study of the Mond conjecture, but the extra work it requires is minimal.

Given a germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ of finite singularity type, consider a representative of an unfolding of $f$ of the form

$$F\colon U \to B_\epsilon \times T.$$

The mapping $F$ is of the form $F = (f_t, t)$ and is assumed to be of finite singularity type.

The $t$ coordinates of a point $(x,t) \in U \subseteq \mathbb{C}^n \times T$ are regarded as deformation parameters and the $x$ coordinates are the spatial coordinates of the original mapping we are deforming. Therefore, the relative version of the sheaf of vector fields on $U$ is

$$\theta_{U,rel} = \mathcal{O}(T\mathbb{C}^n \times T)_{|U}$$

and the sheaf of relative vector fields along $F$ is

$$\theta_{rel}(F) = \mathcal{O}(F^*(TB_\epsilon \times T)).$$

With the same idea in mind, we consider the relative differential of $F$, that is, we only take derivatives with respect to the $x$ coordinates, and define a subsheaf

$$tF_{rel}(\theta_{U,rel}).$$

We can mimic the rest of the non-relative construction, to define a relative sheaf of $\mathcal{O}_{B_\epsilon \times T}$-modules

$$\mathscr{T}^1_{\mathscr{A}_e,rel} F.$$

which is coherent, because $F$ is assumed to be of finite singularity type. The way it is defined, the stalks of the relative version satisfy

$$(\mathscr{T}^1_{\mathscr{A}_e,rel}F)_{(y,t)} = T^1_{\mathscr{A}_e}((f_t)_y).$$

In particular, the support of $\mathscr{T}^1_{\mathscr{A}_e,rel}F$ is the set of points $(y,t)$, such that the multi-germ $(f_t)_y$ is unstable.

Now assume $f$ to be $\mathscr{A}$-finite and assume that the domain and codomain of $F$ are small enough that every perturbation $f_t$ has at most a finite number of instabilities. Then, the projection

$$B_\epsilon \times T \xrightarrow{\pi} T$$

restricts to a finite mapping on the support of $\mathscr{T}^1_{\mathscr{A}_e,rel}F$, which justifies that the pushforward

$$\pi_* \mathscr{T}^1_{\mathscr{A}_e,rel}F$$

is a coherent $\mathscr{O}_T$-module. The support of this module is the set of parameters $t \in T$ for which $f_t$ fails to be infinitessimally stable. Hence, we may define the *bifurcation set of $F$* as

$$\mathscr{B}(F) = \operatorname{supp} \pi_* \mathscr{T}^1_{\mathscr{A}_e,rel}F.$$

This agrees with the definition of bifurcation set we gave in Definition 3.5, in the sense that the germ at $\mathscr{B}(F)$ at $0 \in T$ is exactly the bifurcation set of the germ $(F, S \times \{0\})$. Incidentally, this proves Lemma 3.1.

# Chapter 5

# The image Milnor number

Here we define the image Milnor number of a germ $f \in \mathscr{O}_S(n, n+1)$. The starting point is a general fibration theorem due to Lê. Given $\eta > 0$, we write

$$D_\eta^* = \{t \in \mathbb{C} \mid 0 < |t| < \eta\}.$$

**Theorem 5.1.** *[34] Let $(\mathcal{X}, 0)$ be an analytic set germ embedded in some ambient space $\mathbb{C}^N$ and let $\varphi\colon (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function. For $0 < \eta \ll \epsilon \ll 1$, the restriction*

$$\varphi\colon \mathcal{X} \cap B_\epsilon \cap \varphi^{-1}(D_\eta^*) \longrightarrow D_\eta^* \tag{5.1}$$

*is a locally $C^0$-trivial fibration.*

The fibration (5.1) is known as the Milnor-Lê fibration since it was considered previously by Milnor in his book [46], in the case of $\mathcal{X} = \mathbb{C}^{n+1}$. The proof of Theorem 5.1 is based on the fact that one can choose a small enough representative $\mathcal{X}$ and an analytic Whitney stratification on it which also satisfies the Thom condition. This implies that (5.1) is a proper stratified submersion (with the induced stratification) and hence, a locally $C^0$-trivial fibration by the Thom-Mather first isotopy lemma [43].

The second result is also due to Lê and says that when $(\mathcal{X}, 0)$ is a complete intersection of dimension $n+1$ and $\varphi\colon (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ has isolated critical point in the stratified sense, then the fibre of (5.1) has the homotopy type of a bouquet of $n$-spheres. Again such result is also present in Milnor's book [46] when $\mathcal{X} = \mathbb{C}^{n+1}$ and in that case, the number of such spheres is precisely the classical Milnor number, denoted by $\mu(\varphi)$. Moreover, $\mu(\varphi)$ can be computed algebraically as

$$\mu(\varphi) = \dim_\mathbb{C} \frac{\mathscr{O}_{n+1}}{J(\varphi)}.$$

We present here a more general version, following Hamm and Lê in [28].

**Definition 5.1.** An analytic set germ $(\mathcal{X},0)$ is called a *Milnor space germ* if $\operatorname{rhd}(\mathcal{X},0)=\dim(\mathcal{X},0)$, where $\operatorname{rhd}(\mathcal{X},0)$ is the rectified homotopy depth of $(\mathcal{X},0)$.

The definition and basic properties of $\operatorname{rhd}(\mathcal{X},0)$ can be found in [28]. For instance, we always have $\operatorname{rhd}(\mathcal{X},0)\leq\dim(\mathcal{X},0)$, so Milnor space germs are those with maximal $\operatorname{rhd}(\mathcal{X},0)$. Another well known property is that any complete intersection $(\mathcal{X},0)$ is a Milnor space germ (see [28]). The main result is the following:

**Theorem 5.2.** *[28] Assume that $(\mathcal{X},0)$ is a Milnor space germ of dimension $n+1$ and that $\varphi\colon(\mathcal{X},0)\to(\mathbb{C},0)$ has isolated critical point in the stratified sense. Then the fibre of (5.1) has the homotopy type of a bouquet of spheres of dimension $n$.*

One more ingredient for the recipe of the image Milnor number comes from Siersma in [69]. Suppose now that $(\mathcal{X},0)$ be a hypersurface in $(\mathbb{C}^{n+1}\times\mathbb{C},0)$ and that our function $\varphi$ is the projection $\pi\colon(\mathcal{X},0)\to(\mathbb{C},0)$ given by $\pi(x,t)=t$. We can see $(X,0)$ as a family of hypersurfaces $X_t=\pi^{-1}(t)$ in $\mathbb{C}^{n+1}$. In this situation it is more convenient to consider in $\mathbb{C}^{n+1}\times\mathbb{C}$ a polydisk of the form $B_\epsilon\times D^*_\eta$. Thus, instead of (5.1), we have:

$$\pi\colon\mathcal{X}\cap(B_\epsilon\times D^*_\eta)\longrightarrow D^*_\eta, \tag{5.2}$$

which is also a locally trivial fibration for $0<\eta\ll\epsilon\ll1$ and is equivalent to (5.1). In particular, if $\pi$ has isolated critical point in the stratified sense, the fibre of (5.2) has the homotopy type of a bouquet of $n$-spheres. The number of such spheres is exactly the $n$th Betti number of the fibre, $\beta_n(X_t\cap B_\epsilon)$, and is given by the following formula:

**Theorem 5.3.** *[69] With the above notation, suppose that $G\colon(\mathbb{C}^{n+1}\times\mathbb{C},0)\to(\mathbb{C},0)$ is a reduced equation of $(\mathcal{X},0)$. Then*

$$\beta_n(X_t\cap B_\epsilon)=\sum_{x\in B_\epsilon\setminus X_t}\mu(g_t;x),$$

*where $\mu(g_t;x)$ is the Milnor number of $g_t(x)=G(x,t)$ at $x$.*

Finally, we arrive to the definition of the image Milnor number. Assume that $f\in\mathscr{O}_S(n,n+1)$ has isolated instability and that either $(n,n+1)$ are nice dimensions or $f$ has corank one. By Theorem 3.2, $f$ admits a stabilisation $F(x,t)=(f_t(x),t)$. We denote by $X_t=f_t(U_t)$ the image of a stable perturbation $f_t\colon U_t\to V$, with $t\neq0$.

**Theorem 5.4.** *[50] With the above notation, for $0<\eta\ll\epsilon\ll1$ and $0<|t|<\eta$, $X_t\cap B_\epsilon$ has the homotopy type of a bouquet of $n$-spheres. The number of such spheres, $\beta_n(X_t\cap B_\epsilon)$, is independent of the choice of $\epsilon,\eta$ and the stable perturbation $f_t$.*

**Definition 5.2.** With the notation of Theorem 5.4, $X_t \cap B_\epsilon$ is called the *disentanglement* and $\beta_n(X_t \cap B_\epsilon)$ is called the *image Milnor number* and is denoted by $\mu_I(f)$.

*Sketch of the proof of Theorem 5.4.* [50]. The image $(\mathcal{X}, 0)$ of the stabilisation $F$ has a well defined stratification by stable types. The strata of such stratification are defined as follows: two points in $\mathcal{X}$ belong to the same stratum $A$ if the corresponding map germs at such points are $\mathscr{A}$-equivalent. At a point $(y,t) \in \mathcal{X} \setminus \{0\}$, the germ of $F$ is a trivial unfolding of the stable germ of $f_t$ at $y$, so it is also stable. The stability is used here to prove three essential facts:

1. that each stratum $A$ is a submanifold,
2. that any pair of strata $(A, B)$ satisfy the Whitney condition,
3. that the projection $\pi(y,t) = t$ restricted to each stratum $A \neq \{0\}$ is a submersion.

The hypothesis that $(n, n+1)$ are nice dimensions or $f$ has corank one implies that the stratification by stable types is locally finite, and hence, it is in fact a stratification. We conclude that $\pi\colon (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ has isolated critical point in the stratified sense. Hence, we get the first part of Theorem 5.4 from Theorem 5.2. The second part of the theorem follows by using an argument involving a versal unfolding and the fact that, provided that $(n, n+1)$ are nice dimensions or $f$ has corank one, the bifurcation set (i.e., the subset of parameters $u$ such that $f_u$ is not locally stable) is a proper analytic subset and therefore does not separate its complement in the base space of the unfolding. □

In order to state the Mond conjecture we need one more definition, namely, the notion of weighted homogeneous mapping.

**Definition 5.3.** Let $w_1, \dots, w_n$ and $d$ be positive integers. A function $f\colon \mathbb{C}^n \to \mathbb{C}$ is *weighted homogeneous* of type $(w_1, \dots, w_n; d)$ if it is a polynomial of the form

$$f = \sum_{\alpha_1 w_1 + \dots + \alpha_n w_n = d} a_\alpha x^\alpha,$$

where we are using the multi-index notation $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for each $\alpha = (\alpha_1, \dots, \alpha_n)$. The numbers $w_1, \dots, w_n$ are called the *weights* and $d$ is called the *degree* of $f$.

A mapping $f = (f_1, \dots, f_p)\colon \mathbb{C}^n \to \mathbb{C}^p$ is called *weighted homogeneous* if all the components $f_i$, with $i = 1, \dots, p$, are weighted homogeneous with the same weights $w_1, \dots, w_n$ but possibly different degrees $d_1, \dots, d_p$.

Finally, a map germ $f \in \mathscr{O}_S(n,p)$ at $S = \{x^{(1)}, \dots, x^{(r)}\}$, is called *weighted homogeneous* if all the branches $f|_{\mathbb{C}^n, x^{(j)}}$, with $j = 1, \dots, r$, are weighted homogeneous with the same degrees $d_1, \dots, d_p$ but possibly different weights $w_1^j, \dots, w_n^j$.

When $f \in \mathscr{O}_S(n, n+1)$ is weighted homogeneous and finite, its image $(X, 0)$ is a weighted homogenous hypersurface in $(\mathbb{C}^{n+1}, 0)$, that is, $(X, 0)$ admits a reduced equation $g \in \mathscr{O}_{n+1}$ which is also weighted homogeneous. It follows from Euler's identity that

$$g = \frac{1}{k}\left(d_1 y_1 \frac{\partial g}{\partial y_1} + \cdots + d_{n+1} y_{n+1} \frac{\partial g}{\partial y_{n+1}}\right),$$

where $d_1, \ldots, d_{n+1}$ are the weights and $k$ is the degree of $g$. In particular, $g \in J(g)$.

