SOCIEDADE BRASILEIRA DE MATEMÁTICA

ENSAIOS MATEMÁTICOS
2024, Volume **40**, 1–115
https://doi.org/10.21711/217504322024/em401

The contact process on random graphs

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Abstract. The contact process, introduced by Ted Harris in 1974, is a class of interacting particle systems which can be taken as a model for the spread of epidemics in a population modelled by a graph. Although the process was introduced and first studied in lattices, in the last two decades there has been significant interest and progress on its behavior on random graphs that capture aspects of real-world populations. The aim of these notes is to illustrate this with some of the recent advances, as well as to provide a point of entry into the topic. After giving a crash course on the contact process and related tools, we focus on two subjects: metastability results that hold independently of the geometry of the underlying graph, and the behavior on Bienaymé–Galton–Watson trees.

Keywords. interacting particle systems, contact process, metastability.

2020 Mathematics Subject Classification: 82C22, 60J27.

This work is dedicated to David, who is about to be born.

Contents

Preface

“Contact interactions on a lattice”, by Ted Harris [34], was published fifty years before the time of writing of the present text. The contact process has since become one of the central models in the field of interacting particle systems, and there are good reasons for that. I will name a few. First, it is an individual-based model of epidemics, in which the role of geometry is prominent. Second, it presents rich behavior, some early highlights of which are the phase transition on the Euclidean lattice and the double phase transition on regular trees. Third, it has clear connections to percolation theory. Fourth, it is moldable, lending itself to be changed into variants such as multi-type processes that model competition between species, or multi-stage processes that model diseases with immunization periods. Fifth, it is mathematically tractable, as it enjoys certain key properties such as monotonicity and self-duality.

This text is a guide to some results on the contact process in the last two decades, with a focus on two main topics: metastability and the behavior on Bienaymé–Galton–Watson trees. Admittedly, there is quite some bias towards my own work (with collaborators), and for this reason, I should emphasize that this is not an attempt to provide an exhaustive account of the progress in the topic, in any period of time. Nevertheless, the hope is that, being an aggregation of several related things in one place, it may be useful to readers who are new to the field.

A note about how results are cited is in order, since many of the results mentioned here are not due to my own work. When stating the main results (theorems), I have included the original source(s) and the name of the authors. For smaller results (claims, lemmas, propositions), the source is not always explicitly stated.

Leamington Spa, November 2024

Daniel Valesin

Chapter 1

Introduction

1.1 Background

The *contact process* is a class of interacting particle systems introduced by Ted Harris [34] in 1974. It is usually taken as a model for the spread of an infection in a population. In mathematical epidemiology, similar *compartmental models* (that is, models where a population is split into "compartments", such as "susceptible" and "infected") have been studied for much longer, with an important early reference being Kermack and McKendrik [40].

The dynamics of the contact process can be briefly described as follows. At any point in (continuous) time, vertices of a graph are either healthy or infected. Infected vertices become healthy spontaneously with rate one. Additionally, infected vertices transmit the infection to each of their neighbors with rate $\lambda > 0$.

Interest in the contact process is justified by its mathematical tractability on the one hand, and its rich behavior on the other. It has been the basis for the study of competition models, models with sexual reproduction and maturation, vegetation models and gene regulatory networks. It has also triggered theoretical work on criticality, random walk on dynamic random environments, metastability, shape theorems, and superprocesses etc.

In the 1970's and 1980's, focus was on the study of the phase transition of this process on the Euclidean lattice $\mathbb{Z}^d$. Already in the seminal paper [34], it was proved that there is a critical value λ_c of the infection rate, such that, for the process started from a single infection, if $\lambda < \lambda_c$, the infection eventually disappears with probability one, whereas if $\lambda > \lambda_c$, then it persists forever with positive probability. Several important properties of the subcritical ($\lambda < \lambda_c$) and supercritical ($\lambda > \lambda_c$) were established. An important aspect of the supercritical contact process is its prominence in the theory of *metastability*: if the process evolves on a large box of $\mathbb{Z}^d$ with $\lambda > \lambda_c$, then the infection takes a very long time to die out (exponential in the volume of the box), and while it remains active, it gives the impression of being in equilibrium. This was identified in [17], and many other references followed. In particular, [54] and [63] showed that, as long as the infection rate is above a certain threshold (the critical parameter for the process on the integer lattice; see Section 2.3), the contact process exhibits metastable behavior on very general classes of graphs.

In 1990, Bezuidenhout and Grimmett [7] proved that the process on the Euclidean lattice also dies out when $\lambda = \lambda_c$. In doing so, they introduced an important renormalization technique that made the proof of other interesting results possible, notably the shape theorem in all dimensions.

In 1992, due to an important work by Pemantle [60], the contact process on trees gained attention, with the highlight being the occurrence of a double phase transition (in an intermediate "weak survival" regime, the infection survives forever on the tree, but any given finite region eventually becomes permanently free from it). The contact process on Bienaymé–Galton–Watson trees was also first considered around this time [61], inaugurating the study of the contact process on random graphs.

This study has picked up pace in the following two decades. The key point of interest, which offers insight into real-world epidemics, is how the geometry of the graph affects the propagation of the infection. The important works [6] and [19] showed that on graphs that exhibit a power law degree distribution, the process is in a way "always supercritical": even when the infection rate is very small, the infection is sustained for a long time, due to the presence of very highly-connected vertices ("stars"). The behavior of the process on several classes of random graphs has been studied: the preferential attachment graph [6, 13], the configuration model [19, 55, 14, 9, 58, 18, 28], the random d-regular graph [57, 42], the Erdős–Renyi random graph [9, 58, 18, 2], random hyperbolic graphs [51], and random geometric graphs [31]. See also [64], where a variety of spatial graphs are treated. Additionally, *degree-dependent* versions of the contact process (where the rate at which the infection is transmittedfrom u to v is allowed to depend on the degrees of u and v) has been studied in [3, 15].

For many of the random graphs listed above, the behavior of the contact process on Bienaymé–Galton–Watson trees is very relevant; apart from the aforementioned [61], this behavior is studied specifically in [55, 9, 35].

Recently, there has also been interest in the contact process on dynamical graphs (that is, graphs which evolve simultaneously with the process); see [12, 69, 62, 38, 36, 37, 29, 16, 22, 65].

1.2 Contents and structure of text

In the present text, we bring a magnifying glass over two episodes in the story told above: *metastability on general graphs* and the *behavior on Bienaymé–Galton–Watson trees.* The text is written as a roughly self-contained survey of the material of a number of previous papers, which are listed in the two subsections below. Sometimes we have deviated from the methods of proof from the sources, but there is almost no original mathematical result in what we cover.

We have chosen these two topics because we believe that they provide a useful set of ideas and methods for readers who are interested in engaging with the field. The following are some highlights:

- universal coupling arguments which yield recursive lower bounds for the contact process on graphs (Chapter 3);
- the relevance of star graphs for the analysis of survival and metastability (Chapters 3,4,5), including the proof of the theorem by Huang and Durrett that there is no extinction phase on BGW trees whose offspring distribution has no finite exponential moment;
- the elegant technique by Bhamidi, Nam, Nguyen and Sly to prove the converse of the result mentioned in the previous item: there is an extinction phase on Bienaymé–Galton–Watson trees whose offspring distribution has some finite exponential moment;
- the analysis of the "optimal strategy for survival" for the contact process with small infection rate in power law random graphs (Chapter 5).

Because this text may be of interest to readers with little familiarity with the jargon and methods of interacting particle systems, an effort was made to keep the exposition accessible and self-contained. In particular, we make very little reference to infinitesimal generators, and base most of the arguments on Poisson graphical constructions.

We now elaborate further on our two chosen topics, still informally. For precise mathematical statements, see the opening paragraphs of Chapters 3, 4 and 5.

1.2.1 Metastability on general graphs

Our treatment of this topic is done in **Chapter 3** and is based on the references [55] (Mountford, Mourrat, V. and Yao) and [64] (Schapira and V.).

Metastability is said to happen for dynamical systems that have an equilibrium, but take a very long time to arrive there, persisting in the meantime in a set of states that resembles, but is not, a real equilibrium. See for instance [59], [11] and [45] for treatments of the topic.

For the particular case of the contact process, this is manifested as follows. Assume that the process has sufficiently large infection rate, and that it evolves on a finite graph, started from all vertices infected. Then, for a long time (typically exponentially long in the number of vertices of the graph, at least), the infection stays active in many places in the graph, seemingly in equilibrium. In several cases, the process is metastable on the finite graph if and only if the infection rate is above the critical threshold for a related infinite graph. For instance: if B_n is a box of $\mathbb{Z}^d$ with side length n, then the contact process with rate λ on B_n survives for a long time (with high probability as $n \to \infty$) if and only if λ is above the critical value of the process on $\mathbb{Z}^d$. A detailed statement is given in Section 2.3.5.

The results we will present in Chapter 3 show that if $\lambda > \lambda_c(\mathbb{Z})$ (that is, if the contact process is strong enough to survive on a line, or equivalently, to be metastable on a line segment), then it is strong enough to be metastable on *arbitrary* connected graphs. The metastability is manifested both in the order of magnitude of the extinction time of the infection, and in the fact that the law of this extinction time, when properly normalized, converges to the exponential distribution. This last point is due to a loss of memory due to mixing in the quasi-equilibrium (called *thermalization* in the context of metastability theory).

1.2.2 Behavior on Bienaymé–Galton–Watson trees

Our treatment of this topic is done in **Chapters 4 and 5**, and is based on the references [35] (Huang and Durrett), [9] (Bhamidi, Nam, Nguyen and Sly) and [55] (Mountford, V. and Yao).

Recall that a *Bienaymé–Galton–Watson (BGW) tree* with offspring distribution p (where p is a probability distribution on $\{0, 1, \ldots\}$) is a random rooted tree $\mathcal{T}$ characterized by the recursive prescription that, given the tree up to generation n, the numbers of children of each individual at generation n are independent and all distributed as p. If the mean of p is larger than 1, then the tree is infinite with positive probability; we assume this to be the case in all that follows.

As mentioned above, BGW trees were the first class of random graphs in which the contact process was studied, in [61]. BGW trees are obtained as local limits (in the Benjamini–Schramm sense, [5]) of several important classes of random graphs, such as the configuration model and the Erdős–Renyi random graph (see [71] for a general treatment of random graphs). For this reason, many questions concerning the contact process on these random graphs can be reduced to questions about the process on BGW trees.

If the offspring distribution p is sufficiently heavy-tailed (in particular, if it is a power law), then the contact process on $\mathcal{T}$ can survive forever with positive probability, *even when the rate* λ *is very close to zero* (so, there is no "extinction regime"). This is due to the presence of very-high degree vertices ("stars"), which manage to sustain the infection for a long time, and send the infection to each other. This phenomenon has a counterpart on the aforementioned random graph models that converge to $\mathcal{T}$. Still assuming that the associated offspring distribution is heavy-tailed, the contact process on these graphs survives for a very long time (i.e. it is metastable), for arbitrary λ (no matter how small). This fact was identified by Chatterjee and Durrett for the configuration model in [19], contradicting earlier predictions in the Physics literature, which announced that fast extinction should occur for λ small enough.

In light of this discussion, a natural question is raised: for which choices of the offspring distribution p does the contact process on $\mathcal{T}$ have an extinction regime? This question was answered completely: there is an extinction regime if and only if p has some final exponential moment; the "if" part was proved in [9], and the "only if part" in [35]. The proof is presented fully in Chapter 4.

Next, we focus on the case where p is a power law distribution (that is, $p(m)$ is of order m^α as $m \to \infty$, for some $\alpha > 1$ (still assuming that the mean of p is larger than 1). Such a distribution has no finite exponential moment, so, by the result mentioned in the previous paragraph, the contact process on $\mathcal{T}$ has no extinction regime. Let $\mathcal{P}(\lambda)$ denote the probability that, for the contact process with rate λ on this graph, started with only the root infection, the infection never vanishes from the tree.

While it is true that $\mathcal{P}(\lambda) > 0$ for any λ, this value tends to zero as $\lambda \to 0$ (survival of the infection becomes a rare event). We can then study the speed of this convergence to zero. This is interesting, because it requires us to understand the most likely ways in which the rare event can happen: we think of this intuitively as finding the "best strategy" for the infection to survive. It turns out that as λ tends to zero, $\mathcal{P}(\lambda)$ is of order $\lambda^{f(\alpha)}$, where $f(\alpha)$ is given explicitly, and takes different forms depending on the value of the power law exponent α. This reflects the fact that the presence and relative locations of stars undergoe some dramatic changes as α varies, and this has an impact on survival strategies for the infection. We give the precise statement, and the full proof, in Chapter 5.

In this same chapter, specifically in Section 5.1, we explain that this survival probability on the BGW tree (or on other classes of infinite power law random graphs) can be recovered as *metastable densites* of the contact process on finite random graph models, and the asymptotic behavior of $\mathcal{P}(\lambda)$ is then reencountered in this density. As we explain there, this has been verified for a variety of models (configuration model, preferential attachment graph, random hyperbolic graph, random geometric graph), suggesting universality of the survival strategies.

1.3 Summary of notation

We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let A be a set. We denote by $|A|$ its cardinality, and by $\mathbb{1}_A$ its indicator function.

Lebesgue measure on $\mathbb{R}^d$ is denoted $\mathrm{Leb}(\cdot)$.

Letting X and Y be random elements of a measurable space $(E, \mathcal{E})$ with a partial order, we write $X \preceq Y$ if X is stochastically dominated by Y. The same notation is employed for pairs of measures in $(E, \mathcal{E})$.

All graphs we consider have countable sets of vertices. Given a graph $G = (V, E)$, if $\{u, v\} \in E$, we write $u \sim v$. The *degree* of vertex u, denoted $\deg(u)$, is the number of neighbors of u. We denote by $\mathrm{dist}(u, v)$ the graph distance between u and v. We then let $B(v, r)$ be the ball with centre v and radius r with respect to $\mathrm{dist}(\cdot, \cdot)$. The *diameter* of G is $\mathrm{diam}(G) := \max_{u,v \in V} \mathrm{dist}(u, v)$.

As already mentioned, for any set S, we often identify an element $\xi \in \{0, 1\}^S$ with the set $\{x \in S : \xi(s) = 1\}$. In particular, $|\xi|$ is the cardinality of the set of x such that $\xi(x) = 1$.

We will often employ notation pertaining to the graphical construction of the contact process; this is introduced in Section 2.3. Other notations that are repeated many times in the text are those for the global and local survival events, $\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot)$ and $\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)$, respectively (introduced in Section 2.3.2) and for the extinction time of the contact process, τ_G (introduced in Section 2.3.4).

Acknowledgements. I would like to thank Maria Eulalia Vares for the prolonged encouragement to prepare this work, Isabella Gonçalves de Alvarenga, Leonardo Rolla and Natalia Cardona-Tobón for proof-reading several sections and giving helpful suggestions, and the anonymous referee for providing valuable feedback in a short period of time.

Chapter 2

Toolbox and preliminaries on the contact process

In this chapter, we present a couple of tools that will be useful along the way (the stochastic domination result by Liggett–Schonmann–Stacey [50] and an overview of oriented percolation) and a crash course on the contact process.

2.1 Order and domination in product spaces

Let S be a set. We endow $\{0,1\}^S$ with the partial order given by declaring that $\eta \preceq \eta'$ when $\eta(x) \le \eta'(x)$ for all $x \in S$. A function $f : \{0,1\}^S \to \mathbb{R}$ is called *increasing* if, whenever we have $\eta \preceq \eta'$, we also have $f(\eta) \le f(\eta')$. A set $A \subseteq \{0,1\}^S$ is called *increasing* if its indicator function $\mathbb{1}_A$ is increasing. Given two measures μ, ν on the product σ-algebra of $\{0,1\}^S$, we say $\mu \preceq \nu$ if $\mu(A) \le \nu(A)$ for every increasing measurable set A.

Let π_p denote the Bernoulli product measure on $\{0,1\}^S$. The FKG inequality (see for instance [33]) states that, if A and B are measurable and increasing, then $\pi_p(A \cap B) \ge \pi_p(A) \cdot \pi_p(B)$.

The well-known work of Liggett, Schonmann and Stacey [50] provides tools to establish stochastic domination relations between, on the one hand, a collection of Bernoulli random variables with some mild dependence, and on the other hand, a collection of *independent* Bernoulli random variables. Here, we favour simplicity and give a version of such domination statement containing a non-quantitative bound, and far from the level of generality of the main results of [50].

Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. We say that a collection of random variables $\{X(v) : v \in V\}$ is *k-dependent* if, for every $A, B \subset V$

with $\min_{u\in A, v\in B} \mathrm{dist}(u,v) > k$, the collections $\{X(v) : v \in A\}$ and $\{X(v) : v \in B\}$ are independent.

Theorem 2.1 ([50]). *For any $\Delta \geq 2$, $k \in \mathbb{N}$, and $\varepsilon > 0$, there exists $p > 0$ such that the following holds. Let $G = (V, E)$ be an infinite connected graph in which vertex degrees are at most Δ. Let $\{X(v) : v \in V\}$ be a k-dependent collection of random variables with $\mathbb{P}(X(v) = 1) \geq p$ for every v. Then, $\{X(v) : v \in V\}$ stochastically dominates the Bernoulli product measure with density $1 - \varepsilon$ on V.*

2.2 Oriented percolation

Due to the Poisson graphical construction, which we will introduce in Section 2.3.1, the contact process can be seen as a continuous-time oriented percolation model. Hence, many tools and results from oriented percolation are readily adapted to the contact process.

As with regular percolation, oriented percolation models come in many shapes. Here, we will be interested in models in the same spirit as those treated by Durrett in the important reference [23]. We focus the analysis on oriented bond percolation, but will also need to say a few works about oriented site percolation.

Oriented bond percolation. Define the directed graph with set of vertices

$$\Lambda := \{(m, n) \in \mathbb{Z} \times \mathbb{N}_0 : \ m + n \text{ is even}\}$$

and set of oriented bonds

$$\vec{E}_\Lambda := \{\langle (m, n), (m', n+1)\rangle : \ (m, n) \in \Lambda, \ |m' - m| = 1\}.$$

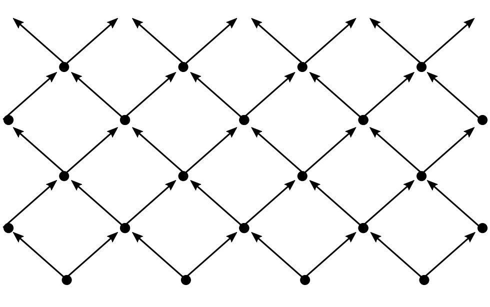


Figure 2.1: The directed graph $(\Lambda, \vec{E}_\Lambda)$

We fix $p \in [0,1]$ and declare that each bond in the above set is open with probability p and closed with probability $1-p$, all independently. The *oriented cluster* of a vertex $(m,n) \in \Lambda$ is defined as

$$\mathcal{C}_{\text{bond}}(m,n) := \{(m,n)\} \cup \left\{ \begin{array}{l} (m',n') : (m',n') \text{ can be reached} \\ \text{from } (m,n) \text{ by a path that} \\ \text{only traverses open edges} \end{array} \right\}.$$

Let $\mathbb{P}_p$ be a probability under which this model is defined. It is proved in [23] that

$$p_c := \sup\{p \in [0,1] : \ \mathbb{P}_p(|\mathcal{C}_{\text{bond}}(0,0)| = \infty) = 0\} \in (0,1). \tag{2.1}$$

It will be useful for us to briefly discuss the proof of this. Showing that $p_c > 0$ is easy; in fact we have

$$\mathbb{P}_p(|\mathcal{C}_{\text{bond}}(0,0)| = \infty) = \lim_{n\to\infty} \mathbb{P}_p(\exists m : (m,n) \in \mathcal{C}_{\text{bond}}(0,0)) \le \lim_{n\to\infty} 2^n p^n,$$

where the inequality is a union bound over all paths; this gives $p_c \ge 1/2$.

The proof of $p_c < 1$ involves a *contour argument*, which we now sketch. We assume that Λ is embedded in $\mathbb{R}^2$ and consider the ℓ_1-balls $B_1(m,n) := \{(x,y) \in \mathbb{R}^d : \|(x,y)-(m,n)\|_1 \le 1\}$. The topological boundary of the set $\cup_{(m,n)\in\mathcal{C}(0,0)} B_1(m,n)$ is the union of line segments of the form

$$\{(m+t, n+t) : t \in [0,1]\} \quad \text{or} \quad \{(m-t,n+t) : t \in [0,1]\}, \tag{2.2}$$

with $(m,n) \in \Lambda \cup \{(m,-1) : m \text{ is odd}\}$. Suppose that $\{|\mathcal{C}_{\text{bond}}(0,0)| < \infty\}$ occurs, and we traverse this boundary in the counterclockwise direction, starting from the origin; when doing so, we construct a sequence $\Gamma = (\vec{e}_1, \dots, \vec{e}_k)$, where each $\vec{e}_i$ is a line segment as in (2.2), together with the orientation in which it is traversed. We call Γ the *contour* of the cluster.

Let γ be one of the possible realizations of the contour Γ (still assuming $\{|\mathcal{C}_{\text{bond}}(0,0)| < \infty\}$ occurs). Note that each of the steps of γ points in one of four directions: north-east, north-west, south-east or south-west. In this order, let $\text{NE}(\gamma)$, $\text{NW}(\gamma)$, $\text{SE}(\gamma)$, and $\text{SW}(\gamma)$ denote the number of steps of γ in each direction.

Noting that γ starts and ends at zero, the number of steps towards the east (either north or south) must equal the number of steps towards the west (either north or south), so

$$\text{NW}(\gamma) + \text{SW}(\gamma) = \text{NE}(\gamma) + \text{SE}(\gamma) = \frac{|\gamma|}{2},$$

where $|\gamma|$ is the total number of steps.

It is easy to see each of the north-west and south-west steps crosses a closed bond of $\vec{E}_\Lambda$. This shows that

$$\mathbb{P}(\Gamma = \gamma) \le (1-p)^{\text{NW}(\gamma)+\text{SW}(\gamma)} = (1-p)^{|\gamma|/2}.$$

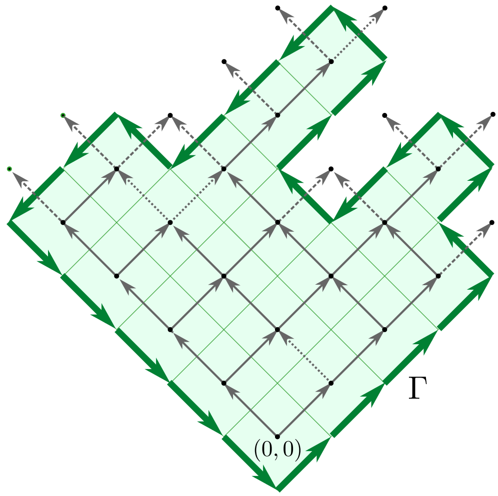


Figure 2.2: The contour of a finite cluster.

Given k, the number of choices for γ with $|\gamma| = k$ is at most 3^k, since in each step of γ we need to choose one out of at most three possible directions.

Putting these considerations together, we obtain

$$\mathbb{P}_p(|\mathcal{C}_{\text{bond}}(0,0)| < \infty) = \sum_{k \in 2\mathbb{N}} \sum_{\gamma : |\gamma| = k} \mathbb{P}_p(\Gamma = \gamma) \leq \sum_{k \in 2\mathbb{N}} 3^k \cdot (1-p)^{k/2}.$$

The right-hand side is smaller than 1 when $p > \frac{17}{18}$, so this number is an upper bound for p_c.

We will also need the following result.

Proposition 2.2. *For any $\varepsilon > 0$ there exists $\bar{p}(\varepsilon) \in (0,1)$ such that, if $p \in (\bar{p}(\varepsilon), 1]$, then with probability above $1 - \varepsilon$, there is an infinite open path started at the origin which never visits $\{(m,n) : m < 0\}$.*

Proof. Let $p \in (0,1)$ be close to 1, in a way that will be chosen later. Define

$$\mathcal{C}'(0,0) := \left\{ \begin{array}{l} (m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 : \text{ there is an open path from } (0,0) \\ \text{to } (m,n) \text{ that never visits the negative half-line} \end{array} \right\}$$

(with $(0,0) \in \mathcal{C}'(0,0)$). Consider the contour Γ of $\cup_{(m,n) \in \mathcal{C}'(0,0)} B_1(m,n)$, oriented counterclockwise. Let Γ' be the portion of this contour starting at $(1,0)$ and ending at the first visit to the y-axis. With same notation as earlier, letting γ' be a possible realization of Γ', we have

$$\begin{aligned}&\mathrm{NW}(\gamma') + \mathrm{SW}(\gamma') + \mathrm{NE}(\gamma') + \mathrm{SE}(\gamma') = |\gamma'|,\\ &\mathrm{NW}(\gamma') + \mathrm{SW}(\gamma') - (\mathrm{NE}(\gamma') + \mathrm{SE}(\gamma')) = 1.\end{aligned}$$

This gives

$$\mathrm{NW}(\gamma') + \mathrm{SW}(\gamma') = \frac{|\gamma'|+1}{2}.$$

Since there is a one-to-one mapping from NW and SW steps of Γ' to closed percolation bonds, we have

$$\begin{aligned}\mathbb{P}_p(|\mathcal{C}'(0,0)| < \infty) &\le \sum_{k\ge 1} \sum_{\gamma' : |\gamma'| = k} \mathbb{P}_p(\Gamma' = \gamma')\\ &\le \sum_{k \ge 1} 3^k \cdot (1-p)^{(k+1)/2}.\end{aligned}$$

The right-hand side above can be made arbitrarily small by taking p close to 1. $\square$

We will need the following consequence of the above findings.

Corollary 2.3. *Given $\varepsilon > 0$, taking $\bar{p}(\varepsilon)$ as in Proposition 2.2, for any $p > \bar{p}(\varepsilon)$ we have*

$$\liminf_{n\to\infty} \mathbb{P}_p\left(\exists \text{ open path from } (0,0) \text{ to } (0,2n)\right) > (1-\varepsilon)^2. \tag{2.3}$$

Proof. For any $n \in \mathbb{N}$, define the events

$$A_n := \left\{ \begin{array}{l} \exists \text{ open path from } (0,0) \text{ to } \mathbb{Z}\times\{2n\} \\ \text{that never touches the negative half-line} \end{array} \right\};$$

$$B_n := \left\{ \begin{array}{l} \exists \text{ open path from } \mathbb{Z}\times\{0\} \text{ to } (0,2n) \\ \text{that never touches the negative half-line} \end{array} \right\}.$$

On $A_n \cap B_n$, there exists an open path from $(0,0)$ to $(0,2n)$. We have $\mathbb{P}_p(A_n) = \mathbb{P}_p(B_n)$ by time reversal, and

$$\mathbb{P}_p(A_n) \ge \mathbb{P}_p\left(\begin{array}{l} \text{there is an infinite open path from } (0,0) \text{ that} \\ \text{never touches the negative half-line} \end{array} \right) > 1-\varepsilon,$$

by Proposition 2.2. By the FKG inequality, we obtain $\mathbb{P}_p(A_n \cap B_n) \ge \mathbb{P}_p(A_n)\cdot\mathbb{P}_p(B_n)$, concluding the proof. $\square$

Oriented site percolation. We define the directed graph $(\Lambda, \vec{E}_\Lambda)$ as above. This time, nothing is done to the edges (say they are all left open), while every vertex is declared open with probability p and closed with probability $1-p$, all independently. The cluster of a vertex (m,n) is defined as

$$\mathcal{C}_{\text{site}}(m,n) := \{(m,n)\} \cup \left\{ \begin{array}{l} (m',n') : \ (m',n') \text{ can be reached} \\ \text{from } (m,n) \text{ by a path that} \\ \text{only visits open vertices} \end{array} \right\}.$$

In order to make the parameter p explicit, we write

$$\mathcal{C}_{\text{bond}}(m,n) = \mathcal{C}_{\text{bond}}(m,n;p), \quad \mathcal{C}_{\text{site}}(m,n) = \mathcal{C}_{\text{site}}(m,n;p).$$

We claim that

$$\mathcal{C}_{\text{site}}(0,0;p^2) \preceq \mathcal{C}_{\text{bond}}(0,0;p). \tag{2.4}$$

To prove this, we construct a coupling as follows. Start with a probability space in which we have defined a collection of random variables $X = \{X(\vec{e}) : \vec{e} \in \vec{E}_\Lambda\}$ with $X(\vec{e}) \sim \text{Bernoulli}(p)$, all independent. Let $\mathscr{C}^X_{\text{bond}}$ be the cluster of the origin in bond percolation obtained from X. Additionally, use X to construct a site percolation configuration, by declaring a vertex to be open if both edges starting from it are open. Let $\mathscr{C}^X_{\text{site}}$ be the cluster of the origin in this site percolation configuration. We then clearly have

$$\mathscr{C}^X_{\text{site}} \subseteq \mathscr{C}^X_{\text{bond}}, \quad \mathscr{C}^X_{\text{site}} \overset{\text{(distr)}}{=} \mathcal{C}_{\text{site}}(0,0;p^2), \quad \mathscr{C}^X_{\text{bond}} \overset{\text{(distr)}}{=} \mathcal{C}_{\text{bond}}(0,0;p).$$

Next, we claim that

$$\mathbb{P}(|\mathcal{C}_{\text{bond}}(0,0;p)| = \infty) \le \mathbb{P}(|\mathcal{C}_{\text{site}}(0,0;1-(1-p)^2)| = \infty). \tag{2.5}$$

To prove this, let X and $\mathscr{C}^X_{\text{bond}}$ be as in the proof of the previous claim. Define a site percolation configuration from X by declaring that a vertex is open if at least one of the edges starting from it is open. Let $\hat{\mathscr{C}}^X_{\text{site}}$ be the cluster of the origin for this configuration. We have

$$\mathscr{C}^X_{\text{bond}} \overset{\text{(distr)}}{=} \mathcal{C}_{\text{bond}}(0,0;p), \quad \hat{\mathscr{C}}^X_{\text{site}} \overset{\text{(distr)}}{=} \mathcal{C}_{\text{site}}(0,0;1-(1-p)^2).$$

Note that we cannot guarantee that $\mathscr{C}^X_{\text{bond}} \subseteq \hat{\mathscr{C}}^X_{\text{site}}$; for instance, in the case where the bonds leaving the origin are open, but the bonds leaving $(-1,1)$ and $(1,1)$ are all closed, we have $\mathscr{C}^X_{\text{bond}} = \{(0,0),(-1,1),(1,1)\}$ and $\hat{\mathscr{C}}^X_{\text{site}} = \{(0,0)\}$. However, it is still true that

$$\{|\mathscr{C}^X_{\text{bond}}| = \infty\} \subseteq \{|\hat{\mathscr{C}}^X_{\text{site}}| = \infty\},$$

and this is sufficient to obtain (2.5).

Now, combining (2.4) and (2.5) with our previous knowledge that bond percolation has a non-trivial phase transition, we conclude that the same is true for site percolation.

2.3 The contact process

In this section, we briefly go over the fundamentals of the theory of the contact process. The aim is to provide an accessible overview, with some proofs along the way (especially the ones that are simple or enlightening), as well as a gathering of material that will be necessary in the next chapters.

The standard references on the topic are the two books by Liggett, "Interacting particle systems" [47] and "Stochastic interacting systems" [49]. The first one is an early account, so it predates many of the most important developments; it also focusses on the one-dimensional case. The second is more recent and contains almost everything that will be mentioned in this section.

2.3.1 Definition and first properties

The formal construction of a process corresponding to the dynamics described in the introduction can be done either via a Poisson graphical construction or via an infinitesimal pre-generator (which produces a Markov semi-group with the Hile-Yosida Theorem). The former is easier to define and more intuitively appealing, and will serve for all our purposes, so we restrict ourselves to it.

Let $G = (V, E)$ be a locally finite graph (that is, a graph in which all vertices have finite degree) and $\lambda > 0$. Let

$$H = \{(\mathcal{R}^x)_{x \in V},\ (\mathcal{T}^{(x,y)})_{x,y \in V:\ \{x,y\} \in E}\}$$

be a family of independent Poisson point processes on $[0, \infty)$, all independent, as follows:

- each $\mathcal{R}^x$ with intensity 1 (*recovery times* at x, to be drawn as "$\times$" marks over x);
- each $\mathcal{T}^{(x,y)}$ with intensity λ (*transmission times* from x to y, to be drawn as arrows from x to y).

The collection H is called *graphical construction*, *Harris construction*, or *Harris system.*

Given H and an initial configuration, we can construct the contact process, using the notion of *infection paths.* An infection path of H is a function $\gamma : I \to V$ (where I is a time interval) such that

- γ does not touch recovery marks, that is,

$$t \notin \mathcal{R}^{\gamma(t)} \text{ for all } t;$$

- γ can only jump by traversing arrows, that is,

$$\gamma(t) \neq \gamma(t-) \quad \Longrightarrow \quad t \in \mathcal{T}^{(\gamma(t-),\gamma(t))}.$$

We adopt the following notation:

- given $x, y \in V$ and $s \leq t$, we write $(x,s) \rightsquigarrow (y,t)$ if there is an infection path $\gamma : [s,t] \to V$ with $\gamma(s) = x$, $\gamma(t) = y$;

- given $A \subseteq V$ and y, s, t as above, we write $A \times \{s\} \rightsquigarrow (y,t)$ if $(x,s) \rightsquigarrow (y,t)$ for some $x \in A$;

- similarly, with obvious meanings, we write $(x,s) \rightsquigarrow B \times \{t\}$ and $A \times \{s\} \rightsquigarrow B \times \{t\}$;

- given $A \subseteq V$, if we can find an infection path γ from (x,s) to (y,t) such that $\gamma(r) \in A$ for all r, we say that $(x,s) \rightsquigarrow (y,t)$ *inside* A;

- we write $(x,t) \rightsquigarrow \infty$ if $(x,t) \rightsquigarrow V \times \{t'\}$ for all $t' \geq t$; we write $A \times \{t\} \rightsquigarrow \infty$ if $(x,t) \rightsquigarrow \infty$ for some $x \in A$.

Then, given $A \subseteq V$, by setting

$$\xi_t(x) := \mathbb{1}\{A \times \{0\} \rightsquigarrow (x,t)\}, \quad x \in V,\ t \geq 0,$$

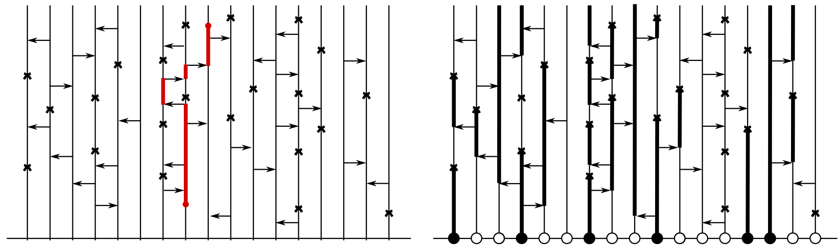
we obtain $(\xi_t)_{t \geq 0}$, the contact process on G with infection rate λ and $\xi_0 = \mathbb{1}_A$.

Let us repeat that we identify an element $\xi \in \{0,1\}^V$ with the set $\{x : \xi(x) = 1\}$. In particular, we write $\xi_0 = \mathbb{1}_A$ or $\xi_0 = A$ interchangeably. We also denote the identically-zero element of $\{0,1\}^V$ by $\varnothing$.

Regarding the notation for the initial configuration: it is a common practice to include the set of initially infected vertices as a super-index in the notation, so that the process started from $\mathbb{1}_A$ would be denoted $(\xi_t^A)_{t \geq 0}$. We sometimes adopt this practice, but most of the times we denote the process by $(\xi_t)_{t \geq 0}$ and explicitly state what the initial configuration is. When it is not explicitly stated, it is unimportant.

When we start the contact process from a random initial configuration, we always assume that this configuration is independent of the graphical construction.

In the figure below (where the graph is $\mathbb{Z}$), the left side shows an infection path, and the right side shows the evolution of the contact process (initially infected sites are represented by black balls, and infected areas are depicted with thicker black lines).



In graphs where degrees grow rapidly, one could encounter situations of "explosion", where the process starts from finitely many infected vertices, and reaches infinitely many infected vertices in a finite amount of time. We do not regard this case as problematic nor attempt to exclude it.

The graphical construction allows us to construct, in a single probability space, contact processes started from all possible initial configurations (*universal coupling*). It also makes many important properties easy to check. We now list some of them.

