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A survey on cobordism of spaces and maps: smooth and singular cases

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Abstract. We present a survey on cobordism of spaces and maps in the smooth and the singular cases. The starting point of the theory of cobordism is the famous Thom theorem saying that a space is null cobordant if and only if all of its Stiefel-Whitney numbers vanish. Stong extended the theory to maps, defining notions of cobordism of maps and Stiefel-Whitney numbers for maps; then he proved a similar result than the Thom one in this situation. Later on, Goresky and Pardon proved a Thom theorem for locally orientable $\mathbb{Z}_2$-Witt (singular) spaces using characteristic classes, as described by Wu, instead of Stiefel-Whitney classes, and in the framework of intersection homology.

Our aim is to introduce in the most elementary way the different notions which are involved in cobordism theory, and to provide to the reader suitable references concerning the Stiefel-Whitney classes and Wu classes, going to the discovery of the cobordism world. A general reference for Stiefel-Whitney classes, Steenrod squares and Wu classes is the Milnor's book on characteristic classes. Of course this survey does not intend to be comprehensive.

Keywords. Cobordism of spaces and maps, Stiefel-Whitney numbers.

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We would like to dedicate this paper to Wu Wen Tsün (in english: Wu Wenjun) who passed away on 7th May 2017, 5 days before getting 98. The first author had the pleasure to meet Wu Wen Tsün at the Chinese Academy of Sciences, in Beijing, in 1992 and to have a very interesting discussion about characteristic classes

Contents

1 Introduction

R. Thom was one of the first to define cobordism of smooth manifolds and he determined the non-oriented cobordism algebra. Many authors worked on the subject, in particular Stong introduced a notion of cobordism for maps $f : X \to Y$ of closed smooth manifolds and defined Stiefel-Whitney numbers for such a map and proved that two maps are cobordant if and only if they have the same characteristic numbers.

Goresky and Pardon discussed four classes of singular spaces for which they constructed characteristic classes such that the respective characteristic numbers determine the cobordism groups. Siegel described the class of $\mathbb{Q}$-Witt spaces and computed the cobordism groups of such spaces, showing that in non trivial cases they are equal to the Witt group.

Characteristic classes of smooth manifolds are defined in usual cohomology groups on which there is a multiplicative structure given by the cup-product. On the other hand, characteristic classes of singular varieties do not exist in cohomology, they lie in homology where there is no product structure. In order to recover a product and define characteristic numbers, one has to consider intersection homology. In general, characteristic classes cannot be lifted to intersection homology and maps $f : X \to Y$ do not provide homomorphism of intersection homology. Fortunately the characteristic classes we consider lie in intersection homology and there are classes of maps, such as normally nonsingular maps or placid maps which provide well-defined homomorphisms in intersection homology. But, unlike homology theories, intersection homology is not homotopy invariant and does not, in general, satisfy the universal coefficient theorem. In this case, the Witt spaces provide an important class of examples defined by a relatively tractable condition concerning intersection homology.

Our aim in this survey, is to present some results on cobordism of maps, by considering pseudomanifolds X and Y which are compact locally orientable Witt spaces [5, 6].

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2 Cobordism of spaces - The smooth case

All authors agree on the fact that the starting point of cobordism theory relies on the Henri Poincaré work [21, §5] where he defines a notion of homology which is basically the one of cobordism [9, p.289]. Bordism was then explicitly introduced by Lev Pontryagin [22] showing that the characteristic numbers of a closed manifold vanish if the manifold is a boundary. The major development of the theory is the paper of René Thom [32] who showed that cobordism groups could be computed by means of homotopy theory, via the Thom construction.

2.1 Cobordism of spaces

One knows that an n-dimensional manifold M is a topological space locally homeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$.

A manifold with boundary is similar, except that a point of M is allowed to have a neighborhood that is homeomorphic to the half-space $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\}$. The boundary of M, denoted by ∂M, is the set of points whose neighborhood is homeomorphic to the half-space. A closed manifold is a compact manifold with empty boundary.

A cobordism between two manifolds M and N is a compact manifold W whose boundary is the disjoint union of M and N, *i.e.* $\partial W = M \sqcup N$.

Two manifolds M and N are called cobordant if such a cobordism exists. All manifolds cobordant to a fixed given manifold M form the cobordism class of M. Cobordism is a fundamental equivalence relation on the class of compact manifolds of the same dimension.

Note that every closed manifold M is the boundary of the non-compact manifold $M \times [0, 1)$; that is the reason for which one has to require W to be compact in the definition of cobordism.

The set of cobordism classes of closed unoriented n-dimensional manifolds is denoted by $\mathfrak{N}_n$; it is an abelian group whose operation is the disjoint union. More specifically, if $[M]$ and $[N]$ denote the cobordism classes of the manifolds M and N respectively, we define $[M] + [N] = [M \sqcup N]$; this is a well-defined operation which turns $\mathfrak{N}_n$ into an abelian group. The identity element of this group is the class $[\emptyset]$ consisting of all closed n-manifolds which are boundaries. Further we have $[M] + [M] = [\emptyset]$ for every M since $M \sqcup M = \partial(M \times [0, 1])$. Therefore, $\mathfrak{N}_n$ is a vector space over $\mathbb{Z}_2$. The cartesian product of manifolds defines a multiplication $[M][N] = [M \times N]$, so $\mathfrak{N}_* = \sum_{n \geqslant 0} \mathfrak{N}_n$ is a graded algebra, with the grading given by the dimension.

The above is the most basic form of the definition. It is also referred to as unoriented bordism. In many situations, the manifolds in question are oriented, or carry some other additional structure referred to as

G-structure. This gives rise to "cobordism with G-structure". Examples include $G = O(k)$, the orthogonal group, giving back the unoriented cobordism, the subgroup $SO(k)$, giving rise to oriented cobordism, the spin group, the unitary group $U(k)$, the trivial group, giving rise to framed cobordism *etc...* In this survey we will deal with unoriented cobordism only. Readers interested in other theories will find complete information in the book of Stong [29], for instance.