Now we have all the ingredients to state the Mond conjecture. As we mention in the introduction, this conjecture appeared in the paper [50] together with a proof of it in the case $n = 2$.

**Conjecture 5.5 (Mond conjecture [50]).** *Let $f \in \mathscr{O}_S(n, n+1)$ be with isolated instability and suppose that either $(n, n+1)$ are nice dimensions or $f$ has corank one. Then,*

$$\operatorname{codim}_{\mathscr{A}_e}(f) \leq \mu_I(f),$$

*with equality if $f$ is weighted homogeneous.*

We give next a couple of examples which have been used frequently by Mond to illustrate the conjecture:

*Example* 5.6 (The Reidemeister moves). The only plane curve singularity which is stable is the node or transverse double point $A_1$ (in Arnold's terminology). Looking at singularities of $\mathscr{A}_e$-codimension 1, we find three types which are called the cusp $A_2$, the tacnode $A_3$ and the triple point $D_4$ (see e.g. [51]). The stabilisations of these three singularities are usually represented by the three Reidemeister moves, as in Figure 5.1. The Reidemeister moves are well known in knot theory, since they are used to recognize when two generic plane projections of space curves belong to the same knot class.

The pictures in Figure 5.1 show the real part of the image of the stable perturbation $f_t$ for $t < 0$ (left) and $t > 0$ (right) and of the unstable germ $f_0$ (center). We observe that, for $t > 0$, the three images present one, two or three nodes, respectively. This is equal to the $\delta$-invariant of the curves and coincides precisely with the number of complex nodes. Moreover, all the images have one 1-cycle, which is also the number of vanishing cycles of their complex images. Observe that the three singularities are weighted homogeneous and hence, $\mu_I(f) = 1$. The three map germs are examples of singularities with real good perturbations in the sense of [39]. This means $f$ admits a stable perturbation $f_t$ given by real polynomials and such that, at least for one of the two values $t < 0$ or $t > 0$, the real image of $f_t$ has the same homology as its complex version.

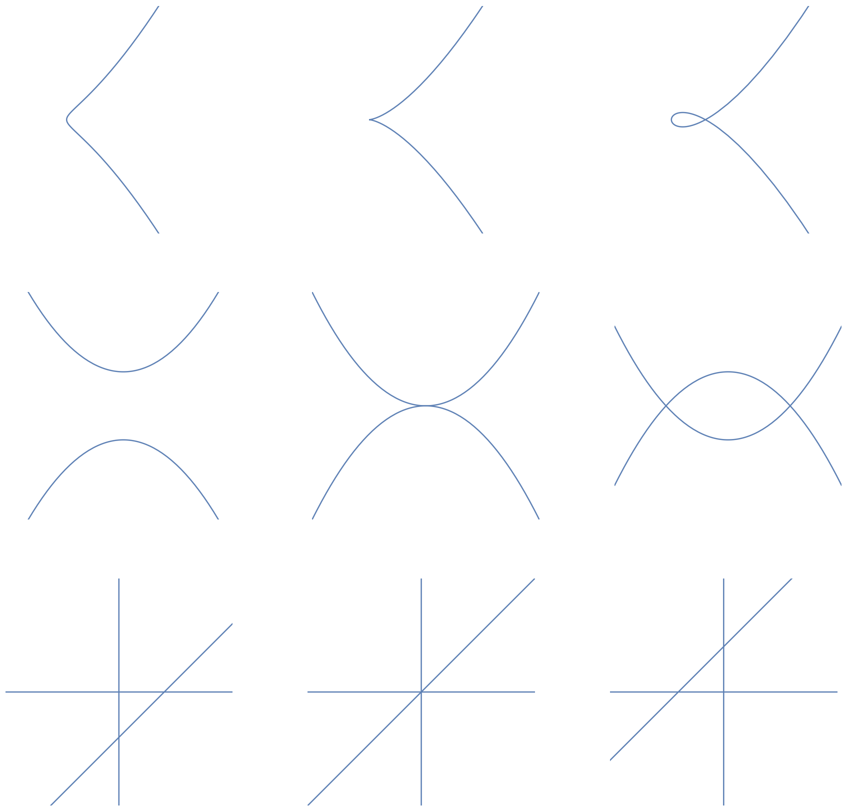

Figure 5.1: The three Reidemeister moves

*Example* 5.7 (The $H_2$ singularity). The map germ $f\colon(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$, given by $f(x,y)=(x,y^3,xy+y^5)$, is the $H_2$ singularity in the Mond list of simple singularities [47] and has $\mathscr{A}_e$-codimension 2. According to [39], a stabilisation of $f$ can be obtained as follows:

$$f_t(x,y)=(x,y^3-ty,xy+y^5-ty^3).$$

Again this is an example of a real good perturbation, since the real projection of the image has the same homology as the complex one. We can see in Figure 5.2 a real picture for $t>0$, in which one appreciates that $f_t$ presents one triple point and two Whitney umbrellas. These are equal to the number of complex triple points and Whitney umbrellas, respectively. The image of $f_t$ has the homotopy type of a bouquet of two 2-spheres (one of them is showed explicitly in the picture and the other one is hidden in the other side). As in the previous example, $f$ is weighted homogeneous and hence, has image Milnor number $\mu_I(f)=2$.

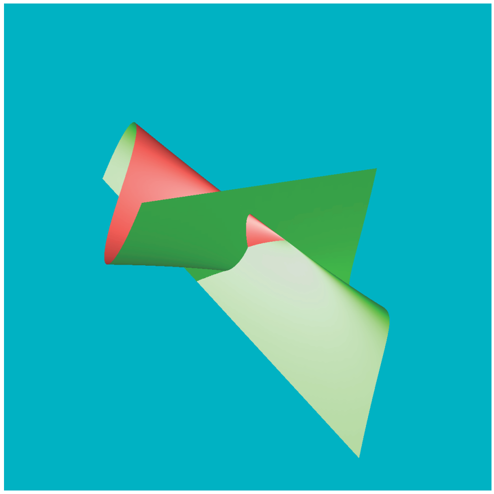

Figure 5.2: Image of the disentanglement of the $H_2$ singularity

We conclude this section with Table 5.1, which serves as a dictionary between the languages of singularities of hypersurfaces with isolated singularity and images of mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ with isolated instability.

Table 5.1: Dictionary between singularities of hypersurfaces and mappings

| **Hypersurface** $(X,0)$ | **Map germ** $f \in \mathscr{O}_S(n,n+1)$ |
|---|---|
| biholomorphism | $\mathscr{A}$-equivalence |
| smooth | stable |
| isolated singularity | isolated instability |
| smoothing | stabilisation |
| $\tau(X,0)$ | $\operatorname{codim}_{\mathscr{A}_e}(f)$ |
| versal deformation | versal unfolding |
| $\dfrac{\mathscr{O}_{n+1}}{J(g)+(g)}$ | $T^1_{\mathscr{A}_e}(f)$ |
| Milnor fibre | disentanglement |
| $\mu(X,0)$ | $\mu_I(f)$ |
| $\dfrac{\mathscr{O}_{n+1}}{J(g)}$ | ??? |

We remark that in the right-hand column, corresponding to singularities of mappings, there is a gap in the last row for the analog of the Jacobian algebra, $\mathscr{O}_{n+1}/J(g)$, whose vector dimension is the Milnor number. In Section 6 we will introduce the Jacobian module $M(g)$, following the approach of [18], whose vector dimension would be equal the image Milnor number, at least in the cases where the Mond conjecture in known to be true.

# Chapter 6

# The Jacobian module

Assume $f \in \mathscr{O}_S(n, n+1)$ is a finite map germ. By the finite mapping theorem (see e.g. [14]), $f$ has a well defined image $(X, 0)$ which is a hypersurface in $(\mathbb{C}^{n+1}, 0)$. Let $g \in \mathscr{O}_{n+1}$ be a reduced equation for $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$. The Jacobian ideal $J(g)$ is the ideal in $\mathscr{O}_{n+1}$ generated by the partial derivatives $\partial g/\partial y_i$ of $g$, $i = 1, \dots, n+1$.

Since $\mathscr{O}_n$ has an $\mathscr{O}_{n+1}$-module structure via the induced ring morphism $f^* : \mathscr{O}_{n+1} \to \mathscr{O}_n$, we can consider $J(g) \cdot \mathscr{O}_n$, which is nothing but the ideal in $\mathscr{O}_n$ generated by $\partial g/\partial y_i \circ f$. Now its inverse image $(f^*)^{-1}(J(g) \cdot \mathscr{O}_n)$ is an ideal in $\mathscr{O}_{n+1}$ containing $J(g)$. Therefore, it makes sense to make the following definition:

**Definition 6.1.** We define the *Jacobian module* as

$$M(g) = \frac{(f^*)^{-1}(J(g) \cdot \mathscr{O}_n)}{J(g)}.$$

Although this is not the original definition of the Jacobian module $M(g)$ given in [18, Definition 3.2], we will see later in Proposition 6.4 that in fact both definitions coincide (see also [18, Proposition 5.1]). In order to see this, we need to introduce the double point spaces of $f$. We start with the multiple point spaces in the target of $f$ with the analytic structure given by Fitting ideals, following Mond and Pellikaan [55].

We recall that a presentation of an $R$-module $M$ is an exact sequence

$$R^p \xrightarrow{\lambda} R^q \longrightarrow M \longrightarrow 0.$$

The morphism $\lambda$ is given by a matrix $\lambda = (\lambda_{ij})$ of size $q \times p$ and with entries in $R$. The $k$'th Fitting ideal of $M$, denoted by $F_k^R(M)$, is the ideal of $R$ generated by the $(q-k) \times (q-k)$ minors of $\lambda$. This makes sense if $1 \leq q-k \leq \min\{p, q\}$. By convention, we put $F_k^R(M) = R$ when $q-k \leq 0$ or $F_k^R(M) = 0$ when $q-k > \min\{p, q\}$. We refer to [27] for basic properties of Fitting ideals, for instance, the fact that they are independent of the choice of the presentation.

If $f \in \mathscr{O}_S(n, n+1)$ is finite, then $\mathscr{O}_n$ is finitely generated as $\mathscr{O}_{n+1}$-module, by the Weierstrass preparation theorem (see e.g. [14]). Since $\mathscr{O}_{n+1}$ is Noetherian, any finitely generated module admits a presentation. In particular, $\mathscr{O}_n$ has well defined Fitting ideals as $\mathscr{O}_{n+1}$-module.

**Definition 6.2.** The *k'th Fitting ideal* of $f$ is defined as the $k$'th Fitting ideal of $\mathscr{O}_n$ as $\mathscr{O}_{n+1}$-module:

$$F_k(f) = F_k^{\mathscr{O}_{n+1}}(\mathscr{O}_n).$$

We also define the *k'th target multiple point space* as the complex subspace germ of $(\mathbb{C}^{n+1}, 0)$ given by $M_k(f) = V(F_{k-1}(f))$, with local ring $\mathscr{O}_{M_k(f)} = \mathscr{O}_{n+1}/F_{k-1}(f)$.

The name for $M_k(f)$ comes from the fact that the underlying set germ of $M_k(f)$ is the set germ in $(\mathbb{C}^{n+1}, 0)$ of points with at least $k$ preimages, counted with multiplicity. More precisely, we have (see [55, Proposition 1.5]):

**Proposition 6.1.** *As a set germ, $M_k(f)$ is given by the points $y$ in a neighbourhood of $0$ in $\mathbb{C}^{n+1}$ such that*

$$\sum_{f(x)=y} \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathbb{C}^n,x}}{f^*\mathfrak{m}_{\mathbb{C}^{n+1},y}} \geq k.$$

*In particular, $M_1(f)$ coincides with $(X, 0)$, the image of $f$.*

When $f$ is generically one-to-one, then $M_1(f)$ is reduced, so we have the equality $M_1(f) = (X, 0)$ also as complex space germs (see [55, Proposition 3.1]). The double point space $M_2(f)$ is usually denoted by $f(D(f))$ and coincides, as a set germ, with the singular locus of the image $(X, 0)$. In fact, $f(D(f))$ is given by the points $y$ in a neighbourhood of 0 in $\mathbb{C}^{n+1}$ such that either $y = f(x)$ and $x$ is a non-immersive point of $f$ or $y = f(x) = f(x')$, with $x \neq x'$.