- *Additivity.* For any $A \subseteq V$, $A \neq \varnothing$, we have

$$\xi_t^A = \cup_{x \in A} \xi_t^{\{x\}}.$$

- *Attractivity.* For any $A, B \subseteq V$ with $A \subseteq B$, we have

$$\xi_t^A \leq \xi_t^B \quad \text{for all } t \geq 0.$$

  Hence, $(\xi_t^A)_{t\geq 0}$ is stochastically dominated by $(\xi_t^B)_{t\geq 0}$. It is also possible to check that if $\lambda < \lambda'$, then $(\xi_t^A)_{t\geq 0}$ with rate λ is stochastically dominated by $(\xi_t^A)_{t\geq 0}$ with rate λ'. To do so, we take a graphical construction $H = \{(\mathcal{R}^x), (\mathcal{T}^{(x,y)})\}$, we sample an independent set of extra arrows $(\tilde{\mathcal{T}}^{(x,y)})$ with rate $\lambda' - \lambda$, and evolve one of the process using only the arrows in $(\mathcal{T}^{(x,y)})$, whereas the larger process is also allowed to use the extra arrows. Similar considerations show that, if G' is a subgraph of G, then the contact process on G' is stochastically dominated by the contact process on G with the same infection rate and same set of initially infected vertices.

- *Absorbing state.* If $\xi_t \neq \varnothing$, then $\xi_s = \varnothing$ for all $s \geq t$.

- *Duality.* For any $A, B \subseteq V$ and any $t \geq 0$, we have

$$\begin{aligned}\mathbb{P}(\xi_t^A \cap B \neq \varnothing) &= \mathbb{P}(A \times \{0\} \rightsquigarrow B \times \{t\}) \\ &= \mathbb{P}(B \times \{0\} \rightsquigarrow A \times \{t\}) = \mathbb{P}(\xi_t^B \cap A \neq \varnothing).\end{aligned} \tag{2.6}$$

Applying this with $A = V$ and $B = \{x\}$ gives

$$\mathbb{P}(\xi_t^V(x) = 1) = \mathbb{P}(\xi_t^{\{x\}} \neq \varnothing).$$

A probability measure μ on $\{0,1\}^V$ is said to be *stationary* for the contact process if, starting the process from a random configuration $\xi_0 \sim \mu$, we have that $\xi_t \sim \mu$ for all t. From the observation above that the all-zero configuration is absorbing, we see that the delta measure associated to this configuration, denoted $\delta_\varnothing$, is stationary. It can be seen as the "lower stationary measure" of the process, because any other stationary measure dominates it from above. We also have:

Proposition 2.4 (Upper stationary measure). *Let $(\xi_t)_{t\geq 0}$ be the contact process on $G = (V,E)$ started from the fully infected configuration. As $t \to \infty$, ξ_t converges in distribution to a probability measure $\bar{\mu}$ on $\{0,1\}^V$. This measure is stationary for the contact process on G. Moreover, if ν is some other stationary measure for the contact process on G, then $\nu \preceq \bar{\mu}$.*

Proof. Since the process is started from the identically-1 configuration, for any $A \subseteq V$ we have

$$\begin{aligned}
\mathbb{P}(\xi_t(x) = 1\ \forall x \in A) &= \mathbb{P}(V \times \{0\} \rightsquigarrow (x,t)\ \forall x \in A) \\
&= \mathbb{P}((x,0) \rightsquigarrow V \times \{t\}\ \forall x \in A) \\
&\xrightarrow{t\to\infty} \mathbb{P}((x,0) \rightsquigarrow \infty\ \forall x \in A),
\end{aligned}$$

where in the second equality we have used duality, (2.6). This shows that ξ_t converges in distribution, as $t \to \infty$, to the distribution of the random configuration $\zeta \in \{0,1\}^V$ given by

$$\zeta(x) := \mathbb{1}\{(x,0) \rightsquigarrow \infty\}, \quad x \in V.$$

We denote this measure by $\bar{\mu}$.

Now assume that (ζ', H) are sampled independently, where $\zeta' \sim \bar{\mu}$ and H is a Harris system. Let $(\xi'_t)_{t\geq 0}$ be the contact process with $\xi'_0 = \zeta'$ and constructed from H. Then, for any $A \subseteq V$,

$$\mathbb{P}(\xi'_t(x) = 1\ \forall x \in A) = \mathbb{P}\left(\bigcap_{x\in A}\bigcup_{y\in V}\{\zeta'(y) = 1,\ (y,0) \rightsquigarrow (x,t)\}\right).$$

Again by duality, the right-hand side equals

$$\mathbb{P}\left(\bigcap_{x\in A}\bigcup_{y\in V}\{(x,0) \rightsquigarrow (y,t),\ \zeta'(y) = 1\}\right).$$

Using the Markov property, this equals

$$\mathbb{P}\left(\bigcap_{x\in A}\bigcup_{y\in V}\{(x,0)\rightsquigarrow(y,t),\ (y,t)\rightsquigarrow\infty\}\right)=\mathbb{P}\left(\bigcap_{x\in A}\{(x,0)\rightsquigarrow\infty\}\right)$$
$$=\bar{\mu}(\{\xi:\xi(x)=1\ \forall x\in A\}).$$

This proves that $\bar{\mu}$ is stationary.

To prove the last statement, again let $(\xi_t)_{t\geq 0}$ be the process started from the identically-1 configuration, and let $(\xi''_t)_{t\geq 0}$ be the process started from $\xi''_0\sim\nu$, both obtained from the same graphical construction. For all t we have $\xi''_t(x)\leq\xi_t(x)$ for all x, so the law of ξ''_t, which is ν, is stochastically dominated by the law of ξ_t. By taking $t\to\infty$, this gives $\nu\preceq\bar{\mu}$. □

The measures $\delta_\varnothing$ and $\bar{\mu}$ are in some cases equal; see Theorem 2.9 below for the case of $G=\mathbb{Z}^d$.

2.3.2 Survival regimes

Let $G=(V,E)$ be a graph and $(\xi_t)_{t\geq 0}$ be the contact process on G (with some arbitrary initial configuration). We define the *global survival event*:

$$\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot):=\{\forall t\geq 0\ \exists v\in V:\ \xi_t(v)=1\}.$$

We also define the *local survival event*:

$$\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot):=\{\forall s\geq 0,\ v\in V\ \exists t\geq s:\ \xi_t(v)=1\}.$$

Note that $\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)\subseteq\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot)$.

Recall that for $A\subset V$, we write (ξ^A_t) for the process with $\xi_0=\mathbb{1}_A$. Under the assumption that G is connected, it is easy to check (using for example the graphical construction) that, for a given choice of λ,

$$\text{either } \mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi^A_\cdot))>0 \text{ for all finite and non-empty } A,$$
$$\text{or } \mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi^A_\cdot))=0 \text{ for all finite and non-empty } A,$$

and similarly for $\mathrm{Surv}_{\mathrm{loc}}(\xi^A_\cdot)$. Still assuming that G is connected, we say that the contact process with rate λ on G

- *dies out* if $\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi^A_\cdot))=0$ for all finite and non-empty A (we also refer to this as the *extinction regime*;
- *survives*, or *survives globally*, if $\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi^A_\cdot))>0$ for all finite and non-empty A (also called the *survival regime*);
- *survives locally* if $\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi^A_\cdot))>0$ for all finite and non-empty A (also called the *local survival regime*)

(we restrict this definition to connected graphs, as it is not very natural to assign one of the above categories to graphs with several connected components).

We also record the following as a lemma:

Lemma 2.5. *If all vertices of G have degree smaller than d and $\lambda < 1/d$, then the contact process with rate λ on G dies out.*

Proof. For $\xi \in \{0,1\}^V$, we write $|\xi| := \sum_x \xi(x)$, the number of infections in ξ. Assume that A is finite and non-empty. The process $(|\xi_t^A|)_{t\geq 0}$ is stochastically dominated by the continuous-time Markov chain $(X_t)_{t\geq 0}$ on $\mathbb{N}_0$ with $X_0 = |A|$ and jump rates

$$r(0,n) = 0\ \forall n, \qquad \text{and for } n \geq 1,\ r(n,n-1) = n,\ r(n,n+1) = d\lambda n.$$

Let $t_0 = 0$ and, for $n \in \mathbb{N}_0$, let us define $t_{n+1} := \inf\{t > t_n : X_t \neq X_{t_n}\}$. Then, $(X_{t_n})_{n\in\mathbb{N}_0}$ is a random walk on $\mathbb{N}_0$ that is absorbed at zero, and otherwise jumps to the left with probability $\frac{1}{1+d\lambda}$ and to the right with probability $\frac{d\lambda}{1+d\lambda}$. Since $1 > d\lambda$, this chain has a bias towards the right, so it is absorbed at zero almost surely. □

Again fix a connected graph G. Note that for any $A \subset V$, the probabilities $\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot^A))$ and $\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot^A))$ are non-decreasing functions of λ. We define

$$\lambda_c(G) = \lambda_c^{(1)}(G) := \inf\left\{\begin{array}{c}\lambda > 0 : \text{the contact process}\\ \text{on } G \text{ with rate } \lambda \text{ survives}\end{array}\right\}$$

and

$$\lambda_c^{(2)}(G) := \inf\left\{\begin{array}{l}\lambda > 0 : \text{the contact process}\\ \text{on } G \text{ with rate } \lambda \text{ survives locally}\end{array}\right\}.$$

2.3.3 Phase transitions on the Euclidean lattice and the regular tree

The following statement is perhaps the most well-known feature of the contact process.

Theorem 2.6 (Harris [34]). *We have $\lambda_c(\mathbb{Z}^d) \in (0,\infty)$.*

Proof. We have $\lambda_c(\mathbb{Z}^d) \geq 1/(2d)$ by Lemma 2.5. Let us prove that $\lambda_c(\mathbb{Z}^d) < \infty$. We want to prove that the process survives when λ is large enough; by monotonicity, it suffices to prove this in dimension $d = 1$. Note that

$$\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot^{\{0\}}) = \{(0,0) \rightsquigarrow \infty\}.$$

Fix $\delta > 0$. Let H be a graphical construction for the contact process on $\mathbb{Z}$. We define an oriented percolation configuration (in the graph $(\Lambda, \vec{E}_\Lambda)$ of Section 2.2) from H by setting, for each $\vec{e} = \langle (x,n),(y,n+1)\rangle \in \vec{E}_\Lambda$:

$$\eta_H(\vec{e}) = \mathbb{1}\left\{\begin{array}{l}\text{in the time interval } [\delta n, \delta(n+1)], \text{ there is no recovery}\\ \text{at } x \text{ or } y, \text{ and at least one transmission from } x \text{ to } y\end{array}\right\}.$$

Then, $\{\eta_H(x,n) : x \in \mathbb{Z},\ n \in \mathbb{N}_0\}$ is a 2-dependent oriented percolation configuration. Moreover,

$$\{\exists \text{ infinite open path from the origin in } \eta_H\} \subseteq \mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot^{\{0\}}).$$

Finally, η_H has density parameter

$$p = (e^{-\delta})^2 \cdot (1 - e^{-\delta\lambda}).$$

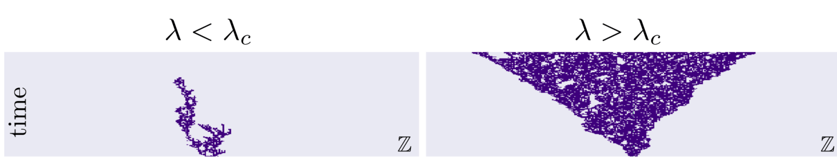
This can be made as close to 1 as desired by first taking δ small and then taking λ large. The proof is now completed using Theorem 2.1 and (2.1). $\square$

We refer to $\lambda < \lambda_c(\mathbb{Z}^d)$, $\lambda = \lambda_c(\mathbb{Z}^d)$ and $\lambda > \lambda_c(\mathbb{Z}^d)$ as the *subcritical*, *critical* and *supercritical* regimes, respectively. As mentioned in the Introduction, Bezuidenhout and Grimmett [7] proved that the contact process on $\mathbb{Z}^d$ with rate $\lambda = \lambda_c(\mathbb{Z}^d)$ dies out. Using the methods developed in that paper, it is also possible to prove that

$$\lambda_c(\mathbb{Z}^d) = \lambda_c^{(2)}(\mathbb{Z}^d).$$

This shows that the process dies out if $\lambda \leq \lambda_c(\mathbb{Z}^d)$, whereas it survives locally if $\lambda > \lambda_c(\mathbb{Z}^d)$.

The figure below shows simulations of the contact process on $\mathbb{Z}$ in the subcritical and supercritical regimes.



The right-hand side of the above figure suggests that when the one-dimensional contact process survives, the infected region grows linearly; that is indeed the case. Given $A \subseteq \mathbb{Z}$, define $R_t^A := \sup\{x : \xi_t^A(x) = 1\}$. That is, R_t^A is the rightmost infected position at time t, for the process started from $\mathbb{1}_A$. In [47, Sections VI.2 and VI.3], it is proved that $\frac{1}{t}R_t^{-\mathbb{N}_0}$ converges almost surely, as $t \to \infty$, to some value $\alpha = \alpha(\lambda) \in [-\infty, \infty)$, and moreover, $\alpha > 0$ if and only if $\lambda > \lambda_c(\mathbb{Z})$. Using an argument involving crossing of infection paths, it is easy to see that

$$\big\{\xi_t^{\{0\}} \neq \varnothing \ \forall t\big\} \subseteq \big\{R_t^{-\mathbb{N}_0} = R_t^{\{0\}} \ \forall t\big\},$$

which shows that $\frac{1}{t}R_t^{\{0\}} \to \alpha$ almost surely as $t \to \infty$ on the event that $(\xi_t^{\{0\}})$ survives. Letting $L_t^{\{0\}}$ be the leftmost infected particle at time t, by symmetry we also obtain that, on the event that $(\xi_t^{\{0\}})$ survives, $\frac{1}{t}L_t^{\{0\}} \to -\alpha$ almost surely as $t \to \infty$. This is the one-dimensional version of the *shape theorem*, Theorem 2.10 below.

With the speed α at hand, we can describe a more sophisticated comparison with oriented percolation than the one that was done in the proof of Theorem 2.6. Fix $\delta > 0$ and, for each $m \in \mathbb{N}$, define the space-time "tunnel"

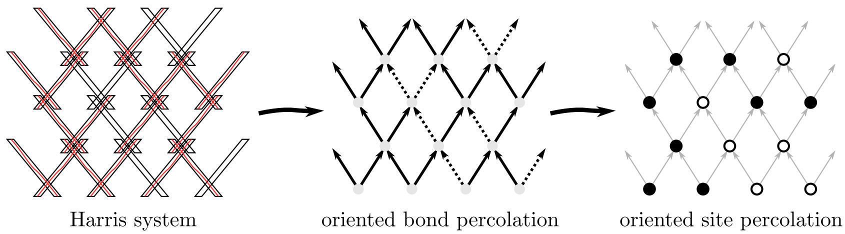
$$\mathscr{R}_m^{(\delta)} := \{(x,t) : \ 0 \le t \le m, \ -\delta m + \alpha t \le x \le \delta m + \alpha t\}.$$

The following is [47, Lemma 3.17, p. 297] (the original argument is due to Durrett in [23]):

Proposition 2.7. *For any $\lambda > \lambda_c(\mathbb{Z})$ and $\delta > 0$, we have*

$$\mathbb{P}\left(\begin{array}{c}\textit{there is an infection path from the bottom}\\ \textit{to the top of } \mathscr{R}_m^{(\delta)} \textit{ that stays inside } \mathscr{R}_m^{(\delta)}\end{array}\right) \xrightarrow{m\to\infty} 1.$$

With this at hand, from the graphical construction H of the contact process, we can define an oriented percolation configuration η_H, and then from this, a site percolation configuration $\sigma(\eta_H)$. See the figure below. The idea is that tunnels are in one-to-one correspondence with oriented bonds; a bond of η_H is declared open if its tunnel is crossed from bottom to top by an infection path that stays inside it. Then, a site is declared open in $\sigma(\eta_H)$ if the two bonds emanating from it are open in η_H, as was done in the proof of (2.4).



The oriented percolation $\sigma(\eta_H)$ produced from this is 2-dependent, and, provided $\lambda > \lambda_c(\mathbb{Z})$, its density of open sites can be made as close to 1 as desired, by taking $m \to \infty$. In particular, by Theorem 2.1 and (2.1), m can be chosen so that with positive probability, the oriented cluster of the origin in $\sigma(\eta_H)$ is infinite. It is easy to see that if the oriented cluster of the origin in $\sigma(\eta_H)$ is infinite, then there is an infinite infection path started from $\{-\delta m, \dots, \delta m\}$ in H.

One nice feature of this construction is that, even if λ is only slightly larger than $\lambda_c(\mathbb{Z})$, the oriented percolation configuration can be made as supercritical as desired.

We will make reference to the following fact:

Proposition 2.8. *We have*

$$\lambda_c(\mathbb{Z}) = \lambda_c(\mathbb{N}_0),$$

that is, the critical value of the contact process on the half-line is the same as that on the line.

The fact that $\lambda_c(\mathbb{Z}) \le \lambda_c(\mathbb{N}_0)$ is obvious, and the reverse inequality can be proved using the comparison with oriented percolation described above, combined with the fact that oriented percolation on $\mathbb{N}_0$ with sufficiently high density of open bonds has infinite clusters, as in Proposition 2.2 above. Alternatively, a different proof can be found in [1, Corollary 2.5].

A lot more is known about the contact process on $\mathbb{Z}^d$. Let us mention two highlights: the *complete convergence theorem* and the *shape theorem*. For the first, we introduce a definition.

A stationary measure for the contact process (or any interacting particle system) is called *extremal* if it cannot be obtained as a non-trivial convex combination of other stationary measures. (This is relevant because the set of stationary measures of an interacting particle system is a compact and convex set in the topology of weak convergence of measures, and thus, by the Krein–Milman theorem, this set is equal to the closed convex hull of its extremal points). The following was proved by Griffeath.

Theorem 2.9 (Complete convergence theorem [32]). *The set of extremal stationary measures of the contact process on $\mathbb{Z}^d$ equals $\{\delta_\varnothing, \bar{\mu}\}$. The two measures in this set are distinct if and only if $\lambda > \lambda_c(\mathbb{Z}^d)$. Moreover, for any $A \subseteq \mathbb{Z}^d$, as $t \to \infty$, the random configuration ξ_t^A converges in distribution to the convex combination $\beta_A \cdot \bar{\mu} + (1-\beta_A)\cdot \delta_\varnothing$, where $\beta_A := \mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot^A))$.*

Concerning the supercritical process, the following is known:

Theorem 2.10 (Shape theorem [25], [24]). *Let $(\xi_t^{\{0\}})_{t\ge 0}$ be the contact process on $\mathbb{Z}^d$ with $\lambda > \lambda_c(\mathbb{Z}^d)$. Define the random subset of $\mathbb{R}^d$:*

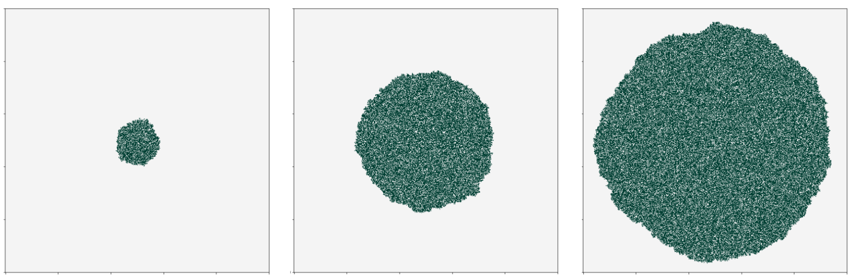
$$K_t := \bigcup_{s \le t} \bigcup_{x \in \xi_s^{\{0\}}} (x + [-\tfrac{1}{2}, \tfrac{1}{2}]^d).$$

Then, there exists a (deterministic) compact and convex set $K \subset \mathbb{R}^d$ such that, for any $\epsilon > 0$,

$$\mathbb{P}\left(\exists T \geq 0 : \forall t \geq T, \ (1-\epsilon)t \cdot K \subseteq K_t \subseteq (1+\epsilon)t \cdot K \mid \xi_s^{\{0\}} \neq \varnothing \ \forall s\right) = 1.$$

A first version for this, which guaranteed the convergence only for λ large enough, was proved by Durrett and Griffeath in [25]. To obtain the result for all $\lambda > \lambda_c(\mathbb{Z}^d)$, the renormalization scheme of Bezuidenhout and Grimmett was needed; the implementation of this was done by Durrett in [24].

Below is a simulation of the contact process on $\mathbb{Z}^2$, illustrating the shape theorem.



We now very briefly turn to the infinite d-regular tree $\mathbb{T}^d$ (with $d \geq 3$). The following has been proved:

Theorem 2.11 (Pemantle [60] and Liggett [48]). *We have $0 < \lambda_c^{(1)}(\mathbb{T}^d) < \lambda_c^{(2)}(\mathbb{T}^d) < \infty$.*

The result was proved first by Pemantle for all $d \geq 4$ and then by Liggett for $d = 3$. It is also known that the process dies out at $\lambda = \lambda_c^{(1)}(\mathbb{T}^d)$ and survives globally but not locally at $\lambda = \lambda_c^{(2)}(\mathbb{T}^d)$. A thorough exposition on this topic is given by Liggett in [49, Section I.4].

2.3.4 Extinction time on finite graphs

If $G = (V, E)$ is a finite graph, then the contact process almost surely reaches the empty configuration. Indeed, in every second of the dynamics, there is a positive probability that all infected vertices recover and none of them transmit the infection; by Borel–Cantelli, this eventually happens with probability one. This argument is given in more details, and quantitatively, in Lemma 2.12 below.

We define the *extinction time of the contact process on G* as the random variable

$$\tau_G := \inf\{t : \xi_t^V = \varnothing\}, \qquad \text{where } (\xi_t^V)_{t\ge 0} \text{ is the process started from } \xi_0 \equiv 1. \tag{2.7}$$

This is one of the central objects of interest in this text. In this section, we give some simple first estimates about it.

Lemma 2.12. *For any graph $G = (V, E)$,*

$$\mathbb{E}[\tau_G] \le \left(\frac{1}{1-e^{-1}}\right)^{|V|} \cdot e^{2|E|\lambda}.$$

Proof. Let X be the smallest natural number so that, on the time interval $[X-1, X]$, the following occurs: there is a recovery mark on every vertex of G, and there is no transmission mark from x to y, for any $\{x, y\} \in E$. We then have $\tau_G \le X$, and $X \sim \mathrm{Geom}(p)$, with $p := (1-e^{-1})^{|V|} \cdot e^{-2|E|\lambda}$, so

$$\mathbb{E}[\tau_G] \le \mathbb{E}[X] = \frac{1}{p} = \left(\frac{1}{1-e^{-1}}\right)^{|V|} \cdot e^{2|E|\lambda}. \qquad \square$$

The following lemma, which gives an inequality that looks like an "upside-down Markov inequality", goes in the opposite direction: it gives an upper bound for the probability that the extinction time is small.

Lemma 2.13. *Let $G = (V, E)$ be a graph and τ_G be the extinction time of the contact process with some rate $\lambda > 0$ on G. For any $t > 0$, we have*

$$\mathbb{P}(\tau_G \le t) \le \frac{t}{\mathbb{E}[\tau_G]}.$$

Proof. Fix $t > 0$ and define

$$\tau_G^{(t)} := \inf\{s \in \{t, 2t, \ldots\} : \ V \times \{s-t\} \not\leftrightarrow V \times \{s\}\}.$$

On $\{\tau_G^{(t)} = s\}$, there is no infection path from $V \times \{s-t\}$ to $V \times \{s\}$, so there is no infection path from $V \times \{0\}$ to $V \times \{s\}$, so $\tau_G \le s$. This proves that $\tau_G \le \tau_G^{(t)}$, so

$$\mathbb{E}[\tau_G] \le \mathbb{E}[\tau_G^{(t)}]. \tag{2.8}$$

Next, letting $X := \tau_G^{(t)}/t$, we have $X \sim \mathrm{Geom}(p)$, with

$$p := \mathbb{P}(V \times \{0\} \not\leftrightarrow V \times \{t\}) = \mathbb{P}(\tau_G < t) = \mathbb{P}(\tau_G \le t).$$

This gives

$$\mathbb{E}[\tau_G^{(t)}] = t \cdot \mathbb{E}[X] = \frac{t}{p} = \frac{t}{\mathbb{P}(\tau_G \le t)}. \tag{2.9}$$

Combining (2.8) and (2.9) and rearranging gives the desired inequality. $\square$

Because of this lemma, when we want to show that the extinction time of the contact process on a graph is large, we often content ourselves with giving a lower bound to its expectation.

2.3.5 Finite-volume phase transition

In several situations, phase transitions of the contact process in infinite graphs can be recovered in finite counterparts of these graphs. The purpose of this section is to give a brief overview of works in this direction. We refrain from including proofs.

The following has been proved about the contact process on boxes of $\mathbb{Z}^d$. The reason for the numerous citations in the statement is that this theorem was proved in several stages (we refrain from listing the names of all authors).

Theorem 2.14 ([26, 17, 66, 27, 53, 56]). *Let B_n be the finite subgraph of $\mathbb{Z}^d$ induced by the vertex set $\{-n, \dots, n\}^d$.*

- *If $\lambda < \lambda_c(\mathbb{Z}^d)$, then there exists $c_1 = c_1(\lambda) \in (0, \infty)$ such that*

$$\frac{\mathbb{E}[\tau_{B_n}]}{\log(|B_n|)} \xrightarrow{n\to\infty} c_1.$$

- *If $\lambda > \lambda_c(\mathbb{Z}^d)$, then there exists $c_2 = c_2(\lambda) \in (0, \infty)$ such that*

$$\frac{\log \mathbb{E}[\tau_{B_n}]}{|B_n|} \xrightarrow{n\to\infty} c_2.$$

  Moreover, $\tau_{B_n}/\mathbb{E}[\tau_{B_n}]$ converges in distribution to the exponential distribution with parameter 1.

This theorem can be seen as a *finite-volume phase transition*, since it is the finite-graph counterpart of the phase transition of the contact process on $\mathbb{Z}^d$.

In this context, the supercritical case is often referred to as the *metastable regime*. In fact, using standard methods of the contact process on $\mathbb{Z}^d$, the following can be proved. Assume that $\lambda > \lambda_c(\mathbb{Z}^d)$, fix $n_0 \in \mathbb{N}$ (to be kept fixed, as a microscopic spatial scale) and $n \in \mathbb{N}$ (to be taken to infinity). Let (t_n) be a sequence that grows faster than polynomially in n, but much slower than $(\mathbb{E}[\tau_{B_n}])_{n\geq 1}$; the idea is that t_n is a time scale for the contact process on B_n, that is enough for some mixing to take place, but far from enough for extinction. Let $(\xi_t)_{t\geq 0}$ be the contact process with rate λ on B_n started from all vertices infected. Then, the restriction of ξ_{t_n} to the microscopic box B_{n_0} has distribution that is very close to the restriction to the same box of $\bar{\mu}$, the upper stationary distribution of the contact process on $\mathbb{Z}^d$ (infinite volume). The same would hold if we translated the box B_{n_0} inside B_n, as long as we don't place it too close to the boundary. This says that the "fake equilibrium" of the metastability is essentially the infinite-volume (upper) equilibrium restricted to the box. Evidently, the "real equilibrium" is the infinite-volume lower equilibrium, $\delta_\varnothing$, restricted to the box. We omit proofs, but refer to the reader to the Appendix A.1 of [30], where similar ideas were pursued.

This kind of relation between finite and infinite volume should be much more general than above case study, as this is reminiscent of the problem of *locality of percolation*, see [4, 52, 20]. Presumably, given a sequence of rooted graphs (G_n, o_n) that converge locally to an infinite graph (G, o) (in the sense of Benjamini–Schramm [5]), one should be able to somehow recover the phase transision(s) of the contact process on (G, o), by looking at it in (G_n, o_n).

Some progress in this direction has been achieved for graphs converging to the infinite d-regular tree $\mathbb{T}^d$. We let G_n denote the *random d-regular graph* on n vertices. We refrain from giving a full definition of this random graph model here, because in Section 5.1 below, we will describe the *configuration model*, of which the random d-regular graph is a particular case (obtained by setting all degrees as d). What is important at the moment is that, letting o_n be a vertex of G_n chosen uniformly at random, then (G_n, o_n) converges locally to $(\mathbb{T}^d, o)$, where o is the root of $\mathbb{T}^d$. The following has been proved independently in two works:

Theorem 2.15 (Lalley and Su [42], Mourrat and V. [57]). *Let G_n be the random d-regular graph, with $d \geq 3$.*

- *If $\lambda < \lambda_c^{(1)}(\mathbb{T}^d)$, then there exists $c_1 = c_1(\lambda) > 0$ such that for any n,*
$$\mathbb{E}[\tau_{G_n}] \leq c_1 \log n.$$
- *If $\lambda > \lambda_c^{(1)}(\mathbb{T}^d)$, then there exists $c_2 = c_2(\lambda) > 0$ such that for any n,*
$$\mathbb{E}[\tau_{G_n}] \geq e^{c_2 n}.$$

Hence in this case, at least for the transition from quick to slow extinction, the relevant infinite-volume threshold value is $\lambda_c^{(1)}(\mathbb{T}^d)$, not $\lambda_c^{(2)}(\mathbb{T}^d)$.

It is interesting to contrast this to the case of *truncated trees.* Given $h \in \mathbb{N}$, let $\mathbb{T}_h^d$ denote the subtree of $\mathbb{T}^d$ induced by the set of vertices that are at distance at most h from the root. The contact process on $\mathbb{T}_h^d$ was first studied by Stacey [68]. Here we mention the following:

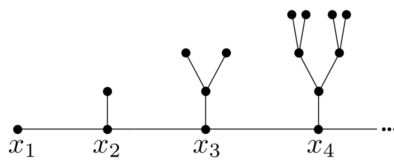
Theorem 2.16 (Cranston, Mountford, Mourrat, V. [21]). *For the contact process on $\mathbb{T}_h^d$, the following holds:*

- *If $\lambda < \lambda_c^{(2)}(\mathbb{T}^d)$, then there exists $c_1 = c_1(\lambda) > 0$ such that for any n,*
$$\mathbb{E}\left[\tau_{\mathbb{T}_h^d}\right] \leq c_1 \log |\mathbb{T}_h^d|.$$

- *If* $\lambda > \lambda_c^{(2)}(\mathbb{T}^d)$*, then there exists* $c_2 = c_2(\lambda) > 0$ *such that for any* n*,*

$$\mathbb{E}\left[\tau_{\mathbb{T}_h^d}\right] \geq e^{c_2|\mathbb{T}_h^d|}.$$

The reader may be puzzled by contrasting this with Theorem 2.15, since this time, it is the second critical value of $\mathbb{T}^d$ that plays a role. The puzzlement disappears once we realize that the Benjamini–Schramm limit of $\mathbb{T}_h^d$ rooted at a uniformly chosen vertex is *not* $(\mathbb{T}^d, o)$. Rather, it is the *canopy tree with random root*, denoted $(\mathcal{C}^d, \mathcal{O})$. The canopy tree is an infinite graph constructed by starting with a half-line with vertices $\{x_1, x_2, \dots\}$ and appending trees of increasing heights, as illustrated below for the case $d = 3$. The random root $\mathcal{O}$ is chosen with $\mathbb{P}(\mathcal{O} = x_m) = \frac{d-1}{d^m}$.



It turns out that $\lambda_c^{(1)}(\mathcal{C}^d) = \lambda_c^{(2)}(\mathcal{C}^d) = \lambda_c^{(2)}(\mathbb{T}^d)$. Hence, in this case we can again say that the threshold between extinction and survival in the limiting graph is the one that appears in the finite-volume phase transition. It would be good to have more general results in this direction.

We come back to the discussion of the interplay between the contact process and local graph convergence in Section 5.1, where we discuss the configuration model and BGW trees.

2.3.6 Domination by process on regular tree

In this section, we prove that the contact process on any graph where all degrees are bounded by d can be stochastically dominated by the contact process on $\mathbb{T}^d$. We will not use the result in the rest of this text, but we find it useful and it is perhaps not so well known, so we decided to include it.

Theorem 2.17 (Mourrat and V. [57]). *Let* $G = (V, E)$ *be a graph in which all vertices have degree at most* d*. Let* $u^* \in V$ *and* $\lambda > 0$*. Let* $(\xi_t)_{t\geq 0}$ *be the contact process on* G *with rate* λ *started from* $\xi_0 = \mathbb{1}_{\{u^*\}}$*, and let* $(\xi'_t)_{t\geq 0}$ *be the contact process on* $\mathbb{T}^d$ *with rate* λ *started from* $\xi'_0 = \mathbb{1}_{\{o\}}$*. Then,*

$$(|\{u : \xi_t(u) = 1\}|)_{t\geq 0} \preceq (|\{v : \xi'_t(v) = 1\}|)_{t\geq 0}.$$

The proof will rely on the notion of *universal covering graph* of a graph, which we now recall. Let $G = (V, E)$ be a connected graph and $u^* \in V$.

The universal covering graph of G with root corresponding to u^* is a pair (T, Φ), where $T = (V_T, E_T)$ is a tree with root $o \in V_T$ and Φ is a surjective map $\Phi : V_T \to V$ satisfying:

- $\Phi(o) = u^*$;
- for any $a \in V_T$, Φ maps $\{a\} \cup \{b \in V_T : b \sim a\}$ isomorphically into $\{\Phi(a)\} \cup \{w \in V : w \sim \Phi(a)\}$.

Let us explain how to construct (T, Φ) with these properties, starting with a definition. We say that a path $\gamma = (u_0, \dots, u_n)$ in G (with $u_{i+1} \sim u_i$ for each i) is *non-backtracking* if $u_{i+1} \neq u_{i-1}$ for $i = 1, \dots, n-1$. This includes paths of the form (u), consisting of a single vertex.

Let V_T be the set of all non-backtracking paths in V started at u^*, including the path (u^*), which we denote by o. We now define the edge set E_T as the set of pairs $\{a, b\}$ for which we can write

$$a = (u_0, \dots, u_n), \quad b = (u_0, \dots, u_n, u_{n+1})$$

for some $n \in \mathbb{N}_0$ and $u_0, \dots, u_{n+1} \in V$ (in words, b is a one-step continuation of a). Finally, given $a = (u_0, \dots, u_n) \in V_T$, we set $\Phi(a) = u_n$, the end-vertex of the path.

It should now be clear that T and Φ satisfy the properties listed earlier.

Proof of Theorem 2.17. It suffices to prove the proposition under the additional assumption that G is connected. We let (T, Φ) be the universal covering graph of G with root corresponding to u^*, constructed in the way explained in the paragraphs before this proof. We assume that the degrees of vertices in G are at most d, so the same must hold true for T. In particular, we can embed a copy of T inside $\mathbb{T}^d$.

We define an auxiliary process $(\zeta_t)_{t \geq 0}$ on $\{0,1\}^T$ as follows. We set $\zeta_0 = \mathbb{1}_{\{o\}}$, and $(\zeta_t)_{t \geq 0}$ evolves as a contact process with rate λ on T, except that it follows the *extra rule*: if the process tries to create a new infection, say at vertex $a \in T$, and there is already some infected vertex $b \in \Phi^{-1}(\Phi(a))$, then the new infection is not created. In particular, there is always at most one infection inside any of the sets $\Phi^{-1}(u)$, with $u \in V$.