2.2 Stiefel-Whitney classes

2.2.1 Obstruction theory

The Stiefel-Whitney classes have been firstly defined by Eduard Stiefel [27] and Hassler Whitney [33, 34] in the context of obstruction theory. If E is a (real) vector bundle over a triangulated space X with n-dimensional fiber, obstruction theory goes as follows: for fixed r, $0 < r \leq n$, aim is to construct r everywhere independent sections of E. That is performed by induction on the dimension of the skeleton of a triangulation K (or a cell decomposition) of X. Note that X does not need to be smooth and can be a CW-complex. On the 0-skeleton of K, the construction is obvious. Let us assume that one has constructed such an r-frame over the $(p-1)$-skeleton and let us consider a p-dimensional simplex σ of K. The r-frame, constructed on $\partial\sigma$ defines a map

$$\gamma_\sigma : \partial\sigma \cong \mathbb{S}^{p-1} \to V_r(\mathbb{R}^n)$$

where $V_r(\mathbb{R}^n)$ is the Stiefel manifold of r-linearly independent vectors in the fibre $F \cong \mathbb{R}^n$ of E. The map γ_σ defines an element $[\gamma_\sigma]$ of the homotopy group $\pi_{p-1}(V_r(\mathbb{R}^n))$ which has been computed as (see [25]):

$$\pi_{p-1}(V_r(\mathbb{R}^n)) = \begin{cases} 0 & \text{for } p < n-r+1, \\ \mathbb{Z} & \text{for } p = n-r+1 \text{ odd or } p = n, \\ \mathbb{Z}_2 & \text{for } p = n-r+1 \text{ even and } p < n. \end{cases} \tag{2.1}$$

Associating to each simplex σ the element $[\gamma_\sigma]$ defines a class $\widehat{w}_p(E)$ in the p-th simplicial (or cellular) cohomology of X with twisted coefficients, the coefficient system being the homotopy group $\pi_{p-1}(V_r(\mathbb{R}^n))$:

$$\widehat{w}_p(E) \in H^p(X; \pi_{p-1}(V_r(\mathbb{R}^n))).$$

Whitney proved that $\widehat{w}_p(E) = 0$ if and only if E, when restricted to the p-th skeleton of X, admits $r = (n-p+1)$ linearly-independent sections.

Since $\pi_{p-1}(V_r(\mathbb{R}^n))$ is either infinite-cyclic or isomorphic to $\mathbb{Z}_2$, there is a canonical reduction of the classes $\widehat{w}_p(E)$ to classes $w_p(E) \in H^p(X; \mathbb{Z}_2)$ which are the Stiefel-Whitney classes.

By definition $w_0(E) = 1$. One notes that $w_1(E) = 0$ if and only if the bundle E is orientable.

2.2.2 Axiomatic definition

The Stiefel-Whitney characteristic classes can also be defined in an axiomatic way as follows (see [19] for example): The (total) Stiefel-Whitney characteristic class $w(E) \in H^*(X; \mathbb{Z}_2)$ of a finite rank real vector bundle E on a paracompact base space X is defined as the unique class such that the following axioms are fulfilled:

1. Normalization: The Whitney class of the tautological line bundle γ_1^1 over the real projective space $\mathbb{P}^1(\mathbb{R})$ is nontrivial, *i.e.* $w(\gamma_1^1) = 1 + a \in H^*(\mathbb{P}^1(\mathbb{R}); \mathbb{Z}_2)$.

2. $w_0(E) = 1 \in H_0(X; \mathbb{Z}_2)$, and for $i >$ rank of E, $w_i(E) = 0 \in H^i(X; \mathbb{Z}_2)$.

3. Whitney product formula: $w(E \oplus F) = w(E) \smile w(F)$ (cup-product in cohomology).

4. Naturality: $w(f^*E) = f^*w(E)$ for any real vector bundle E over X and map $f : Y \to X$, where f^*E denotes the pullback vector bundle.

We notice that the "Whitney product formula" has been firstly provided by H. Whitney [34], but that a simpler proof has been given by Wu Wen Tsün [36].

The Stiefel-Whitney classes of the tangent bundle $E = TM$ of a smooth manifold M are called the Stiefel-Whitney classes of the manifold and denoted by $w_i(M)$, or w_i if there is no ambiguity.

2.2.3 Homology Stiefel-Whitney classes

In the case of singular varieties, there is no more tangent bundle. One cannot use the construction of cohomological Stiefel-Whitney class by obstruction theory to define a Stiefel-Whitney class of a singular variety. However, there is a definition of homological Stiefel-Whitney class valuable for particular varieties as well.

Let us consider a (locally finite) n-dimensional polyedron X. Given a triangulation K of X, one denote by K' its first barycentric subdivision. Each vertex a in K' is the barycenter of a simplex in K whose dimension is denoted by $|a|$. In that way, inclusion of simplices in K provides an order on vertices in K'.

For $i = 0, \dots, n$, one considers the following chain of K':

$$\widehat{c}_i = \sum_{a_0 < \cdots < a_i} (-1)^{|a_0|+\cdots+|a_i|} \langle a_0 \cdots a_i \rangle \in C_i(K';\mathbb{Z}), \tag{2.2}$$

which is the sum (with appropriate sign) of all i-dimensional simplices $\langle a_0 \cdots a_i \rangle$ in K'.

The chain $\widehat{c}_i$ can be infinite if X is not compact. If X is compact and connected, then $\widehat{c}_0$ is an integral cycle whose homology class represents the Euler-Poincaré characteristic $\chi(X)$.

Let M an n-dimensional manifold without boundary, for $0 \leq i < n$, the (infinite) chain $\widehat{c}_i$ is a cycle, integral if $n - i$ is odd or $i = 0$, and (mod 2) if $n - i$ is even.

Theorem 2.1 ([16]). *Let M an n-dimensional manifold without boundary, for $0 < p \leq n$, then the homology class of the cycle $\widehat{c}_{n-p}$, denoted by $\underline{\widehat{w}}_{n-p}(M)$, satisfies*

- **i.** $\underline{\widehat{w}}_{n-p}(M) \in H_{n-p}(M;\mathbb{Z}_2)$ *if p is odd or $p = n$.*
- **ii.** $\underline{\widehat{w}}_{n-p}(M) \in H_{n-p}(M;\mathbb{Z}_2)$ *if p is even and $p < n$.*

It is also Poincaré dual of the cohomology class $\widehat{w}_p(TM)$:

$$\underline{\widehat{w}}_{n-p}(M) = \widehat{w}_p(TM) \cap [M].$$

The theorem was conjectured by Stiefel [27] and proved by Whitney (1940, not published) and Cheeger [8] (1968, sketch of proof). A complete proof has been provided by S. Halperin and D. Toledo [16].

Coming back to the case of an n-dimensional polyedron X and using mod 2 coefficients, the following result is a motivation for the forthcoming definition 2.4.

Proposition 2.2 ([30]). *The chain*

$$c_i = \sum_{a_0 < \cdots < a_i} \langle a_0 \cdots a_i \rangle \in C_i(K';\mathbb{Z}_2), \tag{2.3}$$

sum of all i-dimensional simplices in K' is a mod *2 cycle if and only if, for all $x \in X$, one has:*

$$\chi(X, X \setminus \{x\}) \equiv 1 \ (mod\ 2). \tag{2.4}$$

Here the relative Euler-Poincaré characteristic $\chi(X, X \setminus \{x\})$ is defined as

$$\chi(X, X \setminus \{x\}) = \sum_{i=0}^{n} \dim H^i(X, X \setminus \{x\}; \mathbb{Z}_2).$$

The condition 2.4 is verified by manifolds and one has the following corollary of Theorem 2.1, with mod 2 coefficients:

Corollary 2.3. *If M is an n-dimensional compact manifold, then the Stiefel homology classes $\underline{w}_{n-p}(M)$ of the cycles c_{n-p} are dual to the Stiefel-Whitney cohomology classes $w_p(M)$, by Poincaré duality isomorphism*

$$H^p(M;\mathbb{Z}_2) \to H_{n-p}(M;\mathbb{Z}_2),$$

that is $\underline{w}_{n-p}(M) = w_p(M) \cap [M]$.