We also have a double point space in the source, with analytic structure given by the conductor ideal. The restriction $\overline{f}\colon (\mathbb{C}^n, S) \to (X, 0)$ is the normalization map, hence the induced morphism $\overline{f}^*\colon \mathscr{O}_{X,0} \to \mathscr{O}_n$ is a monomorphism and we may regard $\mathscr{O}_{X,0}$ as a subring of $\mathscr{O}_n$.

**Definition 6.3.** We denote the conductor of $\mathscr{O}_{X,0}$ in $\mathscr{O}_n$ by $C(f)$, that is,

$$C(f) = \{h \in \mathscr{O}_{X,0} \mid h \cdot \mathscr{O}_n \subseteq \mathscr{O}_{X,0}\}.$$

The *source double point space* is the the complex subspace germ of $(\mathbb{C}^n, S)$ given by $D(f) = V(C(f))$, with semi-local ring $\mathscr{O}_{D(f)} = \mathscr{O}_n/C(f)$.

We recall that the conductor $C(f)$ has the property that it is the largest ideal of $\mathscr{O}_{X,0}$ which is also an ideal in $\mathscr{O}_n$. The conductor $C(f)$ can be computed easily by means of the following result due to Piene [66] (see also Bruce and Marar [6]):

**Proposition 6.2.** *Suppose $f \in \mathscr{O}_S(n, n+1)$ is finite and generically one-to-one. There exists a unique $\lambda \in \mathscr{O}_n$ such that*

$$\frac{\partial g}{\partial y_i} \circ f = (-1)^i \lambda \det(df_1, \dots, df_{i-1}, df_{i+1}, \dots, df_{n+1}), \quad i = 1, \dots, n+1.$$

*Moreover, $C(f)$ is generated in $\mathscr{O}_n$ by $\lambda$.*

The relationship between the ideals $C(f)$ and $F_1(f)$ is given in the following proposition due to Mond and Pellikaan [55, Proposition 3.5]:

**Proposition 6.3.** *With the assumptions of Proposition 6.2,*

$$C(f) = F_1(f) \cdot \mathscr{O}_{X,0}.$$

It follows that, as complex space germs, $D(f)$ is the inverse image of $f(D(f))$ by $f$. In particular, the underlying set germ of $D(f)$ is given by points $x$ in a neighbourhood of $S$ in $\mathbb{C}^n$ such that either $f$ is non-immersive at $x$ or $f(x) = f(x')$ for some $x' \neq x$. Moreover, it is also clear from Propositions 6.2 and 6.3 that $f^*$ induces an epimorphism

$$\frac{F_1(f)}{J(g)} \longrightarrow \frac{C(f)}{J(g) \cdot \mathscr{O}_n}$$

whose kernel is precisely $M(g)$ as defined in Definition 6.1. This is in fact the original definition given for $M(g)$ in [18, Definition 3.2]. This can be also expressed by means of an exact sequence as follows:

**Proposition 6.4.** *With the assumptions of Proposition 6.2, we have an exact sequence of $\mathscr{O}_{n+1}$-modules:*

$$0 \longrightarrow M(g) \longrightarrow \frac{F_1(f)}{J(g)} \longrightarrow \frac{C(f)}{J(g) \cdot \mathscr{O}_n} \longrightarrow 0. \tag{6.1}$$

Another important step for the geometrical interpretation of the Jacobian module is given in a second exact sequence, which involves the module $T^1_{\mathscr{A}_e} f$ that controls the $\mathscr{A}_e$-codimension.

**Proposition 6.5.** *Suppose $f \in \mathscr{O}_S(n, n+1)$ is finite, generically one-to-one and the set of non-immersive points has codimension $\geq 2$. There is an exact sequence of $\mathscr{O}_{n+1}$-modules:*

$$0 \longrightarrow K(g) \longrightarrow M(g) \longrightarrow T^1_{\mathscr{A}_e} f \longrightarrow 0,$$

*where* $K(g) = (J(g) + (g))/J(g)$*. In particular, if* $\dim_{\mathbb{C}} M(g) < \infty$*, then* $f$ *is* $\mathscr{A}$*-finite and*

$$\dim_{\mathbb{C}} M(g) = \operatorname{codim}_{\mathscr{A}_e}(f) + \dim_{\mathbb{C}} K(g).$$

*Proof.* We use [50, Proposition 2.1]. A priori we need the hypothesis is that $f$ is $\mathscr{A}$-finite and $n \geq 2$, but a careful revision of the proof shows that the only necessary assumptions are that $f \in \mathscr{O}_S(n, n+1)$ is finite, generically one-to-one and the set of non-immersive points has codimension $\geq 2$. Of course, such conditions are satisfied when $f$ is $\mathscr{A}$-finite and $n \geq 2$. Thus, by [50, Proposition 2.1] we have an isomorphism

$$T^1_{\mathscr{A}_e} f \longrightarrow \frac{J(g)\cdot \mathscr{O}_n}{J(g)\cdot \mathscr{O}_{X,0}},$$

induced by the evaluation of vector fields $\xi \mapsto \xi(g)$. The exact sequence is now given in [18, Proposition 3.3] in the following way:

$$0 \longrightarrow K(g) \longrightarrow M(g) \longrightarrow \frac{J(g)\cdot \mathscr{O}_n}{J(g)\cdot \mathscr{O}_{X,0}} \longrightarrow 0. \qquad \square$$

The following corollary is a direct consequence of Definition 6.1 and Proposition 6.5 and gives an easy procedure to compute the $\mathscr{A}_e$-codimension of a map germ by using a computer algebra system, such as Singular [73].

**Corollary 6.6.** *With the assumptions of Proposition 6.5, we have*

$$\operatorname{codim}_{\mathscr{A}_e}(f) = \dim_{\mathbb{C}} \frac{(f^*)^{-1}(J(g)\cdot \mathscr{O}_n)}{J(g) + (g)}.$$

Another important consequence is that the relationship between the stability and the Jacobian module can be strengthened a little more when either $(n, n+1)$ are nice dimensions or $f$ has corank one. In fact, in both cases any stable singularity is weighted homogeneous, up to $\mathscr{A}$-equivalence. The case of corank one is easy, since any stable singularity of corank one is $\mathscr{A}$-equivalent to the generalised Whitney umbrella, which is weighted homogeneous. The fact that the stable singularities in the range of the nice dimensions $(n, p)$ are weighted homogeneous, up to $\mathscr{A}$-equivalence, comes from Mather's classification of contact algebras. We refer to [53, Theorem 7.6] for a detailed and organised account of the proof of this fact. As a consequence, we get:

**Corollary 6.7.** *With the assumptions of Proposition 6.5, suppose that either* $(n, n+1)$ *are nice dimensions or* $f$ *has corank one. Then:*

1. $f$ *is stable if and only if* $M(g) = 0$*,*
2. $f$ *is* $\mathscr{A}$*-finite if and only if* $\dim_{\mathbb{C}} M(g) < \infty$*.*

# Chapter 7

# The relative Jacobian module

The next step is to define a relative version of the Jacobian module for unfoldings which specialises to $M(g)$ when we make the parameters $u = 0$. In other words, we look for a module $M_{\mathrm{rel}}(G)$ associated to the defining equation $G$ of the image of an unfolding, with the property that

$$\frac{M_{\mathrm{rel}}(G)}{I \cdot M_{\mathrm{rel}}(G)} \cong M(g), \tag{7.1}$$

where $I = (u_1, \ldots, u_r)$ is the ideal in $\mathscr{O}_{n+1+r}$ generated by the parameters. We can consider $M_{\mathrm{rel}}(G)$ as $\mathscr{O}_r$-module via the morphism $\pi^*\colon \mathscr{O}_r \to \mathscr{O}_{n+1+r}$ induced by the projection $\pi : \mathbb{C}^{n+1} \times \mathbb{C}^r \to \mathbb{C}^r$, $(y, u) \to u$. Then, we can see the left hand side of (7.1) as a tensor product

$$\frac{M_{\mathrm{rel}}(G)}{I \cdot M_{\mathrm{rel}}(G)} \cong M_{\mathrm{rel}}(G) \otimes \frac{\mathscr{O}_r}{\mathfrak{m}_r}.$$

Let $F(x, u) = (f_u(x), u)$ be an $r$-parameter unfolding of a finite germ $f \in \mathscr{O}_S(n, n+1)$. Since $F$ is also finite, it has a well defined image which is a hypersurface $(\mathcal{X}, 0)$ in $(\mathbb{C}^{n+1} \times \mathbb{C}^r, 0)$. We choose $G \in \mathscr{O}_{n+1+r}$, such that $G(y, u) = 0$ is a reduced equation for $(\mathcal{X}, 0)$.

The *relative Jacobian ideal* is the ideal $J_{\mathrm{rel}}(G)$ in $\mathscr{O}_{n+1+r}$ generated by $\partial G/\partial y_i$, $i = 1, \ldots, n+1$. So,

$$J_{\mathrm{rel}}(G) \cdot \mathscr{O}_{n+1} = J(g),$$

where $g(y) = G(y, 0)$. Here, $\mathscr{O}_{n+1}$ is considered as $\mathscr{O}_{n+1+r}$-module via the morphism $j^*\colon \mathscr{O}_{n+1+r} \to \mathscr{O}_{n+1}$ induced by the standard inclusion $j\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \times \mathbb{C}^r$, $y \mapsto (y, 0)$. Now the definition of $M_{\mathrm{rel}}(G)$ is analogous to that of $M(g)$ in Definition 6.1:

**Definition 7.1.** The *relative Jacobian module* is

$$M_{\text{rel}}(G) = \frac{(F^*)^{-1}(J_{\text{rel}}(G)\cdot \mathscr{O}_{n+r})}{J_{\text{rel}}(G)}.$$

As it happened with $M(g)$, this is not the original definition given in [18, Definition 4.1], but the equivalence between the two definitions is given in the next proposition, which is obvious from Proposition 6.3.

**Proposition 7.1.** *Let $f \in \mathscr{O}_S(n, n+1)$ be finite and generically one-to-one. We have an exact sequence of $\mathscr{O}_{n+1+r}$-modules:*

$$0 \longrightarrow M_{\text{rel}}(G) \longrightarrow \frac{F_1(F)}{J_{\text{rel}}(G)} \longrightarrow \frac{C(F)}{J_{\text{rel}}(G)\cdot \mathscr{O}_{n+r}} \longrightarrow 0. \quad (7.2)$$

It is well known that the Fitting ideals are well behaved under change of ring (see e.g. [27, Lemma 7.2.5]), which implies that $F_1(F)\cdot\mathscr{O}_{n+1} = F_1(f)$ and the equality $C(F)\cdot \mathscr{O}_n = C(f)$ follows easily from Piene's result (see Proposition 6.2). Here, the $\cdot$ means the product in the corresponding induced structure of modules via the standard inclusions. One could think that after tensoring the exact sequence (7.2) with $\mathscr{O}_r/\mathfrak{m}_r$ we should obtain the exact sequence (6.1) with $M_{\text{rel}}(G)\otimes\mathscr{O}_r/\mathfrak{m}_r$ in the right hand side term. This would imply that

$$\frac{M_{\text{rel}}(G)}{I\cdot M_{\text{rel}}(G)} \cong M_{\text{rel}}(G)\otimes\frac{\mathscr{O}_r}{\mathfrak{m}_r} \cong M(g).$$

Unfortunately, we cannot use this argument, a priori, for two reasons: in general, the quotient of modules is not well behaved under tensor product and the tensor product functor is not exact. We need first some lemmas.