Clearly, $(\zeta_t)_{t \geq 0}$ is stochastically dominated by a contact process on T, which in turn is stochastically dominated by a contact process on $\mathbb{T}^d$. Using the fact that Φ maps 1-neighborhoods of T isomorphically into 1-neighborhoods of G, it is easy to check that

$$\xi'_t(u) := \mathbb{1}\{\exists a \in \Phi^{-1}(u) : \zeta_t(a) = 1\}, \quad u \in V,\ t \geq 0$$

has the same distribution as a contact process on G started from $\mathbb{1}_{\{u^*\}}$. This completes the proof. $\square$

Corollary 2.18. *For any natural number* $d \geq 2$ *and* $\lambda < \lambda_c^{(1)}(\mathbb{T}^d)$*, there exists* $c > 0$ *such that the following holds. Let* G *be a graph in which all vertices have degree at most* d*. Let* $(\xi_t)_{t\geq 0}$ *be the contact process on* G *started from a single vertex infected. Then,*

$$\mathbb{P}(\xi_t \neq \varnothing) \leq e^{-ct}, \quad t \geq 0.$$

Proof. By Theorem 2.17, it suffices to prove that for any natural number $d \geq 2$ and $\lambda < \lambda_c^{(1)}(\mathbb{T}^d)$, there exists $c > 0$ such that, letting $(\xi_t)_{t\geq 0}$ be the contact process with rate λ on $\mathbb{T}^d$ started from $\mathbb{1}_{\{o\}}$, we have

$$\mathbb{P}(\xi_t \neq \varnothing) < Ce^{-ct} \quad \text{for all } t \geq 0. \tag{2.10}$$

In case $d = 2$ (so that $\mathbb{T}^d = \mathbb{Z}$), this is given in [49, Theorem 2.34, p.60]. In case $d \geq 3$, it can be obtained by putting together several results from the literature on the contact process on $\mathbb{T}^d$. Let us give some details on how this collage is done (the purpose is to give a reference and not a full explanation, so we refrain from defining the mathematical objects involved). Consider the function $\phi_\lambda(\rho)$ defined in [49, p.87, Eq.(4.23)] and the function $\beta(\lambda)$ defined in [49, p.96, Eq.(4.48)]. It holds that

$$\beta(\lambda_c^{(1)}(\mathbb{T}^d)) = 1/d; \tag{2.11}$$

see [49, Corollary 4.87(b)] (this was originally proved in [67, Eq.(3.13)]). Next, in [44, Theorem 3']^1

$$\text{if } \lambda < \lambda' \text{ and } \beta(\lambda') < 1/\sqrt{d}, \text{ then } \beta(\lambda) < \beta(\lambda'). \tag{2.12}$$

By putting (2.11) and (2.12) together, we see that

$$\text{if } \lambda < \lambda_c^{(1)}(\mathbb{T}^d), \text{ then } \beta(\lambda) < 1/d. \tag{2.13}$$

Next, we have

$$\phi_\lambda(\beta(\lambda)) = \phi_\lambda\left(\frac{1}{d\beta(\lambda)}\right) = 1; \tag{2.14}$$

see [49, p.107, Eq.(4.85)], or the original reference [43, Corollary 1]). By putting together (2.13) and (2.14), we obtain

$$\text{if } \lambda < \lambda_c^{(1)}(\mathbb{T}^d), \text{ then there exists } \rho > 1 \text{ such that } \phi_\lambda(\rho) = 1. \tag{2.15}$$

[1] This reference is a correction to a previous paper, [43]. This previous paper states that $\lambda \mapsto \beta(\lambda)$ is strictly increasing on $[0, \lambda_c^{(2)})$, but a mistake was later found in the proof. This statement with the incorrect proof is also reproduced in [49, Theorem 4.130]. The weaker result proved in the correction paper [44], expressed here in (2.12), is fortunately good enough for our purposes.

Next, [49, Proposition 4.44(a)] gives

$$\text{if } \lambda < \lambda_c^{(1)}(\mathbb{T}^d) \text{ and } \frac{1}{\sqrt{d}} \le \rho < \rho', \text{ then } \phi_\lambda(\rho) < \phi_\lambda(\rho'). \tag{2.16}$$

Putting together (2.15) and (2.16) gives

$$\text{if } \lambda < \lambda_c^{(1)}(\mathbb{T}^d), \text{ then } \phi_\lambda(1) < 1. \tag{2.17}$$

Finally, letting $(\xi_t)_{t\ge 0}$ be the contact process on $\mathbb{T}^d$ started with only the root infected, we have

$$\mathbb{E}[|\xi_t|] \le C(\lambda) \cdot \phi_\lambda(1)^t \quad \text{for all } t \ge 0, \tag{2.18}$$

by [49, Proposition 4.27(b)]. The desired bound (2.10) follows by putting together (2.17) and (2.18). □

2.3.7 Stars

Throughout this section, let S_n denote a *star graph* with n leaves, that is, a graph isomorphic to $G = (V, E)$ given by

$$V = \{o, x_1, \dots, x_n\}, \quad E = \{\{o, x_1\}, \dots, \{o, x_n\}\}.$$

The vertex o, which has degree n, is called the *hub*.

Provided that n is large (in a way that depends on λ), the contact process survives for a long time on S_n. This is due to the fact that when the hub gets infected, it is likely to transmit the infection to many of the leaves, and moreover, as soon as it recovers, it is likely to receive the infection again from one of the leaves.

We are particularly interested in this phenomenon in situations where λ is small. This is because the role of stars becomes most relevant in the study of the contact process with small λ in graphs with highly inhomogeneous degrees. In some of these settings, the process manages to survive even when λ is very low, because the infection is sustained by stars, and sent from one star to another. Looking at large λ is then not interesting, because the process would have managed to survive even without the stars.

Assume that λ is small; then, how big does n need to be for the infection to survive for a long time on S_n? To answer this, let us do some heuristics. If the hub were kept artificially infected, then the leaves would evolve independently, all as the Markov chain that jumps from 0 to 1 with rate λ, and from 1 to 0 with rate 1. The equilibrium distribution of this chain assigns probability $\frac{1}{1+\lambda}$ to 0 and probability $\frac{\lambda}{1+\lambda}$ to 1. Suppose that this has been running for a long time, so that all leaves are close to equilibrium, and then we suddenly force a recovery at the hub. The probability that a leaf that is infected at that time heals before reinfecting the root is $\frac{1}{1+\lambda}$.

There are about $\frac{\lambda}{1+\lambda}n$ infected leaves at that time, so the probability that they all recover without reinfecting the root is about

$$\left(\frac{1}{1+\lambda}\right)^{\frac{\lambda}{1+\lambda}n} = \exp\left\{\frac{\lambda}{1+\lambda}\cdot\log\left(\frac{1}{1+\lambda}\right)\cdot n\right\}.$$

When λ is small, we have $\frac{\lambda}{1+\lambda}\cdot\log\left(\frac{1}{1+\lambda}\right)\approx -\lambda^2$. Hence, the exponential on the right-hand side above is small when $n \gg 1/\lambda^2$.

The estimates we obtain below confirm this heuristics: they show that when λ is small, then the extinction time of the contact process in S_n is with high probability larger than $\exp\{c\lambda^2 n\}$, for some constant c that does not depend on λ or n.

We follow the approach in [55]. It should be mentioned that Huang and Durrett in [35] and Jo in [39] obtained sharper estimates through different methods, but we will not need them here.

Lemma 2.19. *For any $\varepsilon > 0$ there exists $c > 0$ such that the following holds. Let $(X_t)_{t\geq 0}$ be the continuous-time Markov chain on $\{0,1\}$ with $X_0 = 0$ and jump rates given by $r(1,0) = 1$ and $r(0,1) = \beta \geq 1$. Then,*

$$\mathbb{P}(\mathrm{Leb}(\{t\in[0,1] : X_t = 0\}) > \varepsilon) < e^{-c\beta}.$$

Proof. Define $T_0 := 0$ and let $0 < T_1' < T_1 < T_2' < T_2 < \cdots$ be the times when the chain jumps; that is, for $j \geq 1$,

$$T_j' := \inf\{t > T_{j-1} : X_t = 1\}, \quad T_j := \inf\{t > T_j' :\ X_t = 0\}.$$

For arbitrary $m \in \mathbb{N}$, we have

$$\begin{aligned}\left\{\sum_{i=1}^m (T_i' - T_{i-1}) \leq \varepsilon,\ \sum_{i=1}^m (T_i - T_i') > 1\right\} &\subseteq \left\{\sum_{i=1}^m (T_i' - T_{i-1}) \leq \varepsilon,\ T_m > 1\right\}\\ &\subseteq \{\mathrm{Leb}(\{t\in[0,1] \colon X_t = 0\}) \leq \varepsilon\},\end{aligned}$$

so

$$\begin{aligned}&\mathbb{P}\left(\mathrm{Leb}(\{t\in[0,1] : X_t = 0\}) > \varepsilon\right)\\ &\leq \mathbb{P}\left(\sum_{i=1}^m (T_i' - T_{i-1}) > \varepsilon\right) + \mathbb{P}\left(\sum_{i=1}^m (T_i - T_i') \leq 1\right)\\ &= \mathbb{P}\left(\sum_{i=1}^m Z_i > \beta\varepsilon\right) + \mathbb{P}\left(\sum_{i=1}^m Z_i' \leq 1\right),\end{aligned}$$

where $Z_1, Z_2 \ldots, Z_1', Z_2', \ldots$ are independent Exp(1) random variables. Setting $m := \max(1, \lfloor \beta\varepsilon/2 \rfloor)$, a simple large deviations estimate shows that both the probabilities on the right-hand side are smaller than $\exp\{-c\beta\}$ uniformly in $\beta \geq 1$, where c is a constant depending on ε. $\square$

Lemma 2.20. *There exists $c > 0$ such that the following holds. Let $\lambda > 0$ and $m, n \in \mathbb{N}$ with $n \geq m \geq 2e/\lambda$. Let A be a set of leaves of S_n with $|A| = m$, and let $(\xi_t)_{t \geq 0}$ be the contact process on S_n with rate λ started from A infected. Then,*

$$\mathbb{P}\left(\begin{array}{l} |\xi_1| \geq (\lambda \wedge 1)n/(8e), \\ \mathrm{Leb}(\{t \in [0,1] :\ \xi_t(o) = 0\}) \leq 1/4 \end{array} \right) > 1 - 3e^{-c(\lambda \wedge 1)m}. \tag{2.19}$$

Proof. We denote by E_* be the event inside the probability in (2.19).

Define the random set

$$\mathscr{A} := \{v \in A : v \text{ has no recovery in } [0,1]\}$$

and the event

$$E_1 := \{|\mathscr{A}| \geq m/(2e)\}.$$

We have

$$\mathbb{P}(E_1) = \mathbb{P}(\mathrm{Bin}(m, 1/e) \geq m/(2e)) > 1 - \exp\{-cm\}, \tag{2.20}$$

where the first inequality is a Chernoff bound ($c > 0$ is a constant that does not depend on m, n or λ, and its value may decrease in the rest of the proof).

Next, define the auxiliary process $(Y_t)_{0 \leq t \leq 1}$ on $\{0,1\}$ as follows. Set $Y_0 = 0$; then, let (Y_t) evolve as follows:

- when at state 0, it jumps to state 1 when there is a transmission from an element of $\mathscr{A}$ to o;
- when at state 1, it jumps to state 0 when there is a recovery mark at o.

Clearly, we have

$$Y_t \geq \xi_t(o), \quad 0 \leq t \leq 1.$$

Conditionally on E_1, the process $(Y_t)_{0 \leq t \leq 1}$ stochastically dominates the process $(\tilde{Y}_t)_{0 \leq t \leq 1}$, where $(\tilde{Y}_t)_{t \geq 0}$ is a continuous-time Markov chain on {0,1} that has $\tilde{Y}_0 = 0$, jumps from 1 to 0 with rate 1 and jumps from 0 to 1 with rate $\beta := \lambda \cdot |\mathscr{A}| \geq \frac{\lambda m}{2e} \geq 1$, by the assumption on m. Thus, defining

$$E_2 := \left\{ \mathrm{Leb}(\{t \in [0,1] : Y_t = 1\}) \geq \frac{3}{4} \right\},$$

using Lemma 2.19 and reducing the value of c if necessary, we have

$$\mathbb{P}(E_2 \mid E_1) > 1 - \exp\{-c\lambda m\}. \tag{2.21}$$

We now split the proof into two cases, the first one being when $m \geq n/2$. In this case, we have $m/(2e) \geq n/(4e) \geq (\lambda \wedge 1)n/(4e)$, so $E_* \supseteq E_1 \cap E_2$, and the proof is then complete in this case.

We now turn to the second case, when $m < n/2$. Define the random set

$$\mathscr{B} := \left\{ \begin{array}{l} v \in S_n \backslash (A \cup \{o\}) : \ v \text{ has no recovery in } [0,1], \\ \text{there exists } t \in (0,1) \text{ such that } Y_t = 1 \text{ and} \\ \text{there is a transmission from } o \text{ to } v \text{ at time } t \end{array} \right\}.$$

Note that the number of infected leaves in ξ_1 is bounded from below by $|\mathscr{B}|$. Define the event

$$E_3 := \left\{ |\mathscr{B}| \geq \frac{(\lambda \wedge 1)n}{8e} \right\},$$

so that $E_* \supseteq E_2 \cap E_3$. Let $\mathcal{G}$ denote the σ-algebra generated by the recovery marks at $A \cup \{o\}$ and the transmission times between A and o, all inside the time interval $[0,1]$. Then, $E_2 \in \mathcal{G}$, and on E_2,

$$\mathbb{P}(E_3 \mid \mathcal{G}) = \mathbb{P}\left(\text{Bin}\left(n - |A|, \ \frac{1}{e}(1 - e^{-\frac{3}{4}\lambda}) \right) \geq \frac{(\lambda \wedge 1)n}{8e} \right). \tag{2.22}$$

We bound

$$1 - e^{-\frac{3}{4}\lambda} \geq 1 - e^{-\frac{3}{4}(\lambda \wedge 1)} \geq \frac{3(\lambda \wedge 1)}{8},$$

where in the second inequality we have used the fact that $1 - e^{-x} \geq x/2$ for $x \in [0,1]$. Also using the assumption that $|A| = m < n/2$, the right-hand side of (2.22) is bounded from below by

$$\mathbb{P}\left(\text{Bin}\left(\frac{n}{2}, \ \frac{3(\lambda \wedge 1)}{8e} \right) \geq \frac{(\lambda \wedge 1)n}{8e} \right).$$

Since $\frac{3}{16} > \frac{1}{8}$, we obtain

$$\mathbb{P}(E_3 \mid E_1 \cap E_2) = \frac{\mathbb{E}[\mathbb{P}(E_3 \mid \mathcal{G}) \cdot \mathbb{1}_{E_1 \cap E_2}]}{\mathbb{P}(E_1 \cap E_2)} \geq 1 - e^{-c(\lambda \wedge 1)n} \geq 1 - e^{-c(\lambda \wedge 1)m}.$$

This completes the proof. □

We now see two important consequences of the above lemma.

Lemma 2.21. *For any $\varepsilon > 0$ there exists $C > 0$ such that the following holds. Let $\lambda \in (0,1]$ and $n \geq C/\lambda^2$. Letting $(\xi_t)_{t \geq 0}$ denote the contact process on S_n started from only the hub infected, we have*

$$\mathbb{P}\left(\exists t > 0 : \ |\xi_t| \geq \frac{\lambda n}{8e} \right) \geq 1 - \varepsilon.$$

Proof. Fix $\varepsilon > 0$ and $\lambda \in (0, 1]$. Let

$$\delta := -\log\left(1 - \frac{\varepsilon}{3}\right) \tag{2.23}$$

and, letting c be the constant of Lemma 2.20, let

$$C_0 := -\frac{1}{c} \cdot \log\left(\frac{\varepsilon}{3}\right). \tag{2.24}$$

Now, we can choose $C > 0$ large enough that

$$\mathbb{P}\left(\mathrm{Bin}\left(\left\lfloor \frac{C}{\lambda^2} \right\rfloor,\ 1 - e^{-\lambda\delta}\right) > \frac{C_0}{\lambda}\right) > 1 - \frac{\varepsilon}{3} \tag{2.25}$$

uniformly over $\lambda \in (0, 1]$.

Let E_1 be the event that o has no recovery before time δ. By (2.23), we have $\mathbb{P}(E_1) = \varepsilon/3$. Let E_2 be the event that in the time interval $[0, \delta]$, at least C_0/λ leaves become infected. By (2.25), we have $\mathbb{P}(E_2 \mid E_1) > 1-\varepsilon/3$. Let E_3 be the event that at time $\delta + 1$, there are at least $\lambda n/(8e)$ infected leaves. By (2.24) and Lemma 2.20, we have $\mathbb{P}(E_3 \mid E_2) > 1 - \varepsilon/3$. This completes the proof. $\square$

Proposition 2.22. *There exists $c > 0$ such that the following holds. Let $\lambda \in (0, 1]$ and $n \geq \frac{32e^2}{\lambda^2}$. Let A be a set of leaves of S_n with $|A| \geq \frac{\lambda n}{8e}$. Then, the contact process $(\xi_t)_{t \geq 0}$ on S_n with rate λ and started from A infected satisfies*

$$\mathbb{P}\left(\begin{array}{l} \xi_{\exp\{c\lambda^2 n\}} \neq \varnothing \textit{ and for all } t \in [1, \exp\{c\lambda^2 n\}] \\ \textit{we have } \mathrm{Leb}(\{s \in [t-1, t] : \xi_s(o) = 1\}) \geq \frac{1}{2} \end{array}\right) > 1 - e^{-c\lambda^2 n}.$$

Proof. Let $m := \lfloor \lambda n/(8e) \rfloor$. Note that $m \geq \frac{2e}{\lambda}$. Define the events

$$E_{k,*} := \left\{ \begin{array}{l} \mathrm{Leb}(\{s \in [k, k+1] : \xi_s(o) = 1\}) \geq \frac{3}{4}, \\ \xi_{k+1} \text{ has at least } m \text{ infected leaves} \end{array} \right\}, \quad k \in \mathbb{N}_0.$$

By Lemma 2.20, we have $\mathbb{P}((E_{0,*})^c) \leq e^{-c\lambda^2 n}$ and $\mathbb{P}((E_{k,*})^c \mid E_{k-1,*}) \leq e^{-c\lambda^2 n}$ for any k. The desired claim easily follows, by reducing the value of c. $\square$

We conclude this section with the following result, which goes in the opposite direction as the previous ones, giving a lower bound for the probability that the contact process on S_n dies out in a relatively short period of time.

Proposition 2.23. *Let $\lambda \in (0, \frac{1}{4})$ and S be a star graph with n leaves. Then, the extinction time τ_S of the contact process with rate λ on S satisfies*

$$\mathbb{P}(\tau_S \le 3\log\tfrac{1}{\lambda}) \ge \frac{1}{4}\cdot e^{-16\lambda^2 n}.$$

In particular, for any $t > 0$,

$$\mathbb{P}(\tau_S > t) \le \left(1 - \frac{1}{4}e^{-16\lambda^2 n}\right)^{\lfloor t/(3\log(1/\lambda))\rfloor}. \tag{2.26}$$

Proof. Let σ denote the time of the first recovery mark at the hub o after time $\log\frac{1}{\lambda}$. We have

$$\mathbb{P}(\tau_S \le 3\log\tfrac{1}{\lambda}) \ge \mathbb{P}(\sigma \le 2\log\tfrac{1}{\lambda},\ \tau_S \le \sigma + \log\tfrac{1}{\lambda}). \tag{2.27}$$

For each vertex $v \neq o$, let $A(v)$ denote the event that there exists $t \in [\sigma, \sigma + \log\frac{1}{\lambda}]$ such that there is a recovery mark at v at time t, and there is no transmission from v to o between times σ and t. The probability of $A(v)$ equals

$$\begin{aligned}(1 - e^{-(\lambda+1)\log(1/\lambda)})\cdot\frac{1}{1+\lambda} &\ge (1 - e^{-\log(1/\lambda)})\cdot\frac{1}{1+\lambda}\\ &\ge (1-\lambda)(1-\lambda) \ge 1 - 2\lambda,\end{aligned}$$

where we have used the inequality $1/(1+\lambda) \ge 1-\lambda$, which holds for all $\lambda > 0$. Letting $(\mathcal{F}_t)_{t\ge 0}$ denote the filtration generated by the graphical construction, the probability on the right-hand side of (2.27) is larger than

$$\mathbb{E}\left[\mathbb{1}\{\sigma \le \log\tfrac{1}{\lambda}\}\cdot\mathbb{P}\left(\cap_{v\in\xi_\sigma}A(v)\middle|\ \mathcal{F}_\sigma\right)\right] \ge \mathbb{E}\left[\mathbb{1}\{\sigma \le \log\tfrac{1}{\lambda}\}\cdot(1-2\lambda)^{|\xi_\sigma|}\right]. \tag{2.28}$$

For each $v{\neq}o$, let $(X_t(v))_{t\ge 0}$ be the process on $\{0,1\}$ which has $X_0(v) = 1$ and switches to state 0 whenever there is a recovery mark at v, and switches to state 1 whenever there is a transmission from o to v (even if o is not infected at the time of that transmission). Then, $\xi_t(v) \le X_t(v)$ for any t, and the processes $(X_t(v))$ over different choices of v are independent (and they are independent of σ). Moreover, it is easy to see that

$$\mathbb{P}(X_t(v) = 1) = e^{-(\lambda+1)t} + (1 - e^{-(\lambda+1)t})\cdot\frac{\lambda}{1+\lambda} \le e^{-t} + \lambda.$$

Since $\sigma \ge \log\frac{1}{\lambda}$, we bound

$$\mathbb{P}(X_\sigma(v) = 1) \le 2\lambda.$$

Hence,

$$\mathbb{P}(|\xi_\sigma| \le 4\lambda n \mid \mathcal{F}_\sigma) \ge \mathbb{P}\left(\sum_v X_\sigma(v) \le 4\lambda n \mid \mathcal{F}_\sigma\right)$$
$$\ge \mathbb{P}(\mathrm{Bin}(n, 2\lambda) \le 4\lambda n) \ge \frac{1}{2},$$

by Markov's inequality.

Now, the right-hand side of (2.28) is larger than

$$(1-2\lambda)^{4\lambda n} \cdot \mathbb{P}(\sigma \le 2\log\tfrac{1}{\lambda},\ |\xi_\sigma| \le 4\lambda n)$$
$$\ge (1-2\lambda)^{4\lambda n} \cdot \frac{1}{2} \cdot \mathbb{P}(\sigma \le 2\log\tfrac{1}{\lambda})$$
$$= (1-2\lambda)^{4\lambda n} \cdot \frac{1}{2} \cdot (1 - e^{-\log(1/\lambda)}) \ge \frac{1}{4}(1-2\lambda)^{4\lambda n},$$

since $\lambda < \frac{1}{2}$. Using the inequality $1 - x \ge e^{-2x}$, which holds if $x \in (0, 1/2)$, together with the fact that $\lambda < \frac{1}{4}$, we bound $(1-2\lambda)^{4\lambda n} \ge e^{-16\lambda^2 n}$, completing the proof. □

2.3.8 FKG inequality for the contact process

In Section 2.1, we have already stated the FKG inequality for discrete product measures. We will now state a version of this inequality that pertains to a partial order on the space of realizations of the graphical construction of the contact process. This involves some topological technicalities, which we will only briefly explain, directing the reader to [8] for further details.

Let $\mathscr{X}$ be the collection of infinite subsets of $[0,\infty)$ that intersect every bounded interval only finitely many times. Let φ be the one-to-one mapping from $\mathscr{X}$ to the space of càdlàg functions from $[0,\infty)$ to $\mathbb{N}_0$ given by $(\varphi(\chi))(t) = |\chi \cap [0,t]|$ for all $t \ge 0$. Then, let $\mathcal{O}$ be the Skorohod topology on this set of càdlàg functions (see [10]), and take the topology $\{\varphi^{-1}(O) : O \in \mathcal{O}\}$ on $\mathscr{X}$. Loosely speaking, $\chi, \chi' \in \mathscr{X}$ are close to each other in this topology if, for some large t, $\chi \cap [0,t]$ and $\chi' \cap [0,t]$ have the same number of elements, and letting $x_1 < \cdots < x_m$ be the elements of $\chi \cap [0,t]$ and $y_1 < \cdots < y_m$ the elements of $\chi' \cap [0,t]$, we have that x_i is close to y_i, for each i.

Let us denote by $\mathscr{H}$ the space of realizations of the graphical construction of the contact process on the graph $G = (V, E)$. By definition, $\mathscr{H}$ is the product space consisting of copies of $\mathscr{X}$ indexed by $V \cup \{(x,y) : \{x,y\} \in E\}$. We take the infinite product topology on $\mathscr{H}$ (and the associated Borel σ-algebra, where we define the probability measure $\mathbb{P}$ as prescribed in the definition of the graphical construction). We can now talk about the topological boundary ∂A of an event $A \in \mathscr{H}$.

The last ingredient is a partial order on $\mathscr{H}$. For $\omega, \omega' \in \mathscr{H}$, we say that $\omega \preceq \omega'$ if every arrow in ω is also present in ω', and every recovery mark of ω' is also present in ω. As before, an event $A \in \mathscr{H}$ is increasing if $[\omega \in A,\ \omega \preceq \omega']$ implies $\omega' \in A$. We are now ready to state:

Theorem 2.24 (FKG inequality for the contact process, [8]). *If $A, B \in \mathscr{H}$ are increasing and $\mathbb{P}(\partial A) = \mathbb{P}(\partial B) = 0$, then $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) \cdot \mathbb{P}(B)$.*

For many events of interest, it is not hard to check that the topological boundary has measure zero. This is the case, for instance, for an event of the form $A := \{(x,s) \rightsquigarrow (y,t)\}$. To see this, also define B as the set of ω's for which either: some Poisson process has an arrival exactly at s, or some Poisson process has an arrival exactly at t, or two distinct Poisson processes have arrivals at exactly the same time. Then, $\mathbb{P}(B) = 0$. By considering small perturbations of the point processes, it is easy to check that if $h \in A \cap B^c$, then h is in the topological interior of A, and if $h \in A^c \cap B^c$, then h is in the topological interior of A^c. This shows that $\partial A \subset B$.

We will also use the fact that

$$\mathbb{P}((x,0) \rightsquigarrow \infty, (y,0) \rightsquigarrow \infty) \geq \mathbb{P}((x,0) \rightsquigarrow \infty) \cdot \mathbb{P}((y,0) \rightsquigarrow \infty), \qquad (2.29)$$

which can be obtained from the FKG inequality and an approximation argument, as follows. For each $n \in \mathbb{N}$, let $\Lambda_{x,n}$ be the event that there is an infection path from $(x,0)$ to some point in the set

$$\{(v,s) : \mathrm{dist}(x,v) = n,\ 0 \leq s \leq n\} \cup \{(v,n) : \mathrm{dist}(x,v) \leq n\},$$

and define $\Lambda_{y,n}$ similarly, with x replaced by y. Arguing as above, we can show that $\mathbb{P}(\partial \Lambda_{x,n}) = \mathbb{P}(\partial \Lambda_{y,n}) = 0$ and both events are increasing, so $\mathbb{P}(\Lambda_{x,n} \cap \Lambda_{y,n}) \geq \mathbb{P}(\Lambda_{x,n}) \cdot \mathbb{P}(\Lambda_{y,n})$. We then have

$$\begin{aligned}
\mathbb{P}((x,0) \rightsquigarrow \infty,\ (y,0) \rightsquigarrow \infty) &= \lim_{n\to\infty} \mathbb{P}(\Lambda_{x,n} \cap \Lambda_{y,n}) \\
&\geq \lim_{n\to\infty} \mathbb{P}(\Lambda_{x,n}) \cdot \mathbb{P}(\Lambda_{y,n}) \\
&= \mathbb{P}((x,0) \rightsquigarrow \infty) \cdot \mathbb{P}((y,0) \rightsquigarrow \infty).
\end{aligned}$$

Chapter 3

General metastability results on graphs

In the mathematical treatment of metastability, the task of proving that a particle system is metastable often relies heavily on the specificities of the environment where it evolves. The contact process has a privileged position in this respect: as we will see in this chapter, there are proofs of metastability that apply *at once to all graphs.*

Recall that any infinite connected graph G contains an infinite half-line inside it, which implies by monotonicity that $\lambda_c(G) \le \lambda_c(\mathbb{N}) = \lambda_c(\mathbb{Z})$. In other words, if the infectivity is high enough that the process can survive on a half-line, then it can also survive on any infinite connected graph. As we will see, the counterpart of this idea for finite graphs is that, when $\lambda > \lambda_c(\mathbb{Z})$, the contact process with rate λ survives for very long (exponentially in the number of vertices) on arbitrary connected graphs.

The first result in this direction was the following:

Theorem 3.1 (Mountford, Mourrat, V. and Yao [54]). *For any $\lambda > \lambda_c(\mathbb{Z})$ and $d \in \mathbb{N}$, there exists $c > 0$ such that the following holds. For any connected graph $G = (V, E)$ in which all degrees are smaller than or equal to d, letting τ_G be the extinction time of the contact process with rate λ on G, we have*

$$\mathbb{E}[\tau_G] > \exp\{c|V|\}.$$

Here, the assumption that degrees are bounded may seem puzzling: if the graph has vertices of big degrees, this should only help the infection spread more easily, thus leading to even larger extinction times. This intuition should be correct, and the bounded degree assumption is made for technical reasons. It allows one to split the graph into smaller pieces (see Lemma 3.15 below), and use recursive methods involving the extinction times of the pieces (in the spirit of Proposition 3.13 below).

We believe that a statement like the one of Theorem 3.1, but without any degree boundedness assumption, should be true. The best available progress in this direction is the following.

Theorem 3.2 (Schapira and V. [63]). *For any $\lambda > \lambda_c(\mathbb{Z})$ and $\varepsilon > 0$, there exists $c > 0$ such that the following holds. For any connected graph $G = (V, E)$ with at least two vertices, letting τ_G be the extinction time of the contact process with rate λ on G, we have*

$$\mathbb{E}[\tau_G] > \exp\left\{c\frac{|V|}{(\log|V|)^{1+\varepsilon}}\right\}. \tag{3.1}$$

Apart from the proof of Theorem 3.2, the reference [63] contains a new and shorter proof of Theorem 3.1. In the present text, we will fully prove Theorem 3.1 (following the shorter approach of [63]), but for brevity, rather then proving Theorem 3.2, we will prove the weaker statement given in the proposition below.

Proposition 3.3. *For any $\lambda > \lambda_c(\mathbb{Z})$, there exists $c > 0$ such that the following holds. For any connected graph G, letting τ_G be the extinction time of the contact process with rate λ on G, we have*

$$\mathbb{E}[\tau_G] > \exp\{c|V|^{1/3}\}. \tag{3.2}$$

Concerning all these results, it is worth it to remind the reader that, using the inequality $\mathbb{P}(\tau_G \le t) \le t/\mathbb{E}[\tau_G]$ from Lemma 2.13, we can take advantage of lower bounds on the expectation of the extinction time to obtain upper bounds for the probability that the extinction time is too short.

We will also see a general result pertaining to convergence to the exponential distribution:

Theorem 3.4 (Schapira and V. [63]). *Let $\lambda > 0$ and $(G_n)_{n\ge1} = ((V_n, E_n))_{n\ge1}$ be a sequence of connected graphs with $|V_n| \xrightarrow{n\to\infty} \infty$. Let τ_{G_n} denote the extinction time of the contact process with rate λ on G_n. Then, as $n \to \infty$, $\tau_{G_n}/\mathbb{E}[\tau_{G_n}]$ converges in distribution to the exponential distribution with parameter 1.*

A weaker version of this statement had been established earlier in [54].

A key tool that goes into all these results is a coupling bound on the contact process with $\lambda > \lambda_c(\mathbb{Z})$, which we develop in Section 3.1 (specifically, see Proposition 3.6 there). We then prove Theorem 3.1 and Proposition 3.3 in Section 3.2, and prove Theorem 3.4 in Section 3.3.

3.1 Coupling on general graphs

Let $G = (V, E)$ be a graph and H be a realization of the graphical construction for the contact process on G. We say that H is *fully coupled* on the time interval $[s,t]$ if for every $x \in V$,

$$\text{either } (x,s) \not\rightsquigarrow V \times \{t\} \quad \text{or} \quad \{y : (x,s) \rightsquigarrow (y,t)\} = \{y : V \times s \rightsquigarrow (y,t)\}.$$

The following lemma, whose proof is elementary and left to the reader, gathers some properties of this definition.

Lemma 3.5. (a) *Assume that H is fully coupled on $[0,t]$ and (ξ_t^A) and (ξ_t^V) are both constructed from H, with $\xi_0^A = \mathbb{1}_A$ and $\xi_0^V \equiv 1$; then, either $\xi_t^A = \varnothing$ or $\xi_t^A = \xi_t^V$.*

(b) *H is fully coupled on $[s,t]$ if and only if for every $u, v \in V$ such that $(u,s) \rightsquigarrow V \times \{t\}$ and $V \times \{s\} \rightsquigarrow (v,t)$, we also have $(u,s) \rightsquigarrow (v,t)$.*

(c) *If H is fully coupled on $[s,t]$ and $[s',t'] \supseteq [s,t]$, then H is fully coupled on $[s',t']$.*

(d) *If $s \le t \le s' \le t'$, then the events that H is fully coupled on $[s,t]$ and that H is fully coupled on $[s',t']$ are independent.*

(e) *For $s_1 < s_2 < s_3 < s_4$, assume that $V \times \{s_1\} \rightsquigarrow V \times \{s_3\}$, $V \times \{s_2\} \rightsquigarrow V \times \{s_4\}$ and H is fully coupled on $[s_2,s_3]$; then, $V \times \{s_1\} \rightsquigarrow V \times \{s_4\}$.*

The main goal of this section is to prove the following.

Proposition 3.6. *For any $\lambda > \lambda_c(\mathbb{Z})$, there exists $\mathfrak{c}_1 = \mathfrak{c}_1(\lambda) > 0$ such that for every n and every connected graph G with n vertices, a graphical construction of the contact process with rate λ on G is fully coupled on $[0, n(\log n)^3]$ with probability above $\mathfrak{c}_1$.*

By combining this proposition with Lemma 3.5(c) and (d), we obtain:

Corollary 3.7. *Assume that $\lambda > \lambda_c(\mathbb{Z})$ and G is a connected graph with n vertices. Then, a graphical construction of the contact process with rate λ on G is fully coupled on $[0,t]$ with probability above $(1-\mathfrak{c}_1)^{\lfloor t/(n(\log n)^3)\rfloor}$.*

We now state two lemmas that have similar statements, the first concerning line segments and the second, stars.

Lemma 3.8. *For any $\lambda > \lambda_c(\mathbb{Z})$, there exists $\mathfrak{c}_{\mathrm{line}} = \mathfrak{c}_{\mathrm{line}}(\lambda) > 0$ such that for any $n \in \mathbb{N}$, the graphical construction H for the contact process with rate λ on the line segment $\{1,\dots,n\}$ satisfies*

(a) *for any vertex v,*

$$\mathbb{P}\begin{pmatrix} \exists s_1 < n/\mathfrak{c}_{\mathrm{line}} : (v,0) \rightsquigarrow (1,s_1), \\ \exists s_2 < n/\mathfrak{c}_{\mathrm{line}} : (v,0) \rightsquigarrow (n,s_2), \\ (v,0) \rightsquigarrow \{1,\dots,n\} \times \{n/\mathfrak{c}_{\mathrm{line}}\} \end{pmatrix} > \mathfrak{c}_{\mathrm{line}}$$

and

(b) *H is fully coupled on $[0, n/\mathfrak{c}_{\text{line}}]$ with probability above $1 - e^{-\mathfrak{c}_{\text{line}} n}$.*

Lemma 3.9. *For any $\lambda > 0$, there exists $\mathfrak{c}_{\text{star}} = \mathfrak{c}_{\text{star}}(\lambda) > 0$ such that for any $n \in \mathbb{N}$, the graphical construction H for the contact process with rate λ on S_n, the star graph with n leaves, satisfies*

(a) *for any vertex v, $\mathbb{P}((v, 0) \rightsquigarrow S_n \times \{n\}) > \mathfrak{c}_{\text{star}}$ and*

(b) *H is fully coupled on $[0, n]$ with probability above $1 - e^{-\mathfrak{c}_{\text{star}} n}$.*

Proofs of both these lemmas can be found on the appendix of [63]. Since they are somewhat long and involve mostly routine arguments, we do not repeat them here.

Using Theorem 2.14 and decreasing the value of $\mathfrak{c}_{\text{line}}$ in Lemma 3.8 if necessary (in a way that depends on λ), we can assume that

$$\mathbb{E}[\tau_{\{1,\dots,n\}}] > e^{\mathfrak{c}_{\text{line}} \cdot n} \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Similarly, from our estimates on star graphs in Section 2.3.7, it is easy to see that we can decrease the value of $\mathfrak{c}_{\text{star}}$ if necessary (in a way that depends on λ) to have

$$\mathbb{E}[\tau_{S_n}] > e^{\mathfrak{c}_{\text{star}} \cdot n} \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Define

$$\mathfrak{c}_0 = \mathfrak{c}_0(\lambda) := \tfrac{1}{3} \min(\mathfrak{c}_{\text{line}}(\lambda), \mathfrak{c}_{\text{star}}(\lambda)). \tag{3.5}$$

Let $G = (V, E)$ be a graph consisting of either a line segment or a star, and $\lambda > 0$. We say that G is *lit* in a configuration $\xi \in \{0, 1\}^V$, or simply that ξ is lit, if the contact process on V with rate λ satisfies

$$\mathbb{P}(\xi_{\exp\{\mathfrak{c}_0 \cdot |V|\}} \neq \varnothing \mid \xi_0 = \xi) > 1 - e^{-\mathfrak{c}_0 \cdot |V|}.$$

We emphasize that the property of being lit depends on the value of λ, and on the constants $\mathfrak{c}_{\text{line}}$ and $\mathfrak{c}_{\text{star}}$ that have been fixed. It is also worth noting that if ξ is lit and $\xi' \geq \xi$, then ξ' is lit.