One will see later on (Theorem 4.1) a generalization of the corollary in the case of $\mathbb{Z}_2$-homology manifolds (Definition 2.4).

2.2.4 Euler spaces and Homology Manifolds

Proposition 2.2 motivates the following definitions:

An integral Euler space is a topological space X such that, for all $x \in X$, one has

$$\chi(X, X \setminus \{x\}) = (-1)^n. \tag{2.5}$$

That is equivalent to say that every $x \in X$ admits a neighborhood homeomorphic to a cone over a polyhedron (the link of x) whose Euler-Poincaré characteristic is even [30].

Examples of integral Euler spaces are given by complex analytic spaces (see [30]) and by integral homology manifolds.

An integral homology manifold is a locally compact topological space X such that, for all $x \in X$, the link of x has the same integer homology than an $(n-1)$-sphere, or equivalently:

$$H_p(X, X \setminus \{x\}; \mathbb{Z}) = \begin{cases} 0 & \text{if } p \neq n, \\ \mathbb{Z} & \text{if } p = n. \end{cases}$$

In the following, we will use similar definitions in the case of $\mathbb{Z}_2$ coefficients:

Definition 2.4. A mod 2-Euler space is a topological space X such that, for all $x \in X$, one has (see 2.4):

$$\chi(X, X \setminus \{x\}) \equiv 1 \text{ (mod2)}.$$

Examples of mod 2-Euler spaces are given by real analytic spaces (see [30]) and by $\mathbb{Z}_2$-homology manifolds:

A $\mathbb{Z}_2$-homology manifold is a locally compact topological space X such that, for all $x \in X$, the link of x has the same $\mathbb{Z}_2$ homology than an $(n-1)$-sphere, or equivalently:

$$H_p(X, X \setminus \{x\}; \mathbb{Z}_2) = \begin{cases} 0 & \text{if } p \neq n, \\ \mathbb{Z}_2 & \text{if } p = n. \end{cases}$$

2.3 The Thom theorem

The Stiefel-Whitney numbers of an (unoriented) closed n-dimensional manifold M are defined as

$$\langle w_{i_1}(M) \cup \cdots \cup w_{i_k}(M), [M] \rangle \in \mathbb{Z}_2$$

for any collection $(i_1, \cdots, i_k)$ of k-uples of integers $i \neq 2^j - 1$ such that $i_1 + \cdots + i_k = n$.

By Poincaré duality, the cup-product (in cohomology) corresponds to the intersection product denoted by $\bullet$ (in homology):

$$\begin{array}{ccc} H^i(M;\mathbb{Z}_2) \times H^j(M;\mathbb{Z}_2) & \xrightarrow{\ \cup\ } & H^{i+j}(M;\mathbb{Z}_2) \quad , \\ \simeq \big\downarrow P_M \times P_M & & \simeq \big\downarrow P_M \\ H_{n-i}(M;\mathbb{Z}_2) \times H_{n-j}(M;\mathbb{Z}_2) & \xrightarrow{\ \bullet\ } & H_{n-i-j}(M;\mathbb{Z}_2) \end{array} \tag{2.6}$$

By Corollary 2.3, the Stiefel-Whitney numbers are then defined either using cohomology or homology classes:

$$\langle w_{i_1}(M) \cup \cdots \cup w_{i_k}(M), [M] \rangle = \varepsilon(\underline{w}_{n-i_1}(M) \bullet \cdots \bullet \underline{w}_{n-i_k}(M)) \in \mathbb{Z}_2,$$

where ε is the evaluation map $\varepsilon : H_0(M;\mathbb{Z}_2) \to \mathbb{Z}_2$.

These numbers are known to be cobordism invariants. It was proved by Lev Pontryagin that if M is the boundary of a smooth compact $(n+1)$-dimensional manifold, then the Stiefel-Whitney numbers of M are all zero. Later on, it was proved by René Thom that if all the Stiefel-Whitney numbers of M are zero then M can be realised as the boundary of some smooth compact manifold. So we have:

Theorem 2.5. *A smooth compact manifold M is the boundary of some smooth compact (unoriented) manifold if and only if all the Stiefel-Whitney numbers of M vanish.*

The cobordism class $[M] \in \mathfrak{N}_n$ of a closed unoriented n-dimensional manifold M is determined by the Stiefel-Whitney characteristic numbers of M, which depend on the stable isomorphism class of the tangent bundle. Thus if M has a stably trivial tangent bundle then $[M] = 0 \in \mathfrak{N}_n$. In 1954 René Thom proved that $\mathfrak{N}_*$ is:

$$\mathfrak{N}_* = \mathbb{Z}_2 \left[x_i \mid i \geqslant 1, i \neq 2^j - 1 \right].$$

the polynomial algebra with one generator x_i in each dimension $i \neq 2^j - 1$.

Thus two unoriented closed n-dimensional manifolds M, N are cobordant, $[M] = [N] \in \mathfrak{N}_n$, if and only if for each collection $(i_1, \cdots, i_k)$ of k-uples of integers $i \neq 2^j - 1$ such that $i_1 + \cdots + i_k = n$ the Stiefel-Whitney numbers are equal:

$$\langle w_{i_1}(M) \cup \cdots \cup w_{i_k}(M), [M] \rangle = \langle w_{i_1}(N) \cup \cdots \cup w_{i_k}(N), [N] \rangle \in \mathbb{Z}_2$$

with $w_i(M) \in H^i(M;\mathbb{Z}_2)$ the i-th Stiefel-Whitney class and $[M] \in H_n(M;\mathbb{Z}_2)$ the $\mathbb{Z}_2$-coefficient fundamental class. One has the same result using homology Stiefel-Whitney classes.

3 Cobordism of maps - The smooth case

3.1 Cobordism of maps

A map of dimension (m, n) is a triple (f, M, N) consisting of two closed differentiable manifolds M and N of dimensions m and n respectively and a continuous function $f : M \to N$.

Two maps (f, M, N) and (f', M', N') of dimension (m, n) will be said to be cobordant if there exists a triple (F, V, W) where:

1. V and W are compact differentiable manifolds with boundary of dimensions $m+1$ and $n+1$ respectively; with $\partial V = M \sqcup M', \partial W = N \sqcup N'$, and
2. $F : V \to W$ is a continuous function whose restriction to M is f and whose restriction to M' is f'.