**Lemma 7.2.** *Suppose $f \in \mathscr{O}_S(n, n+1)$ is finite and generically one-to-one. Then $\mathscr{O}_{n+1}/F_1(f)$ is Cohen-Macaulay of dimension $n-1$.*

The proof is given in [55, Theorem 3.4] for mono-germs, based on the fact that $\mathscr{O}_n$ admits a presentation with a square matrix $\lambda = (\lambda_{ij})$ of size $(m+1)\times(m+1)$, with respect to a system of generators $1 = h_0, h_1, \dots, h_m$ of $\mathscr{O}_n$ as $\mathscr{O}_{n+1}$-module. The authors prove that $F_1(f)$ is equal to the ideal generated by the maximal minors of the matrix $\hat\lambda$ obtained by deleting the first row of $\lambda$. Since $\hat\lambda$ has size $m\times(m+1)$ and $\dim f(D(f)) \le n-1$, then $\mathscr{O}_{n+1}/F_1(f)$ is determinantal, and hence, Cohen-Macaulay of dimension $n-1$. However, this argument does not work for multi-germs, as one can see easily with the example of the triple point in $\mathbb{C}^3$. The extension for multi-germs is found in [60].

The hypothesis of Lemma 7.2 are satisfied when $f$ is $\mathscr{A}$-finite and $F$ is any unfolding. In this case, $\mathscr{O}_{n+1+r}/F_1(F)$ is Cohen-Macaulay of dimension $n-1+r$. By using this fact, it can be proved that

$$\frac{F_1(F)}{J_{\mathrm{rel}}(G)} \otimes \frac{\mathscr{O}_r}{\mathfrak{m}_r} \cong \frac{F_1(f)}{J(g)}, \tag{7.3}$$

(see [18, Proposition 4.4] for details). The same idea works for the right hand side of (7.2): $C(F)$ is a principal ideal in $\mathscr{O}_{n+r}$, so $\mathscr{O}_{n+r}/C(F)$ is also Cohen-Macaulay of dimension $n-1+r$. Hence,

$$\frac{C(F)}{J_{\mathrm{rel}}(G)\cdot \mathscr{O}_{n+r}} \otimes \frac{\mathscr{O}_r}{\mathfrak{m}_r} \cong \frac{C(f)}{J(g)\cdot \mathscr{O}_n}. \tag{7.4}$$

**Lemma 7.3.** *Suppose $f \in \mathscr{O}_S(n, n+1)$ is finite, generically one-to-one and let $F$ be any unfolding. Then,*

$$J_{\mathrm{rel}}(G)\cdot \mathscr{O}_{n+r} = J(G)\cdot \mathscr{O}_{n+r}.$$

*Proof.* Suppose $F(x,u) = (f_u(x), u)$, then the Jacobian matrix of $F$ has the form

$$dF = \begin{pmatrix} df_u & * \\ 0 & I_r \end{pmatrix}$$

where $df_u$ is the Jacobian matrix of $f_u$ with respect to the variables $x_1, \dots, x_n$. We denote by $M_1, \dots, M_{n+1}, M'_1, \dots, M'_r$ the $(n+r)$-minors of $dF$ in such a way that $M_1, \dots, M_{n+1}$ are the $n$-minors of $df_u$. Then $M'_1, \dots, M'_r$ can be generated from $M_1, \dots, M_{n+1}$, that is, for each $i = 1, \dots, r$, we can write

$$M'_i = \sum_{j=1}^{n+1} a_{ij} M_j,$$

for some $a_{ij} \in \mathscr{O}_{n+r}$. Now we use Piene's Proposition 6.2:

$$\frac{\partial G}{\partial u_i} \circ F = \Lambda M'_i, \quad \frac{\partial G}{\partial y_j} \circ F = \Lambda M'_j,$$

where $\Lambda$ is the generator of the conductor ideal $C(F)$. We get

$$\frac{\partial G}{\partial u_i} \circ F = \sum_{j=1}^{n+1} a_{ij} \frac{\partial G}{\partial y_j}. \qquad \square$$

**Theorem 7.4.** *Suppose $f \in \mathscr{O}_S(n, n+1)$ is $\mathscr{A}$-finite with $n \geq 2$ and let $F$ be any unfolding. Then,*

$$\frac{M_{\mathrm{rel}}(G)}{I \cdot M_{\mathrm{rel}}(G)} \cong M(g),$$

*where $I = (u_1, \dots, u_r)$ is the ideal in $\mathscr{O}_{n+1+r}$ generated by the parameters.*

*Proof.* The short exact sequence (7.2) induces a long exact Tor-sequence as follows (see e.g. [27, Proposition 7.1.2]):

$$\dots \longrightarrow \operatorname{Tor}_1^{\mathscr{O}_r}\left(\frac{C(F)}{J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r}}, \frac{\mathscr{O}_r}{\mathfrak{m}_r}\right)$$

$$\longrightarrow M_{\mathrm{rel}}(G)\otimes\frac{\mathscr{O}_r}{\mathfrak{m}_r} \longrightarrow \frac{F_1(f)}{J(g)} \longrightarrow \frac{C(f)}{J(g)\cdot\mathscr{O}_n} \longrightarrow 0.$$

We claim that $C(F)/J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r}$ is $\mathscr{O}_r$-flat. In fact, by 7.3

$$\frac{C(F)}{J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r}} = \frac{C(F)}{J(G)\cdot\mathscr{O}_{n+r}} \cong \frac{\mathscr{O}_{n+r}}{R(F)},$$

where $R(F)$ is the ramification ideal of $F$, that is, the ideal generated by the maximal minors of the Jacobian matrix of $F$ and the isomorphism in the right hand side is induced by the multiplication by $\lambda$ in Piene's Proposition 6.2.

The zero locus of $R(F)$ is precisely the set of non-immersive points, which has dimension $\leq n-2+r$ (because $f$ is $\mathscr{A}$-finite and $n \geq 2$). Since the Jacobian matrix has size $(n+1+r)\times(n+r)$, it follows that $\mathscr{O}_{n+r}/R(F)$ is determinantal, and hence, Cohen-Macaulay of dimension $n-2+r$. Moreover, $\mathscr{O}_r$ is regular and

$$\frac{\mathscr{O}_{n+r}}{R(F)}\otimes\frac{\mathscr{O}_r}{\mathfrak{m}_r} \cong \frac{\mathscr{O}_n}{R(f)}$$

has dimension $n-2$, hence $\mathscr{O}_{n+r}/R(F)$ is $\mathscr{O}_r$-flat by [27, Theorem 7.8.2].

Now, by [27, Theorem 7.4.2] the $\mathscr{O}_r$-flatness of $C(F)/J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r}$ implies

$$\operatorname{Tor}_1^{\mathscr{O}_r}\left(\frac{C(F)}{J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r}}, \frac{\mathscr{O}_r}{\mathfrak{m}_r}\right) = 0,$$

and from the exact sequence we get

$$\frac{M_{\mathrm{rel}}(G)}{I\cdot M_{\mathrm{rel}}(G)} \cong M_{\mathrm{rel}}(G)\otimes\frac{\mathscr{O}_r}{\mathfrak{m}_r} \cong M(g). \qquad \square$$

# Chapter 8

# The image Milnor number as a multiplicity

We proved in Corollary 6.7 that if either $(n, n+1)$ are nice dimensions or $f$ has corank one, then $\dim_{\mathbb{C}} M(g) < \infty$ when $f$ is $\mathscr{A}$-finite. In this situation, we have from Theorem 7.4 that

$$\dim_{\mathbb{C}} \frac{M_{\mathrm{rel}}(G)}{I \cdot M_{\mathrm{rel}}(G)} = \dim_{\mathbb{C}} M(g) < \infty, \tag{8.1}$$

and thus, $M_{\mathrm{rel}}(G)$ is finitely generated over $\mathscr{O}_r$ by Nakayama's lemma. It makes sense to consider its Samuel multiplicity with respect to the maximal ideal $\mathfrak{m}_r$, denoted by $e(\mathfrak{m}_r; M_{\mathrm{rel}}(G))$. We refer to [45, §14] for the definition and basic properties of the Samuel multiplicity $e(\mathfrak{q}; M)$ of a finitely generated module $M$ over a local ring $R$ with respect to an $\mathfrak{m}$-primary ideal $\mathfrak{q}$. We use such a multiplicity to state the main theorem of this chapter:

**Theorem 8.1.** *Suppose $f \in \mathscr{O}_S(n, n+1)$ is $\mathscr{A}$-finite with $n \geq 2$ and that either $(n, n+1)$ are nice dimensions or $f$ has corank one. Let $F$ be any $r$-parameter unfolding of $f$ with bifurcation set $\mathscr{B}(F) \subsetneq (\mathbb{C}^r, 0)$. Then,*

$$e(\mathfrak{m}_r; M_{\mathrm{rel}}(G)) = \mu_I(f).$$

*Proof.* Since $\mathscr{B}(F) \subsetneq (\mathbb{C}^r, 0)$, there exists a line $L$ in $\mathbb{C}^r$ such that $\mathscr{B}(F) \cap L = \{0\}$. This is one of the consequences of the Noether normalisation theorem (see e.g. [14, Corollary 3.3.19]). We choose any $u_0 \in L \setminus \{0\}$. If $F(x, u) = (f_u(x), u)$, now we construct a stabilisation $F'$ just by taking $F'(x, t) = (f_{tu_0}(x), t)$. We also take $0 < \eta \ll \epsilon \ll 1$ as in Theorem 5.4, so $\mu_I(f) = \beta_n(X_u \cap B_\epsilon)$, for $u = tu_0$ and $0 < |t| < \eta$.

On the other hand, because of (8.1) we have conservation of multiplicity (see [53, Corollary E.5]), that is,

$$e(\mathfrak{m}_r; M_{\text{rel}}(G)) = \sum_{y \in B_\epsilon} e\left(\mathfrak{m}_{\mathbb{C}^r,u}; \mathscr{M}_{\text{rel}}(G)_{(y,u)}\right),$$

where $\mathscr{M}_{\text{rel}}(G)_{(y,u)}$ is the relative Jacobian module of the multi-germ of $F$ at $(y,u)$. Of course, the sum on the right hand side is finite, since we only consider points $y \in B_\epsilon$ such that $\mathscr{M}_{\text{rel}}(G)_{(y,u)} \neq 0$ and this is a finite set by (8.1).

When $y \in X_u \cap B_\epsilon$, $f_u$ is stable at $y$, so $\mathscr{M}(g_u)_y = 0$, where $g_u$ is the function $g_u(-) = G(-,u)$. By Theorem 7.4,

$$\frac{\mathscr{M}_{\text{rel}}(G)_{(y,u)}}{\mathfrak{m}_{\mathbb{C}^r,u} \cdot \mathscr{M}_{\text{rel}}(G)_{(y,u)}} = \mathscr{M}(g_u)_y = 0,$$

and hence $e\left(\mathfrak{m}_{\mathbb{C}^r,u}; \mathscr{M}_{\text{rel}}(G)_{(y,u)}\right) = 0$, by [53, Proposition C.15].