The following is an easy consequence of previously seen bounds:

Lemma 3.10. *Assume that $\lambda > \lambda_c(\mathbb{Z})$ and G is either a line segment or a star. Then, the fully occupied configuration $\xi \equiv 1$ is lit. Moreover, assuming $(\xi_t)_{t \geq 0}$ starts from $\xi_0 \equiv 1$, we have*

$$\mathbb{P}(\xi_t \textit{ is lit}) > 1 - 2e^{-\mathfrak{c}_0(\lambda) \cdot |V|} \quad \textit{for all } t \in [0, e^{\mathfrak{c}_0 n}].$$

Proof. We assume throughout the proof that $\xi_0 \equiv 1$. For any $t \geq 0$,

$$\mathbb{P}(\xi_t = \varnothing) \leq \frac{t}{\mathbb{E}[\tau_G]} \leq \frac{t}{\exp\{\min(\mathfrak{c}_{\text{line}}, \mathfrak{c}_{\text{star}})|V|\}} \leq t \exp\{-3\mathfrak{c}_0 |V|\}, \tag{3.6}$$

where the first inequality follows from Lemma 2.13, the second inequality from (3.3) and (3.4), and the third from the definition of $\mathfrak{c}_0$. Setting $t = \exp\{\mathfrak{c}_0|V|\}$ gives the first claim of the lemma.

We now turn to the second claim. Let $\mathcal{U}$ be the set of configurations ξ that are not lit. Abbreviate $t_0 := \exp\{\mathfrak{c}_0|V|\}$. For any $t \geq 0$ we have

$$\begin{aligned}\mathbb{P}(\xi_{t+t_0} = \varnothing) &\geq \mathbb{E}[\mathbb{1}\{\xi_t \in \mathcal{U}\} \cdot \mathbb{P}(\xi_{t+t_0} = \varnothing \mid (\xi_s : 0 \leq s \leq t))] \\ &\geq e^{-\mathfrak{c}_0|V|} \cdot \mathbb{P}(\xi_t \in \mathcal{U})\end{aligned}$$

by the Markov property and the definition of being lit. Then,

$$\mathbb{P}(\xi_t \in \mathcal{U}) \leq e^{\mathfrak{c}_0|V|} \cdot \mathbb{P}(\xi_{t+t_0} = \varnothing) \overset{(3.6)}{\leq} (t + t_0) \cdot e^{-3\mathfrak{c}_0|V|}.$$

When $t \leq t_0$, the right-hand side is at most $2e^{-2\mathfrak{c}_0|V|}$, completing the proof. □

Lemma 3.11. *Let $\lambda > 0$, and let G_n be the graph given either by the line segment $\{1, \ldots, n\}$ or a star with n leaves. Then, for the contact process $(\xi_t)_{t \geq 0}$ with rate λ on G_n, we have*

$$\xi_0 \text{ is lit} \quad \Longrightarrow \quad \mathbb{P}(\xi_t \text{ is lit}) > 1 - 4e^{-\mathfrak{c}_0 n} \text{ for any } t \in [n/\mathfrak{c}_0, e^{\mathfrak{c}_0 n}].$$

Proof. Assume that ξ_0 is lit. Let $(\xi_t^1)_{t \geq 0}$ be the process obtained from the same graphical construction as $(\xi_t)_t$, with $\xi_0^1 \equiv 1$. Then, for any $t \geq 0$,

$$\mathbb{P}(\xi_t \text{ is not lit}) \leq \mathbb{P}(\xi_t = \varnothing) + \mathbb{P}(\xi_t \neq \varnothing,\ \xi_t \neq \xi_t^1) + \mathbb{P}(\xi_t^1 \text{ is not lit}).$$

If $t \leq e^{\mathfrak{c}_0 n}$, then $\mathbb{P}(\xi_t = \varnothing) < e^{-\mathfrak{c}_0 n}$ because ξ_0 is lit, and $\mathbb{P}(\xi_t^1 \text{ is not lit}) < 2e^{-\mathfrak{c}_0 n}$ by Lemma 3.10. Finally, the event $\{\xi_t \neq \varnothing,\ \xi_t \neq \xi_t^1\}$ is contained in the event that the graphical construction is not fully coupled on $[0, t]$, by Lemma 3.5(a); if $t \geq n/\mathfrak{c}_0$, then this has probability is smaller than $e^{-\mathfrak{c}_0 n}$, by Lemma 3.8(b) and Lemma 3.9(b). □

Lemma 3.12. *Let $\lambda > \lambda_c(\mathbb{Z})$, and let G_n be the graph given either by the line segment $\{1, \ldots, n\}$ or a star with n leaves. Then, for the contact process $(\xi_t)_{t \geq 0}$ with rate λ on G_n,*

$$\xi_0 \text{ has at least one infection} \quad \Longrightarrow \quad \mathbb{P}(\xi_{n/\mathfrak{c}_0} \text{ is lit}) > \mathfrak{c}_0 - 2e^{-\mathfrak{c}_0 n}.$$

Proof. Let v be a vertex with $\xi_0(v) = 1$. Using a single graphical construction, we couple the contact process $(\xi_t)_{t \geq 0}$ that appears in the statement of the lemma with $(\xi_t^{\{v\}})_{t \geq 0}$ (for which $\xi_0^{\{v\}} = \mathbb{1}_{\{v\}}$) and $(\xi_t^1)_{t \geq 0}$ (for which $\xi_0^1 \equiv 1$). Then, for any t,

$$\mathbb{P}(\xi_t \text{ is lit}) \geq \mathbb{P}(\xi_t^{\{v\}} \text{ is lit}) \geq \mathbb{P}(\xi_t^{\{v\}} = \xi_t^1) - \mathbb{P}(\xi_t^1 \text{ is not lit}).$$

The desired inequality now follows from Lemma 3.8, Lemma 3.9, and Lemma 3.10. □

Proof of Proposition 3.6. It is sufficient to find $\mathfrak{c}_1$ so that the statement of the proposition holds for n large enough; after that, we can reduce $\mathfrak{c}_1$ if necessary, to also cover the remaining values of n.

Fix a connected graph $G = (V, E)$ with n vertices. Let d be the maximum degree of vertices of G, and $\mathrm{diam}(G)$ the diameter of G. We have

$$n \leq \sum_{i=0}^{\mathrm{diam}(G)} d^i = \frac{d^{\mathrm{diam}(G)+1} - 1}{d-1} \leq d^{\mathrm{diam}(G)+1}.$$

If n is large enough, we have $(\sqrt{\log n})^{\sqrt{\log n}+1} < n$; using this together with the above inequality, we obtain

$$\max(d, \mathrm{diam}(G)) \geq \sqrt{\log(n)}. \tag{3.7}$$

This implies that G has a subgraph $G_0 = (V_0, E_0)$ which is either a star or a line segment with

$$N := |V_0| \geq \max(\sqrt{\log(n)}, \mathrm{diam}(G)).$$

We take a graphical construction of the contact process on G. As usual, we write $\xi_t^A := \{u : A \times \{0\} \rightsquigarrow (u, t)\}$.

Claim 1. For any non-empty $A \subseteq V$, we have

$$\mathbb{P}\left(G_0 \text{ is lit in } \xi_t^A \text{ for some } t \leq 2N/\mathfrak{c}_0\right) > \mathfrak{c}_0^2 - 2\mathfrak{c}_0 e^{-\mathfrak{c}_0 N}. \tag{3.8}$$

Proof. Let $\sigma := \inf\{t : \xi_t^A \cap V_0 \neq \varnothing\}$. By Lemma 3.8(a), we have

$$\mathbb{P}(\sigma \leq \mathrm{dist}(A, V_0)/\mathfrak{c}_0) > \mathfrak{c}_0.$$

Since $\mathrm{dist}(A, V_0) \leq \mathrm{diam}(G) \leq N$, this also gives

$$\mathbb{P}(\sigma \leq N/\mathfrak{c}_0) > \mathfrak{c}_0.$$

Next, Lemma 3.12 and the strong Markov property give

$$\mathbb{P}(G_0 \text{ is lit in } \xi_{\sigma+N/\mathfrak{c}_0}^A \mid \sigma \leq N/\mathfrak{c}_0) > \mathfrak{c}_0 - 2e^{-\mathfrak{c}_0 N}. \qquad \square$$

Let $\bar{\mathfrak{c}} := \mathfrak{c}_0^2/2$. When n is large enough (and hence N is large enough), the right-hand side of (3.8) is larger than $\bar{\mathfrak{c}}$. It then follows from Claim 1 that for any $t \geq 0$,

$$\mathbb{P}(\xi_t \neq \varnothing,\ G_0 \text{ is not lit in } \xi_s^A \text{ for any } s \leq t) \leq (1 - \bar{\mathfrak{c}})^{\lfloor \frac{t}{2N/\mathfrak{c}_0} \rfloor}. \tag{3.9}$$

We let $\mathfrak{t}_0 := \frac{30N}{\mathfrak{c}_0 \bar{\mathfrak{c}}}$, which is a choice of t that makes the right-hand side of the above inequality smaller than e^{-10}. If G_0 becomes lit, there is a good chance that the infection inside it remains active for a long time, even without help from outside. We use this in the following claim.

Claim 2. If n is large enough, then for any non-empty $A \subseteq V$,

$$\mathbb{P}(\xi^A_{\mathfrak{t}_0} \neq \varnothing,\ \nexists w \in V_0 :\ A \times \{0\} \rightsquigarrow (w, \mathfrak{t}_0) \overset{G_0}{\rightsquigarrow} V_0 \times \{2\mathfrak{t}_0\}) < 2e^{-10} \tag{3.10}$$

(we write '$\overset{G_0}{\rightsquigarrow}$' meaning that the infection path in question must stay inside G_0).

Proof. Let $\sigma' := \inf\{t : G_0 \text{ is lit in } \xi^A_t\}$. The probability in (3.10) is smaller than

$$\mathbb{P}(\xi^A_{\mathfrak{t}_0} \neq \varnothing,\ \sigma' > \mathfrak{t}_0) \tag{3.11}$$

$$+\,\mathbb{P}\left(\begin{array}{l}\sigma' \leq \mathfrak{t}_0, \text{ there is no infection path that starts}\\ \text{at } V_0 \times \{\sigma'\}, \text{ stays inside } G_0 \text{ and reaches } V_0 \times \{2\mathfrak{t}_0\}\end{array}\right). \tag{3.12}$$

As explained earlier, the choice of $\mathfrak{t}_0$ makes it so that the probability in (3.11) is smaller than e^{-10}. Noting that $e^{\mathfrak{c}_0 N} \gg 2\mathfrak{t}_0$, the probability in (3.12) is smaller than

$$\mathbb{P}\left(\begin{array}{l}\text{there is no infection path that starts at } V_0 \times \{\sigma'\},\\ \text{stays inside } G_0 \text{ and reaches } V_0 \times \{\sigma' + e^{\mathfrak{c}_0 N}\}\end{array}\ \middle|\ \sigma' \leq \mathfrak{t}_0\right).$$

Using the definition of being lit, this is smaller than $e^{-\mathfrak{c}_0 N} \ll e^{-10}$. $\square$

Now define $K := \lfloor (\log n)^2 \rfloor$ and the set of times

$$\mathfrak{s}_k := 3\mathfrak{t}_0 \cdot k, \quad \mathfrak{s}'_k := \mathfrak{s}_k + \mathfrak{t}_0, \quad \mathfrak{s}''_k := \mathfrak{s}_k + 2\mathfrak{t}_0, \qquad k \in \{0, \ldots, K\}.$$

For any $u \in V$ and $k \in \{0, \ldots, K-1\}$, we say that the process $(\xi^u_t)_{t \geq 0}$ *misbehaves for* k if

$$(u, 0) \rightsquigarrow V \times \{\mathfrak{s}'_k\}, \quad \nexists w \in V_0 :\ (u, 0) \rightsquigarrow (w, \mathfrak{s}'_k) \overset{G_0}{\rightsquigarrow} V_0 \times \{\mathfrak{s}''_k\}.$$

Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by the graphical construction. By Claim 2 and the Markov property, we have that

$$\text{on } \{\xi^u_{\mathfrak{s}_k} \neq \varnothing\}, \quad \mathbb{P}((\xi^u_t) \text{ misbehaves for } k \mid \mathcal{F}_{\mathfrak{s}_k}) < 2e^{-10}. \tag{3.13}$$

We say that $u \in V$ is *bad* if the number of $k \in \{0, \ldots, K-1\}$ for which (ξ^u_t) misbehaves is larger than $K/3$.

Claim 3. For any $u \in V$, the probability that u is bad is at most 2^{-K}.

Proof. For any $I \subseteq \{0, \ldots, K\}$, by using (3.13) recursively, we have

$$\mathbb{P}((\xi^u_t) \text{ misbehaves for all } k \in I) < e^{-10|I|}.$$

Hence,

$$\mathbb{P}(u \text{ is bad}) \leq \sum_{\substack{I \subseteq \{0,\ldots,K\}: \\ |I| \geq K/3}} e^{-10|I|} \leq 2^K \cdot e^{-10K/3} < 2^{-K}.$$

Next, we consider dual processes. For $u \in V$ and $k \in \{0, \ldots, K-1\}$, we say that u *misbehaves backwards for* k if

$$V \times \{T - \mathfrak{s}'_k\} \rightsquigarrow (u, T), \quad \nexists w \in V_0 : \; V_0 \times \{T - \mathfrak{s}''_k\} \overset{G_0}{\rightsquigarrow} (w, T - \mathfrak{s}'_k) \rightsquigarrow (u, T).$$

We say that u is *bad backwards* if the number of $k \in \{0, \ldots, K-1\}$ for which $(\hat{\xi}^u_t)$ misbehaves backwards is larger than $K/3$. By Claim 3 and self-duality of the contact process, for any u, this happens with probability smalelr than 2^{-K}. □

We now define the event $\mathcal{A}$ as the event that no vertex is bad, no vertex is bad backwards, and for every $k \in \{0, \ldots, K-1\}$, the graphical construction inside V_0 is fully coupled on the interval $[\mathfrak{s}'_k, \mathfrak{s}''_k]$.

Claim 4. The probability of $\mathcal{A}^c$ can be made as small as desired by taking n large.

Proof. Recall that G_0 is either a star graph or a line segment with cardinality N, and that $\mathfrak{s}''_k - \mathfrak{s}'_k = \mathfrak{t}_0 = 30N/(\mathfrak{c}_0\bar{\mathfrak{c}}) > N/\mathfrak{c}_0$. Hence, by Lemma 3.9(a) and Lemma 3.8(b), the graphical construction inside G_0 is fully coupled on any one of the intervals $[\mathfrak{s}'_k, \mathfrak{s}''_k]$ with probability above $1 - e^{-\mathfrak{c}_0 N}$. Using a union bound over all vertices and all time intervals, we get

$$\mathbb{P}(\mathcal{A}^c) \leq 2 \cdot 2^{-K} n + K \cdot e^{-\mathfrak{c}_0 N} = 2^{-\lfloor (\log n)^2 \rfloor + 1} \cdot n + \lfloor (\log n)^2 \rfloor \cdot e^{-\mathfrak{c}_0 \sqrt{\log n}},$$

which can be made small by taking n large. □

Claim 5. If $\mathcal{A}$ occurs, then the graphical construction on G is fully coupled on $[0, T]$, where $T := K\mathfrak{t}_0$.

Proof. We will verify the condition given in Lemma 3.5. Fix a realization of the graphical construction for which $\mathcal{A}$ occurs, and let u, v be vertices of G such that $(u, 0) \rightsquigarrow V \times \{T\}$ and $V \times \{0\} \rightsquigarrow (v, T)$. Since neither u nor v are bad, there exists $k^* \in \{0, \ldots, K-1\}$ such that u does not misbehave for k^* and v does not misbehave backwards for $K - 1 - k^*$. This implies that there exist $w', w'' \in V_0$ such that

$$(u, 0) \rightsquigarrow (w', \mathfrak{s}'_{k^*}) \overset{G_0}{\rightsquigarrow} V_0 \times \{\mathfrak{s}''_{k^*}\}, \quad V_0 \times \{\mathfrak{s}'_{k^*}\} \overset{G_0}{\rightsquigarrow} (w'', \mathfrak{s}''_{k^*}) \rightsquigarrow (v, T).$$

Since the graphical construction inside G_0 is fully coupled in $[\mathfrak{s}'_k, \mathfrak{s}''_k]$, we must also have $(w', \mathfrak{s}'_{k^*}) \rightsquigarrow (w'', \mathfrak{s}''_{k^*})$, hence $(u, 0) \rightsquigarrow (v, T)$ as required. □

Finally, note that $T = \lfloor (\log n)^2 \rfloor \cdot 30N/(\mathfrak{c}_0\bar{\mathfrak{c}}) \leq n(\log n)^3$, so if the graphical construction is fully coupled in $[0, T]$, it is also fully coupled in $[0, n(\log n)^3]$, by (3.5)(c). $\square$

3.2 Long survival on general graphs

3.2.1 Product bounds on extinction time

The following proposition gives a first idea of the application we have in mind for the notion of the contact process to be fully coupled.

Proposition 3.13. *For any $\lambda > \lambda_c(\mathbb{Z})$, there exists $c_{\text{split}} = c_{\text{split}}(\lambda) > 0$ such that the following holds. Let $m \geq 2$, and $G = (V, E)$ be a graph containing m disjoint and connected subgraphs $G_1 = (V_1, E_1), \ldots, G_m = (V_m, E_m)$. Then,*

$$\mathbb{E}[\tau_G] \geq \frac{c_{\text{split}}}{(2|V|^4)^m} \cdot \prod_{i=1}^{m} \mathbb{E}[\tau_{G_i}]. \tag{3.14}$$

Proof. It suffices to prove the statement for $G = (V, E)$ with $|V|$ large enough. Once this is done, we can reduce the value of c_{split} to take care of smaller $|V|$.

Let $(\xi_t)_{t \geq 0}$ be the contact process on G obtained from a graphical construction H, starting from all vertices infected. Fix $s > 0$, to be chosen later. For each $k \in \mathbb{N}_0$, define the events

$$\text{Cross}_k := \bigcup_{i=1}^{m} \{V_i \times \{ks\} \rightsquigarrow V_i \times \{(k+2)s\} \text{ inside } G_i\},$$

and

$$\text{Coup}_k := \{H \text{ is fully coupled in } [ks, (k+1)s]\}.$$

Using Lemma 3.5(e), we see that, for any $K \in \mathbb{N}$,

$$\{\xi_{sK} \neq \varnothing\} \supseteq \bigcap_{k=0}^{K} (\text{Cross}_k \cap \text{Coup}_k).$$

Lemma 2.13 and independence give

$$\mathbb{P}((\text{Cross}_k)^c) \leq \frac{(2s)^m}{\prod_{i=1}^{m} \mathbb{E}[\tau_{G_i}]}, \tag{3.15}$$

and Corollary 3.7 gives

$$\mathbb{P}((\text{Coup}_k)^c) \leq \exp\left\{ -\mathfrak{c}_1 \left\lfloor \frac{s}{|V|(\log(|V|))^3} \right\rfloor \right\}. \tag{3.16}$$

Combining these observations with a union bound gives, for all $t \geq s$,

$$\mathbb{P}(\tau_G \leq t) \leq \left\lceil \frac{t}{s} \right\rceil \cdot \left(\frac{(2s)^m}{\prod_{i=1}^m \mathbb{E}[\tau_{G_i}]} + \exp\left\{ -\mathfrak{c}_1 \left\lfloor \frac{s}{|V|(\log(|V|))^3} \right\rfloor \right\} \right). \quad (3.17)$$

We now set

$$s = |V|^4, \qquad t = \frac{\prod_{i=1}^m \mathbb{E}[\tau_{G_i}]}{2^{m+3} s^{m-1}}.$$

For now, let us assume that $s \leq t$; the case $s > t$ will be treated later. When $s \leq t$, we have $\lceil t/s \rceil \leq 2t/s$, and by (3.17),

$$\mathbb{P}(\tau_G \leq t) \leq \frac{1}{4} + \frac{\prod_{i=1}^m \mathbb{E}[\tau_{G_i}]}{2^{m+2}|V|^{4m}} \cdot \exp\left\{ -\mathfrak{c}_1 \left\lfloor \frac{|V|^3}{(\log(|V|))^3} \right\rfloor \right\} \quad (3.18)$$

If $|V|$ is large enough we have $2^{m+2}|V|^{4m} > 1$. Additionally, by Lemma 2.12, we can bound

$$\mathbb{E}[\tau_{G_i}] \leq ((1 - e^{-1})^{-1} \vee e^{2\lambda})^{|V_i|^2},$$

so

$$\prod_{i=1}^m \mathbb{E}[\tau_{G_i}] \leq ((1 - e^{-1})^{-1} \vee e^{2\lambda})^{|V|^2}.$$

This shows that, if $|V|$ is large enough, the expression on the right-hand side of (3.18) is smaller than $1/2$. Hence,

$$\mathbb{E}[\tau_G] \geq t \cdot \mathbb{P}(\tau_G > t) \geq \frac{\prod_{i=1}^m \mathbb{E}[\tau_{G_i}]}{2^{m+4}|V|^{4m-4}} \geq \frac{\prod_{i=1}^m \mathbb{E}[\tau_{G_i}]}{2^{m+4}|V|^{4m}},$$

so the desired bound holds in this case.

It remains to treat the case where $t < s$. Writing $t/s < 1$ and bounding $\mathbb{E}[\tau_G] \geq 1$ (since any fixed vertex recovers after an amount of time with expectation 1), we have

$$\mathbb{E}[\tau_G] \geq 1 > \frac{t}{s} = \frac{\prod_{i=1}^m \mathbb{E}[\tau_{G_i}]}{2^{m+3}|V|^{4m}},$$

giving the bound in this case as well. □

For situations where the subgraphs $G_1, \ldots, G_m$ are small compared to G, the following proposition can be more useful than the previous one.

Proposition 3.14. *Assume that $\lambda > \lambda_c(\mathbb{Z})$. Let $m \geq 2$, and G be a graph containing m disjoint and connected subgraphs $G_1 = (V_1, E_1), \ldots, G_m = (V_m, E_m)$. Then, for any s, t with $0 < s < t$, we have*

$$\mathbb{P}(\tau_G \leq t)$$
$$\leq \left\lfloor \frac{t}{s} \right\rfloor \cdot \left(\frac{(2s)^m}{\prod_{i=1}^m \mathbb{E}[\tau_{G_i}]} + \sum_{1 \leq i < j \leq m} \exp\left\{ -c_{\text{coup}} \cdot \left\lceil \frac{s}{\sigma_{i,j}(\log \sigma_{i,j})^3} \right\rceil \right\} \right), \quad (3.19)$$

where

$$\sigma_{i,j} := |V_i| + |V_j| + \mathrm{dist}(V_i, V_j) - 1.$$

Proof. Let $G_{i,j}$ be the subgraph of G consisting of G_i, G_j, and a path of minimal length connecting the two. Note that the number of vertices of $G_{i,j}$ is $\sigma_{i,j}$.

As in the previous proof, for $k \in \mathbb{N}_0$, define

$$\mathrm{Cross}_k := \bigcup_{i=1}^{m} \{V_i \times \{ks\} \rightsquigarrow V_i \times \{(k+2)s\} \text{ inside } G_i\};$$

also define

$$\mathrm{Coup}_k := \bigcap_{1 \le i < j \le m} \left\{ \begin{array}{l} \text{The restriction of } H \text{ to } G_i \\ \text{is fully coupled in } [ks, (k+1)s] \end{array} \right\}.$$

We then have

$$\{\xi_{sK} \neq \varnothing\} \supseteq \bigcap_{k=0}^{K} (\mathrm{Cross}_k \cap \mathrm{Coup}_k).$$

The proof now follows by bounding as in (3.15) and (3.16). □

3.2.2 Recursive proofs of long survival

In this section, we prove Theorem 3.1 and Proposition 3.3. The following lemma about splitting trees will be useful. We write $|T|$ for the number of vertices of a tree T.

Lemma 3.15. *Let $d \ge 2$ and let T be a finite and connected tree in which all vertices have degree smaller than or equal to d. Then, there exists an edge of e whose removal produces two subtrees, both of which contain at least $\lceil (|T| - 1)/d \rceil$ vertices.*

Proof. Given an edge $e = \{u, v\}$ of T, the removal of e breaks T into two subtrees, which we denote by $T_{e,u} \ni u$ and $T_{e,v} \ni v$. We then define $f(e) := \max(|T_{e,u}|, |T_{e,v}|)$. Let $\bar{e} = \{\bar{u}, \bar{v}\}$ be an edge for which f is minimized; assume that $f(\bar{e}) = |T_{\bar{e},\bar{u}}| \ge |T_{\bar{e},\bar{v}}|$. Let $v_1, \ldots, v_m$ denote the neighbors of $\bar{u}$ other than $\bar{v}$. For any i we have

$$|T_{\{\bar{u},v_i\},v_i}| < |T_{\bar{e},\bar{u}}|$$

and

$$\max(|T_{\{\bar{u},v_i\},v_i}|, |T_{\{\bar{u},v_i\},\bar{u}}|) = f(\{\bar{u}, v_i\}) \ge f(\bar{e}) = |T_{\bar{e},\bar{u}}|.$$

where the last inequality follows from the minimality of $\bar{e}$. Putting these two inequalities together gives

$$|T_{\{\bar{u},v_i\},\bar{u}}| \ge |T_{\bar{e},\bar{u}}| \;\Rightarrow\; |T| - |T_{\{\bar{u},v_i\},\bar{u}}| \le |T| - |T_{\bar{e},\bar{u}}| \;\Rightarrow\; |T_{\{\bar{u},v_i\},v_i}| \le |T_{\bar{e},\bar{v}}|.$$

Then,

$$|T| = 1 + |T_{\bar{e},\bar{v}}| + \sum_i |T_{\{\bar{u},v_i\},v_i}| \leq 1 + d \cdot |T_{\bar{e},\bar{v}}|.$$

We have thus obtained $|T_{\bar{e},\bar{u}}| \geq |T_{\bar{e},\bar{v}}| \geq (|T|-1)/d$. □

Lemma 3.16. *For any $\lambda > \lambda_c(\mathbb{Z})$ and $\alpha > 0$, there exists n_1 such that for every $n \geq n_1$, on every connected graph G with n vertices, the expectation of the extinction time of the contact process with rate λ is at least n^α.*

Proof. Let n be large and G be a connected graph with n vertices. It suffices to prove the statement under the additional assumption that G is a tree (otherwise, we can pass to a spanning tree and use monotonicity).

Recall the constant $\mathfrak{c}_0$ from (3.5), and let $N := \lfloor \frac{2\alpha}{\mathfrak{c}_0} \log n \rfloor$. In case G has either a vertex of degree at least N, or a subgraph consisting of a line segment of length at least N, then $\mathbb{E}[\tau_G] > n^\alpha$, by (3.3), (3.4), and the choice of $\mathfrak{c}_0$. For the rest of the proof, we assume that

$$\operatorname{diam}(G) \leq N, \qquad \max_{v \in G} \deg(v) \leq N. \tag{3.20}$$

Consider a collection

$$\{G'_1 = (V'_1, E'_1), \dots, G'_k = (V'_k, E'_k)\}$$

of disjoint and connected subtrees of G such that

$$\cup_{i=1}^k V'_i = V \qquad \text{and} \qquad |V'_i| \geq \sqrt{n} \text{ for all } i.$$

If some of the trees in this collection have more than $2N\sqrt{n}$ vertices, then we can use Lemma 3.15 to split each of them into two disjoint and connected subtrees, each with at least $\lceil (2N\sqrt{n}-1)/N \rceil \geq \sqrt{n}$ vertices.

Using this observation, we see that we can obtain a collection

$$\{G''_1 = (V''_1, E''_1), \dots, G''_m = (V''_m, E''_m)\}$$

of disjoint and connected subtrees of G such that

$$\cup_{i=1}^m V''_i = V \qquad \text{and} \qquad \sqrt{n} \leq |V''_i| \leq 2N\sqrt{n} \text{ for each } i.$$

In particular, $n = \sum_{i=1}^m |V''_i| \leq 2N\sqrt{n} \cdot m \leq n^{2/3} \cdot m$, so

$$m \geq n^{1/3}. \tag{3.21}$$

Arguing as in (3.7), we deduce from $|V''_i| \geq \sqrt{n}$ that G''_i has a connected subtree $G_i = (V_i, E_i)$ which is either a star or a line segment and has

$$|V_i| = \lfloor (\tfrac{1}{2} \log n)^{1/2} \rfloor. \tag{3.22}$$

By (3.3) and (3.4),

$$\mathbb{E}[\tau_{G_i}] \geq \exp\{\mathfrak{c}_0|V_i|\} \geq \exp\{\tfrac{\mathfrak{c}_0}{4}\sqrt{\log n}\}, \quad i = 1, \ldots, m,$$

and so,

$$\prod_{i=1}^{m} \mathbb{E}[\tau_{G_i}] \geq \exp\{\tfrac{\mathfrak{c}_0}{4}\sqrt{\log n} \cdot m\}.$$

We apply Proposition 3.14 to G and the subgraphs $G_1, \ldots, G_N$ (with s and t still to be chosen). Note that

$$\sigma_{i,j} \leq |V_i| + |V_j| + \mathrm{diam}(G) \leq 2(\tfrac{1}{2}\log n)^{1/2} + N \leq (\log n)^2,$$

where the second inequality follows from (3.20) and (3.22). We further bound

$$\sigma_{i,j}(\log \sigma_{i,j})^3 \leq (\log n)^3,$$

when n is large. Then, the expression in (3.19) is smaller than

$$\left\lceil \frac{t}{s} \right\rceil \cdot \left(\left(\frac{2s}{\exp\{\frac{\mathfrak{c}_0}{4}\sqrt{\log n}\}} \right)^m + n^2 \cdot \exp\left\{ -c_{\mathrm{coup}} \left\lfloor \frac{s}{(\log n)^3} \right\rfloor \right\} \right).$$

We now take $s = (\log n)^5$ and $t = 2n^\alpha$. Using (3.21), it is easy to see that the above expression can be made as small as desired by taking n large. Then, bounding $\mathbb{E}[\tau_G] \geq t \cdot \mathbb{P}(\tau_G \geq t)$ gives the desired bound.

□

Proof of Proposition 3.3. The proof is similar to that of the previous lemma, only simpler. Again, it suffices to prove the lemma for trees. Additionally, it suffices to prove the proposition for n sufficiently large; afterwards we can reduce the value of the constant c in the bound $\mathbb{E}[\tau_G] > e^{cn^{1/3}}$ to take care of other values of n.

Let $G = (V, E)$ be a connected tree with n vertices. In case G has either a vertex with degree above $n^{1/3}$, or a subgraph consisting of a line segment of length at least $n^{1/3}$, then $\mathbb{E}[\tau_G] \geq \exp\{\mathfrak{c}_0 n^{1/3}\}$, by (3.3), (3.4), and the choice of $\mathfrak{c}_0$. We assume for the rest of the proof that

$$\mathrm{diam}(G) \leq n^{1/3}, \qquad \max_{v \in G} \deg(v) \leq n^{1/3}. \tag{3.23}$$

Arguing as in the proof of Lemma 3.16, we can obtain a collection

$$\{G_1 = (V_1, E_1), \ldots, G_k = (V_k, E_k)\}$$

of disjoint and connected subtrees of G such that

$$\cup_{i=1}^k V_i = V \qquad \text{and} \qquad n^{1/3} \leq |V_i| \leq n^{2/3} \text{ for each } i.$$

In particular, $n = \sum_{i=1}^{k} |V_i| \le kn^{2/3}$, so $k \ge n^{1/3}$. By Lemma 3.16 with $\alpha = 15$, when n is large we have

$$\mathbb{E}[\tau_{G_i}] \ge |V_i|^{15} \ge (n^{1/3})^{15} = n^5.$$

We now apply Proposition 3.13, which gives

$$\mathbb{E}[\tau_G] \ge \frac{c_{\text{split}}}{(2n^4)^k} \prod_{i=1}^{k} \mathbb{E}[\tau_{G_i}] \ge c_{\text{split}} \cdot \left(\frac{n^5}{2n^4}\right)^{n^{1/3}} = c_{\text{split}} \cdot \left(\frac{n}{4}\right)^{n^{1/3}}. \qquad \square$$

Proof of Theorem 3.1. Once more, it suffices to prove the result for trees. For any integer $r \ge 2$, define

$$\alpha_r := \inf\left\{\frac{\log \mathbb{E}[\tau_G]}{|V|} : G = (V,E) \text{ is a connected graph: } 2 \le |V| \le 2^r\right\};$$

the expectation inside the infimum is of the extinction time of the contact process with parameter λ on G. The result will follow if we prove that α_r is bounded away from zero as $r \to \infty$.

Fix $r \in \mathbb{N}$ (to be assumed large), and also a connected graph $G = (V,E)$ with $n := |V| \in \{2^r + 1, \ldots, 2^{r+1}\}$. Using Lemma 3.15, there exists a constant K_d (depending only on d) such that there exists a collection $\{G_1 = (V_1, E_1), \ldots, G_k = (V_k, E_k)\}$ of connected subtrees of G with

$$k \le K_d, \qquad \cup_{i=1}^{k} V_i = V, \qquad |V_i| \le 2^r \text{ for each } i.$$

By Proposition 3.13, we have

$$\begin{aligned} \log \mathbb{E}[\tau_G] &\ge \sum_{i=1}^{k} \log \mathbb{E}[\tau_{G_i}] - 4k \log |V| - k \log 2 \\ &\ge \sum_{i=1}^{k} \log \mathbb{E}[\tau_{G_i}] - 5K_d(r+1) \log 2, \end{aligned} \tag{3.24}$$

assuming that r (and thus also $|V|$) is large enough. Since $|V_i| \le 2^r$, we have $\log \mathbb{E}[\tau_{V_i}]/|V_i| \ge \alpha_r$. Using this in (3.24), and also dividing by $|V|$, gives

$$\frac{\log \mathbb{E}[\tau_G]}{|V|} \ge \alpha_r - 5K_d \log 2 \cdot (r+1)2^{-r}$$

Taking the infimum over all $G = (V,E)$ with $2 \le |V| \le 2^{r+1}$, we obtain

$$\alpha_{r+1} \ge \alpha_r - C(r+1)2^{-r},$$

with $C := 5K_d \log 2$. Using this and induction, it follows that for any r_0 and any $r \ge r_0$, we have

$$\alpha_r \ge \alpha_{r_0} - C \sum_{i=r_0}^{\infty} (i+1)2^{-i} \ge \alpha_{r_0} - C' 2^{-r_0}, \tag{3.25}$$

for some constant C'.

Next, note that Proposition 3.3 gives, for any graph $G = (V, E)$,

$$\frac{\log \mathbb{E}[\tau_G]}{|V|} \geq \frac{c|V|^{1/3}}{|V|} = c|V|^{-2/3}$$

which gives

$$\alpha_r \geq c2^{-\frac{2}{3}r}. \tag{3.26}$$

We can now put together (3.25) and (3.26) to conclude. We fix r_0 large enough that $c2^{-\frac{2}{3}r_0} - C'2^{-r_0} > 0$, and then we obtain

$$\alpha_r \geq c2^{-\frac{2}{3}r_0} - C'2^{-r_0} > 0 \quad \text{for all } r \geq r_0. \qquad \square$$

3.3 Convergence to exponential distribution

In this section, we prove Theorem 3.4. It will be obtained as a consequence of the following proposition, combined with findings from the previous two sections.