The set of equivalence classes under this relation of maps of dimension (m, n) will be denoted $\mathfrak{N}_{(m,n)}$.

3.2 Stiefel-Whitney numbers of maps

3.2.1 The Gysin map

Let (f, M, N) be a map of dimension (m, n). One defines the Gysin map (sometimes called transfer, or Umkehrungs, or push-forward)

$$f_! : H^k(M;\mathbb{Z}_2) \to H^{k+n-m}(N;\mathbb{Z}_2),$$

by commutativity of the following diagram:

$$\begin{array}{ccc} H_{m-k}(M;\mathbb{Z}_2) & \xrightarrow{f_*} & H_{m-k}(N;\mathbb{Z}_2) \\ P_M \uparrow \simeq & & P_N \uparrow \simeq \\ H^k(M;\mathbb{Z}_2) & \xrightarrow{f_!} & H^{k+n-m}(N;\mathbb{Z}_2), \end{array} \tag{3.1}$$

where P_M and P_N are respective Poincaré duality isomorphisms. Note that, using $\mathbb{Z}_2$ coefficients, they are isomorphisms for closed but not necessarily oriented manifolds.

This map coincides with the "Umkehrungs" homomorphism used by Stong [28, p.255]: For $\alpha \in H^i(M;\mathbb{Z}_2)$, one considers the composition

$$H^{m-i}(N;\mathbb{Z}_2) \xrightarrow{f^*} H^{m-i}(M;\mathbb{Z}_2) \xrightarrow{\cup\alpha} H^m(M;\mathbb{Z}_2) \xrightarrow{[M]} \mathbb{Z}_2$$

and hence an element $f_!(\alpha) \in H^{n+i-m}(N;\mathbb{Z}_2)$. More precisely, for any $\alpha \in H^i(M;\mathbb{Z}_2)$ and $\beta \in H_{i+n-m}(N;\mathbb{Z}_2)$, one has $\langle f_!(\alpha),\beta\rangle = \langle f^*(\tilde{\beta}) \cup \alpha, [M]\rangle \in \mathbb{Z}_2$, where $\tilde{\beta} = P_N^{-1}(\beta) \in H^{m-i}(N;\mathbb{Z}_2)$ and $[M]$ is the $\mathbb{Z}_2$-fundamental class of M. Note that the homomorphism $f_!$ is only additive.

Remark 3.1. We will use an analogous description of the map $f_!$, due to Atiyah and Hirzebruch [1]: let us consider $h : M \to \mathbb{S}^s$ an imbedding of M in some s-dimensional sphere $\mathbb{S}^s$ and T a tubular neighborhood of $(f \times h)(M)$ in $N \times \mathbb{S}^s$, then $f_!$ is the composition of the maps:

$$\begin{array}{r} H^i(M;\mathbb{Z}_2) \xrightarrow{\varphi} H^{i+s+n-m}(T/\partial T;\mathbb{Z}_2) \xrightarrow{c^*} H^{i+s+n-m}(N \times \mathbb{S}^s;\mathbb{Z}_2) \\ \downarrow \quad\quad \\ H^{i+n-m}(N;\mathbb{Z}_2), \end{array}$$

where φ denotes the Thom isomorphism, the map $c : N\times\mathbb{S}^s \to T/\partial T$ is the collapsing map and the last map is the projection (denoted as isomorphism in Stong [28, p.255]):

$$\begin{array}{r} H^{i+s+n-m}(N \times \mathbb{S}^s;\mathbb{Z}_2) \cong H^{i+n-m}(N;\mathbb{Z}_2) \oplus H^{i+s+n-m}(N;\mathbb{Z}_2) \\ \downarrow \quad\quad \\ H^{i+n-m}(N;\mathbb{Z}_2). \end{array}$$

3.2.2 Stiefel-Whitney numbers of maps

Let us consider a map (f, M, N) of dimension (m, n). For any collection of partitions $\omega, \omega_1, \cdots \omega_r$ such that $|\omega| + |\omega_1| + \ldots + |\omega|_r + r(n-m) = n$ one may define a Stiefel-Whitney number of (f, M, N) by

$$\langle w_\omega(N) \cup f_!(w_{\omega_1}(M)) \cup \ldots \cup f_!(w_{\omega_r}(M)), [N]\rangle,$$

where for $\mu = (i_1, \ldots, i_s)$, w_μ is the product of Stiefel-Whitney classes $w_{i_1} \cdots w_{i_s}$.

The Stiefel-Whitney numbers of (f, M, N) can also be defined in homology, in the following way: for any collection of partitions $\omega, \omega_1, \cdots \omega_r$ such that $|\omega|+|\omega_1|+\ldots+|\omega_r|+r(n-m) = n$ one considers the intersection product

$$\underline{w}_{n-\omega}(N) \bullet f_*(\underline{w}_{n-\omega_1}(M)) \bullet \ldots \bullet f_*(\underline{w}_{n-\omega_r}(M)) \in H_0(N;\mathbb{Z}_2).$$

By Corollary 2.3 and commutativity of the diagrams (2.6) and (3.1), one obtains equality:

$$\begin{aligned}&\langle w_\omega(N) \cup f_!(w_{\omega_1}(M)) \cup \ldots \cup f_!(w_{\omega_r}(M)), [N]\rangle \\ &= \varepsilon\left(\underline{w}_{n-\omega}(N) \bullet f_*(\underline{w}_{n-\omega_1}(M)) \bullet \ldots \bullet f_*(\underline{w}_{n-\omega_r}(M))\right)\end{aligned}$$

where $\varepsilon : H_0(N;\mathbb{Z}_2) \to \mathbb{Z}_2$ is the evaluation map.

3.3 The Stong theorem

Theorem 3.2 ([28, Theorem p.255]). *Two maps of dimension (m, n) are cobordant if and only if they have the same Stiefel-Whitney numbers.*

4 The Wu classes and the Steenrod square

Wu classes v_i are defined implicitly in terms of Steenrod squares, as the cohomology class representing the Steenrod squares. More precisely, if X is an n-dimensional $\mathbb{Z}_2$-homology manifold, for any cohomology class x of degree $n-i$, one has $v_i \cup x = Sq^i(x)$. The Stiefel-Whitney classes w_i are the Steenrod squares of the Wu classes v_i, defined by Wu Wen Tsün in [37]. Most simply, the total Stiefel-Whitney class w is the total Steenrod square of the total Wu class v: $Sq(v) = w$.

4.1 Steenrod squares

The i-products and in particular, the Steenrod squares of locally compact and paracompact spaces are defined by Steenrod in [24]. We will follow the presentation provided by Cartan in his famous Cartan seminar [7, exp. 14 and 15]. The name, Steenrod squares, comes from the fact that Sq^n restricted to classes of degree n is the cup (square) product.