Otherwise, if $y \in B_\epsilon \setminus X_u$, then the right hand side of (7.2) is 0 at $(y,u)$, since such a module is supported only at the points in the image of $F$. Moreover, the zero locus of $F_1(F)$ is also contained in the image of $F$. Therefore,

$$\mathscr{M}_{\text{rel}}(G)_{(y,u)} \cong \left(\frac{F_1(F)}{J_{\text{rel}}(G)}\right)_{(y,u)} = \frac{\mathscr{O}_{\mathbb{C}^{n+1}\times\mathbb{C}^r,(y,u)}}{J_{\text{rel}}(G)_{(y,u)}}. \tag{8.2}$$

But $J_{\text{rel}}(G)_{(y,u)}$ is generated by $n+1$ elements and (8.2) has dimension $\leq r$. Hence, (8.2) is a complete intersection and thus, Cohen-Macaulay of dimension $r$. By [53, Proposition C.15],

$$\begin{aligned} e\left(\mathfrak{m}_{\mathbb{C}^r,u}; \mathscr{M}_{\text{rel}}(G)_{(y,u)}\right) &= \dim_{\mathbb{C}} \frac{\mathscr{M}_{\text{rel}}(G)_{(y,u)}}{\mathfrak{m}_{\mathbb{C}^r,u} \cdot \mathscr{M}_{\text{rel}}(G)_{(y,u)}} \\ &= \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathbb{C}^{n+1}\times\mathbb{C}^r,(y,u)}}{J_{\text{rel}}(G)_{(y,u)} + \mathfrak{m}_{\mathbb{C}^r,u}} \\ &= \dim_{C} \frac{\mathscr{O}_{\mathbb{C}^{n+1}}}{J(g_u)_y} \\ &= \mu(g_u; y), \end{aligned}$$

the Milnor number of $g_u$ at $y$. Finally, Siersma's Theorem 5.3 implies

$$e(\mathfrak{m}_r; M_{\text{rel}}(G)) = \sum_{y \in B_\epsilon \setminus X_u} \mu(g_u; y) = \beta_n(X_u \cap B_\epsilon) = \mu_I(f). \qquad \square$$

In the proof of the theorem we have used a well known property of the multiplicity, namely that

$$e(\mathfrak{m}; M) \leq \dim_{\mathbb{C}} \frac{M}{\mathfrak{m} \cdot M},$$

with equality if and only if $M$ is Cohen-Macaulay (see [53, Proposition C.15]). This gives the following consequence:

**Corollary 8.2.** *With the hypothesis of Theorem 8.1, we have*

$$\mu_I(f) \leq \dim_{\mathbb{C}} M(g),$$

*with equality if and only if* $M_{\text{rel}}(G)$ *is Cohen-Macaulay.*

It follows that the fact that $M_{\text{rel}}(G)$ is Cohen-Macaulay is independent of the choice of the unfolding $F$ such that $\mathscr{B}(F) \subsetneq (\mathbb{C}^r, 0)$ and only depends on the map germ $f$ and the defining equation $g$ of its image. This also suggests to state the following strong version of Mond conjecture:

**Conjecture 8.3 (Strong Mond conjecture).** *Let* $f \in \mathscr{O}_S(n, n+1)$ *be with isolated instability and suppose that either* $(n, n+1)$ *are nice dimensions or* $f$ *has corank one. Then,*

$$\mu_I(f) = \operatorname{codim}_{\mathscr{A}_e}(f) + \dim_{\mathbb{C}} K(g),$$

*where* $g$ *is a reduced equation of the image of* $f$.

Obviously, this strong version implies the Mond conjecture. Moreover, the strong version is equivalent to the fact that $\mu_I(f) = \dim_{\mathbb{C}} M(g)$ or that $M_{\text{rel}}(G)$ is Cohen-Macaulay, for some (and hence for any) unfolding $F$ such that $\mathscr{B}(F) \subsetneq (\mathbb{C}^r, 0)$, by Proposition 6.5 and Corollary 8.2.

In the case that $(X, 0)$ is a hypersurface with isolated singularity and $g$ is a reduced equation, the Milnor number is given by

$$\mu(X, 0) = \dim_{\mathbb{C}} \mathscr{O}_{n+1}/J(g),$$

so the dimension of the $\mathbb{C}$-vector space $\mathscr{O}_{n+1}/J(g)$ is independent of the choice of the equation $g$. Such statement is not true in general if we remove the isolated singularity condition. If we had a positive answer to the strong Mond conjecture, then we could also claim that the dimension of the $\mathbb{C}$-vector space $M(g)$ is independent of the choice of the equation $g$ of the image of $f$ when it has isolated instabillity.

# Chapter 9

# Proof of the Mond conjecture for surfaces

In this chapter we explain how to prove the Mond conjecture for $\mathscr{A}$-finite map germs $f \in \mathscr{O}_S(2,3)$. As we mention in Chapter 2, the first proof of this result was found by de Jong and Pellikaan (unpublished), but the first published proofs are due to Jong and van Straten [15] and, independently, Mond [50]. Here we propose a different argument, based on the fact that the relative Jacobian module $M_{\mathrm{rel}}(G)$ is Cohen-Macaulay. Our proof uses an argument due to Pellikaan [64] which shows that certain modules of the form $I/J$, for ideals $J \subseteq I$ are Cohen-Macaulay. A more general version of our proof which also works for mappings on ICIS can be found in the recent paper [19] (see Chapter 10).

We first present a lemma which is valid also for any $n \geq 2$.

**Lemma 9.1.** *Let $f \in \mathscr{O}_S(n, n+1)$ be such that $\dim_{\mathbb{C}} M(g) < \infty$ with $n \geq 2$ and let $F$ be an $r$-parameter unfolding. If $\operatorname{depth}(F_1(F)/J_{\mathrm{rel}}(G)) \geq r$, then $M_{\mathrm{rel}}(G)$ is Cohen-Macaulay of dimension $r$.*

*Proof.* This is based on the depth lemma [17, Corollary 18.6], applied to the short exact sequence (7.2). We get

$$\operatorname{depth} M_{\mathrm{rel}}(G) \geq \min\left\{\operatorname{depth}\left(\frac{F_1(F)}{J_{\mathrm{rel}}(G)}\right), \operatorname{depth}\left(\frac{C(F)}{J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r}}\right)+1\right\}.$$

But, as we saw in the proof of Theorem 7.4, $C(F)/(J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r})$ is isomorphic to $\mathscr{O}_{n+r}/R(F)$ and hence, is Cohen-Macaulay of dimension $n-2+r$. Hence,

$$\operatorname{depth}\left(\frac{C(F)}{J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r}}\right) = \dim\left(\frac{C(F)}{J_{\mathrm{rel}}(G)\cdot\mathscr{O}_{n+r}}\right) = n-2+r \geq r.$$

We deduce that $M_{\rm rel}(G)$ has also depth $\geq r$. On the other hand, $\dim_{\mathbb{C}} M(g) < \infty$ implies that $\operatorname{depth} M_{\rm rel}(G) \leq \dim M_{\rm rel}(G) \leq r$, so necessarily

$$\operatorname{depth} M_{\rm rel}(G) = \dim M_{\rm rel}(G) = r$$

and $M_{\rm rel}(G)$ is Cohen-Macaulay. □

**Theorem 9.2.** *Let $f \in \mathscr{O}_S(2,3)$ be $\mathscr{A}$-finite and let $F$ be an $r$-parameter unfolding. Then $M_{\rm rel}(G)$ is Cohen-Macaulay of dimension $r$.*

*Proof.* By Lemma 9.1, we only need to prove that $\operatorname{depth}(F_1(F)/J_{\rm rel}(G)) \geq r$. We follow the argument of [64, Proposition 3.4] to show that $F_1(F)/J_{\rm rel}(G)$ is in fact Cohen-Macaulay of dimension $r$.

We know by Lemma 7.2 that $\mathscr{O}_{3+r}/F_1(F)$ is Cohen-Macaulay of codimension 2. By the Hilbert-Burch theorem [17, Theorem 20.15], it has a free resolution of length 2 as follows:

$$0 \longrightarrow \mathscr{O}_{3+r}^m \xrightarrow{A_2} \mathscr{O}_{3+r}^{m+1} \xrightarrow{A_1} \mathscr{O}_{3+r} \longrightarrow \mathscr{O}_{3+r}/F_1(F) \longrightarrow 0.$$

This implies that $F_1(F)$ admits a presentation

$$\mathscr{O}_{3+r}^m \xrightarrow{A_2} \mathscr{O}_{3+r}^{m+1} \xrightarrow{A_1} F_1(F) \longrightarrow 0. \tag{9.1}$$

Now we only have to add the generators of $J_{\rm rel}(G)$ as relations in (9.1) in order to obtain a presentation of the quotient $F_1(F)/J_{\rm rel}(G)$. In other words, let $\alpha_1, \dots, a_{m+1}$ be the generators of $F_1(F)$ given by the components of $A_1$. Since $J_{\rm rel}(G) \subseteq F_1(F)$ we have

$$\frac{\partial G}{\partial y_i} = \sum_{j=1}^{m+1} b_{ij} a_j, \;\; i = 1, 2, 3,$$

for some matrix $B = (b_{ij})$ with entries in $\mathscr{O}_{3+r}$. Then,

$$\mathscr{O}_{3+r}^m \oplus \mathscr{O}_{3+r}^3 \xrightarrow{A_2+B} \mathscr{O}_{3+r}^{m+1} \xrightarrow{A_1} F_1(F)/J_{\rm rel}(G) \longrightarrow 0, \tag{9.2}$$

provides a presentation of $F_1(F)/J_{\rm rel}(G)$.

We use now a theorem by Buchsbaum-Rim [7] which says that if a module $M$ admits a presentation over a Cohen-Macaulay ring with a matrix of size $p \times q$, with $q \geq p$, and it has codimension $\geq q - p + 1$, then $M$ is Cohen-Macaulay of codimension $q - p + 1$. In our case the presentation matrix has size $(m+1) \times (m+3)$, so $m + 3 - (m+1) + 1 = 3$.

From the $\mathscr{A}$-finiteness of $f$ we have that $M(g)$ and $C(f)/(J(g) \cdot \mathscr{O}_n)$ both have finite $\mathbb{C}$-dimension. The exact sequence (6.1) implies that $F_1(f)/J(g)$ also has finite $\mathbb{C}$-dimension. Now, by (7.3), $F_1(F)/J_{\rm rel}(G)$ has (Krull) dimension $\leq r$, and hence codimension $\geq 3$ in $\mathscr{O}_{3+r}$. By the Buchsbaum-Rim theorem, $F_1(F)/J_{\rm rel}(G)$ is Cohen-Macaulay of codimension 3, that is, of dimension $r$. □

*Remark* 9.3. When $n \geq 3$, $F_1(F)/J_{\text{rel}}(G)$ is never Cohen-Macaulay. In fact, the same argument with the depth lemma used in the proof of Lemma 9.1 gives that

$$\operatorname{depth} M_{\text{rel}}(G) \geq \operatorname{depth}(F_1(F)/J_{\text{rel}}(G)).$$

But we know that $\operatorname{depth} M_{\text{rel}}(G) \leq \dim M_{\text{rel}}(G) \leq r$, hence $F_1(F)/J_{\text{rel}}(G)$ has depth $\leq r$. On the other hand, $F_1(F)/J_{\text{rel}}(G)$ has codimension 3 and hence, dimension $n-2+r > r$, if $n \geq 3$. In particular, the argument used in the proof of Theorem 9.2 does not work for $n \geq 3$.

Now the strong version of the Mond conjecture (and hence the Mond conjecture) follows for map germs $f \in \mathscr{O}_S(2,3)$.

**Corollary 9.4.** *Let $f \in \mathscr{O}_S(2,3)$ be $\mathscr{A}$-finite. Then,*

$$\mu_I(f) = \operatorname{codim}_{\mathscr{A}_e}(f) + \dim_{\mathbb{C}} K(g),$$

*where $g$ is a reduced equation of the image of $f$. In particular, we have*

$$\mu_I(f) \geq \operatorname{codim}_{\mathscr{A}_e}(f),$$

*with equality if $f$ is weighted homogeneous.*

# Chapter 10

# Additional comments

## 10.1 $\mu = \tau$ implies weighted homogeneity

A celebrated theorem of Saito [67] says that if $(X, 0)$ is a hypersurface with isolated singularity, then the equality $\tau(X, 0) = \mu(X, 0)$ implies that $(X, 0)$ is weighted homogeneous, up to a coordinate change in $(\mathbb{C}^{n+1}, 0)$. It seems natural to investigate the analogous question for $\mathscr{A}$-finite map germs $f \in \mathscr{O}_S(n, n+1)$: *Does the equality*

$$\mu_I(f) = \operatorname{codim}_{\mathscr{A}_e}(f) \tag{10.1}$$

*imply that $f$ is weighted homogeneous, up to $\mathscr{A}$-equivalence?*

The case $n = 1$ is easy, since the image is a curve with isolated singularity, so Saito's theorem can be applied in this situation. However, this question is open in higher dimensions, even for $n = 2$, as far as we know. A positive answer to the strong Mond conjecture would imply that the equality (10.1) is equivalent to that $g \in J(g)$. So, another natural question, independent of the strong Mond conjecture, is: *Does $g \in J(g)$ imply that $f$ is weighted homogeneous, up to $\mathscr{A}$-equivalence?*

## 10.2 Mappings on ICIS

The Thom-Mather theory of singularities of mappings was generalised by Mond and Montaldi in [52] for map germs of the form $f\colon (X, 0) \to (\mathbb{C}^p, 0)$, where $(X, 0)$ is an isolated complete intersection singularity (ICIS) of dimension $n$. They decide to denote such map germs as a pair $(X, f)$. This notation may seem a bit redundant, since the ICIS $(X, 0)$ is actually the domain of the mapping $f$, but they do this in order to highlight the fact that both the ICIS and the mapping contribute to the singularity. In fact, the unfoldings are defined is such a way that we deform $(X,0)$ and $f$ simultaneously.