Proposition 3.17 (Mountford [53]). *For each $n \in \mathbb{N}$, let $\lambda_n > 0$ and $G_n = (V_n, E_n)$ be a finite graph. Let τ_{G_n} denote the extinction time of the contact process with rate λ on G_n. Assume that there is a sequence of positive real numbers $(a_n)_{n\geq 0}$ such that*

$$\frac{a_n}{\mathbb{E}[\tau_{G_n}]} \xrightarrow{n\to\infty} 0 \tag{3.27}$$

and

$$\delta_n := \sup_{A\subseteq V_n} \mathbb{P}(\xi^A_{a_n} \neq \varnothing,\ \xi^A_{a_n} \neq \xi^{V_n}_{a_n}) \xrightarrow{n\to\infty} 0. \tag{3.28}$$

Then, as $n \to \infty$, $\tau_{G_n}/\mathbb{E}[\tau_{G_n}]$ converges in distribution to the exponential distribution with parameter 1.

Before addressing the proof of Proposition 3.17, we show how it can be used to prove Theorem 3.4.

Proof of Theorem 3.4. We fix $\lambda > \lambda_c(\mathbb{Z})$ a sequence of graphs $(G_n) = ((V_n, E_n))$ as in the statement of the theorem. We apply Proposition 3.17 with $\lambda_n \equiv \lambda$. We set $a_n := \exp\{|V_n|^{1/6}\}$. This sequence satisfies (3.27) by Proposition 3.3. Moreover, letting H denote a graphical construction for the process,

$$\delta_n \leq \mathbb{P}(H \text{ is fully coupled on } [0, a_n]).$$

By Corollary 3.7, the right-hand side is smaller than

$$(1 - \mathfrak{c}_1)^{\lfloor a_n/(|V_n|(\log|V_n|)^3)\rfloor} \xrightarrow{n\to\infty} 0,$$

giving (3.28) and completing the proof. $\square$

For the proof of Proposition 3.17, we will need the following.

Lemma 3.18. *Under the assumptions of Proposition 3.17, for any $s \geq a_n$ and any $t > s + a_n$,*

$$\mathbb{P}\begin{pmatrix} V_n \times \{0\} \rightsquigarrow V_n \times \{s\}, \\ V_n \times \{s\} \rightsquigarrow V_n \times \{t\}, \\ V_n \times \{0\} \not\rightsquigarrow V_n \times \{t\} \end{pmatrix} < 2\delta_n + \frac{2a_n}{\mathbb{E}[\tau_{G_n}]}. \tag{3.29}$$

Proof. We write

$$\xi_t^{A,s}(v) := \mathbb{1}\{A \times \{s\} \rightsquigarrow (v,t)\}, \quad t \geq s.$$

Since $t > s + a_n$, the probability on the left-hand side of (3.29) is smaller than

$$\mathbb{P}\left(\xi_s^{V_n} \neq \varnothing,\ \xi_{s+a_n}^{V_n,s} \neq \varnothing,\ \xi_{s+a_n}^{V_n} \neq \xi_{s+a_n}^{V_n,s}\right). \tag{3.30}$$

As a first step to bound this, we bound:

$$\begin{aligned}\mathbb{P}(\tau_{G_n} \in (s, s+a_n)) \leq \mathbb{P}&\left(\xi_{s+a_n}^{V_n,s-a_n} = \varnothing\right) \\ &+ \mathbb{P}\left(\xi_s^{V_n} \neq \varnothing,\ \xi_{s+a_n}^{V_n,s-a_n} \neq \varnothing,\ \xi_{s+a_n}^{V_n} \neq \xi_{s+a_n}^{V_n,s-a_n}\right) \\ &\leq \frac{2a_n}{\mathbb{E}[\tau_{G_n}]} + \delta_n, \end{aligned} \tag{3.31}$$

where in the second inequality we have bounded the first probability using Lemma 2.13 and the second probability using the definition of δ_n in (3.28) and the Markov property. Next, the probability in (3.30) is smaller than

$$\begin{aligned}&\mathbb{P}(\tau_{G_n} \in (s, s+a_n)) + \mathbb{P}\left(\xi_{s+a_n}^{V_n} \neq \varnothing,\ \xi_{s+a_n}^{V_n,s} \neq \varnothing,\ \xi_{s+a_n}^{V_n} \neq \xi_{s+a_n}^{V_n,s}\right) \\ &\quad\leq \frac{2a_n}{\mathbb{E}[\tau_{G_n}]} + 2\delta_n,\end{aligned}$$

where we have bounded the first probability using (3.31) and the second by again using the definition of δ_n and the Markov inequality. $\square$

Proof of Proposition 3.17. For $n \in \mathbb{N}$ and $\varepsilon > 0$, let $w_{n,\varepsilon}$ be the positive real number satisfying $\mathbb{P}(\tau_{G_n} < w_{n,\varepsilon}) = \varepsilon$. To justify the existence and uniqueness of this value, we note that τ_{G_n} is the absorption time of a continuous-time Markov chain on a finite state space, so its distribution function is continuous and strictly monotone.

It will be useful to note that for any ε, we have $a_n < w_{n,\varepsilon}$ if n is large enough. This follows from

$$\mathbb{P}(\tau_{G_n} < a_n) \leq \frac{a_n}{\mathbb{E}[\tau_{G_n}]} \xrightarrow{n\to\infty} 0,$$

where the inequality is from Lemma 2.13 and the convergence is being assumed.

Similarly to the proof of Lemma 2.13, define

$$\hat{\tau}_{n,\varepsilon} := \inf\{t \in \{w_{n,\varepsilon}, 2w_{n,\varepsilon}, \ldots\} : \; V_n \times \{t - w_{n,\varepsilon}\} \not\rightsquigarrow V_n \times \{t\}\}.$$

Note that $\hat{\tau}_{n,\varepsilon} \sim \mathrm{Geom}(\varepsilon)$ and $\tau_{G_n} \le \hat{\tau}_{n,\varepsilon}$. In particular, for any $k \in \mathbb{N}$ we have

$$\begin{aligned}(1-\varepsilon)^k &= \mathbb{P}(\hat{\tau}_{n,\varepsilon} > k \cdot w_{n,\varepsilon}) \\ &= \mathbb{P}(\tau_{G_n} > k \cdot w_{n,\varepsilon}) + \mathbb{P}(\hat{\tau}_{n,\varepsilon} > k \cdot w_{n,\varepsilon},\; \tau_{G_n} \le k \cdot w_{n,\varepsilon}).\end{aligned} \tag{3.32}$$

We now claim that for any $k \in \mathbb{N}$,

$$\mathbb{P}(\tau_{G_n} \le k \cdot w_{n,\varepsilon},\; \hat{\tau}_{n,\varepsilon} > k \cdot w_{n,\varepsilon}) \xrightarrow{n \to \infty} 0. \tag{3.33}$$

This is evident for $k = 1$, and for $k > 1$ it follows from bounding

$$\begin{aligned}&\mathbb{P}(\tau_{G_n} \le k \cdot w_{n,\varepsilon},\; \hat{\tau}_{n,\varepsilon} > k \cdot w_{n,\varepsilon}) \\ &\le \sum_{j=0}^{k} \mathbb{P}\begin{pmatrix} V_n \times \{0\} \rightsquigarrow V_n \times \{j \cdot w_{n,\varepsilon}\}, \\ V_n \times \{0\} \not\rightsquigarrow V_n \times \{(j+1) \cdot w_{n,\varepsilon}\}, \\ V_n \times \{j \cdot w_{n,\varepsilon}\} \rightsquigarrow V_n \times \{(j+1) \cdot w_{n,\varepsilon}\} \end{pmatrix} \xrightarrow{n \to \infty} 0\end{aligned}$$

by Lemma 3.18 (using the fact that $w_{n,\varepsilon} > a_n$).

Combining (3.32) and (3.33) gives

$$\text{for any } k \in \mathbb{N}, \quad \mathbb{P}(\tau_{G_n} > k \cdot w_{n,\varepsilon}) \nearrow (1-\varepsilon)^k \text{ as } n \to \infty. \tag{3.34}$$

Abbreviating $\bar{F}_n(t) := \mathbb{P}(\tau_{G_n} > t)$, we now bound

$$w_{n,\varepsilon} \cdot \sum_{j=1}^{\infty} \bar{F}_n(j \cdot w_{n,\varepsilon}) \le \mathbb{E}[\tau_{G_n}] \le w_{n,\varepsilon} \cdot \sum_{j=0}^{\infty} \bar{F}_n(j \cdot w_{n,\varepsilon}),$$

so

$$\sum_{j=1}^{\infty} \bar{F}_n(j \cdot w_{n,\varepsilon}) \le \frac{\mathbb{E}[\tau_{G_n}]}{w_{n,\varepsilon}} \le \sum_{j=0}^{\infty} \bar{F}_n(j \cdot w_{n,\varepsilon}),$$

Using (3.34) and the Monotone Convergence Theorem, we obtain

$$\frac{1}{\varepsilon} - 1 \le \liminf_{n \to \infty} \frac{\mathbb{E}[\tau_{G_n}]}{w_{n,\varepsilon}} \le \limsup_{n \to \infty} \frac{\mathbb{E}[\tau_{G_n}]}{w_{n,\varepsilon}} \le \frac{1}{\varepsilon}. \tag{3.35}$$

Now fix $t > 0$, and let ε be small, to be chosen later (depending on t). By (3.35), if n is large enough we have

$$\left\lfloor \left(\frac{1}{\varepsilon} - 1 - \varepsilon\right) \cdot t \right\rfloor < \frac{\mathbb{E}[\tau_{G_n}] \cdot t}{w_{n,\varepsilon}} < \left\lfloor \left(\frac{1}{\varepsilon} + \varepsilon\right) \cdot t \right\rfloor. \tag{3.36}$$

We now write

$$\mathbb{P}\left(\frac{\tau_{G_n}}{\mathbb{E}[\tau_{G_n}]} > t\right) = \bar{F}_n\left(\frac{\mathbb{E}[\tau_{G_n}]\cdot t}{w_{n,\varepsilon}}\cdot w_{n,\varepsilon}\right);$$

by (3.36), this is in the interval

$$\left(\bar{F}_n\left(\left\lfloor\left(\frac{1}{\varepsilon}+\varepsilon\right)\cdot t\right\rfloor\cdot w_{n,\varepsilon}\right),\ \bar{F}_n\left(\left\lfloor\left(\frac{1}{\varepsilon}-1-\varepsilon\right)\cdot t\right\rfloor\cdot w_{n,\varepsilon}\right)\right).$$

By (3.34), as $n\to\infty$ this approaches

$$\left((1-\varepsilon)^{\lfloor(\frac{1}{\varepsilon}+\varepsilon)\cdot t\rfloor},\ (1-\varepsilon)^{\lfloor(\frac{1}{\varepsilon}-1-\varepsilon)\cdot t\rfloor}\right).$$

By taking ε small enough, both extremities can be made arbitrarily close to e^{-t}. This completes the proof. □

3.4 Convergence of exponential rate

We conclude this chapter by briefly describing results from [64], by Schapira and V., which are further applications of Proposition 3.13. We will not present proofs in this section.

Let us first give some motivation. Earlier in this section, we have seen that, given $\lambda > \lambda_c(\mathbb{Z})$ and a sequence of connected graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, the value of $\mathbb{E}[\tau_{G_n}]$ typically grows at least exponentially with $|V_n|$. For certain choices of the sequence (G_n), this can be improved to the statement that

$$\frac{\log\mathbb{E}[\tau_{G_n}]}{|V_n|} \xrightarrow{n\to\infty} c \in (0,\infty). \tag{3.37}$$

This was done by Mountford in [56] for boxes of $\mathbb{Z}^d$ (under the assumption that $\lambda > \lambda_c(\mathbb{Z}^d)$, which is at least as good as $\lambda > \lambda_c(\mathbb{Z})$, and presumably better). The approach of Mountford was to use a sub-additivity argument, breaking a box with side length 2^m into 2^d boxes of side length 2^{m-1}, and arguing that the contact processes in these sub-boxes "help each other out", with a kind of argument that is similar to our Proposition 3.13.

As a side remark, we find it quite puzzling that (3.37) is done for boxes, but there is no obvious way to obtain the same convergence for d-dimensional tori. The issue is that the torus lacks the "self-similarity" of the box, in the sense that $(\mathbb{Z}/2^m\mathbb{Z})^d$ cannot be broken into 2^d tori isomorphic to $(\mathbb{Z}/2^{m-1}\mathbb{Z})^d$.

The purpose of [64] was to prove (3.37) for the sequence (G_n) taken from certain classes of random graphs. The graphs treated all have the aforementioned "self-similarity" property – but now in a distributional sense – and this allows for the application of Proposition 3.13. The assumption that $\lambda > \lambda_c(\mathbb{Z})$ is always made (mostly for allowing the application of Proposition 3.13); this assumption is probably not optimal in any of the cases being considered. The following are a few of the possible choices of (G_n) taken in [64] (we refer the reader to this paper for further details on the models):

- G_n = the giant component given by supercritical *Bernoulli bond percolation* inside a box of $\mathbb{Z}^d$ ($d \geq 2$) with side length n;
- G_n = the giant component given by the supercritical *random geometric graph* inside a box of $\mathbb{R}^d$ ($d \geq 2$) with side length n;
- G_n = a supercritical Bienaymé–Galton–Watson tree, conditioned to being infinite, but truncated at height n.

Chapter 4

Existence of an extinction regime on BGW trees

For supercritical BGW trees in which the offspring distribution has sufficiently heavy tail, the contact process has no extinction regime: the infection survives locally, even if the infection rate is very small. This is because, even if we choose the infection rate λ very small, the tree will have very high-degree vertices (where "high-degree" depends on λ) which sustain the infection for a long time, and send it to each other.

Due to the combined works of Huang and Durrett [35] and Bhamidi, Nam, Nguyen and Sly [9], we know exactly in which BGW trees the contact process has no extinction regime: those where the offspring distribution has no exponential moment. We formally state this in the two theorems below.

Theorem 4.1 (Huang and Durrett [35]). *Let $\mathcal{T}$ be a BGW tree with offspring distribution p. Assume that p has expectation larger than 1 and that*

$$\sum_k e^{ck} \cdot p(k) = \infty \quad \textit{for all } c > 0. \tag{4.1}$$

Then, for any $\lambda > 0$, for almost every realization of $\mathcal{T}$ on the event $\{\mathcal{T}$ is infinite$\}$, the contact process with rate λ can survive locally on $\mathcal{T}$.

Theorem 4.2 (Bhamidi, Nam, Nguyen and Sly [9]). *Let $\mathcal{T}$ be a BGW tree with offspring distribution p. Assume that*

$$\sum_k e^{ck} \cdot p(k) < \infty \quad \textit{for some } c > 0. \tag{4.2}$$

Then, for λ small enough, on almost every realization of $\mathcal{T}$, the contact process with rate λ dies out.

We give the proof of Theorem 4.1 in Section 4.2 and the proof of Theorem 4.2 in Section 4.3. Before those, we briefly discuss the critical values of the contact process on BGW trees.

4.1 Preliminaries on the contact process on BGW trees

The main result we are aiming for in this section is the following.

Proposition 4.3. *Let $\mathcal{T}$ be a BGW tree. Then, almost surely on the event that $\mathcal{T}$ is infinite, $\lambda_c^{(1)}(\mathcal{T})$ and $\lambda_c^{(2)}(\mathcal{T})$ are constant.*

A result in this spirit was proved by Pemantle in [60] (see Proposition 3.1 there), but the proof presented here is somewhat different from the treatment there.

We obtain Proposition 4.3 as a direct consequence of the following lemma.

Lemma 4.4. *Let $\mathcal{T}$ be a BGW tree and $\lambda > 0$. Let $(\xi_t)_{t\geq 0}$ be the contact process on $\mathcal{T}$ with rate λ, with only the root infected at time 0. Then,*

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot)) > 0 \tag{4.3}$$

if and only if

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot) \mid \mathcal{T}) > 0 \quad \textit{almost surely on } \{\mathcal{T} \textit{ is infinite}\}. \tag{4.4}$$

The same statement holds with $\mathrm{Surv}_{\mathrm{glob}}$ *replaced by* $\mathrm{Surv}_{\mathrm{loc}}$.

Proof. The proof is the same for global survival and local survival, so we only do the former. The fact that (4.4) implies (4.3) is easily seen by integration. We prove the reverse implication.

Given a (deterministic) tree T, let

$$f(T) := \sup_{v\in T} \mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot^v)),$$

where $(\xi_t^v)_{t\geq 0}$ is the contact process on T with $\xi_0^v = \mathbb{1}_{\{v\}}$. We will consider the random variable $f(\mathcal{T})$, where $\mathcal{T}$ is the BGW tree in the statement of the lemma.

Let ρ denote the probability in (4.3). The desired statement will follow from proving that

$$f(\mathcal{T}) \geq \rho \quad \text{almost surely on } \{\mathcal{T} \text{ is infinite}\}. \tag{4.5}$$

For each $n \in \mathbb{N}_0$, let $\mathcal{T}_n$ denote the sub-tree of the BGW tree $\mathcal{T}$ obtained by excluding all vertices at distance larger than n from the root. Let $\mathcal{F}_n$ be the σ-algebra generated by $\mathcal{T}_n$. Also let $\mathcal{F}$ be the σ-algebra generated by all the $\mathcal{F}_n$, which is the same as the σ-algebra generated by $\mathcal{T}$. It follows from a convergence result for martingales ([46, Theorem 14.3.4]) that

$$\mathbb{E}[f(\mathcal{T}) \mid \mathcal{F}_n] \xrightarrow[\text{a.s.}]{n\to\infty} \mathbb{E}[f(\mathcal{T}) \mid \mathcal{F}] = f(\mathcal{T}). \tag{4.6}$$

We now note that, for each n,

$$\text{on } \{\exists v \in \mathcal{T}:\ \mathrm{dist}(o, v) = n\}, \quad \mathbb{E}[f(\mathcal{T}) \mid \mathcal{F}_n] \geq \rho,$$

since any vertex at distance n from o is the root of a (yet unrevealed, under $\mathcal{F}_n$) new BGW tree with the same distribution as that of $\mathcal{T}$. This implies in

$$\text{on } \{\mathcal{T} \text{ is infinite}\}, \quad \mathbb{E}[f(\mathcal{T}) \mid \mathcal{F}_n] \geq \rho \text{ for all } n. \tag{4.7}$$

Clearly, (4.6) and (4.7) together give (4.5). □

The following may also be of interest: in a BGW tree, if there is a chance of local survival, then there cannot be global survival without local survival.

Proposition 4.5. *Let $\mathcal{T}$ be a BGW tree and $\lambda > 0$. Let $(\xi_t)_{t\geq 0}$ be the contact process on $\mathcal{T}$ with rate λ, with only the root infected at time* 0. *Assume that*

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) > 0. \tag{4.8}$$

Then,

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot) \cap (\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot))^c) = 0. \tag{4.9}$$

Remark 4.6. Let $G = (V, E)$ be a connected graph and consider the contact process on G with rate $\lambda > 0$ (in the following discussion, λ will be kept fixed, and the initial configuration will vary). Recall that the probability $\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot^A))$ is positive for *some* finite and non-empty $A \subset V$ if and only if it is positive for *all* finite and non-empty $A \subset V$. Assume that this is the case. It is easy to see that the conditional probabilities

$$\{\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot^A) \mid \mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot^A)) : A \subset V,\ A \text{ finite and non-empty}\}$$

are either all zero or all positive. Guided by this consideration, the above proposition could be replaced by a stronger version in which the initial configuration is a finite and non-empty set (which may depend on the realization of the tree). For simplicity, we stated it (and will prove it) only for the case where $\xi_0 = \mathbb{1}_{\{o\}}$.

Proof of Proposition 4.5. Define

$$L_t := \max\{\mathrm{dist}(o, v) : v \in \xi_s \text{ for some } s \leq t\}, \qquad t \geq 0$$

that is, L_t is the farthest from the root the infection has reached by time t. Also define the filtration

$$\mathcal{G}_t = \sigma((\xi_s(o), L_s) : s \leq t), \qquad t \geq 0.$$

We emphasize that this filtration only incorporates the information about whether the root is infected and the farthest the infection has reached up to a given time. In particular, $\mathcal{G}_0$ is the trivial σ-algebra, and does not include information about the realization of the random tree $\mathcal{T}$. We also note that, letting $\mathcal{G}_\infty$ be the σ-algebra generated by all the $\mathcal{G}_t$, we have

$$\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) = \left\{ \limsup_{t\to\infty} \xi_t(o) = 1 \right\} \in \mathcal{G}_\infty. \tag{4.10}$$

It follows from a convergence result for martingales ([46, Theorem 14.3.4]) that

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid \mathcal{G}_t) \xrightarrow[\text{a.s.}]{t\to\infty} \mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid \mathcal{G}_\infty) \overset{(4.10)}{=} \mathbb{1}_{\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)}. \tag{4.11}$$

We claim that

$$\text{on } \mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot), \quad \limsup_{t\to\infty} \mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid \mathcal{G}_t) > 0. \tag{4.12}$$

To justify this, we argue as follows. Let σ_n be the first time when $L_t = n$. On the event $\{\sigma_n < \infty\}$, at time σ_n, the infection has reached a vertex X_n in generation n. The σ-algebra $\mathcal{G}_{\sigma_n}$ includes no information about the subtree $\mathcal{T}_{X_n}$ which includes X_n and its descendants. Then, there is a chance given by the left-hand side of (4.8) that the contact process spreading from X_n alone survives locally in $\mathcal{T}_{X_n}$, and if this happens, there is local survival on the whole tree. This proves that

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid \mathcal{G}_{\sigma_n}) \cdot \mathbb{1}_{\{\sigma_n < \infty\}} \geq \rho > 0,$$

where ρ is the left-hand side of (4.8). Hence,

$$\text{on } \{\sigma_n < \infty \text{ for all } n\}, \quad \limsup_{t\to\infty} \mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid \mathcal{G}_t) > 0. \tag{4.13}$$

Now, since the contact process cannot survive forever on a finite set, we have

$$\mathbb{P}(\sigma_n < \infty \text{ for all } n \mid \mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) = 1. \tag{4.14}$$

Now, (4.13) and (4.14) together give (4.12).

Let us now see how (4.12) allows us to conclude. We have

$$\begin{aligned}\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) &\overset{(4.11)}{=} \mathbb{P}\left(\limsup_{t\to\infty} \mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid \mathcal{G}_t) > 0 \right) \\ &\overset{(4.12)}{\geq} \mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot)).\end{aligned}$$

Since $\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot) \supseteq \mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)$, this gives (4.9). $\square$

4.2 Offspring distribution with no finite exponential moment

In this section, we prove Theorem 4.1. The proof given here differs significantly from the one of [35]. We argue that, under the assumptions of the theorem, when $\mathcal{T}$ is infinite, it almost surely contains a subgraph consisting of a half-line of stars connected to each other by line segments, with sizes of the stars and lengths of segments chosen carefully, depending on λ, so that the subgraph can sustain the infection, by a comparison with oriented percolation.

We start by introducing some notation. Let $(d_n)_{n\in\mathbb{N}}$ and $(\ell_n)_{n\in\mathbb{N}}$ be sequences of natural numbers with $d_0 \geq 1$ and $d_n \geq 2$ for all $n \geq 1$. Let $G((d_n),(\ell_n))$ be the graph defined as follows. Start with an infinite half-line, with vertices $v_0, v_1, \ldots$ and all edges of the form $\{v_i, v_{i+1}\}$. Then, letting $j_0 := 0$ and $j_n := \sum_{i=1}^n \ell_i$ for $n \in \mathbb{N}$, we append leaves neighboring each v_{j_n}, until the degree of v_{j_n} becomes d_n. In summary, this is a graph consisting of stars with hubs

$$h_n := v_{j_n}, \quad n \in \mathbb{N}_0,$$

with

$$\deg(h_n) = d_{j_n}, \quad \mathrm{dist}(h_n, h_{n+1}) = \ell_n, \quad n \in \mathbb{N}_0.$$

Proposition 4.7. *For any $\varepsilon > 0$ there exists $\mathfrak{C}(\varepsilon) > 0$ such that the following holds. Let $\lambda > 0$ and $\ell \in \mathbb{N}$; also let (d_n) and (ℓ_n) be sequences such that*

$$\ell_n \leq \ell, \quad d_n \geq \mathfrak{C}(\varepsilon) \cdot \max\left(1, \frac{\log(1/\lambda)}{\lambda^2}\right) \cdot \ell \textit{ for all } n.$$

Letting $(\xi_t)_{t\geq 0}$ denote the contact process on $G((d_n),(\ell_n))$ with rate λ so that $\xi_0(v_0) = 1$, we have

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) > 1 - \varepsilon.$$

Before we prove this proposition, we show how it can be used to prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.4, it suffices to prove that

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) > 0 \quad \text{for any } \lambda > 0 \tag{4.15}$$

(where the contact process is started from only the root infected, and the probability includes randomness both of the graph and of the contact process). Fix $\lambda > 0$. Letting $\mathfrak{C}(1/2)$ be as in Proposition 4.7 (with $\varepsilon = 1/2$), define

$$\mathfrak{m}_\lambda := \left(\mathfrak{C}(1/2) \cdot \max\left(1, \frac{\log(1/\lambda)}{\lambda^2}\right)\right)^{-1},$$

The assumption that p has no finite exponential moment implies that for any $a > 0$, there are infinitely many values of $d \in \mathbb{N}$ such that $p(d) > \exp\{-ad\}$. In particular, we take $d \in \mathbb{N}$ such that

$$p(d) > \exp\left\{-\tfrac{1}{2}\log(\mu)\cdot \mathfrak{m}_\lambda \cdot d\right\},$$

where $\mu := \sum_{k\geq 1} kp(k)$ is the expectation of p (which is larger than 1 by assumption), and assume that d is large enough that

$$\ell := \lfloor \mathfrak{m}_\lambda d \rfloor > 1.$$

Let A be the event that $\mathcal{T}$ contains a subgraph of the form $G((d_n),(\ell_n))$, where the sequences $(d_n),(\ell_n)$ satisfy $d_n \geq d$ and $\ell_n \leq \ell$ for all n. By Proposition 4.7 and our choice of λ and ℓ, we have

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid A) > 0,$$

so (4.15) will follow once we prove that $\mathbb{P}(A) > 0$.

To do so, we introduce some definitions. Let $\mathcal{Z}_0 := \{o\}$, and inductively, for $m \in \mathbb{N}$, let $\mathcal{Z}_m$ be the set of vertices that have degree above d, are at distance ℓm from the root, and descend from some vertex of $\mathcal{Z}_{m-1}$. Then, $A \supseteq \{\mathcal{Z}_m \neq \varnothing\ \forall m\}$. To show that the latter event has positive probability, we note that $(|\mathcal{Z}_m|)_{m\geq 0}$ is a branching process, except that the law of $|\mathcal{Z}_1|$ is different from the offspring distribution of further generations. Both the law of $|\mathcal{Z}_1|$ and the offspring distribution of further generations have expectation above

$$\mu^\ell \cdot p([d,\infty)) \geq \mu^\ell \cdot p(\{d\}) \geq \exp\left\{\log(\mu)\cdot \ell - \tfrac{1}{2}\log(\mu)\cdot \mathfrak{m}_\lambda \cdot d\right\} > 1,$$

by the definition of ℓ. This shows that the process $(|\mathcal{Z}_m|)_{m\geq 0}$ is supercritical, and concludes the proof of $\mathbb{P}(A) > 0$. □

We now start the work towards proving Proposition 4.7. The following lemma contains a lower bound for the probability of infecting many leaves in a star.

Lemma 4.8. *Let $\lambda \in (0,1]$ and $n \geq 1$. Let $(\xi_t)_{t\geq 0}$ be the contact process on S_n with rate λ and started from o infected. Then,*

$$\mathbb{P}\left(|\xi_1| > \left\lfloor \frac{\lambda n}{2e} \right\rfloor\right) \geq \frac{1}{2e}.$$

Proof. Let E denote the event that o has no recovery mark on $[0,1]$. On E, let $\mathscr{B}$ be the set of leaves of S_n that have no recovery mark on $[0,1]$, and that receive at least one transmission from o during this time interval. Then, for any $m \geq 0$,

$$\begin{aligned}\mathbb{P}(|\xi_1| \geq m) &\geq \mathbb{P}(E)\cdot \mathbb{P}(|\mathscr{B}| \geq m \mid E)\\ &= \frac{1}{e}\cdot \mathbb{P}\left(\mathrm{Bin}\left(n, e^{-1}(1-e^{-\lambda})\right) \geq m\right).\end{aligned}$$

Using $1 - e^{-\lambda} \geq \lambda/2$ for $\lambda \in (0, 1]$, the probability on the right-hand size is at least

$$\mathbb{P}\left(\mathrm{Bin}\left(n, \frac{\lambda}{2e}\right) \geq m\right).$$

Plugging in $m = \lfloor \lambda n/(2e) \rfloor$, this is at least $1/2$. □

Taking the notion of "igniting" a vertex from the previous lemma, the next lemma gives a bound on the probability that the contact process runs for a given amount of time on a graph without ever igniting one of its vertices.

Lemma 4.9. *Let G be a connected graph with diameter ℓ and $\lambda \in (0, 1]$. Let $(\xi_t)_{t \geq 0}$ be the contact process with rate λ on G started from an arbitrary initial configuration. Then, for any vertex v of G, defining*

$$\sigma_v := \inf\{t \geq 0 : |\xi_t \cap B(v, 1)| \geq \lfloor \lambda \cdot \deg(v)/(2e) \rfloor\},$$

for any $t > 0$ we have

$$\mathbb{P}(\xi_t \neq \varnothing,\ \sigma_v > t) \leq \exp\left\{-\frac{\lambda^\ell}{(2e^2)^{\ell+1}} \cdot \left\lfloor \frac{t}{\ell + 1} \right\rfloor\right\}. \tag{4.16}$$

Proof. Fix a vertex v of G. Also fix some arbitrary $\xi_0 \neq \varnothing$ (the statement is trivial if $\xi_0 = \varnothing$). Take a vertex u such that $\xi_0(u) = 1$. In case $u = v$, we have

$$\mathbb{P}(\sigma_v \leq 1) > \frac{1}{2e} \tag{4.17}$$

by Lemma 4.8. In case $u \neq v$, take a sequence $u_0, \ldots, u_m$ of vertices so that $u_0 = u$, $u_m = v$, $u_j \sim u_{j+1}$ for each j, and $m \leq \ell$. Let E_j be the event that there are no recoveries at u_{j-1} or u_j in the time interval $[j-1, j]$, and there is a transmission from $j - 1$ to j in this time interval. Let E be the event that the graphical construction is such that, if there is an infection at v at time m, then this infection reaches $\lfloor \frac{\lambda}{2e} \deg(v) \rfloor$ neighbors of v by time $m + 1$. Then,

$$\mathbb{P}(\sigma_v \leq m + 1) \geq \left(\prod_{j=1}^{m} \mathbb{P}(E_j)\right) \cdot \mathbb{P}(E) \geq (e^{-2}(1 - e^{-\lambda}))^m \cdot \frac{1}{2e}. \tag{4.18}$$

By (4.17) and (4.18) (together with $m \leq \ell$ and $1 - e^{-\lambda} \geq \lambda/2$ for $\lambda \in [0, 1]$), we obtain that in all cases,

$$\mathbb{P}(\sigma_v \leq \ell + 1) \geq \frac{\lambda^\ell}{(2e^2)^{\ell+1}}.$$

Combining this with the Markov property and using the inequality $1 - x \leq e^{-x}$ gives (4.16). □

We now turn to the last preparatory lemma for the proof of Proposition 4.7, which is useful to argue that the infection in a star can reach and ignite another vertex that is not too far away.

Lemma 4.10. *There exists $M_0 > 0$ such that the following holds. Let $\lambda < \frac{1}{2e^2}$, and let G be a connected graph with distinct vertices u and v satisfying*

$$\deg(u) \geq M_0 \cdot \frac{\log(1/\lambda)}{\lambda^2} \cdot \operatorname{dist}(u, v). \tag{4.19}$$

Let $(\xi_t)_{t \geq 0}$ be the contact process with rate λ on G with initial condition satisfying

$$|\xi_0 \cap B(u, 1)| > \frac{\lambda}{8e} \cdot \deg(u).$$

Then, defining

$$\sigma_v := \inf\{t \geq 0 : |\xi_t \cap B(v, 1)| \geq \lfloor \lambda \cdot \deg(v)/(2e) \rfloor\},$$

and letting c be the constant of Proposition 2.22, we have

$$\mathbb{P}(\sigma_v > \exp\{c\lambda^2 \deg(u)\}) < 2\exp\{-c\lambda^2 \deg(u)\}.$$

Proof. We abbreviate

$$m_u := \deg(u), \quad m_v := \deg(v), \quad \ell := \operatorname{dist}(u, v).$$

By monotonicity considerations, it suffices to treat the case where the graph G consists of a line segment of length ℓ with extremities u and v, together with extra $m_u - 1$ leaves attached to u, and $m_v - 1$ leaves attached to v. Throughout the proof, we will assume that M_0 is large enough.

We bound

$$\begin{aligned}\mathbb{P}(\sigma_v > e^{c\lambda^2 m_u}) &\leq \mathbb{P}(\tau_G \leq e^{c\lambda^2 m_u}) + \mathbb{P}(\tau_G > e^{c\lambda^2 m_u},\ \sigma_v > e^{c\lambda^2 m_u}) \\ &\leq \exp\{-c\lambda^2 m_u\} + \exp\left\{-\frac{\lambda^{\ell+2}}{(2e^2)^{\ell+3}} \cdot \left\lfloor \frac{\exp\{c\lambda^2 m_u\}}{\ell+3} \right\rfloor\right\}\end{aligned}$$

by Proposition 2.22 and Lemma 4.9. We now need to show that the second probability on the right-hand side is smaller than the first, that is, that

$$\frac{\lambda^{\ell+2}}{(2e^2)^{\ell+3}} \cdot \left\lfloor \frac{\exp\{c\lambda^2 m_u\}}{\ell+3} \right\rfloor > c\lambda^2 m_u. \tag{4.20}$$

Since $\lambda < \frac{1}{2e^2}$, we have $\frac{\lambda^{\ell+2}}{(2e^2)^{\ell+3}} > \lambda^{2\ell+5}$. Moreover, $\frac{\exp\{c\lambda^2 m_u\}}{\ell+3}$ can be assumed large (uniformly in $\lambda < 1/(2e^2)$ and $\ell \in \mathbb{N}$), since

$$\frac{\exp\{c\lambda^2 m_u\}}{\ell+3} > \frac{c\lambda^2 m_u}{\ell+3} \overset{(4.19)}{>} \frac{M_0 \log(1/\lambda)\ell}{\ell+3};$$

in particular, we have

$$\left\lfloor \frac{\exp\{c\lambda^2 m_u\}}{\ell+3} \right\rfloor \geq \frac{\exp\{c\lambda^2 m_u\}}{\ell+3} - 1 \geq \frac{\exp\{c\lambda^2 m_u\}}{2\ell+6}.$$

Hence, the left-hand side of (4.20) is larger than

$$\lambda^{2\ell+5} \cdot \frac{\exp\{c\lambda^2 m_u\}}{2\ell+6}.$$

Again assuming that $c\lambda^2 m_u$ is large, and using the fact that $e^x > xe^{x/2}$ when x is large, the above is larger than

$$\begin{aligned}
&\frac{\lambda^{2\ell+5}}{2\ell+6} \cdot \exp\left\{\frac{c}{2}\lambda^2 m_u\right\} \cdot c\lambda^2 m_u \\
&\overset{(4.19)}{\geq} \frac{\lambda^{2\ell+5}}{2\ell+6} \cdot \exp\left\{\frac{c}{2}\lambda^2 \cdot M_0 \frac{\log(1/\lambda)}{\lambda^2} \cdot \ell\right\} \cdot c\lambda^2 m_u \\
&= \frac{1}{2\ell+6} \cdot \left(\frac{1}{\lambda}\right)^{\frac{cM_0\ell}{2} - 2\ell - 5} \cdot c\lambda^2 m_u.
\end{aligned}$$

When M_0 is large enough, the expression $\frac{1}{2\ell+6} \cdot \left(\frac{1}{\lambda}\right)^{\frac{cM_0\ell}{2} - 2\ell - 5}$ is larger than 1 uniformly over $\lambda \in (0, 1/(2e^2))$ and $\ell \in \mathbb{N}$, completing the proof of (4.20). □

Proof of Proposition 4.7. Let $\mathcal{P}(\lambda, (d_n), (\ell_n))$ be the probability that the contact process with rate λ on $G((d_n), (\ell_n))$ started from only v_0 infected survives locally. For $d, \ell > 0$, define

$$\mathbf{p}(\lambda, d, \ell) := \inf\{\mathcal{P}(\lambda, (d_n), (\ell_n)) : d_n \geq d \text{ and } \ell_n \leq \ell \text{ for all } n\}.$$

We claim that for any $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that

$$\mathbf{p}\left(\lambda, \frac{M_0 \log(1/\lambda)}{\lambda^2} \cdot \ell, \ell\right) > 1 - \varepsilon \quad \forall \lambda \in (0, \lambda_\varepsilon],\ \ell \in \mathbb{N}, \tag{4.21}$$

where M_0 is the constant of Lemma 4.10.