In Cartan seminar a general multiplication, denoted by $\varphi : G\times G \to G'$, where G and G' are abelian groups, is considered. Here, we will use the groups of coefficients $G = G' = \mathbb{Z}_2$ and usual multiplication $\times$. The use of $\mathbb{Z}_2$ is motivated by the fact that for the usual multiplication in $\mathbb{Z}$, one has

$$\alpha \times \alpha \equiv \alpha \mod 2.$$

As the multiplication in $\mathbb{Z}_2$ is symmetric, the bilinear application denoted by $\overline{\varphi}$ in [7] and defined by $\overline{\varphi}(\alpha, \beta) = \varphi(\beta, \alpha)$ coincides with φ. Also the maps denoted by $\overline{p_i}$ in [7] coincide with the p_i (defined below).

Let X a locally compact and paracompact space. The Alexander-Spanier, or Čech-Alexander, cohomology of X is defined in the following way: the groups of r-cochains $C^r(X;\mathbb{Z}_2)$ is the set of functions $f : X^{r+1} \to \mathbb{Z}_2$ with $f(x_0, x_1, \ldots, x_r) \in \mathbb{Z}_2$, on which addition is the natural one.

The coboundary $\delta : C^r(X;\mathbb{Z}_2) \to C^{r+1}(X;\mathbb{Z}_2)$ is defined by

$$(\delta f)(x_0, x_1, \dots, x_{r+1}) = \sum_{i=0}^{r+1} (-1)^i f(x_0, x_1, \dots, \hat{x}_i \dots x_{r+1}).$$

There is a subcomplex C_0^* of C^* such that functions of C_0^r are those that vanish in a neighborhood of the diagonal $X \subset X^{r+1} = X \times \cdots \times X$. The Čech-Alexander cohomology groups of X are defined as the cohomology groups of the complex C^*/C_0^*. They coincide with simplicial or singular cohomology groups for locally finite complexes.

The product

$$p : C^r(X;\mathbb{Z}_2) \times C^s(X;\mathbb{Z}_2) \to C^{r+s}(X;\mathbb{Z}_2)$$

is defined, for two cochains f and g, by:

$$p(f,g)(x_0, x_1, \dots, x_{r+s}) = f(x_0, x_1, \dots, x_r) \times g(x_r, \dots, x_{r+s}).$$

On the one hand, the product passes to the quotient by the complex C_0^*, on the other hand, one has

$$\delta p(f,g) = p(\delta f, g) + (-1)^r p(f, \delta g). \tag{4.1}$$

The induced product in cohomology is the cup-product

$$H^r(X;\mathbb{Z}_2) \times H^s(X;\mathbb{Z}_2) \xrightarrow{\cup} H^{r+s}(X;\mathbb{Z}_2).$$

4.1.1 *i*-products

For all $i \in \mathbb{Z}$, we now define products

$$p_i : H^r(X;\mathbb{Z}_2) \times H^s(X;\mathbb{Z}_2) \to H^{r+s-i}(X;\mathbb{Z}_2).$$

First of all one defines $p_i = 0$ if $i < 0$ and $p_0 = p$. Before giving the precise definition, let us give an intuitive idea of what is $p_i(f,g)$ at the level of cochains: The product $p_i(f,g)(x_0, x_1, \dots, x_{r+s-i})$ will be a sum of elements of the type

$$\pm f(x_{i_0}, \dots, x_{i_r}) \times g(x_{j_0}, \dots, x_{j_s})$$

where the sequences $\{i_0, \dots, i_r\}$ and $\{j_0, \dots, j_s\}$ satisfy

$$i_0 < i_1 < \cdots < i_r, \quad j_0 < j_1 < \cdots < j_s,$$

and

$$\{i_0, \dots, i_r, j_0, \dots, j_s\} = \{0, 1, \cdots, r+s-i\}.$$

However, not all sequences satisfying these conditions appear in the sum.

Let us provide now the precise definition of $p_i(f,g)$. Let $a \in X$ be a fixed point, one defines the operator $T_a : C^r(X;\mathbb{Z}_2) \to C^{r-1}(X;\mathbb{Z}_2)$ by

$$T_a(f)(x_0, x_1, \ldots, x_{r-1}) = f(a, x_0, x_1, \ldots, x_{r-1}), \quad \text{for } r \geq 1,$$

and $T_a(f) = 0$ for $f \in C^0(X;\mathbb{Z}_2)$. One has $\delta T_a(f) + T_a(\delta f) = f$. The i-products are defined by $p_0(f,g) = p(f,g)$ and, for $i \geq 1$ and $f \in C^0(X;\mathbb{Z}_2)$, by $p_i(f,g) = 0$, and then by induction by the formula

$$T_a p_i(f,g) = p_i(T_a f, g) + (-1)^{r(s+1)} p_{i-1}(T_a g, T_a f).$$

Note that, as the product $\times$ is a bilinear symmetric map, one has $p_i = \overline{p}_i$ with the Cartan notations, that is the case "1)" in [7, §4].

The i-products verify the previous "intuitive idea", in particular, one has $p_i(f,g) = 0 \quad \text{if } i > \min(r,s)$ and

$$p_r(f,f)(x_0, x_1, \ldots, x_r) = f(x_0, x_1, \ldots, x_r) \times f(x_0, x_1, \ldots, x_r).$$

One has the fundamental formula for the coboundary

$$\begin{aligned} \delta p_i(f,g) = p_i(\delta f, g) &+ (-1)^r p_i(f, \delta g) \\ &+ (-1)^{r+s+i} p_{i-1}(f,g) + (-1)^{r+s+rs} p_{i-1}(g,f). \end{aligned} \tag{4.2}$$

which reduces to (4.1) for $i = 0$. That is the formula labelled as (I) in [7] with $p_i = \overline{p}_i$ (see above).

The formula (4.2) shows that the i-products are defined on the quotient in cohomology and they define maps

$$p_i : H^r(X;\mathbb{Z}_2) \times H^s(X;\mathbb{Z}_2) \to H^{r+s-i}(X;\mathbb{Z}_2).$$

The Steenrod squares are now defined by

$$Sq^i(f) = p_{r-i}(f,f), \qquad Sq^i : H^r(X;\mathbb{Z}_2) \to H^{r+i}(X;\mathbb{Z}_2).$$

4.1.2 Axiomatic definition

In 1962, Steenrod and Epstein [26] showed that the Steenrod squares can be defined in an axiomatic way. Namely, the Steenrod squares $Sq^i : H^k(X;\mathbb{Z}_2) \to H^{k+i}(X;\mathbb{Z}_2)$ are characterized by the following axioms :

1. Naturality: Sq^i is an additive homomorphism
$$H^k(X;\mathbb{Z}_2) \to H^{k+i}(X;\mathbb{Z}_2)$$
such that for any map $f : X \to Y$, then $f^*(Sq^i(x)) = Sq^i f^*(x)$.
2. $Sq^0 = id$.
3. Sq^n is the cup product $Sq^n(x) = x \cup x$ for $x \in H^n(X;\mathbb{Z}_2)$.
4. If $n > \deg(x)$, then $Sq^n(x) = 0$.
5. Cartan formula: $Sq^n(x \cup y) = \sum_{i+j=n} Sq^i(x) \cup Sq^j(y)$.