According to [52], an unfolding of $(X,f)$ is a map germ $F\colon (\mathcal{X},0)\to(\mathbb{C}^p\times\mathbb{C}^r,0)$ together with a flat projection $\pi\colon(\mathcal{X},0)\to(\mathbb{C}^r,0)$ and an isomorphism $j\colon (X,0)\to\big(\pi^{-1}(0),0\big)$ such that the following diagram commutes

$$\begin{array}{ccccc} & & (X,0) & & \\ & \swarrow^{j} & & \searrow^{f\times\{0\}} & \\ (\pi^{-1}(0),0) & & & & \big(\mathbb{C}^p\times\{0\},0\big) \\ \downarrow & & & & \downarrow \\ (\mathcal{X},0) & & \xrightarrow{\quad F \quad} & & (\mathbb{C}^p\times\mathbb{C}^r,0) \\ & \searrow_{\pi} & & \swarrow_{\pi_2} & \\ & & (\mathbb{C}^r,0) & & \end{array},$$

where $\pi_2 : \mathbb{C}^p\times\mathbb{C}^r\to\mathbb{C}^r$ is the Cartesian projection. The unfolding is denoted by the pair $(\mathcal{X},F)$ and, for each $u$ in a neighbourhood of 0 in $\mathbb{C}^r$, we have a pair $(X_u,f_u)$, where $X_u=\pi^{-1}(u)$ and $f_u\colon X_u\to\mathbb{C}^p$ is the restriction of $\pi_1\circ F$.

The $\mathscr{A}_e$-codimension of $(X,f)$ is defined as the number of parameters of a miniversal unfolding. The starting point used by Mond and Montaldi is the fact that

$$\operatorname{codim}_{\mathscr{A}_e}(X,f)=\operatorname{codim}_{\mathscr{K}_{D,e}}g,$$

where $g$ induces $(X,f)$ from a stable unfolding $(\mathbb{C}^N,F)$ of $(X,f)$ by fibre product, as described by Damon in [12]. Then, Mond and Montaldi proved that

$$\operatorname{codim}_{\mathscr{A}_e}(X,f)=\dim_{\mathbb{C}}\frac{\theta(f)}{tf(\theta_{X,0})+\omega f(\theta_p)}+\tau(X,0),\tag{10.2}$$

where $\theta_{X,0}$ is the module of tangent vector fields on $(X,0)$, $\theta(f)$ is module of vector fields along $f$ and the morphisms $tf$ and $\omega f$ are defined in an analogous way to the smooth case. The term $\tau(X,0)$ is the Tjurina number of the ICIS, which can be seen as the number of parameters of a miniversal deformation of $(X,0)$. In particular, $(X,f)$ is stable in this setting if and only if $(X,0)$ is smooth and $f$ is stable in the usual sense.

As in the smooth case, a stabilisation of $(X,f)$ is a 1-parameter unfolding with the property that $(X_t,f_t)$ is stable if $t\neq 0$. This means that $X_t$ is a smoothing of $(X,0)$ and $f_t\colon X_t\to V$ is a locally stable mapping, for some open neighbourhood $V$ of 0 in $\mathbb{C}^p$. Since $(X,0)$ is an ICIS, the smoothing $X_t$ is obtained as a Milnor fibre of $(X,0)$ and the mapping $f$ can be approximated by a stable mapping $f_t$ on $X_t$ if $(n,p)$ are nice dimensions and $(X,f)$ has isolated instability.

Following the same script as in the smooth case, they also showed in [52] that if $(n,p)$ are nice dimensions, with $n \geq p$, and $(X_t, f_t)$ is a stable perturbation, then $\Delta(f_t) \cap B_\epsilon$ has the homotopy type of a bouquet of $(p-1)$-spheres, where $\Delta(f_t)$ is the discriminant. The *discriminant Milnor number* is well defined as the number of such spheres, i.e.,

$$\mu_\Delta(X, f) = \beta_{p-1}(\Delta(f_t) \cap B_\epsilon).$$

Then, they generalised the theorem of Damon and Mond [13] for mappings on ICIS with $\dim X \geq p$, as expected:

$$\operatorname{codim}_{\mathscr{A}_e}(X, f) \leq \mu_\Delta(X, f),$$

with equality when $(X, f)$ is weighted homogeneous.

In the case $(n, n+1)$, the definition of the image Milnor number $\mu_I(X, f)$ appears in the paper [22] and is given in the obvious way by taking the image instead of the discriminant, i.e.,

$$\mu_I(X, f) = \beta_n(f_t(X_t) \cap B_\epsilon)$$

and is well defined provided that $(n, n+1)$ are nice dimensions or $f$ has corank one. Now it makes sense to consider a generalised version of the Mond conjecture in this setting:

$$\operatorname{codim}_{\mathscr{A}_e}(X, f) \leq \mu_I(X, f), \tag{10.3}$$

with equality when $(X, f)$ is weighted homogeneous.

An important reason to consider the generalised Mond conjecture for mappings on ICIS is that it also generalises the $\mu \geq \tau$-inequality in the classical case of a hypersurface with isolated singularity, as the following example shows:

*Example* 10.1. We consider the pair $(X, i)$, where $(X, 0)$ is a hypersurface with isolated singularity in $(\mathbb{C}^{n+1}, 0)$ and $i\colon (X, 0) \hookrightarrow (\mathbb{C}^{n+1}, 0)$ is the inclusion. By construction, any vector field along $i$ is the restriction to $(X, 0)$ of a vector field on $(\mathbb{C}^{n+1}, 0)$ and thus, $\omega i(\theta_{n+1}) = \theta(i)$. Hence, (10.2) gives

$$\operatorname{codim}_{\mathscr{A}_e}(X, i) = \tau(X, 0).$$

On the other hand, let $g \in \mathscr{O}_{n+1}$ be a reduced equation of $(X, 0)$. A stabilisation of $(X, i)$ is given by the pair $(\mathbb{C}^{n+1}, F)$, where $F = (\mathrm{id}, g)\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ and $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is the flat projection. For each $t$ in a neighborhood of 0 in $\mathbb{C}$, $X_t = g^{-1}(t)$, which is smooth if $t \neq 0$ and $f_t\colon X_t \to \mathbb{C}^{n+1}$ is the inclusion, which is locally stable, also for $t \neq 0$. By definition, the image Milnor number is

$$\mu_I(X, i) = \beta_n(X_t \cap B_\epsilon) = \mu(X, 0).$$

The conjecture (10.3) has been proved when $n = 1$ in the particular cases that either $(X, 0)$ is a plane curve (see [3]) or $(X, 0)$ is irreducible and $(X, f)$ is weighted homogeneous (see [4]).

*Example* 10.2. [3] Let $(X, 0)$ be the plane curve given by $x^5+y^6+x^2y^2 = 0$ and let $f\colon (X, 0) \to (\mathbb{C}^2, 0)$ be the map germ $f(x, y) = (x, y^3)$. The image can be obtained easily with the aid of Singular and gives the plane curve $(Y, 0)$ with equation

$$u^{15} + 3u^{10}v^2 + 3u^5v^4 + u^6v^2 + v^6 = 0.$$

It follows that $\mu(Y, 0) = 46$ and $\tau(Y, 0) = 41$. The $\mathscr{A}_e$-codimension and the image Milnor number are computed in [3] and give:

$$\operatorname{codim}_{\mathscr{A}_e}(X, f) = 24, \quad \mu_I(X, f) = 29.$$

We see that $(X, f)$ is not weighted homogeneous, up to $\mathscr{A}$-equivalence. The difference between the two invariants coincides with $\mu(Y, 0) - \tau(Y, 0)$.

In a recent paper [19], the authors generalise the construction of the Jacobian module $M(g)$ for mappings on ICIS. They show that the conjecture (10.3) follows for $n \geq 2$, provided that the relative Jacobian module $M_{\text{rel}}(G)$ is Cohen-Macaulay, as it happens in the smooth case. Moreover, they also obtain a proof of the conjecture for the case $n = 2$, following similar arguments to those of the proof given in Chapter 9 for the smooth case.

*Example* 10.3. [19] We consider the pair $(X, f)$, where $(X, 0)$ is the surface in $(\mathbb{C}^3, 0)$ with equation $x^3 + y^3 - z^2 = 0$ and $f : (X, 0) \to (\mathbb{C}^3, 0)$ is the mapping $f(x, y, z) = (x, y, z^3 + xz + y^2)$. With the results of [19] and the aid of Singular, we can see that $\operatorname{codim}_{\mathscr{A}_e}(X, f) = \mu_I(X, f) = 6$.

## 10.3 Frontals

In [56], the authors propose a $\mu \geq \tau$-conjecture analogous to the Mond conjecture, but for frontals. We recall that a map germ $f \in \mathscr{O}_S(n, n+1)$ is called a frontal if it admits a lifting to a Legendrian mapping $\tilde{f}\colon (\mathbb{C}^n, S) \to PT^*\mathbb{C}^{n+1}$, where $\pi\colon PT^*\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the projectivised cotangent bundle, together with the standard contact structure given by the standard contact form $\alpha$ on $PT^*\mathbb{C}^{n+1}$. The fact that $\tilde{f}$ is Legendrian means that $\tilde{f}^*\alpha = 0$. In other words, $\tilde{f} = (f, [\nu])$, where $\nu\colon (\mathbb{C}^n, S) \to T^*\mathbb{C}^{n+1}$ is a holomorphic, everywhere non-zero 1-form along $f$ such that $\nu(df \circ \xi) = 0$, for all $\xi \in \theta_n$. If $\nu$ is given in coordinates by $\nu = \sum_{i=1}^{n+1} \nu_i dy_i$, this is also equivalent to

$$\sum_{i=1}^{n+1} \nu_i \frac{\partial f_i}{\partial x_j} = 0, \quad \forall j = 1, \dots, n. \tag{10.4}$$

Geometrically, a frontal can be understood as a mapping whose image has a well defined tangent hyperplane at every point. The notion of frontal was introduced for the first time by Fujimori, Saji, Umehara and Yamada in [20] (see also the paper of Zakalyukin and Kurbatskiĭ [74]) as a natural extension of the notion of wavefront, which is the particular case where the lifting $\tilde{f}$ is an immersion.

*Example* 10.4. It is easy to check that any plane curve $f \in \mathscr{O}_S(1,2)$ is a frontal. The swallowtail and the folded Whitney umbrella are two examples of map germs in $\mathscr{O}_S(2,3)$ which are frontals. They are defined as

$$f(x,y) = \left(x, y^3 + xy, 3y^4 + 2xy^2\right), \quad f'(x,y) = \left(x, y^2, xy^3\right),$$

respectively (see Figure 10.1). We have

$$\frac{\partial f}{\partial x} \wedge \frac{\partial f}{\partial y} = (x+3y^2)(2y^2, -4y, 1), \quad \frac{\partial f'}{\partial x} \wedge \frac{\partial f'}{\partial y} = y(2y^3, -3xy, 2).$$

In both cases, we can take $(2y^2, -4y, 1)$ or $(2y^3, -3xy, 2)$ respectively, as the coordinates of a 1-form $\nu$ which is everywhere non-zero and satisfies (10.4). Hence, $f$ and $f'$ are frontals. The Whitney umbrella, given by $(x,y) \mapsto (x, y^2, xy)$, is not a frontal, as the reader can check easily.

We remark that the frontals are not $\mathscr{A}$-finite when $n \geq 2$, so the classical theory of singularities of mappings cannot be used directly. The Thom-Mather theory of frontals has been developed in the recent paper [57], based on the previous work of Ishikawa [33] on Legendrian singularities. An important point is that the frontality is preserved under $\mathscr{A}$-equivalence, so we do not need to change the equivalence relation. However, we have to restrict ourselves to deformations which preserve the frontal structure. By definition, a frontal unfolding of $f$ is an unfolding $F$ which is also a frontal as a map germ. This ensures that its lifting $\tilde{F}$ induces a Legendrian deformation of $\tilde{f}$. And conversely, any Legendrian deformation of $\tilde{f}$ induces a frontal unfolding $F$ of $f$. The notions of frontal stability or versality can be adapted easily in this setting.