Note that, once we have (4.21), for any $\lambda \geq \lambda_\varepsilon$ we have

$$\mathbf{p}\left(\lambda, \frac{M_0 \log(1/\lambda_\varepsilon)}{(\lambda_\varepsilon)^2} \cdot \ell, \ell\right) \geq \mathbf{p}\left(\lambda_\varepsilon, \frac{M_0 \log(1/\lambda_\varepsilon)}{(\lambda_\varepsilon)^2} \cdot \ell, \ell\right) > 1 - \varepsilon,$$

so the statement of the proposition easily follows.

Let us prove (4.21); fix $\varepsilon > 0$. Let $\lambda > 0$ be small, to be chosen later. Fix $\ell \in \mathbb{N}$ and let (d_n) and (ℓ_n) be sequences satisfying

$$d_n \geq d := \frac{M_0 \log(1/\lambda)}{\lambda^2} \cdot \ell, \quad \ell_n \leq \ell \quad \text{for all } n,$$

and let $G = G((d_n),(\ell_n))$. We take the contact process $(\xi_t)_{t\geq 0}$ on G with rate λ started from only v_0 infected. By Lemma 2.21, if λ is small enough (so that $\frac{1}{\lambda^2}\log\left(\frac{1}{\lambda}\right) \gg \frac{1}{\lambda^2}$), with probability above $1-\varepsilon/2$, there is $s>0$ such that more than $\lambda d_0/(8e)$ neighbors of v_0 are infected at time s. Hence, we can switch to the contact process $(\xi'_t)_{t\geq 0}$ started from a configuration in which at least $\lambda d_0/(8e)$ neighbors of v_0 are infected, and then it suffices to prove that

$$\liminf_{t\to\infty} \mathbb{P}(\xi'_t \cap B(v_0,1) \neq \varnothing) > 1-\varepsilon/2. \tag{4.22}$$

To do so, we will couple $(\xi'_t)_{t\geq 0}$ with an oriented percolation process.

Let $\bar{t} := \exp\{c\lambda^2 d\}$, where c is the constant of Proposition 2.22. We will split time into the intervals $[\bar{t}n, \bar{t}(n+1)]$, $n \in \mathbb{N}_0$.

We will also divide space into pieces, with some overlap between neighboring ones, as we now explain. We abbreviate $S_j := B(h_j,1)$, the subgraph of G induced by the j-th hub, h_j, and its neighbors. Write $\mathcal{N}_0 := \{0,1\}$ and $\mathcal{N}_j := \{j-1, j, j+1\}$ for $j \geq 1$. For $j \in \mathbb{N}_0$, we let $\bar{S}_j$ denote the subgraph of G containing S_i for all $i \in \mathcal{N}_j$, together with line segments of length at most ℓ connecting h_j to h_i for each $i \in \mathcal{N}_j \backslash \{j\}$.

Given $(j,n) \in \mathbb{N}_0 \times \mathbb{N}_0$, define the process $(\zeta_t^{(j,n)})_{\bar{t}n \leq t \leq \bar{t}(n+1)}$ on $\{0,1\}^{\bar{S}_j}$ as follows. We let

$$\zeta_{\bar{t}n}^{(j,n)}(v) = \xi'_{\bar{t}n}(v) \quad \text{for all } v \in \bar{S}_j,$$

that is, we initialize the process at time $\bar{t}n$ inside $\bar{S}_j$ as the same as the contact process at that time inside that set. Then, we let $(\zeta_t^{(j,n)})_{\bar{t}n \leq t \leq \bar{t}(n+1)}$ evolve as a contact process on $\bar{S}_j$, using the same graphical construction as the contact process (inside $\bar{S}_j$ only).

Define

$$\begin{aligned} \Lambda_0 &:= \{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 : m+n \text{ is even}\}, \\ \vec{E}_{\Lambda_0} &:= \{\langle (m,n),(m',n+1)\rangle :\ (m,n) \in \Lambda_0,\ |m'-m| = 1\}, \end{aligned}$$

so that $(\Lambda_0, \vec{E}_{\Lambda_0})$ is the subgraph of the directed graph studied in Section 2.2 induced by the set of vertices with non-negative first coordinate. We will define an oriented percolation configuration $X = \{X(\vec{e}) : \vec{e} \in \vec{E}_{\Lambda_0}\}$ on this subgraph as a function of the graphical construction of the contact process.

For $\vec{e} = \langle (i,n),(j,n+1)\rangle \in \vec{E}_{\Lambda_0}$, we apply the following rules:

(a) if $|\zeta_{\bar{t}n}^{(i,n)} \cap S_i| \geq \frac{\lambda}{8e} \cdot \deg(h_i)$, set $X(\vec{e}) = \mathbb{1}\left\{|\zeta_{\bar{t}(n+1)} \cap S_j| \geq \frac{\lambda}{8e} \cdot \deg(h_j)\right\}$;

(b) if, on the other hand, $|\zeta_{\bar{t}n}^{(i,n)} \cap S_i| < \frac{\lambda}{8e} \cdot \deg(h_i)$, set $X(\vec{e}) = 1$.

Case (b) is artificially introduced to guarantee that the probability that $X(\vec{e}) = 1$ is never small, regardless of the information revealed in levels below $\vec{e}$; we elaborate on this further below.

Let us make two observations that follow from this construction. First,

$$\{\xi'_{\bar{t}n} \cap S_0 \neq \varnothing\} \supseteq \{\exists \text{ an open path from } (0,0) \text{ to } (0,n) \text{ in } X\}\,. \quad (4.23)$$

Second, letting $\mathcal{G}_n$ denote the σ-algebra generated by the graphical construction up to time $\bar{t}n$, we have

$$\begin{aligned}&\text{conditionally on } \mathcal{G}_n,\\ &\qquad \{X(\langle(i,n),(j,n+1)\rangle) : i,j \in \mathbb{N}_0,\ |i-j|=1\}\\ &\text{is a 1-dependent collection of Bernoulli random variables,}\\ &\text{each of which is equal to 1 with probability} > q(d,\lambda),\end{aligned} \quad (4.24)$$

where $q(d,\lambda) := 1-(d+1)\exp\{-c\lambda^2 d\}$, and c is again the constant of Proposition 2.22. The meaning of 1-dependence here is with respect to the distance $\mathrm{dist}(\langle(i,n),(j,n+1)\rangle, \langle(i',n),(j',n+1)\rangle) = |i-i'|+|j-j'|$. Note that the presence of rule (b) above is made so that (4.24) holds, and this rule has no effect for the inclusion (4.23).

Now we use Corollary 2.3: fix $p \in (0,1)$ close enough to 1 that the probability on the left-hand side of (2.3) is above $1-\varepsilon/2$. Using Theorem 2.1, choose q_0 large enough that every 1-dependent collection of Bernoulli random variables dominates a collection of independent Bernoulli(p) random variables.

Let $\{X'(\vec{e}) : \vec{e} \in \vec{E}_{\Lambda_0}\}$ be independent, Bernoulli(p). We claim that, if λ is small enough that $q(d,\lambda) > q_0$, then for any $n \in \mathbb{N}_0$,

$$\{X'(\vec{e}) : \vec{e} \in \vec{E}_{\Lambda_0}\} \overset{\text{stoch.}}{\preceq} \{X(\vec{e}) : \vec{e} \in \vec{E}_{\Lambda_0}\}. \quad (4.25)$$

This is easy to prove using (4.24), the choice of q_0, and induction on $m \in \{0,\dots,n\}$.

We can now conclude the proof of (4.22): (4.23) and (4.25) imply that

$$\mathbb{P}(\xi'_{\bar{t}n} \cap S_0 \neq \varnothing) \geq \mathbb{P}(\exists \text{ an open path from } (0,0) \text{ to } (0,n) \text{ in } X'),$$

and Corollary 2.3 implies that the right-hand side above is larger than $1-\varepsilon/2$, uniformly in n. $\square$

4.3 Offspring distribution with a finite exponential moment

In this section, we give the proof of Theorem 4.2. The proof presented here is the same as in [9], only with different notation and terminology.

The argument given actually establishes a stronger result, Theorem 4.11 below. Before we state it, we give some definitions.

Let T be a tree with root o, and $\lambda > 0$. We define the *reinforced contact process* $(\zeta_t)_{t\geq 0}$ on (T, o) as the process with state space $\{0,1\}^T$ that evolves exactly like the contact process, except that it obeys the rule that *recoveries can only occur at a vertex when none of the descendants of that vertex are infected.* We always start this process from the configuration where only the root is infected. As with the classical contact process, the all-healthy configuration is absorbing for (ζ_t), and we say that (ζ_t) dies out if this configuration is reached almost surely.

Theorem 4.11 (Bhamidi, Nam, Nguyen and Sly [9]). *Let $\mathcal{T}$ be a BGW tree with offspring distribution p. Assume that*

$$\sum_k e^{ck} \cdot p(k) < \infty \quad \textit{for some } c > 0. \tag{4.26}$$

Then, if λ is small enough, the reinforced contact process $(\zeta_t)_{t\geq 0}$ defined above with rate λ dies out on almost every realization of $\mathcal{T}$.

Before we prove this, let us state and prove an auxiliary lemma. Given a tree T with root o, let $\tau^\zeta = \tau^\zeta(T, o)$ denote the time when (ζ_t) on (T, o) reaches the all-healthy configuration.

Lemma 4.12. *Let (T, o) be a finite tree, and for each $v \sim o$, let T_v be the sub-tree containing v and all its descendants. Then,*

$$\mathbb{E}[\tau^\zeta(T, o)] = \prod_{v\sim o} (1 + \lambda \mathbb{E}[\tau^\zeta(T_v, v)]). \tag{4.27}$$

Proof. Let $(\zeta'_t)_{t\geq 0}$ be the auxiliary process that evolves exactly like (ζ_t), except that for (ζ'_t), *the root never recovers.* We also start (ζ'_t) from the configuration where only the root is infected, and let $\tau^{\zeta'}$ denote the first time, after the first jump of (ζ'_t), that this process returns to its initial configuration. We claim that

$$\mathbb{E}[\tau^\zeta] = \lambda \deg(o) \cdot \mathbb{E}[\tau^{\zeta'}]. \tag{4.28}$$

To prove this, first define

$$\bar{t} := \mathbb{E}[\tau^{\zeta'}] - \frac{1}{\lambda \deg(o)}.$$

Note that $\bar{t}$ is the expected time until (ζ'_t) returns to its initial configuration, but excluding the time until its first jump. Now, let K be a random variable with the Geometric($\frac{1}{\lambda \deg(o)+1}$) distribution. Then, it is easy to convince oneself that

$$\mathbb{E}[\tau^\zeta] = \sum_{k=1}^{\infty} \mathbb{P}(K = k) \cdot \left(\left(\frac{1}{\lambda \deg(o) + 1} + \bar{t} \right) \cdot (k-1) + \frac{1}{\lambda \deg(o) + 1} \right).$$

The right-hand side equals

$$\left(\frac{1}{\lambda\deg(o)+1}+\bar{t}\right)\cdot\mathbb{E}[K]-\bar{t}=\left(\frac{1}{\lambda\deg(o)+1}+\bar{t}\right)\cdot(\lambda\deg(o)+1)-\bar{t}$$
$$=1+\lambda\deg(o)\cdot\bar{t}=\lambda\deg(o)\cdot\mathbb{E}[\tau^{\zeta'}],$$

where the last equality follows from the definition of $\bar{t}$. This completes the proof of (4.28).

We now observe that (ζ'_t) is a recurrent Markov chain on a finite state space (since we assume that the tree T is finite); we let π' be its stationary distribution. We can naturally think of (ζ'_t) as a product chain, that is, apart from the root (which is always infected, and hence irrelevant), it consists of independent Markov chains $(\zeta'_{v,t})_{t\geq 0}$, each evolving on one of the sub-trees that descends from a child of o. We let π'_v denote the stationary distribution of $(\zeta'_{v,t})$. Note that

$$\pi'(\{o\})=\bigotimes_{v\sim o}\pi'_v(\varnothing). \tag{4.29}$$

We now compute both sides of this equality, using elementary considerations from Markov chain theory:

$$\pi'(\{o\})=\frac{1/(\lambda\deg(o))}{\mathbb{E}[\tau^{\zeta'}]}\overset{(4.28)}{=}\frac{1}{\mathbb{E}[\tau^{\zeta}]} \tag{4.30}$$

and

$$\pi'_v(\varnothing)=\frac{1/\lambda}{(1/\lambda)+\mathbb{E}[\tau^{\zeta}_{(T_v,v)}]}=\frac{1}{1+\lambda\cdot\mathbb{E}[\tau^{\zeta}_{(T_v,v)}]} \tag{4.31}$$

Putting together (4.29), (4.30) and (4.31), we obtain the desired equality. □

Proof of Theorem 4.11. For each $L\in\mathbb{N}_0$, let $\mathcal{T}_L$ denote the sub-tree of the BGW tree $\mathcal{T}$ obtained by excluding all vertices at distance larger than L from the root. Define

$$f(L):=\mathbb{E}[\tau^{\zeta}_{(\mathcal{T}_L,o)}],\qquad L\in\mathbb{N}_0$$

(note that the expectation on the right-hand side involves both the randomness of the choice of the graph and of the reinforced contact process). We will show that, if λ is small enough, we have $f(L)\leq e$ for all L. Note that $f(0)=1$ regardless of λ. For now assume that λ is small enough (we will see how small in the course of the proof), and assume that $f(L)\leq e$ has already been established.

For a rooted tree (T, o), we write $h(T, o) := \mathbb{E}[\tau^{\zeta}_{(T,o)}]$. For each $v \sim o$ in $\mathcal{T}_{L+1}$, let $\mathcal{T}_{L,v}$ denote the sub-tree rooted at v. Using the above lemma, we can bound

$$h(\mathcal{T}_{L+1}, o) = \prod_{v \sim o} (1 + \lambda \cdot h(\mathcal{T}_{L,v}, v)),$$

In the above equality, we take $\mathbb{E}[\cdot \mid \deg(o) = d]$ on both sides to obtain

$$\begin{aligned}\mathbb{E}[h(\mathcal{T}_{L+1}, o) \mid \deg(o) = d] &= \mathbb{E}\left[\prod_{v \sim o} (1 + \lambda \cdot h(\mathcal{T}_{L,v}, v)) \;\middle|\; \deg(o) = d \right] \\ &= (1 + \lambda \cdot \mathbb{E}[h(\mathcal{T}_L, o)])^d = (1 + \lambda \cdot f(L))^d.\end{aligned}$$

Integrating both sides now gives

$$\begin{aligned}f(L+1) = \mathbb{E}[h(\mathcal{T}_{L+1}, o)] &= \mathbb{E}\left[(1 + \lambda f(L))^{\deg(o)}\right] \\ &\leq \mathbb{E}\left[\exp\{\lambda f(L) \cdot \deg(o)\}\right] \\ &= \mathbb{E}\left[\exp\left\{\frac{\lambda f(L)}{c} \cdot c \deg(o)\right\}\right],\end{aligned}$$

where c is the positive constant such that $\mathbb{E}[e^{c \deg(o)}] < \infty$. Now, if $\lambda \leq \frac{c}{e}$, then $\frac{\lambda f(L)}{c} \leq 1$ by the induction hypothesis. Then, the function $x \mapsto x^{\lambda f(L)/c}$ is concave, and by Jensen's inequality,

$$\begin{aligned}\mathbb{E}\left[\exp\left\{\frac{\lambda f(L)}{c} \cdot c \deg(o)\right\}\right] &\leq \mathbb{E}[\exp\{c \deg(o)\}]^{\lambda f(L)/c} \\ &= \exp\left\{\lambda \cdot \frac{\log \mathbb{E}[e^{c \deg(o)}] \cdot f(L)}{c}\right\}.\end{aligned}$$

If $\lambda < \frac{c}{\log \mathbb{E}[e^{c \deg(o)}] \cdot e}$, the right-hand side is smaller than e, completing the induction step (note that the restriction we have found for λ does not involve L).

Now that we have proved that $f(L) < e$ for all L when $\lambda < \frac{c}{\log \mathbb{E}[e^{c \deg(o)}] \cdot e}$, we note that, by monotone convergence,

$$\mathbb{E}[\tau^{\zeta}_{(\mathcal{T},o)}] = \lim_{L \to \infty} f(L) \leq e;$$

in particular, $\tau^{\zeta}_{(\mathcal{T},o)} < \infty$ almost surely, completing the proof. $\square$

Chapter 5

Small-λ survival on power law BGW trees

Let $\mathcal{T}$ be a BGW tree whose offspring distribution has no exponential moment, and consider the contact process on $\mathcal{T}$, started from only the root infected. By Theorem 4.1, the process survives (locally) with positive probability, even when λ is very small. It should be clear that, although always positive, the survival probability does tend to zero as $\lambda \to 0$: for instance, if the root has degree below $1/\sqrt{\lambda}$, which happens with high probability when λ is small, then the initial infection at the root is likely to vanish before spreading to any other vertex.

In other words, when λ is small, survival of the infection is possible but rare. The goal of this chapter is to study this rare event. This is interesting, because it requires one to capture the most likely way in which it happens, and thus has an optimization flavour. We restrict our attention to the case where the offspring distribution is a power law (as in (5.2) below). As we will see, interesting transitions will arise, as one varies the power law exponent.

Let p be a probability measure on $\mathbb{N}_0$. We will abuse notation and write $p(k)$ instead of $p(\{k\})$, for each k. Throughout this section, p will be taken as the offspring distribution of a BGW tree. We assume that

$$p(0) = 0,\ p(1) < 1. \tag{5.1}$$

and that there exists $\alpha \in (1, \infty)$ such that

$$0 < \liminf_{k\to\infty} \frac{p(k)}{k^\alpha} \le \limsup_{k\to\infty} \frac{p(k)}{k^\alpha} < \infty. \tag{5.2}$$

The assumption (5.1) is made as a simple way to guarantee that the associated BGW tree is almost surely infinite and not equal to a half-line.

This is done for convenience; one could instead take the assumption that the mean offspring $\sum kp(k) > 1$, and state the results of this section for the contact process on the BGW tree conditioned on being infinite.

Theorem 5.1 (Mountford, V., Yao [55]). *Let $\mathcal{T}$ be a BGW tree whose offspring distribution p satisfies* (5.1) *and* (5.2). *There exist constants c and C such that for $\lambda > 0$ small enough, the following holds. Let $(\xi_t)_{t\geq 0}$ be the contact process on $\mathcal{T}$ started from only the root infected. Then,*

$$cf(\alpha,\lambda) \leq \mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) \leq Cf(\alpha,\lambda),$$

where

$$f(\alpha,\lambda) = \begin{cases} \lambda^{\frac{\alpha-1}{2-\alpha}} & \text{if } \alpha \in (1,\frac{3}{2}]; \\ \lambda^{2\alpha-2}\cdot(\log\frac{1}{\lambda})^{-(\alpha-1)} & \text{if } \alpha\in(\frac{3}{2},2]; \\ \lambda^{2\alpha-2}\cdot(\log\frac{1}{\lambda})^{-2(\alpha-1)} & \text{if } \alpha > 2. \end{cases} \tag{5.3}$$

Remark 5.2. The above statement is a little different from the statement of the main theorem of [55]; we explain this further in Section 5.1 below.

As mentioned earlier, this theorem has a flavor of optimization: when λ is small, survival of the infection is a rare event, and its probability reflects the best survival strategy, involving the combined randomness of the graph and of the infection. There is no precise definition of "best strategy"; rather, we give a lower bound for the survival probability by presenting an event that guarantees survival (and estimating the probability of this event), and then give a matching upper bound.

Let us briefly describe these survival-guaranteeing events that are employed in our lower bounds; later we will discuss which event is taken as the "best strategy" for each value of α.

Key event 1: there exists a path $o = v_0 \sim v_1 \sim v_2 \sim \cdots$ of distinct vertices such that v_0 infects v_1 before recovering, then v_1 infects v_2 before recovering etc.

Using a comparison with a branching process, it can be seen that this event has positive probability for all $\lambda > 0$ if and only if p has infinite expectation, that is, if and only if $\alpha \in (1,2]$. Hence, this event will not be a valid strategy for $\alpha > 2$. For $\alpha \in (1,2]$, with the method we will employ in Section 5.2.1, it can be proved that

$$\text{when } \alpha\in(1,2],\quad \mathbb{P}(\text{Key event 1}) > c\lambda^{\frac{\alpha-1}{2-\alpha}} \tag{5.4}$$

(where c is a constant that depends on α, and the inequality holds for λ small enough).

For the second key event, recall the definition of the stars-and-line-segments graph $G((d_n),(\ell_n))$ from the beginning of Section 4.2. The event will depend on suitable choices of $d(\alpha, \lambda)$ – a lower bound for the degrees of stars – and $\ell(\alpha, \lambda)$ – an upper bound for the lengths of line segments–, which we leave unspecified for now.

Key event 2: $\mathcal{T}$ contains a subgraph $G((d_n), (\ell_n))$, with the first star being centered at the root, and with $d_n \geq d(\alpha, \lambda)$ and $\ell_n \leq \ell(\alpha, \lambda)$ for all n.

Let us comment on the choice of $d(\alpha, \lambda)$ and $\ell(\alpha, \lambda)$. These values should be related as imposed by Proposition 4.7: the stars should be big enough so that the infection is sustained there long enough to traverse the line segments. This is where $\alpha = 2$ becomes relevant: when $\alpha \in (1, 2]$, a vertex with big degree typically has a neighbor with degree at least as big, whereas when $\alpha > 2$, this is no longer the case: stars are then typically isolated. The following can be proved – see Sections 5.2.2 and 5.2.3 (the constants that appear need to be suitably chosen and may depend on α, but not on λ; the inequalities hold for λ small):

$$\begin{gathered}\text{when } \alpha \leq 2, \text{ taking } d(\alpha, \lambda) = C \tfrac{1}{\lambda} \log \tfrac{1}{\lambda} \text{ and } \ell = 1, \\ \text{we have } \mathbb{P}(\text{Key event 2}) > c\lambda^{2\alpha-2}(\log(1/\lambda))^{-(\alpha-1)}\end{gathered} \tag{5.5}$$

and

$$\begin{gathered}\text{when } \alpha > 2, \text{ taking } d(\alpha, \lambda) = C \tfrac{1}{\lambda} (\log \tfrac{1}{\lambda})^2 \text{ and } \ell = C' \log \tfrac{1}{\lambda}, \\ \text{we have } \mathbb{P}(\text{Key event 2}) > c\lambda^{2\alpha-2}(\log(1/\lambda))^{-2(\alpha-1)}\end{gathered} \tag{5.6}$$

These lower bounds are essentially the price paid for the root to have degree $d(\alpha, \lambda)$; once this happens, the rest of the requirement of Key event 2 (namely, the appearance of the rest of $G((d_n), (\ell_n))$) happens more or less for free.

Now, noting that $\frac{\alpha-1}{2-\alpha} \leq 2\alpha - 2$ when $\alpha \in (1, \frac{3}{2}]$, by putting together (5.4), (5.5) and (5.6), we see that:

- when $\alpha \in (1, \frac{3}{2}]$, the first key event is more likely than the second;
- when $\alpha \in (\frac{3}{2}, 2]$, the second key event, with $d(\alpha, \lambda) = C \frac{1}{\lambda} \log \frac{1}{\lambda}$ and $\ell = 1$, is more likely than the first;
- when $\alpha > 2$, the first key event has probability zero when λ is small, and we can take the second key event with $d(\alpha, \lambda) = C \frac{1}{\lambda} (\log \frac{1}{\lambda})^2$ and $\ell = C' \log \frac{1}{\lambda}$.

This already gives the expressions in (5.3), as lower bounds.

Upper bounds are harder work: in order to prove them, we have to consider many different cases in which the infection could survive forever, and to show that each of them has smaller probability than the key event of choice.

The proof of Theorem 5.1 is quite long, and we break it into six parts (we give lower and upper bounds separately, for each of the three cases for α). Some preliminary results for the upper bounds are also needed. Before turning to all this, we make a digression about finite random graphs.

5.1 Metastable densities in power law random graphs

The original motivation behind Theorem 5.1 was the study the contact process on a class of random graphs called the *configuration model*, in [55], following up on bounds obtained earlier in [19]. As we will explain, *metastable densities* of the contact process on these graphs are related to survival probabilities on BGW trees, as the ones found in Theorem 5.1. In this section, we give a brief outline of this direction of research, stating some of the main findings, but omitting proofs.

5.1.1 Configuration model with power law degree distribution

Let us start by briefly describing the class of random graphs known as the configuration model. Let $n \in \mathbb{N}$ and let $D_1, \dots, D_n$ be independent random variables, all with distribution p. We want $D_1 + \cdots + D_n$ to be even; this can be achieved for instance by artificially adding one unit to D_n in case the sum is odd. We then use this random sequence to construct a random graph $G_n = (V_n, E_n)$ as follows. We take $V_n = \{1, \dots, n\}$. We define the set of *half-edges* as $\mathcal{H}_n := \{(v, a) : v \in V_n,\ 1 \le a \le D_v\}$. Uniformly at random, we choose a *perfect matching* of $\mathcal{H}_n$ (that is, a partition of $\mathcal{H}_n$ into sets of two elements), and declare that if $(u, a), (v, b) \in \mathcal{H}_n$ are together in the perfect matching that was chosen, then an edge between u and v is included in the graph. This gives rise to the set of edges E_n. It should be noted that this procedure actually produces a *multi-graph*, that is, it is possible to have multiple edges between two vertices, and also edges from a vertex to itself.

This is one of the most well-studied classes of random graphs, see for instance [70, 71]. Taking the degree distribution p as a power law is typical, as it is supposed to model the degree distribution of certain real-world networks. Specificall in the study of epidemics, it ensures that the graph has so-called "superspreaders", that is, individuals of exceptionally high degree that do the heavy lifting in sustaining and spreading the infection.

Let τ_n denote the extinction time of the contact process on G_n (recall that this is the time until the infection disappears, for the process started from all vertices infected). In [19], Chatterjee and Durrett proved that

$$\text{for any } \lambda > 0 \text{ and } \delta > 0,\ \mathbb{P}(\tau_n > \exp\{n^{1-\delta}\}) \xrightarrow{n\to\infty} 1. \tag{5.7}$$

A result in the same spirit had been proved earlier by Berger, Borgs, Chayes and Saberi [6] for the preferential attachment graph. This means that on these power law graphs, the contact process is "always supercritical": there is no regime where the infection disappears quickly. The result of [19] contradicted earlier predictions in the Physics literature, obtained from non-rigorous methods, to the effect that there was a fast extinction regime in the configuration model, at least for some values of the power law exponent.

In [54], (5.7) was improved to

$$\text{for any } \lambda > 0, \text{ there exists } c > 0 \text{ such that } \mathbb{P}(\tau_n > \exp\{cn\}) \xrightarrow{n\to\infty} 1. \tag{5.8}$$

Suppose that we want to take a snapshot of the contact process with small λ on G_n at a "typical time" (meaning: not too soon after the start of the dynamics, and long before the infection is extinct). To this end, we take a sequence of times $(t_n)_{n\geq 1}$ such that $t_n \to \infty$ and $\log(t_n) \ll n$ as $n \to \infty$. We fix $\lambda > 0$ (small) and let $(\xi_t^{(n)})_{t\geq 0}$ denote the contact process on G_n with rate λ, started from all infected. The quantity

$$\frac{1}{n}|\{v \in V_n : \xi_{t_n}^{(n)}(v) = 1\}|$$

represents the density of infection at the "typical time" t_n. Using a variance computation, one can show that the above quantity is close to

$$\mathbb{P}(\xi_{t_n}^{(n)}(\mathcal{U}_n) = 1),$$

where $\mathcal{U}_n$ is a vertex of V_n chosen uniformly at random. By duality, the above probability is equal to

$$\mathbb{P}(\xi_{t_n}^{(\mathcal{U}_n)} \neq \varnothing), \tag{5.9}$$

where $(\xi_t^{(\mathcal{U}_n)})_{t\geq 0}$ is a contact process with rate λ on G_n, started from a single infection at $\mathcal{U}_n$.

In case $\alpha > 2$, G_n has a *local graph limit* (in the sense of Benjamini–Schramm, [5]). This limiting graph $\widetilde{\mathcal{T}}$ is a BGW tree, except that it has two stages: the root has offspring distribution p (same as the degree distribution of G_n), and all other vertices have degree distribution q, which is the size biasing of p, that is, q is given by $q(n) := np(n)/\sum_k kp(k)$. Then, the probability in (5.9) can be shown to approach, as $n \to \infty$, the probability of (global) survival of the contact process on $\widetilde{\mathcal{T}}$ (with same infection rate, starting from only the root infected).

Keeping these observations in mind, we now state the main theorem of [55].

Theorem 5.3 (Mountford, V., Yao [55]). *Let p be a probability on $\mathbb{N}$ satisfying (5.1) and (5.2), let G_n be the configuration model with degree distribution p and n vertices, and let $(\xi_t^{(n)})_{t\geq 0}$ be the contact process with rate $\lambda > 0$ on G_n, started from all vertices infected. For any sequence (t_n) with $t_n \to \infty$ and $\log(t_n)/n \to 0$ and for any $\varepsilon > 0$, we have*

$$\mathbb{P}\left(\left|\frac{|\{v \in V_n:\ \xi_{t_n}^{(n)}(v) = 1\}|}{n} - \tilde{f}(\alpha,\lambda)\right| > \varepsilon\right) \xrightarrow{n\to\infty} 0,$$

where $\tilde{f}(\alpha,\lambda)$ is the probability of global survival of the contact process with rate λ started from only the root infected on $\widetilde{\mathcal{T}}$, the two-stage BGW in which the root has offspring distribution p and other vertices have degree distribution q, the size biasing of p. Moreover, as $\lambda \to 0$,

$$\tilde{f}(\alpha,\lambda) \sim \begin{cases} \lambda^{\frac{1}{3-\alpha}} & \text{if } \alpha \in (1, \frac{3}{2}]; \\ \lambda^{2\alpha-3}\cdot(\log\frac{1}{\lambda})^{-(\alpha-2)} & \text{if } \alpha \in (\frac{3}{2}, 2]; \\ \lambda^{2\alpha-3}\cdot(\log\frac{1}{\lambda})^{-2(\alpha-2)} & \text{if } \alpha > 2. \end{cases} \tag{5.10}$$

Note that $\tilde{f}(\alpha,\lambda)$ is different from $f(\alpha,\lambda)$ given in (5.3), which is natural, since $\widetilde{\mathcal{T}}$ is different from $\mathcal{T}$. In fact, the relation

$$\tilde{f}(\alpha,\lambda) = \lambda\cdot f(\alpha-1,\lambda) \tag{5.11}$$

holds; let us explain it. As mentioned earlier, these functions capture the "best strategy" for survival, and typically, survival becomes much easier when the degree at the root, or near the root, is big. Note that it is easier for offspring with law q to be big than it is for offspring with law p, because q has power law exponent $\alpha - 1$ rather than α. Hence, it is not surprising that the best strategy in $\widetilde{\mathcal{T}}$ is the same as in $\mathcal{T}$, except that in the former, this strategy is followed by one of the subtrees rooted at the children of the root. This explains the $\alpha - 1$ in (5.11). The factor λ multiplying in the right-hand side has to do with the requirement that the infection initially at the root must pass to the child whose subtree follows the strategy.

In this survey, we have opted to include the proof of Theorem 5.1, and not the proof of Theorem 5.3. This way, we are able to fully illustrate the richness of behavior of the survival probability, while avoiding extra technical complications that arise in the two-stage tree and the configuration model.

To conclude this section, let us mention that probabilities p satisfying (5.2) with $\alpha \in (1,2]$ are also valid degree distributions for the configuration model, and the reason this case is not included in Theorem 5.3

is that there is no Benjamini–Schramm convergence in that case (the size biasing of p is not well defined). Nevertheless, the metastable density can still be computed: Can and Schapira found in [14] that it is equal to $\lambda^{\alpha-1}$ when $\alpha \in (1, 2)$ and $\lambda \log(1/\lambda)$ when $\alpha = 2$.

5.1.2 Other graphs and settings

We now mention other works related to the theme of metastable densities and survival of the contact process with small λ, exploring other settings or graphs.

In the recent reference [28], Fernley and Jacob studied the contact process on BGW trees with power law degree distribution (exactly as in the setting treated in this chapter), adding *immunization* at vertices of big degree. In their setup, this means that the degree distribution (say, represented by a random variable $X \sim p$) is replaced by a truncated distribution $X\mathbb{1}\{X \leq k_\lambda\}$, where k_λ is a λ-dependent truncation level. They find very precise results about how different levels of truncation produce different expressions for the survival probability (in a way that depends on the power law exponent α).

Further work has revealed that the scheme of optimization of survival outlined above (after the statement of Theorem 5.1) is also manifested on other classes of power law random graphs.

In [13], improving on previous bounds from [6], Can considered the *preferential attachment graph*, in a setup where the power law exponent α is strictly larger than 3. Here, in contrast to the configuration model, stars are within order 1 distance to each other, so the density is given by the middle expression in (5.10).

In [51], Linker, Mitsche, Schapira and V. studied the contact process on a power law random graph that arises as the local limit of *random hyperbolic graphs*, a model introduced in [41]. This graph has a power law degree distribution with exponent $\alpha \in (1, 2)$, and the same densities from (5.10) are found there (with the same corresponding survival strategies). One highlight of this case is that the graph is not a tree, so new techniques had to be developed to handle loop traversal.

The more recent work [31] considers the contact process on a class of geometric random graphs obtained from Poisson point processes in $\mathbb{R}^d$, in situations that yield power law degree distributions with parameter $\alpha \in (1, 2)$. Interestingly, in their graphs, the "Key event 1" mentioned above has lower probability, due to a slightly larger spacing between stars, so they find a density matching the middle case in (5.10) for all their range of α. See the discussion following the statement of Theorem 2.1 in that paper.

Let us finally mention that in a series of works [38, 36, 37] (subsets of) Jacob, Linker and Mörters study the contact process on *dynamic* power law random graphs. That is, they allow the edge set to evolve simultaneously with the infection, albeit autonomously, that is, transitions in the graph do not depend on the current state of the infection. For certain choices of the parameters that define their model, the infection may die quickly, in contrast with (5.8), due to stars being disintegrated long before they have a chance to sustain the infection. They also study metastable densities, finding a much richer variety of regimes than what we present here for static graphs.

5.2 Lower bounds

We now jump straight into the proofs of the lower bounds in the different regimes mentioned in Theorem 5.1.

5.2.1 Case $\alpha \in (1, \frac{3}{2}]$

Proof of Theorem 5.1, lower bound for $\alpha \in (1, \frac{3}{2}]$. We already know from Theorem 4.1 that $\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) > 0$, and then it follows from Proposition 4.5 that $\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) = \mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot))$. Hence, to prove the desired bound, it suffices to prove that there exists $c > 0$ such that

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot)) > c\lambda^{\frac{\alpha-1}{2-\alpha}}.$$

This reduction will be important, because unlike the other two regimes $\alpha \in (\frac{3}{2}, 2]$ and $\alpha > 2$, the proof below will guarantee global, not local, survival.

Let $(\zeta_t)_{t\geq 0}$ be the modification of the contact process on $\mathcal{T}$ defined as follows. At time zero, only the origin is infected. Moreover, after a site changes from infected to healthy for the first time, it cannot become infected again. In particular, it is impossible for any site to cause its parent to become infected.