4.1.3 Geometrical interpretation

Geometrical interpretation of the Steenrod squares (coming from Thom) is the following: if M is an n-dimensional manifold, let us consider a proper immersion $f : V \to M$, where V is an $(n-d)$-dimensional manifold, with normal bundle ν. Let 1 be the unit class in $H^0(V;\mathbb{Z}_2)$ and consider the pushforward (3.2.1) $f_!(1) \in H^n(M;\mathbb{Z}_2)$, then $Sq^i(x) = f_!(w_i(\nu))$ (see below the Wu formula (4.5)).

Mosher and Tangora in [20] provide an introduction to Steenrod squares using Eilenberg-MacLane $K(\pi, n)$ spaces.

4.2 Stiefel-Whitney classes, Steenrod square and Thom class

Let E be an n-dimensional vector bundle over a paracompact space X with projection map $\pi : E \to X$. Let $E \setminus s_0(X)$ be the complement of the zero section in E. Then there exists a unique cohomology class $u_E \in H^n(E, E \setminus s_0(X);\mathbb{Z}_2)$, called the Thom class, such that $u_E|_{(F_x, F_x\setminus\{0\})} \neq 0$ for all fibers.

The Thom isomorphism

$$\varphi : H^i(X;\mathbb{Z}_2) \to H^{i+n}(E, E \setminus s_0(X);\mathbb{Z}_2)$$

is defined by $\varphi(x) = \pi^* x \cup u_E$. One has $\varphi(1) = u_E$.

Let $1 \in H^0(X;\mathbb{Z}_2)$, then the i-th Stiefel-Whitney class $w_i(E) \in H^i(X;\mathbb{Z}_2)$ is equal to

$$w_i(E) = \varphi^{-1} Sq^i \varphi(1).$$

That is

$$\pi^* w_i(E) \cup u_E = Sq^i(u_E), \qquad Sq^i u_E = \varphi(w_i(X)).$$

4.3 Wu classes

Let X be a topological space with fundamental class $[X] \in H_n(X;\mathbb{Z}_2)$ and such that one has Poincaré isomorphism $z \mapsto z \cap [X]$. That is the case of n-dimensional compact manifold and more generally $\mathbb{Z}_2$-homology manifolds (see §2.2.3). Via the Kronecker pairing

$$\langle \cdot, \cdot \rangle : H^i(X;\mathbb{Z}_2) \times H_i(X;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2.$$

one obtains an isomorphism

$$\mathrm{Hom}(H^{n-i}(X;\mathbb{Z}_2), \mathbb{Z}_2) \cong H_{n-i}(X;\mathbb{Z}_2) \cong H^i(X;\mathbb{Z}_2).$$

Under this isomorphism, the homomorphism $x \mapsto \langle Sq^i(x), [X] \rangle$ from $H^{n-i}(X; \mathbb{Z}_2)$ to $\mathbb{Z}_2$ corresponds to a well defined cohomology class $v_i(X) \in H^i(X; \mathbb{Z}_2)$, such that

$$Sq^i(x) = v_i \cup x, \qquad \text{for any} \quad x \in H_c^{n-i}(X; \mathbb{Z}_2) \tag{4.3}$$

(cohomology with compact supports).

The class $v_i(X)$ is called the i-th Wu class of X, denoted by v_i if there is no ambiguity. In the original Wu's papers [37, 35] the class was denoted by U^i (see also Milnor [19, §11]). One says that the Wu class v_i is the class that represents Sq^i under the cup product.

According to a result of Wu (Wu wrote that this result comes from Cartan), in the case of an n-dimensional orientable manifold, then $v_{2k+1} = 0$ for all k.

4.4 Siefel-Whitney classes and Wu classes

Let X be an n-dimensional $\mathbb{Z}_2$-homology manifold, one can define the following classes:

$$\tilde{w}_i(X) = \sum_{k=0}^{i} Sq^k(v_{i-k}(X)), \qquad \text{for } 0 \leq r \leq n. \tag{4.4}$$

These classes are those denoted by W^i and called W-classes in the original Wu's paper [37].

Theorem 4.1.

a) (Wu [37] and [4, §3.4]). *Let M be a compact n-dimensional manifold, then the classes $\tilde{w}_i(M)$ coincide with the Stiefel-Whitney classes $w_i(M)$ of the tangent bundle to M. One has:*

$$\underline{w}_{n-i}(M) = w_i(M) \cap [M] = \tilde{w}_i(M) \cap [M].$$

b) (Taylor [31]). *Let X be an n-dimensional $\mathbb{Z}_2$-homology manifold without boundary, then the classes $\tilde{w}_i(X)$ and $\underline{w}_{n-i}(X)$ are Poincaré dual, i.e.*

$$\underline{w}_{n-i}(X) = \tilde{w}_i(X) \cap [X].$$

Considering Corollary 2.3 and Theorem 4.1 the classes $\tilde{w}_i(X)$ are called Stiefel-Whitney classes of the n-dimensional $\mathbb{Z}_2$-homology manifold and, according to the most used notation, we denote them also by $w_i(X)$ or w_i if there is no ambiguity.

Let X be an n-dimensional $\mathbb{Z}_2$-homology manifold, the Steenrod squares of Stiefel-Whithey classes are given by the famous Wu's formula [38]:

$$Sq^k(w_i) = \sum_{t=0}^{k} \binom{i-k+t-1}{t} w_{k-t} \cup w_{i+t}. \tag{4.5}$$

One deduces the following relations between Stiefel-Whitney and Wu classes:

$$\begin{aligned} w_1 &= Sq^0(v_1) = v_1 \\ w_2 &= Sq^0(v_2) + Sq^1(v_1) = v_2 + v_1 \cup v_1 \\ w_3 &= Sq^0(v_3) + Sq^1(v_2) = v_3 + Sq^1(v_2) \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= w_2 + w_1^2 \\ v_3 &= w_1 \cup w_2 \\ v_4 &= w_4 + w_3 \cup w_1 + w_2^2 + w_1^4 \\ &\vdots \end{aligned}$$

5 The singular case - Intersection homology

In the singular case, several problems arise. The first one is that there is no more tangent bundle to a singular variety, then no more cohomology Stiefel-Whitney classes for singular spaces. One can define Stiefel-Whitney in homology [17], but intersection of cycles and Poincaré-Lefschetz duality do not exist with ordinary homology, so it is not possible to construct Stiefel-Whitney numbers with ordinary homology. The idea is to use intersection homology defined by Goresky and MacPherson for which intersection of cycles is well defined. In fact, Goresky and Pardon showed that Stiefel-Whitney classes do not lie in intersection homology, but Wu classes can be lifted to intersection homology. One can define Wu numbers and obtain an equivalent of Thom theorem for cobordism of spaces.