For technical reasons, in order to use Ishikawa's result on infinitesimal deformations, we have to consider only frontals whose Legendrian lifting has corank 1. The frontal codimension of $f$ is defined as

$$\operatorname{codim}_{\mathscr{F}_e}(f) = \dim_{\mathbb{C}} \frac{\mathscr{F}(f)}{tf(\theta_n) + \omega f(\theta_{n+1})},$$

where $\mathscr{F}(f)$ is the subspace of frontal infinitesimal deformations, that is,

$$\mathscr{F}(f) = \left\{ \left.\frac{df_t}{dt}\right|_{t=0} : (f_t, t) \text{ is frontal, } f_0 = f \right\}.$$

Then, it follows that $\operatorname{codim}_{\mathscr{F}_e}(f)$ is the number of parameters of a miniversal frontal unfolding (see [57, Theorem 3.21]). In particular $\operatorname{codim}_{\mathscr{F}_e}(f) =$

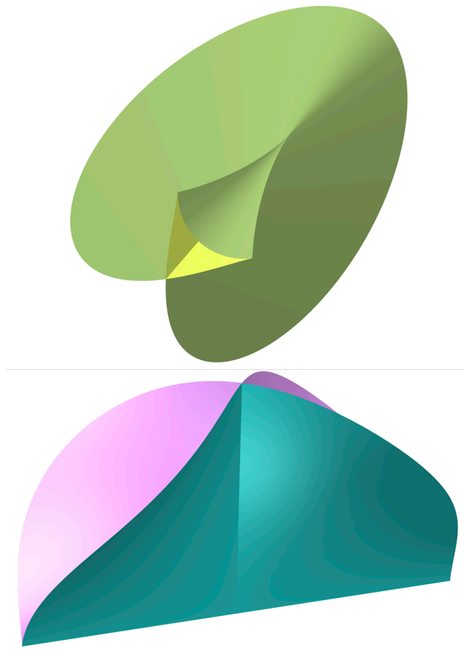

Figure 10.1: The swallowtail and the folded Whitney umbrella

0 if and only if $f$ is frontal stable. We also deduce a frontal version of the Mather-Gaffney criterion: $f$ is $\mathscr{F}$-finite (i.e., it has finite frontal codimension) if and only if it has isolated frontal instability.

The classification of stable frontal singularities is known up to dimension $n \leq 3$ (see [57, Theorem 6.10]). The stable mono-germs are:

- $n = 1$, the regular point and the cusp;
- $n = 2$, the regular point, the cuspidal edge, the swallowtail and the folded Whitney umbrella;
- $n = 3$, apart from the singularities coming from trivial unfoldings of singularities in dimensions $n = 1, 2$, we have two new mono-germs $A_{3,1}$ and $A_{4,0}$ in Ishikawa's notation [33].

All stable multi-germs are obtained by taking transverse combinations of the mono-germs. The classification in higher dimensions is open, as far as we know.

As in the case of $\mathscr{A}$-finite germs, if $f$ is $\mathscr{F}$-finite and either $(n, n+1)$ are "frontal nice dimensions" or $f$ has corank one, then $f$ can be approximated by a frontal stable mapping. It follows that $f$ admits a frontal stabilisation, that is, a 1-parameter unfolding $F(x,t) = (f_t(x), t)$ such that $f_t$ is locally frontal stable, for $t \neq 0$. Let $X_t$ be the image of such $f_t$. Then, $X_t \cap B_\epsilon$ has the homotopy type of a bouquet of $n$-spheres and the number of such spheres is called the *frontal Milnor number*:

$$\mu_{\mathscr{F}}(f) = \beta_n(X_t \cap B_\epsilon),$$

(see [56, Definition 5.2]). The frontal version of the Mond conjecture is now natural:

$$\operatorname{codim}_{\mathscr{F}_e}(f) \leq \mu_{\mathscr{F}}(f),$$

with equality if $f$ is weighted homogeneous [56, Conjecture 5.1]. The frontal Mond conjecture has been proved for $n = 1$ in [57, Corollary 5.13]), but the case $n \geq 2$ is still open.

*Example* 10.5. Consider the plane curve $E_6$, $f \in \mathscr{O}(1,2)$ given by $f(x) = (x^3, x^4)$. We know that the image has Milnor number $\mu(X,0) = 6$. Now we compare the stable perturbation (Figure 10.2, left)

$$f_t(x) = (x^3 + tx, x^4 + (4/3)tx^2),$$

with the frontal stable perturbation (Figure 10.2, right)

$$f'_t(x) = (x^3 + tx, x^4 + (2/3)tx^2).$$

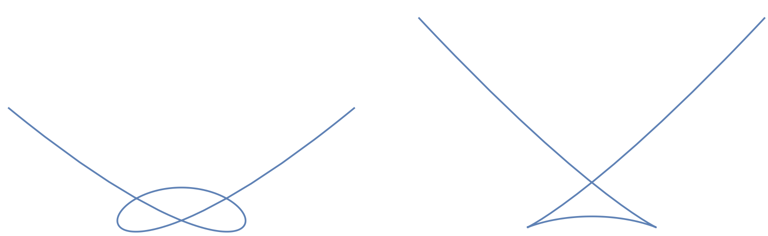

Figure 10.2: Stable and frontal stable perturbations of $E_6$

We remark that the pictures in Figure 10.2 are real representations of the corresponding complex curves. Nevertheless, they are real good pictures, in the sense that they present the expected homotopy type of its complex model.

In the left hand side, the curve has three 1-cycles, which is compatible with $\mu_I(f) = 3$. In the right hand side, in order to preserve the frontal structure, two of the 1-cycles have collapsed to cusps, which are frontal stable and must be preserved. Thus, only one 1-cycle survives, as it should be expected, since $\mu_{\mathscr{F}}(f) = 1$. By the way, we can observe that the frontal stabilisation $F'(x,t) = (f'_t(x), t)$ is the swallowtail (see Example 10.4).

## 10.4 The far side of Mond

For isolated hypersurface singularities, the usual $\mu \geq \tau$ inequality can be expressed as $1 \leq \frac{\mu}{\tau}$. In [35], Liu showed that this quotient can also be bounded above.

**Theorem 10.6.** *Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be an isolated hypersurface singularity. Then,*

$$1 \leq \frac{\mu}{\tau} \leq n+1.$$

The proof is not hard, and is based on the fact that any equation $h$ of $(X,0)$ satisfies $h^n \subseteq Jh$ [70]. This bound is not meant to be sharp, the relevance of the result is the existence of a bound. For curves, we know a sharp bound due to Almirón [1], and can be summarised as follows:

**Theorem 10.7.** *Let $(X,0)$ be a reduced plane curve singularity. Then,*

$$1 \leq \frac{\mu}{\tau} < \frac{4}{3}.$$

The fact that the bound is sharp follows from the existence of a family of singularities, found by Dimca and Greuel [16, Example 4.1], whose $\mu/\tau$ values converge to $3/4$. Now we shall show that something similar applies to curve map-germs. The following result is an improved version for multi-germs of a result found in [5].

**Theorem 10.8.** *All $\mathscr{A}$-finite germs $f\colon (\mathbb{C},S) \to (\mathbb{C}^2,0)$ satisfy*

$$1 \leq \frac{\mu_I}{\operatorname{codim}_{\mathscr{A}_e}(f)} < \frac{5}{2}.$$

*For mono-germs, a sharp upper bound is*

$$\frac{\mu_I}{\operatorname{codim}_{\mathscr{A}_e}(f)} < 2.$$

*Proof.* The lower bound corresponds to the Mond conjecture and, as we explained in Chapter 2, Section 2.1, it is a trivial consequence of the equalities

$$\mu = \mu_I(f) + \delta, \quad \tau = \operatorname{codim}_{\mathscr{A}_e}(f) + \delta,$$

where $\mu, \tau$ and $\delta$ are the corresponding invariants of the image of $f$. For the upper bound we use Hironaka's formula

$$\mu = 2\delta - r + 1,$$

where $r$ stands for the number of branches. Together with the $\frac{\mu}{\tau} < \frac{4}{3}$ inequality, this gives

$$\operatorname{codim}_{\mathcal{A}_e}(f) = \tau - \frac{\mu + r - 1}{2} > \frac{3}{4}\mu - \frac{\mu + r - 1}{2} = \frac{\mu - 2r - 2}{4}$$

and

$$\mu_I = \mu - \delta = \frac{\mu - 2r - 2}{2} + \frac{r-1}{2}.$$

Therefore,

$$\frac{\mu_I}{\operatorname{codim}_{\mathcal{A}_e}(f)} < 2 + \frac{r-1}{2\operatorname{codim}_{\mathcal{A}_e}(f)}.$$

This gives the stated bound for mono-germs. To see that it is sharp just observe that, for the parametrizations of the family of Dimca and Greuel, the $\mu_I/\operatorname{codim}_{\mathscr{A}_e}(f)$ quotient converges to 2.

For multi-germs, observe that if $f$ is unstable, then

$$\operatorname{codim}_{\mathscr{A}_e}(f) > \operatorname{codim}_{\mathscr{A}_e}(f'), \tag{10.5}$$

where $f'$ is the multi-germ of $r-1$ branches obtained after eliminating one of the branches of $f$ (this inequality can be deduced, for instance, from the exact sequence that appears in the proof of [62, Theorem 4.3]).

When $r = 4$, the simplest case is when we consider 4 distinct lines, which has $\mathscr{A}_e$-codimension 3. Moreover, it is 4-determined and hence, any other given by 4 smooth branches which are pairwise transverse has also $\mathscr{A}_e$-codimension 3. In the general case, any $f$ with 4 branches can be deformed to such singularity, so $\operatorname{codim}_{\mathcal{A}_e}(f) \geq 3$, by the upper semicontinuity. From (10.5) we deduce, by induction on $r$, that $\operatorname{codim}_{\mathcal{A}_e}(f) \geq r-1$, if $r \geq 4$.

The same inequality $\operatorname{codim}_{\mathcal{A}_e}(f) \geq r-1$ holds obviously when $r = 2$ or when $r = 3$ and $\operatorname{codim}_{\mathcal{A}_e}(f) \geq 2$. Thus, the only remaining case is $r = 3$ and $\operatorname{codim}_{\mathcal{A}_e}(f) = 1$. But this implies that $f$ is an ordinary triple point, which is weighted homogeneous and hence,

$$\frac{\mu_I}{\operatorname{codim}_{\mathcal{A}_e}(f)} = 1 \qquad \square$$

This bound for multi-germs is not meant to be sharp. Based on Liu's Theorem 10.6, we propose what follows:

**Conjecture 10.9.** *Let $f \in \mathscr{O}_S(n, n+1)$ be with isolated instability and suppose that either $(n, n+1)$ are nice dimensions or $f$ has corank one. Then,*

$$\frac{\mu_I(f)}{\operatorname{codim}_{\mathscr{A}_e}(f)} \leq n+1.$$

# Bibliography


[1] P. Almirón. On the quotient of Milnor and Tjurina numbers for two-dimensional isolated hypersurface singularities. *Math. Nachr.*, 295:1254–1263., 2022.

[2] A. Altıntaş Sharland. Examples of finitely determined map-germs of corank 2 from $n$-space to $(n+1)$-space. *Internat. J. Math*, 25(5):1–17, 2014.

[3] D. Ament, H. Alice and J. J. Nuño Ballesteros. Mond's conjecture for maps between curves. *Math. Nachr.*, 290(17-18):2845–2857, 2017.

[4] D. A. H. Ament, J. J. Nuño-Ballesteros, and J. N. Tomazella. Image Milnor number and $\mathcal{A}_e$-codimension for maps between weighted homogeneous irreducible curves. *Adv. Geom.*, 20(3):319–330, 2020.

[5] I. Breva Ribes and R. Oset Sinha. Augmentation of singularities: $\mu/\tau$-type conjectures and simplicity. arXiv:2305.13811, 2023.

[6] J. W. Bruce and W. L. Marar. Images and varieties. *J. Math. Sci., New York*, 82(5):3633–3641, 1996.