Clearly, this auxiliary process can be constructed with the same graphical construction as the contact process $(\xi_t)_{t\geq 0}$ on $\mathcal{T}$, and, recalling that we take $\xi_0 = \mathbb{1}_{\{o\}}$ as well, we have $\xi_t \geq \zeta_t$ for every t. In particular, defining the survival event for ζ in the same way as it is defined by ξ,

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\xi_\cdot)) \geq \mathbb{P}(\mathrm{Surv}_{\mathrm{glob}}(\zeta_\cdot)).$$

Let

$$Z_n := |\{v : \mathrm{dist}(o, v) = n,\ \exists t \geq 0 \text{ such that } \zeta_t(v) = 1\}|, \quad n \in \mathbb{N}_0.$$

It is easy to convince oneself that $(Z_n)_{n\geq 0}$ is a branching process, and $\mathrm{Surv}_{\mathrm{glob}}(\zeta_\cdot) \supseteq \{Z_n > 0 \text{ for all } n\}$.

We now make an estimate involving the tail of the offspring distribution of this branching process. Fix $x > 0$ and define the event

$$A_x := \{\deg(o) > 2x/(1-e^{-\lambda})\} \cap \{\text{the root does not recover before time 1}\}.$$

Conditionally on A_x, Z_1 stochastically dominates a

$$\text{Binomial}(\lceil 2x/(1-e^{-\lambda})\rceil, 1-e^{-\lambda})$$

random variable, and so, $\mathbb{P}(Z_1 \geq x \mid A_x) \geq 1/2$. We then have

$$\mathbb{P}(Z_1 \geq x) \geq \frac{1}{2e}\mathbb{P}(\deg(o) > 2x/(1-e^{-\lambda})) \geq c\left(\frac{\lambda}{x}\right)^{\alpha-1} \tag{5.12}$$

for some $c > 0$, uniformly over $\lambda < 1$ and $x \geq 1$.

We now want to use this bound to give an upper bound for the extinction probability of the branching process, which we denote by q_λ. Define the probability generating function

$$\Psi_\lambda(q) := \mathbb{E}[q^{Z_1}], \quad q \in [0,1].$$

Recall that q_λ equals the largest $q \in [0,1]$ such that $\Psi_\lambda(q) = q$. For any $q \in [0,1]$ and $x > 0$ we can bound

$$\Psi_\lambda(q) \leq \mathbb{P}(Z_1 \leq x) + q^x \cdot \mathbb{P}(Z_1 > x) = 1 - \mathbb{P}(Z_1 > x) \cdot (1-q^x).$$

Assuming $q > 1/2$, we take $x = 1/\log(1/q)$, so that $q^x = e^{-1}$; also using (5.12) and changing the value of $c > 0$ in each step, we obtain

$$\Psi_\lambda(q) \leq 1 - c(\lambda \log(1/q))^{\alpha-1} \leq 1 - c\lambda^{\alpha-1}(1-q)^{\alpha-1}, \tag{5.13}$$

where the second inequality follows from $\lim_{q\to 1-} \dfrac{\log(1/q)}{1-q} = 1$. Next, note that

$$\text{if } 1-q < \left(c\lambda^{\alpha-1}\right)^{\frac{1}{2-\alpha}}, \quad \text{then} \quad c\lambda^{\alpha-1}(1-q)^{\alpha-1} > 1-q.$$

Together with (5.13), this implies that

$$\text{if } 1-q < \left(c\lambda^{\alpha-1}\right)^{\frac{1}{2-\alpha}}, \quad \text{then} \quad \Psi_\lambda(q) < q,$$

so Ψ_λ has no fixed point in the interval $(1-\left(c\lambda^{\alpha-1}\right)^{\frac{1}{2-\alpha}}, 1)$. This implies that the extinction probability q_λ is smaller than $1-\left(c\lambda^{\alpha-1}\right)^{\frac{1}{2-\alpha}}$, so the survival probability $1-q_\lambda$ is larger than $\left(c\lambda^{\alpha-1}\right)^{\frac{1}{2-\alpha}} = c\lambda^{\frac{\alpha-1}{2-\alpha}}$. $\square$

5.2.2 Case $\alpha \in (\frac{3}{2}, 2]$

The lower bound in this regime and the next will be obtained as a consequence of Proposition 4.7: we will show that with a probability suitably bounded away from zero, $\mathcal{T}$ has a subgraph consisting of a half-line with stars, as required by that proposition.

Lemma 5.4. *Let $\mathcal{T}$ be a BGW tree with offspring distribution p satisfying* (5.2) *with $\alpha \in (1, 2]$. Then,*

$$\lim_{d\to\infty} \mathbb{P}\left(\begin{array}{c} \textit{at least two vertices at distance 2} \\ \textit{from the root have degree above } d \end{array} \;\middle|\; \deg(o) = d \right) = 1.$$

Proof. For $d \in \mathbb{N}$ and $\beta > 0$, define the events

$$\begin{aligned} A(\beta, d) &:= \{|\{v : \mathrm{dist}(o, v) = 2\}| \geq \beta d\}, \\ B(d) &:= \{|\{v : \mathrm{dist}(o, v) = 2,\ \deg(v) > d\}| \geq 2\}. \end{aligned}$$

We have

$$\mathbb{P}(A(\beta, d) \mid \deg(o) = d) = \mathbb{P}(X_1 + \cdots + X_d \geq \beta d),$$

where $X_1, \ldots, X_d$ are independent, all with law p. Since p has infinite expectation when $\alpha \in (1, 2]$, the law of large numbers implies that the right-hand side above tends to 1 as $d \to \infty$, regardless of the value of β.

Next, note that

$$\mathbb{P}(B(d) \mid A(\beta, d)) \geq \mathbb{P}(\mathrm{Binomial}(\lfloor \beta d \rfloor, p([d, \infty))) \geq 2).$$

When $\alpha \in (1, 2]$, there exists $c > 0$ such that $p([d, \infty)) \geq c/d$ when d is large enough; the right-hand side above is then larger than

$$\mathbb{P}(\mathrm{Binomial}(\lfloor \beta d \rfloor, c/d) \geq 2).$$

By the convergence of binomial to Poisson, when $d \to \infty$, this probability converges to $1 - e^{-c\beta} - (c\beta)e^{-c\beta}$.

We have thus proved that, for arbitrary $\beta > 0$,

$$\liminf_{d\to\infty} \mathbb{P}(B(d) \mid \deg(o) = d) \geq 1 - e^{-c\beta} - (c\beta)e^{-c\beta}.$$

Since the right-hand side can be made arbitrarily close to 1 by taking β large, the proof is complete. $\square$

Proof of Theorem 5.1, lower bound for $\alpha \in (\frac{3}{2}, 1]$. Using the above lemma, choose d_0 large enough that for all $d \geq d_0$,

$$\mathbb{P}(|\{v : \mathrm{dist}(o, v) = 2,\ \deg(v) > d\}| \geq 2 \mid \deg(o) = d) > 3/4. \tag{5.14}$$

Let $\mathfrak{C}(1/2)$ be the constant given in Proposition 4.7, with $\varepsilon = 1/2$. Choose λ small enough that $\frac{\log(1/\lambda)}{\lambda^2} > 1$ and $d_1 = d_1(\lambda) := \frac{2\mathfrak{C}(1/2)\log(1/\lambda)}{\lambda^2}$ is larger than d_0.

Let A be the event that $\deg(o) \geq d_1$ and $\mathcal{T}$ has a sequence of vertices $o = u_0, u_1, u_2, \ldots$, all with degree above d_1, and so that u_{j+1} is a grandchild of u_j for each j. Then, for the contact process $(\xi_t)_{t\geq 0}$ on $\mathcal{T}$ with rate λ, we have

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) \geq \mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid A) \cdot \mathbb{P}(A) \geq \frac{1}{2} \cdot \mathbb{P}(A),$$

by Proposition 4.7.

To give a lower bound to $\mathbb{P}(A)$, we use a branching process. On the event $\{\deg(o) \geq d_1\}$, let $\mathcal{Z}_0 := \{o\}$, and inductively, let $\mathcal{Z}_{k+1}$ be the set of grandchildren with degree above d_1 of vertices of $\mathcal{Z}_k$ (with $\mathcal{Z}_{k+1} = \varnothing$ if $\mathcal{Z}_k = \varnothing$). Then, by (5.14) and the choice of d_1, conditionally on $\{\deg(o) \geq d_1\}$, the process $(|\mathcal{Z}_k|)_{k\geq 0}$ stochastically dominates a branching proces in which each individual has two children with probability larger than $3/4$. In particular, the mean offspring of this process is larger than one, so its survival probability ρ is positive. This proves that

$$\mathbb{P}(A) \geq \rho \cdot \mathbb{P}(\deg(o) \geq d_1) = \rho \cdot p([d_1, \infty)) \geq c\frac{\lambda^{2\alpha-2}}{(\log(1/\lambda))^{\alpha-1}}. \qquad \square$$

5.2.3 Case $\alpha > 2$

Lemma 5.5. *Let $\mathcal{T}$ be a BGW tree with offspring distribution p satisfying* (5.2) *with $\alpha > 2$. Then, there exists $b > 0$ such that*

$$\lim_{d\to\infty} \mathbb{P}\left(\begin{array}{c} \textit{at least two vertices at distance } \lfloor b \log d \rfloor \\ \textit{from the root have degree above } d \end{array} \middle| \deg(o) = d \right) = 1.$$

Proof. Let $\mu := \sum_{k\geq 1} kp(k) > 1$ be the mean offspring of $\mathcal{T}$, let $X_n := |\{v \in \mathcal{T} : \mathrm{dist}(o, v) = n\}|$ be the number of vertices at height n, and let $\sigma := \mathbb{P}(\mathcal{T}$ is infinite$)$.

Fix $\mu' \in (1, \mu)$. We claim that, if n is large enough, we have

$$\mathbb{P}(X_n > (\mu')^n) > \sigma/2.$$

To see this, note that by the Kesten-Stigum theorem (using the fact that $\sum_k k \log(k) p(k) < \infty$ when $\alpha > 2$), we have that on $\{\mathcal{T}$ is infinite$\}$, almost surely, X_n/μ^n converges almost surely to a strictly positive random variable; hence,

$$\mathbb{P}(X_n > (\mu')^n) \geq \mathbb{P}(X_n > (\mu')^n,\ \mathcal{T} \text{ is infinite}) \xrightarrow{n\to\infty} \mathbb{P}(\mathcal{T} \text{ is infinite}).$$

Let $b := 2\alpha/\log(\mu')$. Using the above claim, choose d large enough that, letting $\ell_d := \lfloor b \log d \rfloor$, we have

$$\mathbb{P}(X_{\ell_d - 1} > (\mu')^{\ell_d - 1}) > \sigma/2.$$

Then, letting $Y_1, \ldots, Y_d$ be independent, all with the same law as X_{ℓ_d-1}, we have

$$\begin{aligned}\mathbb{P}(X_{\ell_d} > (\mu')^{\ell_d-1} \mid \deg(o) = d) &\geq \mathbb{P}(Y_1 + \cdots + Y_d > (\mu')^{\ell_d-1}) \\ &\geq \mathbb{P}\left(\max_{1\leq j\leq d} Y_j > (\mu')^{\ell_d-1}\right) \\ &\geq 1 - (1-\sigma/2)^d.\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathbb{P}\left(\begin{array}{l}\text{at least two vertices at distance } \ell_d \\ \text{from the root have degree above } d\end{array} \middle| \deg(o) = d,\ X_{\ell_d} > (\mu')^{\ell_d-1}\right) \\ &\geq \mathbb{P}(\mathrm{Binomial}(\lfloor(\mu')^{\ell_d-1}\rfloor,\ p([d,\infty))) \geq 2) =: \beta(d).\end{aligned}$$

Bounding

$$\lfloor(\mu')^{\ell_d-1}\rfloor \geq \lfloor(\mu')^{b\log(d)/2}\rfloor = \lfloor d^\alpha \rfloor$$

and $p([d,\infty)) > c/d^{\alpha-1}$, we see that we $\beta(d) \xrightarrow{d\to\infty} 1$.

We have thus proved that the probability that appears in the statement of the lemma is larger than $\beta(d)\cdot(1-(1-\sigma/2)^d) \xrightarrow{d\to\infty} 1$. $\square$

Proof of Theorem 5.1, lower bound for $\alpha > 3$. Using Lemma 5.5, we take d_0 large enough that, for any $d \geq d_0$,

$$\mathbb{P}(|\{v \in \mathcal{T}:\ \mathrm{dist}(o,v) = \lfloor b\log d\rfloor,\ \deg(v) > d\}| \geq 2 \mid \deg(o) = d) > \frac{3}{4}. \tag{5.15}$$

Let $\mathfrak{C}(1/2)$ be the constant given in Proposition 4.7, with $\varepsilon = 1/2$. We abbreviate

$$\mathfrak{c}_\lambda := \frac{\mathfrak{C}(1/2)\log(1/\lambda)}{\lambda^2}$$

and take $\lambda < \lambda_0$ small enough that

$$d_\lambda := 3b\mathfrak{c}_\lambda \log(b\mathfrak{c}_\lambda) > d_0.$$

We also define

$$\ell_\lambda := \lfloor b \log d_\lambda \rfloor.$$

The quantities d_λ and ℓ_λ will respectively play the role of degree and distance in an application of Proposition 4.7. For this application to be valid, we will first verify that, reducing λ if necessary, we have

$$d_\lambda \geq \mathfrak{c}_\lambda \ell_\lambda. \tag{5.16}$$

Indeed,

$$\mathfrak{c}_\lambda \ell_\lambda \le \mathfrak{c}_\lambda b \log d_\lambda = \mathfrak{c}_\lambda b \log(3b\mathfrak{c}_\lambda \log(b\mathfrak{c}_\lambda)).$$

When λ is small, we can bound $b\mathfrak{c}_\lambda > 3$ and $b\mathfrak{c}_\lambda > \log(b\mathfrak{c}_\lambda)$, so the right-hand side above is smaller than

$$\mathfrak{c}_\lambda b \log((b\mathfrak{c}_\lambda)^3) = d_\lambda.$$

Now, let A be the event that $\deg(o) \ge d_\lambda$ and $\mathcal{T}$ has a sequence of vertices $o = u_0, u_1, u_2, \dots$, all with degree above d_λ, and so that u_{j+1} is a descendant of u_j at distance at most ℓ_λ from u_j. For the contact process $(\xi_t)_{t\ge 0}$ with rate λ on $\mathcal{T}$, we bound

$$\mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)) \ge \mathbb{P}(\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot) \mid A) \cdot \mathbb{P}(A)$$

and note that $\mathbb{P}(\mathbb{1}_{\mathrm{Surv}_{\mathrm{loc}}(\xi_\cdot)} \mid A) \ge 1/2$ by Proposition 4.7 (which can be applied now that we have verified (5.16)).

It remains to prove that $\mathbb{P}(A) > 0$. To this end, we bound

$$\mathbb{P}(A) \ge \rho \cdot \mathbb{P}(\deg(o) \ge d_\lambda),$$

where ρ is the survival probability of a branching process in which each individual has two children with probability $\frac{3}{4}$. The proof of this step is done in the same way as in the case $\alpha \in (\frac{3}{2}, 1]$, except that this time we use Lemma 5.5. Finally, it is easy to see that there is a constant $C > 0$ such that $d_\lambda \le C\frac{(\log(1/\lambda))^2}{\lambda^2}$, so

$$\mathbb{P}(\deg(o) \ge d_\lambda) = p([d_\lambda, \infty)) \ge c d_\lambda^{-(\alpha-1)} \ge c' \frac{\lambda^{2\alpha-2}}{(\log(1/\lambda))^{2\alpha-2}}. \qquad \square$$

5.3 Upper bounds

Proving the upper bounds for Theorem 5.1 will be considerably more involved than the lower bounds. The bounds themselves will be established in Sections 5.3.3 (case $\alpha \in (1, \frac{3}{2}]$), 5.3.4 (case $\alpha \in (\frac{3}{2}, 2]$) and 5.3.5 (case $\alpha > 2$). Before we get to these, some preliminary work is required. In Section 5.3.1, using martingale methods, we prove estimates on the probability that the infection travels from one vertex to another on trees with degree smaller than a λ-dependent threshold. In Section 5.3.2, we apply these estimates to BGW trees, which are truncated to make degrees fall below the necessary threshold.

5.3.1 Martingale estimates

In this section, we obtain some preliminary upper bounds. The main message here is that, if λ is small, then a tree in which all vertices have degree much smaller than $1/\lambda^2$ is "hostile environment" for the contact process, in the sense that the infection does not manage to sustain itself locally, and is unlikely to find its way from a fixed vertex u to another fixed vertex v which is far from u.

Lemma 5.6. *Assume that $\lambda < \frac{1}{2\sqrt{2}}$, and let T be a tree in which all vertices have degree smaller than $\frac{1}{8\lambda^2}$. Let u and v be two (not necessarily distinct) vertices of T. Then, for the contact process on T with rate λ,*

$$\mathbb{P}((u,0) \rightsquigarrow \{v\} \times [t,\infty)) \leq e^{-t/4} \cdot (2\lambda)^{\mathrm{dist}(u,v)} \quad \text{for any } t \geq 0. \tag{5.17}$$

In particular,

$$\mathbb{P}((u,0) \rightsquigarrow \{v\} \times [0,\infty)) \leq (2\lambda)^{\mathrm{dist}(u,v)}. \tag{5.18}$$

Proof. Using graph truncation and monotonicity, it is sufficient to prove the lemma with the additional assumption that the tree is finite. With this assumption, some of the steps of the proof (such as exchanging a derivative with a sum, see below) become easier to check.

Let u and v be vertices as in the statement of the lemma, and define

$$f(\xi) := \sum_{w \in T} \xi(w) \cdot (2\lambda)^{\mathrm{dist}(w,v)}, \quad \xi \in \{0,1\}^T.$$

Let $(\xi_t)_{t \geq 0}$ be the contact process with rate λ on T, started from only u infected, and let

$$B_t := e^{t/4} \cdot f(\xi_t), \qquad t \geq 0.$$

We claim that (B_t) is a supermartingale with respect to $(\mathcal{F}_t)$, the natural filtration of (ξ_t). Before proving the claim, let us see how it allows us to conclude. Define $\sigma_t := \inf\{s \geq t : \xi_s(v) = 1\}$, and note that on $\{\sigma_t < \infty\}$, we have $B_{\sigma_t} \geq e^{t/4}$, so

$$\mathbb{P}((u,0) \rightsquigarrow \{v\} \times [t,\infty)) = \mathbb{P}(\sigma_t < \infty) \leq \mathbb{E}[e^{-t/4} \cdot B_{\sigma_t} \cdot \mathbb{1}\{\sigma_t < \infty\}].$$

Using the optional stopping theorem, the right-hand side is smaller than

$$e^{-t/4} \cdot B_0 = e^{-t/4} \cdot (2\lambda)^{\mathrm{dist}(u,v)}.$$

To prove the claim, we fix $r \geq 0$ and prove that the function $[0,\infty) \ni s \mapsto \mathbb{E}[B_{r+s} \mid \mathcal{F}_r]$ is non-increasing. To do so, we fix $s \geq 0$ and start computing

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[B_{r+s+a} \mid \mathcal{F}_r]\bigg|_{a=0+} \\ &\quad = \sum_{\xi'} \mathbb{P}(\xi_{r+s} = \xi' \mid \mathcal{F}_r) \cdot \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[B_{r+s+a} \mid \xi_{r+s} = \xi']\bigg|_{a=0+}. \end{aligned} \tag{5.19}$$

For any ξ', the right derivative on the right-hand side is

$$\frac{1}{4}e^{(r+s)/4}\cdot f(\xi') + e^{(r+s)/4}\cdot \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[f(\xi_{r+s+a}) \mid \xi_{r+s} = \xi']\Big|_{a=0+}. \tag{5.20}$$

The right derivative in the right-hand side is

$$\begin{aligned}
&\sum_{w\in\xi'}\left(-(2\lambda)^{\mathrm{dist}(w,v)} + \lambda\sum_{z\sim w, z\notin\xi'}(2\lambda)^{\mathrm{dist}(z,v)}\right)\\
&\le \sum_{w\in\xi'}\left(-(2\lambda)^{\mathrm{dist}(w,v)} + \lambda\left((2\lambda)^{\mathrm{dist}(w,v)-1} + \frac{1}{8\lambda^2}\cdot(2\lambda)^{\mathrm{dist}(w,v)+1}\right)\right)\\
&\le \sum_{w\in\xi'}(2\lambda)^{\mathrm{dist}(w,v)}\left(-1+\frac{1}{2}+\frac{1}{4}\right) = -\frac{1}{4}f(\xi').
\end{aligned}$$

This shows that the expression in (5.20) is ≤ 0, so the expression in (5.19) is ≤ 0. We have thus proved that the function $[0,\infty)\ni s\mapsto \mathbb{E}[B_{r+s}\mid \mathcal{F}_r]$ has a non-positive right derivative. Since this function is continuous and its right derivative is continuous, it is differentiable, and so we conclude that it is non-increasing, as required. □

We will now see two corollaries of Lemma 5.6. The first is about the extinction time of the contact process on finite trees.

Corollary 5.7. *Assume that $\lambda < \frac{1}{2\sqrt{2}}$, and let T be a finite tree in which all vertices have degree smaller than $\frac{1}{8\lambda^2}$. Lettting τ_T be the extinction time of the contact process with rate λ on T, for any $t>0$ we have*

$$\mathbb{P}(\tau_T \ge t) \le |T|^2\cdot e^{-t/4}.$$

Proof. We bound

$$\mathbb{P}(\tau_T\ge t) \le \sum_{u,v\in T}\mathbb{P}((u,0)\rightsquigarrow \{v\}\times[t,\infty)) \ \le |T|^2\cdot e^{-t/4},$$

where the second inequality follows from (5.17). □

The second corollary is about the extinction of the contact process with rate λ on trees whose branching number is below $\frac{1}{2\lambda}$.

Corollary 5.8. *Assume that $\lambda < \frac{1}{2\sqrt{2}}$. Let T be a tree with root o, and assume that all vertices of T, apart possibly from o, have degree smaller than $\frac{1}{8\lambda^2}$. Also assume that*

$$\limsup_{k\to\infty}|\{v : \mathrm{dist}(o,v) = k\}|^{1/k} < \frac{1}{2\lambda}. \tag{5.21}$$

Then, the contact process on T with rate λ dies out.

Proof. Let v be a non-root vertex of T. For $k \in \mathbb{N}$, define the set of vertices

$$\mathcal{S}_k(v) := \{u: \ \mathrm{dist}(o,u) = k, \text{ the geodesic from } u \text{ to } v \text{ does not visit } o\}.$$

We claim that, for any $k > \mathrm{dist}(o,v)$, we have

$$\mathbb{P}(\xi_t^v \neq \varnothing \ \forall t) \leq \mathbb{P}((v,0) \rightsquigarrow (\{o\} \cup \mathcal{S}_k(v)) \times [0,\infty)).$$

To see this, note that when $k > \mathrm{dist}(o,v)$, the set of vertices that can be reached from v by a path that does not visit o or $\mathcal{S}_k(v)$ is finite. So, there is almost surely no infinite infection path that stays inside this set.

We now use (5.18) in the subgraph of T induced by $\{o\}$ and all vertices that can be reached from v without passing by o to obtain

$$\begin{aligned}\mathbb{P}\left((v,0) \rightsquigarrow (\{o\} \cup \mathcal{S}_k(v)) \times [0,\infty)\right) &\leq \sum_{u \in \{o\} \cup \mathcal{S}_k(v)} (2\lambda)^{\mathrm{dist}(v,u)} \\ &\leq (2\lambda)^{\mathrm{dist}(o,v)} + |\mathcal{S}_k(v)| \cdot (2\lambda)^{k-\mathrm{dist}(o,v)},\end{aligned}$$

since $\mathrm{dist}(v,u) \geq k - \mathrm{dist}(o,v)$ for any $u \in \mathcal{S}_k(v)$. Taking k large and using (5.21), it is easy to see that

$$\lim_{k\to\infty} |\mathcal{S}_k(v)| \cdot (2\lambda)^k = 0,$$

so we obtain

$$\mathbb{P}(\xi_t^v \neq \varnothing \ \forall t) \leq (2\lambda)^{\mathrm{dist}(o,v)} \quad \text{for all } v \neq o. \tag{5.22}$$

Now fix a finite set of vertices A. We have

$$\mathbb{P}(\exists t: \ \xi_t^A = \varnothing) \geq \prod_{v \in A} \mathbb{P}(\exists t: \xi_t^v = \varnothing)$$

by the FKG inequality (as in (2.29)). The right-hand side is larger than

$$\prod_{v \in T} \mathbb{P}(\exists t: \xi_t^v = \varnothing) \overset{(5.22)}{\geq} \mathbb{P}(\exists t: \xi_t^o = \varnothing) \cdot \prod_{k=1}^{\infty} (1-(2\lambda)^k)^{|\{v:\mathrm{dist}(o,v)=k\}|}.$$

Again using the assumption (5.21), we have $\sum_{k=1}^{\infty} (2\lambda)^k \cdot |\{v: \mathrm{dist}(o,v) = k\}| < \infty$, so the right-hand side above is strictly positive.

We have now proved that

$$\gamma := \inf_A \mathbb{P}(\exists t: \ \xi_t^A = \varnothing) > 0,$$

where the infimum is taken over all finite sets of vertices. For any given finite set of vertices B, we then have

$$M_s := \mathbb{P}(\xi_t^B \neq \varnothing \ \forall t \mid \mathcal{F}_s) \leq 1 - \gamma < 1. \tag{5.23}$$

By the martingale convergence theorem,

$$M_s \xrightarrow{s\to\infty} \mathbb{1}\{\xi_t^B \neq \varnothing \ \forall t\} \quad \text{almost surely,}$$

so

$$\mathbb{P}(\xi_t^B \neq \varnothing \ \forall t) = \mathbb{P}\left(\omega : M_s(\omega) \xrightarrow{s\to\infty} 1\right) \overset{(5.23)}{=} 0. \qquad \square$$

We now give a version of Lemma 5.6 that involves paths that travel from u to v and then return to u.

Lemma 5.9. *Assume that $\lambda < \frac{1}{2\sqrt{2}}$, and let T be a tree in which all vertices have degree smaller than $\frac{1}{8\lambda^2}$. Let u and v be two distinct vertices of T. Then, for the contact process on T with rate λ,*

$$\mathbb{P}(\exists t, t' : 0 < t < t' :\ (u,0) \rightsquigarrow (v,t) \rightsquigarrow (u,t')) \leq (2\lambda)^{2\text{dist}(u,v)}. \tag{5.24}$$

Proof. We define an auxiliary Markov process $(\zeta_t)_{t\geq 0}$ on $\{0,1,2\}^T$. We think of a site in state 0 as empty, and a site in state $i \in \{1,2\}$ as being occupied by an animal of species i. The process starts from the configuration ζ_0 which has an individual of type 1 at u and is empty everywhere else. The dynamics follows the rules:

- individuals of both types 1 and 2 die with rate 1, leaving their site empty;
- an individual of type 2 sends a child to each neighboring site (possibly overwriting a previous occupant) with rate λ, and the type of the child is also 2;
- an individual of type 1 sends a child to each *empty* neighboring site with rate λ; the type of the child is 2 if it is born at v, and 1 if it is born anywhere else.

These rules are encoded by the generator:

$$\begin{aligned}\mathcal{L}f(\zeta) = &\sum_{\substack{w\in T:\\ \zeta(w)\in\{1,2\}}} (f(\zeta^{w\leftarrow 0}) - f(\zeta))\\ &+ \lambda \sum_{\substack{w\in T:\\ \zeta(w)=2}} \sum_{\substack{z\sim w:\\ \zeta(z)\in\{0,1\}}} (f(\zeta^{z\leftarrow 2}) - f(\zeta))\\ &+ \lambda \sum_{\substack{w\in T:\\ \zeta(w)=1}} \sum_{\substack{z\sim w:\\ z\neq v,\\ \zeta(z)=0}} (f(\zeta^{z\leftarrow 1}) - f(\zeta))\\ &+ \lambda \cdot \mathbb{1}\{\zeta(v)=0\} \sum_{\substack{w\in T:\\ w\sim v,\\ \zeta(w)=1}} (f(\zeta^{v\leftarrow 2}) - f(\zeta)),\end{aligned}$$

where $\zeta^{w\leftarrow i}$ is the configuration obtained from ζ by putting state i at site w and leaving the states of other sites unaltered.

We observe that $(\zeta_t)_{t\geq 0}$ can be constructed using the same graphical construction as the contact process. In fact, letting $(\xi_t)_{t\geq 0}$ be the contact process obtained from this common graphical construction and started from a single infection at u, we have the relation

$$\xi_t(w) = \mathbb{1}\{\zeta_t(w) \in \{1,2\}\}, \quad w \in T.$$

Moreover,

$$\{\exists t, t' : 0 < t < t' :\ (u,0) \rightsquigarrow (v,t) \rightsquigarrow (u,t')\} = \{\exists t \geq 0 :\ \zeta_t(u) = 2\}.$$

We want to bound the probability of the event on the right-hand side. Define, for $\zeta \in \{0,1,2\}^T$,

$$f_1(\zeta) := (2\lambda)^{\mathrm{dist}(u,v)} \sum_{\substack{w\in T:\\ \zeta(w)=1}} (2\lambda)^{\mathrm{dist}(w,v)}, \qquad f_2(\zeta) := \sum_{\substack{w\in T:\\ \zeta(w)=2}} (2\lambda)^{\mathrm{dist}(w,u)},$$

and let $A_t := f_1(\zeta_t) + f_2(\zeta_t)$, for $t \geq 0$. The idea here is that all animals in the configuration ζ receive a weight, depending on their types and positions. To clarify this, fix $w \in T$ and first assume that there is a type-1 animal at w (in particular, $w \neq v$, since only animals of type 2 can exist at v). Then, letting

$$\begin{aligned} &w = w_0, w_1, \ldots, w_k = v \quad \longrightarrow \quad \text{geodesic from } w \text{ to } v,\\ &v = v_0, v_1, \ldots, v_\ell = u \quad \longrightarrow \quad \text{geodesic from } v \text{ to } u, \end{aligned}$$

the most direct way that a type-1 animal at w eventually produces a type-2 animal at u is that births occur along the path obtained by concatenating the above two geodesics. This path has length $\mathrm{dist}(w,v)+\mathrm{dist}(v,u)$; guided by this, we give weight $(2\lambda)^{\mathrm{dist}(w,v)+\mathrm{dist}(v,u)}$ to the type-1 animal at w. Now assume that there is a type-2 animal at w. Then, this animal could produce a type-2 animal at u by following the geodesic from w to u, so we attribute to it the larger weight $(2\lambda)^{\mathrm{dist}(w,u)}$.

We claim that $(A_t)_{t\geq 0}$ is a supermartingale. Once this claim is proved, we can let $\sigma := \inf\{t : \zeta_t(u) = 2\}$, note that on $\{\sigma < \infty\}$ we have $A_\sigma \geq 1$, and bound

$$\begin{aligned} \mathbb{P}(\exists t : \zeta_t(u) = 2) &= \mathbb{P}(\sigma < \infty)\\ &\leq \mathbb{E}[A_\sigma \cdot \mathbb{1}\{\sigma < \infty\}] \leq A_0 = (2\lambda)^{2\mathrm{dist}(u,v)}. \end{aligned}$$

To prove the claim, as in the proof of Lemma 5.6, it is sufficient to show that for every $\zeta' \in \{0,1,2\}^T$ and every $t \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}[A_{t+s} \mid \zeta_t = \zeta']\bigg|_{s=0+} \leq 0.$$

We write the right derivative on the right-hand side as $\tilde{f}_1(\zeta') + \tilde{f}_2(\zeta')$, where

$$\tilde{f}_1(\zeta') := \sum_{w:\zeta'(w)=1} \Big(-(2\lambda)^{\mathrm{dist}(w,v)+\mathrm{dist}(u,v)} + \lambda \sum_{\substack{z\sim w:\\ \zeta'(z)=0}} (2\lambda)^{\mathrm{dist}(z,v)+\mathrm{dist}(u,v)} \Big)$$

and

$$\tilde{f}_2(\zeta') := \sum_{w:\zeta'(w)=2} \Big(-(2\lambda)^{\mathrm{dist}(w,u)} + \lambda \sum_{\substack{z\sim w:\\ \zeta'(z)\in\{0,1\}}} (2\lambda)^{\mathrm{dist}(z,u)} \Big).$$

Using the assumption that the degrees are bounded by $\frac{1}{8\lambda^2}$ as in the proof of Lemma 5.6, it can now be seen that $\tilde{f}_1(\zeta') \le -\frac{1}{4}f_1(\zeta')$ and $\tilde{f}_2(\zeta') \le -\frac{1}{4}f_2(\zeta')$. This concludes the proof. $\square$

Finally, we give versions of the inequalities (5.18) and (5.24) where the starting time of the infection path is allowed to take values different from zero.

Lemma 5.10. *Assume that $\lambda < \frac{1}{2\sqrt{2}}$, and let T be a tree in which all vertices have degree smaller than $\frac{1}{8\lambda^2}$. Let u and v be two distinct vertices of T. Then, for the contact process on T with rate λ, and any $t > 0$, we have*

$$\mathbb{P}(\{u\}\times[0,t] \rightsquigarrow \{v\}\times[0,\infty)) \le (t+1)\cdot(2\lambda)^{\mathrm{dist}(u,v)}. \tag{5.25}$$

and

$$\mathbb{P}(\exists s,s',s'':\ s\le t,\ (u,s)\rightsquigarrow(v,s')\rightsquigarrow(u,s'')) \le (t+1)\cdot(2\lambda)^{2\mathrm{dist}(u,v)}. \tag{5.26}$$

Proof. We will prove (5.25) using (5.18) and an additional argument. One can then prove (5.26) using (5.24) and the same additional argument, so we omit this.

For $k\in\mathbb{N}$, let $A_k := \{\{u\}\times[k,k+1]\rightsquigarrow\{v\}\times[0,\infty)\}$, and let A_k' be the event that A_k occurs and there is no recovery mark at u in the time interval $[k-1,k+1]$. By the FKG inequality,

$$\mathbb{P}(A_k') \ge \mathbb{P}(A_k)\cdot e^{-2}.$$

The left-hand side of (5.25) is equal to

$$\mathbb{P}(\{u\}\times[1,t+1]\rightsquigarrow\{v\}\times[0,\infty)) \le \sum_{k=1}^{\lfloor t\rfloor+1}\mathbb{P}(A_k) \le e^2\cdot\sum_{k=1}^{\lfloor t\rfloor+1}\mathbb{P}(A_k'). \tag{5.27}$$

On the event A'_k we have $(u,s) \rightsquigarrow \{v\} \times [0,\infty)$ for any $s \in [k-1,k]$. In particular,

$$\int_{k-1}^{k} \mathbb{1}\{(u,s) \rightsquigarrow \{v\} \times [0,\infty)\}\, \mathrm{d}s \geq \mathbb{1}_{A'_k}.$$

The right-hand side of (5.27) is then smaller than

$$\begin{aligned} &e^2 \cdot \sum_{k=1}^{\lfloor t \rfloor +1} \mathbb{E}\left[\int_{k-1}^{k} \mathbb{1}\{(u,s) \rightsquigarrow \{v\} \times [0,\infty)\}\, \mathrm{d}s\right] \\ &= e^2 \cdot \int_0^{\lfloor t \rfloor +1} \mathbb{P}((u,s) \rightsquigarrow \{v\} \times [0,\infty))\, \mathrm{d}s \\ &\overset{(5.25)}{\leq} e^2(t+1) \cdot (2\lambda)^{\mathrm{dist}(u,v)}. \end{aligned}$$

□

5.3.2 Truncation and escape probabilities

We now obtain consequences of the bounds from the previous section for the particular case of the contact process on BGW trees. We first explain a very natural procedure to *truncate* the tree, in order to make all vertex degrees smaller than a threshold value M. Namely, we explore the tree generation by generation from the root, and whenever a vertex tries to have too many children, making its degree M or larger, we turn that vertex into a leaf instead (in particular, we do not explore its descendancy any further). Vertices that have been affected in this way are then considered "danger points". The idea is then that the probability that the contact process (started from only the root infected) survives forever on the original tree is smaller than the probability that in the new tree, it either survives forever or infects any of the danger points (we label the latter case as an "escape").