Now that we passed this first obstacle, a second one appears: intersection homology is not functorial, so that, in intersection homology, there are neither covariant nor contravariant maps associated to a map of spaces. Fortunately, for some classes of maps, such covariant and contravariant maps exist (see [2, 14]). That is the case, for instance, of so-called placid maps and of normally nonsingular maps. That allows us to formulate some results in this situation.

5.1 Pseudomanifolds

Definition 5.1 ([15, 2.1]). An n-dimensional pseudomanifold (without bou- ndary) is a purely n-dimensional piecewise linear (PL for short) polyhedron which admits a triangulation such that each $(n-1)$ simplex is a face of exactly two n-simplices.

A pseudomanifold admits a piecewise linear stratification [3, I.1.4], which is a filtration by closed subspaces $\emptyset \subset X_0 \subset X_1 \subset \ldots \subset X_{n-2} \subset X_n = X$, with the singular part $\Sigma(X)$ of X being (included in) the element X_{n-2} of the filtration and such that for each point $x \in X_i - X_{i-1}$ there is a neighborhood U and a PL stratum preserving homeomorphism between U and $\mathbb{R}^{n-i} \times \mathrm{c}(L)$, where $\mathrm{c}(L)$ denotes the (open) cone on the link L of the stratum $X_i - X_{i-1}$, itself a stratified pseudomanifold (see [3, I.1.1]). If $X_i - X_{i-1}$ is non empty, it is a (non necessarily connected) manifold of dimension i, and is called the i-dimensional stratum of the stratification.

Definition 5.2 ([15, 2.3]). An n-pseudomanifold X with boundary ∂X is an n-dimensional compact PL space such that $X - \partial X$ is a pseudomanifold and ∂X is a compact $(n-1)$-dimensional PL subspace of X which has a collared neighborhood U in X, i.e. there is a PL homeomorphism $\varphi : U \cong \partial U \times [0,1)$ such that the restriction $\varphi|_{\partial X}$ is the identity map.

Definition 5.3 ([10], [14, 5.3.1]). A map $f : X \to Y$ between pseudomanifolds is normally nonsingular if there exists a diagram

$$\begin{array}{ccc} N_f & \overset{i}{\hookrightarrow} & Y \times \mathbb{R}^k \\ {\scriptstyle\pi}\downarrow \uparrow{\scriptstyle s} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

where $\pi : N_f \longrightarrow X$ is a vector bundle with zero-section s, i is an open embedding, p is the first projection and one has $f = p \circ i \circ s$. The bundle N_f is called the normal bundle.

Remark 5.4. According to Fulton-MacPherson [10], this definition says that *geometrically the singularities of X at any point x are no better or worse than the singularities of Y at $f(x)$*. In particular if the target space Y is smooth, then the domain is smooth after crossing with some $\mathbb{R}^k$, so it is a homology manifold, *i.e.*

$$H_p(X, X \setminus \{x\}; \mathbb{Z}) = \begin{cases} 0 & \text{if } p \neq n, \\ \mathbb{Z} & \text{if } p = n. \end{cases}$$

5.2 Intersection homology and cohomology

Reference for this section is Goresky-MacPherson original paper [13].

The notion of perversity is fundamental for the definition of intersection homology and cohomology. A perversity $\bar{p}$ is a multi-index sequence of integers $(p(2), p(3), \ldots)$ such that $p(0) = p(1) = p(2) = 0$ and $p(c) \leq p(c+1) \leq p(c) + 1$, for $c \geq 2$. Any perversity $\bar{p}$ lies between the zero perversity $\bar{0} = (0, 0, 0, \ldots)$ and the total perversity $\bar{t} = (0, 1, 2, 3, \ldots)$. In particular, we will use the lower middle perversity, denoted $\overline{m}$ and the upper middle perversity, denoted $\overline{n}$, such that

$$\overline{m}(c) = \left[\frac{c-2}{2}\right] \quad \text{and} \quad \overline{n}(c) = \left[\frac{c-1}{2}\right], \qquad \text{for } c \geq 2.$$

Let $\overline{p}$ be a perversity, the complementary perversity $\overline{q}$ is defined by

$$q(c) + p(c) = t(c) = c - 2 \qquad \text{for all } c \geq 2.$$

Let $\overline{p}$ and $\overline{r}$ be two perversities, if, for every $c \geq 2$, one has $p(c) \leq r(c)$, one will write $\overline{p} \leq \overline{r}$.

Let X be an n-dimensional pseudomanifold and $\overline{p}$ a perversity. The intersection homology groups with $\mathbb{Z}_2$ coefficients, denoted $IH_i^{\overline{p}}(X)$, are the homology groups of the chain complex

$$IC_i^{\overline{p}}(X) = \left\{ \xi \in C_i(X) \mid \begin{matrix} \dim(|\xi| \cap X_{n-c}) \leq i - c + p(c) \text{ and} \\ \dim(|\partial\xi| \cap X_{n-c}) \leq i - 1 - c + p(c), \ \forall c \geq 2 \end{matrix} \right\},$$

where $C_i(X)$ denotes the group of compact i-dimensional PL chains ξ of X with $\mathbb{Z}_2$ coefficients and $|\xi|$ denotes the support of ξ.

The intersection cohomology groups with $\mathbb{Z}_2$ coefficients, denoted by $IH^{n-i}_{\overline{p}}(X)$, are defined as the groups of the cochain complex (see also [15])

$$IC^{n-i}_{\overline{p}}(X) = \left\{ \gamma \in C^{n-i}(X) \mid \begin{matrix} \dim(|\gamma| \cap X_{n-c}) \leq i - c + p(c) \text{ and} \\ \dim(|\partial\gamma| \cap X_{n-c}) \leq i - 1 - c + p(c), \ \forall c \geq 2 \end{matrix} \right\},$$

where $C^{n-i}(X)$ denotes the abelian group, with $\mathbb{Z}_2$ coefficients, of $(n-i)$-dimensional PL cochains of X with closed supports in X.