[7] D. A. Buchsbaum and D. S. Rim. A generalized Koszul complex. II. Depth and multiplicity. *Trans. Amer. Math. Soc.*, 111:197–224, 1964.

[8] R. O. Buchweitz and G.M. Greuel. The Milnor number and deformations of complex curve singularities. *Invent. Math.*, 58(3):241–281, 1980.

[9] J. L. Cisneros-Molina and D. Mond. Multiple points of a simplicial map and image-computing spectral sequences. *J. Singul.*, 24:190–212, 2022.

[10] T. Cooper, D. Mond, and R. Wik Atique. Vanishing topology of codimension 1 multi-germs over $\mathbb{R}$ and $\mathbb{C}$. *Compositio Mathematica*, 131:121–160, 2002.

[11] J. Damon. The unfolding and determinacy theorems for subgroups of $\mathscr{A}$ and $\mathscr{K}$. *Mem. Amer. Math. Soc.*, 50(306):x+88, 1984.

[12] J. Damon. $\mathscr{A}$-equivalence and the equivalence of sections of images and discriminants. In *Singularity theory and its applications, Part I (Coventry, 1988/1989)*, volume 1462 of *Lecture Notes in Math.*, pages 93–121. Springer, Berlin, 1991.

[13] J. Damon and D. Mond. $\mathscr{A}$-codimension and the vanishing topology of discriminants. *Invent. Math.*, 106(2):217–242, 1991.

[14] T. de Jong and G. Pfister. *Local analytic geometry.* Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 2000. Basic theory and applications.

[15] T. de Jong and D. van Straten. Disentanglements. In *Singularity theory and its applications, Part I (Coventry, 1988/1989)*, volume 1462 of *Lecture Notes in Math.*, pages 199–211. Springer, Berlin, 1991.

[16] A. Dimca and G. M. Greuel. On 1-forms on isolated complete intersection curve singularities. *J. Singul.*, 18:114–118, 2018.

[17] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[18] J. Fernández de Bobadilla, J. J. Nuño-Ballesteros, and G. Peñafort Sanchis. A Jacobian module for disentanglements and applications to Mond's conjecture. *Rev. Mat. Complut.*, 32(2):395–418, 2019.

[19] A. Fernández-Hernández and J. J. Nuño Ballesteros. Disentangling mappings defined on ICIS. arXiv:2309.16193 [math.AG], 09 2023.

[20] S. Fujimori, K. Saji, M. Umehara, and K. Yamada. Singularities of maximal surfaces. *Math. Z.*, 259(4):827–848, 2008.

[21] R. Giménez Conejero and J. J. Nuño Ballesteros. The image Milnor number and excellent unfoldings. *Q. J. Math.*, 73(1):45–63, 2022.

[22] R. Giménez Conejero and J. J. Nuño Ballesteros. Singularities of mappings on ICIS and applications to Whitney equisingularity. *Adv. Math.*, 408(part B):Paper No. 108660, 41, 2022.

[23] R. Giménez Conejero and J. J. Nuño Ballesteros. A weak version of the Mond conjecture. *Collect. Math.*, 75(3):753–770, 2024.

[24] V. Goryunov and D. Mond. Vanishing cohomology of singularities of mappings. *Compositio Math.*, 89(1):45–80, 1993.

[25] G. M. Greuel. On deformation of curves and a formula of Deligne. In *Algebraic geometry (La Rábida, 1981)*, volume 961 of *Lecture Notes in Math.*, pages 141–168. Springer, Berlin, 1982.

[26] G. M. Greuel, C. Lossen, and E. I. Shustin. *Introduction to singularities and deformations.* Springer, 2007.

[27] G. M. Greuel and G. Pfister. *A* **Singular** *introduction to commutative algebra.* Springer, Berlin, extended edition, 2008. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh and UNIX).

[28] H. A. Hamm and Lê D. T. The Lefschetz theorem for hyperplane sections. In *Handbook of geometry and topology of singularities. I*, pages 491–540. Springer, Cham, [2020] ©2020.

[29] K. Houston. On singularities of folding maps and augmentations. *Math. Scand.*, 82:191–206, 1998.

[30] K. Houston. Bouquet and join theorems for disentanglements. *Invent. Math.*, 147:471–485, 2002.

[31] K. Houston. Augmentation of singularities of smooth mappings. *International Journal of Mathematics*, 15(2):111–124, 2004.

[32] K. Houston and N. Kirk. On the classification and geometry of corank 1 map-germs from three-space to four-space. *London Math. Soc. Lecture Note Ser.*, pages 325–351, 1999.

[33] G. Ishikawa. Infinitesimal deformations and stability of singular Legendre submanifolds. *Asian J. Math.*, 9(1):133–166, 2005.

[34] Lê D. T. Some remarks on relative monodromy. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 397–403. Sijthoff & Noordhoff, Alphen aan den Rijn, 1977.

[35] Y. Liu. Milnor and Tjurina numbers for a hypersurface germ with isolated singularity. *C. R. Acad. Sci. Paris, Ser. I*, 356:963–966, 2018.

[36] E. J. N. Looijenga. *Isolated singular points on complete intersections*, volume 77 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1984.

[37] E. J. N. Looijenga and J. Steenbrink. Milnor number and Tjurina number of complete intersections. *Math. Ann.*, 271:121–124, 1985.

[38] W. L. Marar and D. Mond. Multiple point schemes for corank 1 maps. *J. London Math. Soc.*, 2(3):553–567, 1989.

[39] W. L. Marar and D. Mond. Real map-germs with good perturbations. *Topology*, 35(1):157–165, 1996.

[40] J. Martinet. *Singularities of smooth functions and maps*, volume 58 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1982. Translated from the French by Carl P. Simon.

[41] J. N. Mather. Stability of $C^\infty$ mappings. II. Infinitesimal stability implies stability. *Ann. of Math. (2)*, 89:254–291, 1969.

[42] J. N. Mather. Stability of $C^\infty$ mappings. VI: The nice dimensions. In *Proceedings of Liverpool Singularities—Symposium, I (1969/70)*, Lecture Notes in Math., Vol. 192, pages 207–253. Springer, Berlin-New York, 1971.

[43] J. N. Mather. Notes on topological stability. *Bull. Amer. Math. Soc. (N.S.)*, 49(4):475–506, 2012.

[44] J. N. N. Mather. Stability of $C^\infty$ mappings. IV. Classification of stable germs by $R$-algebras. *Inst. Hautes Études Sci. Publ. Math.*, (37):223–248, 1969.

[45] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[46] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1968.

[47] D. Mond. On the classification of germs of maps from $\mathbf{R}^2$ to $\mathbf{R}^3$. *Proc. London Math. Soc. (3)*, 50(2):333–369, 1985.

[48] D. Mond. Some remarks on the geometry and classification of germs of maps from surfaces to 3-space. *Topology*, 26(3):361–383, 1987.

[49] D. Mond. The number of vinishing cycles for a quasihomogeneous mapping from $\mathbb{C}^2$ to $\mathbb{C}^3$. *Quart. J. Math*, 2(42):335–345, 1991.

[50] D. Mond. Vanishing cycles for analytic maps. In *Singularity theory and its applications, Part I (Coventry, 1988/1989)*, volume 1462 of *Lecture Notes in Math.*, pages 221–234. Springer, Berlin, 1991.

[51] D. Mond. Looking at bent wires—$\mathscr{A}_e$-codimension and the vanishing topology of parametrized curve singularities. *Math. Proc. Cambridge Philos. Soc.*, 117(2):213–222, 1995.

[52] D. Mond and J. Montaldi. Deformations of maps on complete intersections, Damon's $\mathscr{K}_V$-equivalence and bifurcations. In *Singularities*

*(Lille, 1991)*, volume 201 of *London Math. Soc. Lecture Note Ser.*, pages 263–284. Cambridge Univ. Press, Cambridge, 1994.

[53] D. Mond and J. J. Nuño Ballesteros. *Singularities of mappings—the local behaviour of smooth and complex analytic mappings*, volume 357 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, [2020] ©2020.

[54] D. Mond and J. J. Nuño-Ballesteros. Singularities of mappings. In *Handbook of geometry and topology of singularities III*, pages 81–144. Springer, Cham, [2022] ©2022.

[55] D. Mond and R. Pellikaan. Fitting ideals and multiple points of analytic mappings. In *Algebraic geometry and complex analysis (Pátzcuaro, 1987)*, volume 1414 of *Lecture Notes in Math.*, pages 107–161. Springer, Berlin, 1989.

[56] C. Muñoz Cabello, J. J. Nuño Ballesteros, and R. Oset Sinha. Singularities of frontal surfaces. *Hokkaido Math. J.*, 53(1):175–208, 2024.

[57] C. Muñoz-Cabello, J. J. Nuño-Ballesteros, and R. Oset Sinha. Deformations of corank 1 frontals. To appear in Proc. Roy. Soc. Edinburgh Sect. A, https://doi.org/10.1017/prm.2023.55, 2023.

[58] J. J. Nuño Ballesteros, B. Oréfice-Okamoto, and J. N. Tomazella. Equisingularity of families of isolated determinantal singularities. *Math. Z.*, 289(3-4):1409–1425, 2018.

[59] J. J. Nuño Ballesteros and G. Peñafort Sanchis. Multiple point spaces of finite holomorphic maps. *Q. J. Math.*, 68(2):369–390, 2017.

[60] J. J. Nuño Ballesteros, G. Peñafort Sanchis, and C. Villa. On the analytic structure of double and triple points in the target of finite holomorphic multi-germs. arXiv:2405.12974, 2024.

[61] T. Ohmoto. Singularities of maps and characteristic classes. *Advanced Studies in Pure Mathematics*, 68:191–265, 2016.

[62] R. Oset Sinha, M. A. S. Ruas, and R. Wik Atique. Classifying codimension two multigerms. *Mathematische Zeitschrift*, 278(1):547–573, 2014.

[63] I. Pallarés and G. Peñafort. Image milnor number formulas for weighted-homogeneous map-germs. *Results Math*, 76(152), 2021.

[64] R. Pellikaan. Projective resolutions of the quotient of two ideals. *Nederl. Akad. Wetensch. Indag. Math.*, 50(1):65–84, 1988.

[65] G. Peñafort Sanchis. Reflection maps. *Mathematische Annalen*, 378:559–598, 2020. arXiv:1405.3403.

[66] R. Piene. Ideals associated to a desingularization. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 503–517. Springer, Berlin, 1979.

[67] K. Saito. Quasihomogene isolierte Singularitäten von Hyperflächen. *Invent. Math.*, 14:123–142, 1971.

[68] A. A. Sharland. Examples of finitely determined map-germs of corank 3 supporting mond's $\mu \geq \tau$-type conjecture. *Experimental Mathematics*, 2017.

[69] D. Siersma. Vanishing cycles and special fibres. In *Singularity theory and its applications, Part I (Coventry, 1988/1989)*, volume 1462 of *Lecture Notes in Math.*, pages 292–301. Springer, Berlin, 1991.

[70] H. Skoda and J. Briançon. Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbf{C}^n$. *C. R. Acad. Sci. Paris Sér. A*, 278:949–951, 1974.

[71] H. Vosegaard. A characterization of quasihomogeneous complete intersection singularities. *J. Algebraic Geometry*, 11:581–597, 2002.

[72] C. T. C. Wall. Finite determinacy of smooth map-germs. *Bull. London Math. Soc.*, 13(6):481–539, 1981.

[73] D. Wolfram, G. M. Greuel, G. Pfister, and H. Schönemann. Singular 4-3-0 — A computer algebra system for polynomial computations. `http://www.singular.uni-kl.de`, 2022.

[74] V. M. Zakalyukin and A. N. Kurbatskiĭ. Envelope singularities of families of planes in control theory. *Tr. Mat. Inst. Steklova*, 262(Optim. Upr.):73–86, 2008.

Juan J. Nuño-Ballesteros
Departament de Matemàtiques
Universitat de València
Campus de Burjassot, 46100 Burjassot, Spain
Juan.Nuno@uv.es

Guillermo Peñafort Sanchis
Departament de Matemàtiques
Universitat de València
Campus de Burjassot, 46100 Burjassot, Spain
Guillermo.Penafort@uv.es