Let $\mathcal{T}$ be a BGW tree with root o and offspring distribution p supported on $\{2,3,\ldots\}$. Also let $M \in \mathbb{N}$. We define the tree $\widehat{\mathcal{T}}_M$ as the subgraph of $\mathcal{T}$ induced by the set of vertices

$$\{o\} \cup \left\{ \begin{array}{l} u \in \mathcal{T} : \text{all vertices in the geodesic from } o \text{ to } u, \\ \text{apart possibly from } o \text{ and } u, \text{ have degree} < M \end{array} \right\}.$$

We think of $\widehat{\mathcal{T}}_M$ as the result of exploring $\mathcal{T}$ from the root upwards, and turning every (non-root) vertex that tries to have offspring $\geq M$ into leaves. This point of view makes it clear that $\widehat{\mathcal{T}}_M$ is a BGW tree with offspring distribution p_M given by

$$\begin{aligned} &p_M(0) = p([M,\infty)), \\ &p_M(j) = p(j) \text{ for } 1 \leq j < M, \\ &p_M(j) = 0 \text{ for } j \geq M. \end{aligned} \tag{5.28}$$

We now define

$$\partial\widehat{\mathcal{T}}_{M,k} := \{u \in \widehat{\mathcal{T}}_M :\ u \text{ is a leaf of } \widehat{\mathcal{T}}_M \text{ and } \mathrm{dist}(o,u) = k\}, \quad k \in \mathbb{N}$$

and

$$\partial\widehat{\mathcal{T}}_M := \bigcup_{k=1}^{\infty} \partial\widehat{\mathcal{T}}_{M,k}.$$

These are sets of vertices of $\widehat{\mathcal{T}}_M$, hence also of $\mathcal{T}$.

Now take a graphical construction H for the contact process on $\mathcal{T}$, and let $\widehat{H}_M$ be its restriction to $\widehat{\mathcal{T}}_M$. Note that H has an infection path starting at $(o,0)$ and reaching some vertex of $\partial\widehat{\mathcal{T}}_M$ if and only if $\widehat{H}_M$ has an infection path satisfying the same properties. Hence, defining

$$\mathrm{Escape}(M) := \{(o,0) \rightsquigarrow \partial\widehat{\mathcal{T}}_M \times [0,\infty)\},$$

there is no ambiguity with respect to which of the two graphical constructions (H or $\widehat{H}_M$) we are using.

Lemma 5.11. *Let $\lambda < 1/4$. Let $\mathcal{T}$ be a BGW tree with root o and offspring distribution p supported on $\{2,3,\dots\}$. Let $M \in \mathbb{N}$ and assume that*

$$M \leq \frac{1}{8\lambda^2} \tag{5.29}$$

and

$$\sum_{j=1}^{M} j \cdot p(j) < \frac{1}{2\lambda}. \tag{5.30}$$

Then, for the contact process $(\xi_t^o)_{t\geq 0}$ on $\mathcal{T}$ with rate λ started with only the root infected, defining the escape event Escape(M) *as above, we have*

$$\mathbb{P}(\{\xi_t^o \neq \varnothing\ \forall t\} \cap (\mathrm{Escape}(M))^c) = 0.$$

Proof. The result will follow from proving that the contact process dies out on the truncated tree $\widehat{\mathcal{T}}_M$. Recall that p_M given in (5.28) gives the offspring distribution of $\widehat{\mathcal{T}}_M$. For any $m' > \sum_{j\leq M} jp_M(j)$, by the Markov inequality we have

$$\mathbb{P}(|\{v \in \widehat{\mathcal{T}}_M : \mathrm{dist}(o,v) = k\}| > (m')^k) \leq \frac{\left(\sum_{j\leq M} jp_M(j)\right)^k}{(m')^k};$$

then, also using Borel-Cantelli, we obtain that almost surely,

$$\limsup_{k\to\infty} |\{v \in \widehat{\mathcal{T}}_M : \mathrm{dist}(o,v) = k\}|^{1/k} \leq \sum_{j\leq M} jp_M(j) \overset{(5.30)}{<} \frac{1}{2\lambda}.$$

Then, $\widehat{\mathcal{T}}_M$ almost surely satisfies the assumptions of Corollary 5.8, so the result follows. $\square$

We would now like to use this lemma to prove upper bounds for Theorem 5.1. For each fixed value of the power law exponent $\alpha > 1$, the idea is to take λ small, and choose M in the lemma depending on α and λ. We would like M to be chosen as large as possible: the larger it is, the smaller the probability of the Escape(M) event, so the better the upper bound.

To explain how M will be chosen in each case, note that Lemma 5.11 has two conditions on M: (5.29) and (5.30). We now observe:

- in case $\alpha > 2$, we have $\sum_{j=1}^\infty jp(j) < \infty$, so when λ is small enough (smaller than $(2\sum_{j=1}^\infty jp(j))^{-1}$), (5.30) is satisfied automatically, regardless of M. Then, (5.29) is the only condition that matters in this case, and we can take $M = 1/(8\lambda^2)$;

- if $\alpha \le 2$, then $\sum_{j=1}^\infty jp(j) = \infty$, so (5.30) becomes relevant. Still assuming λ to be small, (5.30) is satisfied when $M \ll \lambda^{-1/(2-a)}$ in case $\alpha \in (1,2)$, or when $\log M \ll \lambda^{-1}$ in case $\alpha = 2$. Hence, the value of α for which the two bounds (5.29) and (5.30) become comparable is $\alpha = \frac{3}{2}$. Recalling that we want to take M as large as possible, this suggests taking $M = \delta\lambda^{-1/(2-a)}$ (with δ small) if $\alpha \in (1, \frac{3}{2}]$ and $M = 1/(8\lambda^2)$ when $\alpha \in (\frac{3}{2}, 2]$.

Recall that $\partial\widehat{\mathcal{T}}_{M,k}$ is the set of leaves of $\widehat{\mathcal{T}}_M$ at distance k from o. Our tool to bound the probability of the Escape(M) event will be Lemma 5.6. In case $M \le 1/(8\lambda^2)$ and $d \le 1/(8\lambda^2)$, we have by (5.18) that

$$\begin{aligned}\mathbb{P}(\mathrm{Escape}(M) \mid \deg(o) = d) &\le \mathbb{E}\left[\sum_{k=1}^\infty \sum_{v \in \partial\widehat{\mathcal{T}}_{M,k}} (2\lambda)^k \,\middle|\, \deg(o) = d\right] \\ &= \sum_{k=1}^\infty (2\lambda)^k \cdot \mathbb{E}\left[|\partial\widehat{\mathcal{T}}_{M,k}| \,\middle|\, \deg(o) = d\right].\end{aligned} \tag{5.31}$$

Letting

$$m_M := \sum_{j \le M} j \cdot p(j),$$

we have, for any $k \ge 1$,

$$\mathbb{E}\left[|\partial\widehat{\mathcal{T}}_{M,k}| \,\middle|\, \deg(o) = d\right] = d \cdot m_M^{k-1} \cdot p([M,\infty)). \tag{5.32}$$

We can then bound

$$\begin{aligned}
&\mathbb{P}(\mathrm{Escape}(M) \cap \{\deg(o) \le M\}) \\
&\quad = \sum_{d \le M} p(d) \cdot \mathbb{P}(\mathrm{Escape}(M) \mid \deg(o) = d) \\
&\quad \overset{(5.31),(5.32)}{\le} \sum_{d \le M} p(d) \sum_{k=1}^{\infty} (2\lambda)^k \cdot d \cdot m_M^{k-1} \cdot p([M,\infty)) \\
&\quad = p([M,\infty)) \cdot \sum_{k=1}^{\infty} (2\lambda m_M)^k.
\end{aligned}$$

If $2\lambda m_M < 1$, then the series equals $\frac{2\lambda m_M}{1-2\lambda m_M}$. In particular, we have obtained:

$$\begin{aligned}
&\text{if } M \le 1/(8\lambda^2) \text{ and } \lambda m_M < 1/4, \text{ then} \\
&\qquad \mathbb{P}(\mathrm{Escape}(M) \cap \{\deg(o) \le M\}) \le 4\lambda m_M \cdot p([M,\infty)).
\end{aligned} \tag{5.33}$$

In applying this bound, it will be useful to note that, by comparison with an integral, there exists $C > 0$ such that

$$p([M,\infty)) \le CM^{-(\alpha-1)} \tag{5.34}$$

and

$$m_M \le \begin{cases} CM^{2-\alpha} & \text{if } \alpha \in (1,2); \\ C \log M & \text{if } \alpha = 2; \\ \sum_{j=1}^{\infty} j \cdot p(j) < \infty & \text{if } \alpha > 2. \end{cases} \tag{5.35}$$

5.3.3 Case $\alpha \in (1, \frac{3}{2}]$

We are finally ready to address the upper bounds for the three regimes of Theorem 5.1; we do this in this and the next two sections.

Proof of Theorem 5.1, upper bound for $\alpha \in (1, \frac{3}{2}]$. We will use (5.33) with the choice

$$M = \delta\lambda^{-1/(2-\alpha)}$$

with $\delta > 0$ small. Note that $M \le 1/(8\lambda^2)$ (since $\frac{1}{2-\alpha} \le 2$ when $\alpha \in (1, \frac{3}{2}]$) and, by (5.35),

$$\lambda m_M \le \lambda \cdot C(\delta\lambda^{-1/(2-\alpha)})^{2-\alpha} < 1/4 \tag{5.36}$$

if δ is small. We then bound:

$$
\begin{aligned}
\mathbb{P}(\xi_t^o \neq \varnothing\ \forall t) &\leq \mathbb{P}(\deg(o) > M) + \mathbb{P}(\{\deg(o) \leq M\} \cap \mathrm{Escape}(M)) \\
&\overset{(5.33)}{\leq} p([M,\infty)) + 4\lambda m_M \cdot p([M,\infty)) \\
&\overset{(5.36)}{\leq} 2p([M,\infty)) \\
&\overset{(5.34)}{\leq} 2CM^{-(\alpha-1)} \\
&= 2C\delta^{-(\alpha-1)}\lambda^{(\alpha-1)/(2-\alpha)}.
\end{aligned}
$$

□

5.3.4 Case $\alpha \in (\frac{3}{2}, 2]$

This case and the next are more involved, because we will need to allow for cases where the degree of the root is a bit larger than $1/(8\lambda^2)$. Here this is handled in the following proposition.

Proposition 5.12. *Let $\mathcal{T}$ be a BGW tree with root o and offspring distribution p satisfying* (5.1) *and* (5.2) *with $\alpha \in (\frac{3}{2}, 2]$. Then, there exist $\delta > 0$ and $\varepsilon > 0$ such that, for λ small enough,*

$$
d \leq \delta \frac{1}{\lambda^2} \log \frac{1}{\lambda} \quad \Longrightarrow \quad \mathbb{P}\left(\mathrm{Escape}\left(\frac{1}{8\lambda^2}\right) \middle| \deg(o) = d\right) < \lambda^{\varepsilon}.
$$

We postpone the proof of this proposition, and for now show how it gives the upper bound in Theorem 5.1 in this case.

Proof of Theorem 5.1, upper bound for $\alpha \in (\frac{3}{2}, 2]$. We take

$$
M = \frac{1}{8\lambda^2} \qquad \text{and} \qquad M' = \delta \frac{1}{\lambda^2} \log \frac{1}{\lambda},
$$

where δ is as in Proposition 5.12. We use Lemma 5.11 and bound

$$
\begin{aligned}
\mathbb{P}(\xi_t^o \neq \varnothing\ \forall t) \leq &\mathbb{P}(\deg(o) > M') \\
&+ \mathbb{P}(\{\deg(o) \leq M\} \cap \mathrm{Escape}(M)) \\
&+ \mathbb{P}(\{M < \deg(o) \leq M'\} \cap \mathrm{Escape}(M)).
\end{aligned} \tag{5.37}
$$

and treat the three terms on the right-hand side separately.

By (5.34),

$$
\mathbb{P}(\deg(o) > M') = p([M', \infty)) \leq C \frac{\lambda^{2\alpha-2}}{(\log \frac{1}{\lambda})^{\alpha-1}}. \tag{5.38}
$$

For the second term, we first use (5.34) to bound

$$
p([M, \infty)) \leq C\lambda^{2\alpha-2}. \tag{5.39}
$$

Next, defining $m_M = \sum_{j \le M} jp(j)$ as before (with our current choice $M = 1/(8\lambda^2)$), by (5.35) we have

$$\lambda m_M \le \begin{cases} C\lambda^{2\alpha-3} & \text{if } \alpha \in (\frac{3}{2}, 2); \\ C\lambda \log \frac{1}{\lambda} & \text{if } \alpha = 2. \end{cases} \tag{5.40}$$

In particular, $\lambda m_N < 1/4$ when λ is small, so we can apply (5.33). Together with the bounds obtained above, it yields

$$\mathbb{P}(\mathrm{Escape}(M) \cap \{\deg(o) \le M\}) \le \begin{cases} C\lambda^{4\alpha-5} & \text{if } \alpha \in (\frac{3}{2}, 2), \\ C\lambda^3 \log \frac{1}{\lambda} & \text{if } \alpha = 2. \end{cases} \tag{5.41}$$

Note that this bound is of smaller order than the one obtained in (5.38), since $4\alpha - 5 > 2\alpha - 2$ when $\alpha > \frac{3}{2}$.

Finally, by Proposition 5.12 and another use of (5.39) we have

$$\mathbb{P}(\{M < \deg(o) \le M'\} \cap \mathrm{Escape}(M)) \le \lambda^{\varepsilon}\mathbb{P}(\deg(o) > M) \le C\lambda^{2\alpha-2+\varepsilon};$$

the factor ε in the exponent saves the day!

Hence, all three terms on the right-hand side of (5.37) are smaller than $C\frac{\lambda^{2\alpha-2}}{(\log(1/\lambda))^{\alpha-1}}$, completing the proof. □

Proof of Proposition 5.12. Fix a small constant $\delta > 0$ to be chosen later, let $d \le \delta \frac{1}{\lambda^2} \log \frac{1}{\lambda}$, and condition on $\{\deg(o) = d\}$. We let

$$t^* := \left(\frac{1}{\lambda}\right)^{17\delta}$$

(the reason for this choice will be clear later, but briefly put, it is taken as an amount of time that is very likely to be larger than the extinction time of the contact process with rate λ on a star with $\delta \frac{1}{\lambda^2} \log \frac{1}{\lambda}$ leaves, using the bound of Proposition 2.23).

Again let $M = 1/(8\lambda^2)$. We will define three "bad events", show that Escape(M) does not happen if neither of them happens, and bound their probabilities.

The first bad event, denoted E_1, is the event that there is an infection path that starts at o at time 0, goes up to time t^*, and only visits o and neighbors of o.

As the second bad event, we define

$$E_2 := \{\{o\} \times [0, t^*] \rightsquigarrow \partial\widehat{\mathcal{T}}_M \times [0, \infty)\}.$$

The third bad event, E_3, is the event that there exist times s, s', t with $s < s' < t$, $s \le t^*$ and an infection path $\gamma : [s,t] \to \widehat{\mathcal{T}}_M$ such that

$$\begin{aligned}
&\gamma(r) = o \text{ for all } r \in [s, s'),\\
&\gamma(r) \neq o \text{ for all } r \in [s', t),\\
&\mathrm{dist}(\gamma(r), o) \ge 2 \text{ for some } r \in (s', t), \text{ and}\\
&\gamma(t) = o.
\end{aligned}$$

In words, E_3 is the event that there is an infection path inside $\widehat{\mathcal{T}}_M$ starting from the root at some time before t^*, reaching some vertex at distance at least two from the root, and then going back to the root.

We claim that $\mathrm{Escape}(M) \subseteq E_1 \cup E_2 \cup E_3$. To see this, assume that $\mathrm{Escape}(M)$ occurs, so that there is an infection path γ in $\widehat{\mathcal{T}}_M$ from $(o, 0)$ to $\partial\widehat{\mathcal{T}}_M \times [0, \infty)$. Define

$$\begin{aligned}
\sigma &:= \inf\{t : \ \mathrm{dist}(\gamma(t), o) \ge 2 \text{ or } \gamma(t) \in \partial\widehat{\mathcal{T}}_M\},\\
\sigma_0 &:= \sup\{t \le \sigma : \ \gamma(t) = o\}.
\end{aligned}$$

Now, if $\sigma > t^*$, then E_1 occurs. If $\sigma \le t^*$, then we can obtain a sub-path of γ that starts at o at some time slightly before σ_0, and either goes directly from o to a boundary vertex (so that E_2 occurs), or reaches distance 2 from o and then returns to o (so that E_3 occurs).

It remains to bound the probabilities of the bad events. Note that

$$d \le \delta \frac{1}{\lambda^2} \log \frac{1}{\lambda} \quad \Longrightarrow \quad e^{-16\lambda^2 d} \ge \lambda^{16\delta},$$

so, by (2.26) and $1 - x \le e^{-x}$, we have

$$\mathbb{P}(E_1 \mid \deg(o) = d) \le \exp\left\{ -\frac{1}{4} \lambda^{16\delta} \cdot \left\lfloor \frac{(1/\lambda^{17\delta})}{3\log(1/\lambda)} \right\rfloor \right\}.$$

As $\lambda \to 0$, the above tends to zero faster than any positive power of λ.

In order to bound the probability of E_2, we first note that E_2 occurs if and only if there exists a vertex $u \in \partial\widehat{\mathcal{T}}_M$, time instants $s < s' < t$ with $s \le t^*$ and an infection path $\gamma : [s,t] \to \widehat{\mathcal{T}}_M$ such that

$$\begin{aligned}
&\gamma(r) = o \text{ for all } r \in [s, s'),\\
&\gamma(r) \in \widehat{\mathcal{T}}_M \backslash \left(\{o\} \cup \partial\widehat{\mathcal{T}}_M \right) \text{ for all } r \in [s', t), \text{ and}\\
&\gamma(t) = u.
\end{aligned}$$

In particular, after γ jumps away from o, it stays in one of the subtrees that descends from o (the one that contains u), which is a subgraph in which all degrees are bounded by $1/(8\lambda^2)$. Due to this observation, we can use (5.25) to bound

$$\mathbb{P}(E_2 \mid \deg(o) = d) \leq \mathbb{E}\left[\sum_{k=1}^{\infty} \sum_{u \in \partial\widehat{\mathcal{T}}_{M,k}} (t^* + 1) \cdot (2\lambda)^k \,\middle|\, \deg(o) = d\right]$$

$$\leq (t^* + 1) \sum_{k=1}^{\infty} (2\lambda)^k \cdot \mathbb{E}\left[|\partial\widehat{\mathcal{T}}_{M,k}| \,\middle|\, \deg(o) = d\right].$$

Again letting $m_M := \sum_{j \leq M} jp(j)$, this is at most

$$(t^* + 1) \sum_{k=1}^{\infty} (2\lambda)^k \cdot d \cdot (m_M)^{k-1} \cdot p([M, \infty))$$

$$= (t^* + 1) \cdot 2\lambda \cdot d \cdot p([M, \infty)) \cdot \sum_{k=1}^{\infty} (2\lambda m_M)^{k-1}.$$

Bounding λm_M as in (5.40), the sum converges if λ is small. Using $t^* + 1 \leq 2t^* = 2(1/\lambda)^{17\delta}$ (for λ small, depending on δ), $d \leq \delta \frac{1}{\lambda^2} \log \frac{1}{\lambda}$ and the bound $p([M, \infty)) \leq C\lambda^{2\alpha - 2}$ from (5.39), this gives

$$\mathbb{P}(E_2 \mid \deg(o) = d) \leq C\lambda^{2\alpha - 3 - 17\delta} \cdot \log \frac{1}{\lambda}.$$

We note that $2\alpha - 3 - 17\delta > 0$ if $\delta < (2\alpha - 3)/17$.

Next, using (5.26), we bound

$$\mathbb{P}(E_3 \mid \deg(o) = d)$$

$$\leq \mathbb{E}\left[\sum_{u \in \widehat{\mathcal{T}}_M : \mathrm{dist}(o,u) \geq 2} (t^* + 1) \cdot (2\lambda)^{2\mathrm{dist}(o,u)} \,\middle|\, \deg(o) = d\right]$$

$$= (t^* + 1) \sum_{k=2}^{\infty} (2\lambda)^{2k} \cdot \mathbb{E}\left[|\{u \in \widehat{\mathcal{T}}_M : \mathrm{dist}(o,u) = k\}| \,\middle|\, \deg(o) = d\right].$$

For each $k \geq 2$, the expectation on the right-hand side equals $d(m_M)^{k-1}$, so the right-hand side is smaller than

$$(t^* + 1) \cdot d \cdot 4\lambda^2 \sum_{k=2}^{\infty} (4\lambda^2 m_M)^{k-1} \leq (t^* + 1) \cdot d \cdot 4\lambda^2 \cdot \frac{4\lambda^2 m_M}{1 - 4\lambda^2 m_M}.$$

By bounding $\lambda m_M \leq 1$ and bounding $t^* + 1$ and using the choice of d as before, this is smaller than

$$C\lambda^{1 - 17\delta} \cdot \log \frac{1}{\lambda},$$

which is smaller than a positive power of λ if $\delta < 1/17$. This completes the proof. $\square$

5.3.5 Case $\alpha > 2$

Again, we state a proposition which handles the case where the degree of the root is large, use this statement to prove the upper bound in the theorem, and then turn to the proof of the proposition.

Proposition 5.13. *Let $\mathcal{T}$ be a BGW tree with root o and offspring distribution p satisfying* (5.1) *and* (5.2) *with $\alpha > 2$. Then, there exist $\delta > 0$ and $\varepsilon > 0$ such that, for λ small enough,*

$$d \le \delta \frac{1}{\lambda^2}\left(\log \frac{1}{\lambda}\right)^2 \quad \Longrightarrow \quad \mathbb{P}(\xi_t^o \neq \varnothing \ \forall t \mid \deg(o) = d) < \lambda^\varepsilon.$$

Proof of Theorem 5.1, upper bound for $\alpha > 2$. We take

$$M = \frac{1}{8\lambda^2} \quad \text{and} \quad M' = \delta\left(\frac{1}{\lambda}\log\frac{1}{\lambda}\right)^2$$

with δ as in Proposition 5.13, and bound, using Lemma 5.11,

$$\begin{aligned}\mathbb{P}(\xi_t^o \neq \varnothing \ \forall t) \le &\mathbb{P}(\deg(o) > M') \\ &+ \mathbb{P}(\{\deg(o) \le M\} \cap \mathrm{Escape}(M)) \\ &+ \mathbb{P}(\{M < \deg(o) \le M'\} \cap \{\xi_t^o \neq \varnothing \ \forall t\}).\end{aligned} \tag{5.42}$$

By (5.34), we have

$$\mathbb{P}(\deg(o) > M') \le C\left(\frac{\lambda}{\log(1/\lambda)}\right)^{2\alpha-2}.$$

Next, by (5.33) and (5.35), when λ is small enough we have

$$\mathbb{P}(\{\deg(o) \le M\} \cap \mathrm{Escape}(M)) \le 4\lambda \textstyle\sum_{j=1}^\infty jp(j) \cdot p([M,\infty)) \overset{(5.34)}{\le} C\lambda^{2\alpha-1}.$$

Finally, by Proposition 5.13,

$$\mathbb{P}(\{M < \deg(o) \le M'\} \cap \{\xi_t^o \neq \varnothing \ \forall t\}) \le \lambda^\varepsilon \mathbb{P}(\deg(o) > M) \overset{(5.34)}{\le} C\lambda^{2\alpha-2+\varepsilon}.$$

We have thus proved that the three terms in the right-hand side of (5.42) are smaller than $C(\frac{\lambda}{\log(1/\lambda)})^{2\alpha-2}$, completing the proof. □

This time, the proof of the proposition will require some preliminary results, which we now state and prove.

Lemma 5.14. *Assume that $\lambda < 1/4$. Let T be a finite tree with root o, and assume that all vertices of T, apart possibly from o, have degree smaller than $\frac{1}{8\lambda^2}$. Let*

$$\nu := \sum_{i \ge 2} (2\lambda)^{2i} \cdot |\{v : \mathrm{dist}(o,v) = i\}|.$$

Then, for any $t > 0$,

$$\mathbb{P}\left(\tau_T \leq 3\log\frac{1}{\lambda} + 2t\right)$$
$$\geq e^{-16\lambda^2 \deg(o)}\left(1 - |T|^2 e^{-t/4}\right)^2 \left(1 - \nu \cdot \left(3\log\frac{1}{\lambda} + 2t + 1\right)\right). \quad (5.43)$$

Proof. Fix $t > 0$, and define

$$t_1 := t, \qquad t_2 := t_1 + 3\log\tfrac{1}{\lambda}, \qquad t_3 := t_2 + t.$$

Letting S denote the subgraph induced by o and its neighbors, define the events

$$E_1 := \{\text{any infection path from } T \times \{0\} \text{ to } T \times \{t_1\} \text{ visits } o\},$$

$$E_2 := \{\text{any infection path from } T \times \{t_1\} \text{ to } T \times \{t_2\} \text{ visits } S^c\},$$

$$E_3 := \{\text{any infection path from } T \times \{t_2\} \text{ to } T \times \{t_3\} \text{ visits } o\}.$$

Also defining

$$E_4 := \left\{\begin{array}{l} \text{there is no infection path } \gamma \text{ from } T \times \{0\} \\ \text{to } T \times \{t_3\} \text{ for which there exist } s_1 < s_2 < s_3 \\ \text{such that } \gamma(s_1) = o,\ \gamma(s_2) \in S^c,\ \gamma(s_3) = o. \end{array}\right\},$$

it is clear that $\{\tau_T \leq t_3\} \supseteq E_1 \cap E_2 \cap E_3 \cap E_4$. We will now give a lower bound for the probability of the intersection of the four events. We first give individual lower bounds for each of them.

Let T' denote the graph obtained by removing o (and all the edges that contain o) from T. Then, T' is a (non-connected) graph in which all vertices have degree bounded by $1/(8\lambda^2)$. Then,

$$\mathbb{P}(E_1) = \mathbb{P}(E_3) = \mathbb{P}(\tau_{T'} \geq t) \geq 1 - |T'|^2 \cdot e^{-t/4} \geq 1 - |T|^2 \cdot e^{-t/4},$$

where the first inequality follows from Corollary 5.7. Next,

$$\mathbb{P}(E_2) > \mathbb{P}(\tau_S < 3\log\tfrac{1}{\lambda}) > e^{-16\lambda^2 \deg(o)},$$

where the second inequality follows from (2.26). To deal with E_4, we start with a union bound,

$$\mathbb{P}(E_4) > 1 - \sum_{v:\mathrm{dist}(o,v)\geq 2} \mathbb{P}\left(\begin{array}{l} \exists s_1 < s_2 < s_3 \leq t_3 : \\ (o, s_1) \rightsquigarrow (v, s_2) \rightsquigarrow (o, s_3) \end{array}\right).$$

By (5.26), this is larger than

$$1 - \sum_{v:\mathrm{dist}(o,v)\geq 2} (t_3 + 1) \cdot (2\lambda)^{2\mathrm{dist}(o,v)} = 1 - \nu \cdot (t_3 + 1).$$

Now consider the partial order on graphical constructions as in the proof of Corollary 5.8. The four events E_1, E_2, E_3, E_4 are all non-increasing with respect to this partial order, so by the FKG inequality, we obtain

$$\mathbb{P}(\cap_{i=1}^4 E_i) \geq \prod_{i=1}^4 \mathbb{P}(E_i).$$

Combining this with the bounds obtained above, the proof is complete. $\square$

Lemma 5.15. *If $\lambda > 0$ is small enough, the following holds. Let T be a finite tree with root o. Assume that all vertices of T, apart possibly from o, have degree smaller than $1/(8\lambda^2)$, and that $|T| \leq 1/\lambda^3$. Then,*

$$\mathbb{P}\left(\tau_T \leq 59 \log \frac{1}{\lambda}\right) > \frac{1}{2} e^{-16\lambda^2 \deg(o)}.$$

In particular, for any $t > 0$,

$$\mathbb{P}(\tau_T > t) \leq \left(1 - \frac{1}{2} e^{-16\lambda^2 \deg(o)}\right)^{\lfloor t/(59 \log(1/\lambda)) \rfloor}. \tag{5.44}$$

Proof. We apply Lemma 5.14 with $t = 28 \log \frac{1}{\lambda}$ (so that $3 \log \frac{1}{\lambda} + 2t = 59 \log \frac{1}{\lambda}$). Note that

$$|T|^2 \cdot e^{-t/4} \leq \frac{1}{\lambda^6} \cdot e^{-7 \log \frac{1}{\lambda}} = \lambda$$

and

$$\begin{aligned} \nu \cdot \left(3 \log \frac{1}{\lambda} + 2t + 1\right) &\leq (2\lambda)^4 \cdot |T| \cdot \left(59 \log \frac{1}{\lambda} + 1\right) \\ &\leq 16\lambda \left(59 \log \frac{1}{\lambda} + 1\right). \end{aligned}$$

Hence, when λ is small enough, we have

$$\left(1 - |T|^2 e^{-t/4}\right)^2 \left(1 - \nu \cdot \left(3 \log \frac{1}{\lambda} + 2t + 1\right)\right) > \frac{1}{2},$$

so the desired inequality follows from (5.43). $\square$

Proof of Proposition 5.13. This time, we do not use the truncated tree $\widehat{\mathcal{T}}_M$ or the escape event.

Let $\delta > 0$ and $\eta > 0$ be small constants to be chosen later: η will be chosen first, and the choice of δ will depend on it. Again, the idea is that after these constants are chosen, we take λ small (depending on δ and η). Fix $d \leq \delta \frac{1}{\lambda^2} (\log \frac{1}{\lambda})^2$. We will condition on $\deg(o) = d$.

Define

$$t^* := \left(\frac{1}{\lambda}\right)^{17\delta \log(1/\lambda)}$$

and the two bad events

$$E_1 := \left\{ \begin{array}{l} \text{there is an infection path } \gamma : [0, t^*] \to \mathcal{T} \\ \text{with } \gamma(0) = o \text{ and } \operatorname{dist}(o, \gamma(t)) \le \eta \log \frac{1}{\lambda} \text{ for all } t \end{array} \right\},$$

$$E_2 := \{\{o\} \times [0, t^*] \rightsquigarrow \{v \in \mathcal{T} : \operatorname{dist}(o, v) = \lfloor \eta \log \tfrac{1}{\lambda} \rfloor\} \times [0, \infty)\},$$

so that we clearly have $\{\xi_t^o \neq \varnothing \ \forall t\} \subseteq E_1 \cup E_2$.

In order to bound the probability of $E_1 \cup E_2$, it is helpful to define further bad events, which only involve the BGW tree $\mathcal{T}$:

$\mathcal{B}_1 := \{$there are more than $\frac{1}{\lambda^3}$ vertices within distance $\eta \log \frac{1}{\lambda}$ of $o\}$,
$\mathcal{B}_2 := \{$there is a vertex $v \in \mathcal{T} \backslash \{o\}$: $\operatorname{dist}(v, o) \le \eta \log \frac{1}{\lambda}$ and $\deg(v) > \frac{1}{8\lambda^2}\}$.

We will bound:

$$\begin{aligned} \mathbb{P}(\xi_t^o \neq \varnothing \ \forall t \mid \deg(o) = d) \le \mathbb{P}(\mathcal{B}_1 \mid \deg(o) = d) + \mathbb{P}(\mathcal{B}_2 \mid \deg(o) = d) \\ + \mathbb{P}(E_1 \mid \{\deg(o) = d\} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c) \\ + \mathbb{P}(E_2 \mid \{\deg(o) = d\} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c). \end{aligned} \tag{5.45}$$

Let

$$m := \sum_{j=1}^{\infty} j p(j).$$

Conditionally on $\{\deg(o) = d\}$, the expected number of vertices within distance $\eta \log \frac{1}{\lambda}$ of o is smaller than

$$1 + d \sum_{k=1}^{\lceil \eta \log(1/\lambda) \rceil} m^{k-1} \le 2d m^{2\eta \log(1/\lambda)} \le 2\delta \frac{1}{\lambda^{2+2\eta \log m}} \left(\log \frac{1}{\lambda}\right)^2, \tag{5.46}$$

where the first inequality holds for λ small, and the second inequality follows from the choice of d. Hence, Markov's inequality gives

$$\mathbb{P}(\mathcal{B}_1 \mid \deg(o) = d) \le 2\delta \frac{1}{\lambda^{2+2\eta \log m}} \left(\log \frac{1}{\lambda}\right)^2 \cdot \lambda^3.$$

If $\eta < \frac{1}{2 \log m}$, the right-hand side tends to zero faster than a positive power of λ as $\lambda \to 0$.

Next, conditionally on $\{\deg(o) = d\}$, the expected number of vertices of degree above $1/(8\lambda^2)$ within distance $\eta \log \frac{1}{\lambda}$ of o is smaller than

$$d \sum_{k=1}^{\lceil \eta \log(1/\lambda) \rceil} m^{k-1} \cdot p([1/(8\lambda^2), \infty)) \overset{(5.39),(5.46)}{\leq} C \frac{1}{\lambda^{2+2\eta \log m}} \left(\log \frac{1}{\lambda} \right)^2 \cdot \lambda^{2\alpha - 2}.$$

This expectation is an upper bound for $\mathbb{P}(\mathcal{B}_2 \mid \deg(o) = d)$:

$$\mathbb{P}(\mathcal{B}_2 \mid \deg(o) = d) \leq C \left(\log \frac{1}{\lambda} \right)^2 \cdot \lambda^{2\alpha - 4 - 2\eta \log m}.$$

Since $\alpha > 2$, we have $2\alpha - 4 > 0$, so, if $\eta < \frac{2\alpha - 4}{2 \log m}$, the right-hand side tends to zero faster than a positive power of λ as $\lambda \to 0$.

We can now choose η satisfying

$$\eta \leq \frac{\min(1,\ 2\alpha - 4)}{2 \log m}.$$

On the event $\mathcal{B}_1^c \cap \mathcal{B}_2^c$, the assumptions of Lemma 5.15 are satisfied, so by (5.44) we have

$$\begin{aligned} &\mathbb{P}(E_1 \mid \{\deg(o) = d\} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c) \\ &\quad \leq \exp\left\{ -\frac{1}{2} e^{-16\lambda^2 d} \cdot \left\lfloor \frac{t^*}{59 \log(1/\lambda)} \right\rfloor \right\} \\ &\quad \leq \exp\left\{ -\frac{1}{2} e^{-16\delta(\log(1/\lambda))^2} \cdot \left\lfloor \frac{e^{17\delta(\log(1/\lambda))^2}}{59 \log(1/\lambda)} \right\rfloor \right\}. \end{aligned}$$

As $\lambda \to 0$, this tends to zero faster than any power of λ.

Finally, we turn to the fourth term in the right-hand side of (5.45). The event E_2 occurs if and only if there exist $u \in \mathcal{T}$ with $\mathrm{dist}(o, u) = \lfloor \eta \log \frac{1}{\lambda} \rfloor$, times $s < s' < t$ with $s \leq t^*$, and an infection path $\gamma : [s, t] \to \mathcal{T}$ such that

$$\begin{aligned} &\gamma(r) = o \text{ for } r \in [s, s'), \text{ and } \gamma(r) \neq o \text{ for } r \in [s', t], \\ &\mathrm{dist}(\gamma(r), o) < \lfloor \eta \log \tfrac{1}{\lambda} \rfloor \text{ for } r \in [s, t), \\ &\gamma(t) = u. \end{aligned}$$

Also, on $\mathcal{B}_1^c \cap \mathcal{B}_2^c$, there are fewer than $1/\lambda^3$ choices for u, and all non-root vertices within distance $\lfloor \eta \log \frac{1}{\lambda} \rfloor$ of o have degree at most $1/(8\lambda^2)$. Then, using (5.25) and a union bound over the choice of u gives

$$\mathbb{P}(E_2 \mid \{\deg(o) = d\} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c) \leq \frac{1}{\lambda^3} \cdot (t^* + 1)(2\lambda)^{\lfloor \eta \log(1/\lambda) \rfloor}.$$

Bounding $t^* + 1 \le 2t^*$, using $\lfloor \eta \log(1/\lambda) \rfloor \ge \frac{1}{2}\eta \log(1/\lambda)$ for λ small, and plugging in the choice of t^*, the right-hand side is smaller than

$$\frac{2}{\lambda^3} \cdot \exp\left\{ 17\delta \left(\log \frac{1}{\lambda} \right)^2 + \frac{\eta}{2} \log 2 \cdot \log \frac{1}{\lambda} - \frac{\eta}{2} \left(\log \frac{1}{\lambda} \right)^2 \right\}.$$

Now, we choose δ small so that $17\delta < \frac{\eta}{2}$, and then take λ small so that $\log(1/\lambda) \ll (\log(1/\lambda))^2$. With these choices, among the terms in the exponential, the last one is larger (in absolute value) than the sum of the first two. Then, the whole expression tends to zero, as $\lambda \to 0$, faster than any power of λ. $\square$

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