From now on, all homology and cohomology groups will be with $\mathbb{Z}_2$ coefficients, that we omit. The main properties of intersection homology that we use are the following:

Let X be a compact n-dimensional pseudomanifold, then, for any perversity $\bar{p}$, the Poincaré map P_X, cap-product by the fundamental class of X, naturally factorizes in the following way [13]:

$$\begin{array}{ccc} H^{n-i}(X) & \xrightarrow{\quad P_X \quad} & H_i(X) \\ & \searrow^{\alpha_X} \quad \nearrow_{\omega_X} & \\ & IH_i^{\bar p}(X) & \end{array} \tag{5.1}$$

where α_X is induced by the cap-product by the fundamental class $[X]$ and ω_X is induced by the inclusion $IC_i^{\overline{p}}(X) \hookrightarrow C_i(X)$.

For perversities $\bar p$ and $\bar r$ such that $\bar p + \bar r \le \bar t$, the intersection product

$$IH_i^{\bar p}(X) \times IH_j^{\bar r}(X) \to IH_{(i+j)-n}^{\bar p + \bar r}(X)$$

is well defined.

If X is a compact pseudomanifold, the Poincaré homomorphism

$$IH_{\bar p}^{n-i}(X) \to IH_i^{\bar p}(X), \tag{5.2}$$

is an isomorphism.

6 Cobordism of spaces - The singular case

M. Goresky and W. Pardon define four classes of singular spaces for which they define various characteristic numbers and for which these characteristic numbers determine the cobordism groups. In the four cases, they construct characteristc numbers by lifting Wu classes to intersection homology. Then they can multiply them. In this survey we will mention one of these classes, the one of Locally Orientable Witt spaces.

In the singular case, the mod 2 *Steenrod square operations* have been defined in intersection cohomology by M. Goresky in [12] (see also[15, §4]), as operations

$$Sq^i \colon IH_{\bar c}^j(X) \to IH_{2\bar c}^{i+j}(X)$$

for perversities $\bar c$ such that $2\bar c \le \bar t$. Via the above Poincaré duality (5.2), one has similar operations in intersection homology (with compact supports).

Definition 6.1 ([15, §5.1]). Let X be an n-dimensional pseudomanifold. Suppose $\bar c$ is a perversity such that $2\bar c \le \bar t$. Let $\bar b = \bar t - \bar c$ be the complementary perversity. For any i with $0 \le i \le [n/2]$ the Steenrod square operation

$$Sq^i \colon IH_i^{\bar c}(X) \to IH_0^{2\bar c}(X) \to \mathbb{Z}_2$$

is given by multiplication with the intersection cohomology i^{th}-Wu class of X:

$$v^i(X) = v_{\bar b}^i(X) \in IH_{\bar b}^i(X).$$

One defines $v^i(X) = 0$, for $i > [n/2]$.

Definition 6.2 ([11, §5.1],[23]). A stratified pseudomanifold X is a $\mathbb{Z}_2$-Witt space if, for each stratum of odd codimension $2k+1$, then $IH_k^{\bar{m}}(L) = 0$, where L is the link of the stratum and $\bar{m}$ the lower middle perversity.

If X is a $\mathbb{Z}_2$-Witt space, then the middle intersection homology group is self-dual, *i.e.,* satisfies the Poincaré duality over $\mathbb{Z}_2$. Also the natural homomorphism

$$IH^i_{\bar{m}}(X) \to IH^i_{\bar{n}}(X)$$

is an isomorphism.

Definition 6.3 ([15, §8.1]). A stratified pseudomanifold X is locally orientable if, for each stratum, the link is an orientable pseudomanifold.

Definition 6.4 ([15, §10.2]). A stratified pseudomanifold X is a locally orientable Witt space if it is both locally orientable and a $\mathbb{Z}_2$-Witt space.

Lemma 6.5 ([15, §10.2]). *If X is a locally orientable Witt space then $Sq^1 Sq^{2i} = Sq^{2i+1}$ as homomorphisms*

$$IH_j^{\bar{m}}(X) \to IH_{j-2i-1}^{\bar{m}}(X).$$

In the situation of a locally orientable Witt space, the Wu classes which are defined as middle intersection homology classes, can be multiplied to construct characteristic Wu numbers

$$\varepsilon(v_i(X) \bullet v_j(X)) = \langle v^{n-i}(X) \cup v^{n-j}(X), [X] \rangle \in \mathbb{Z}_2$$

where $i + j = n$, the map $\varepsilon : H_0(X, \mathbb{Z}_2) \to \mathbb{Z}_2$ denotes the augmentation and the following diagram commutes:

$$\begin{array}{ccccc} IH_i^{\bar{m}}(X) \times IH_j^{\bar{m}}(X) & \xrightarrow{\ \bullet\ } & IH_0^{\bar{t}}(X) & \xrightarrow{\ \varepsilon\ } & \mathbb{Z}_2 \\ \cong\times\cong \uparrow & & \cong \uparrow & & \\ IH^{n-i}_{\bar{m}}(X) \times IH^{n-j}_{\bar{m}}(X) & \xrightarrow{\ \cup\ } & IH^n_{\bar{0}}(X). & & \end{array}$$

Theorem 6.6 ([15, Theorem 10.5]). *A locally orientable Witt space X of dimension n is a boundary of a locally orientable Witt space Y if and only if each of the characteristic Wu numbers*

$$v^{ij}(X) = \varepsilon(v^i(X) \bullet v^j(X) \bullet v^1(X)^{n-i-j}) \in \mathbb{Z}_2$$

vanish, where $\varepsilon : H_0(X; \mathbb{Z}_2) \to \mathbb{Z}_2$ denotes the augmentation.

Here, the class v^1 is a cohomology class and $v^i v^j$ is a (intersection) homology class, so the product is a well defined cobordism invariant.

The reader will find in [15] the discussion and results concerning cobordism of singular spaces considered by these authors and other than locally orientable Witt spaces.

7 Cobordism of maps - The singular case

In the singular case, we are able to prove in [5] vanishing theorems, *i.e.* if maps are null-cobordant, then some Stiefel-Whitney numbers of maps vanish. We are not able to prove the converse, that could be a conjecture. However, that seems very dificult and it is possible that some definitions should be modified. For our "defense" note that:

1. In the case of cobordism of manifolds, after Pontryagin proved the cobordism theorem in that sense, it has been necessary to wait 7 years for Thom proving the converse!

2. In the case of cobordism of singular varieties, Goresky and Pardon proved a cobordism theorem in some specific situations: in particular, for locally $\mathbb{Z}_2$-Witt spaces, their Stiefel-Whitney numbers include only two terms, we have the same restriction in the (general) case of maps.

Definition 7.1 ([5]). Let $f : X \to Y$ be a map between pseudomanifolds of dimensions m and n respectively. The triple (f, X, Y) is null-cobordant if there exist:

1. pseudomanifolds V and W with dimensions $m+1$ and $n+1$, respectively, and $\partial V = X$ and $\partial W = Y$.

2. a map $F : V \to W$ such that the following diagram commutes

$$\begin{array}{ccc} U_X & \xrightarrow{F_{|U_X}} & U_Y \\ \cong \downarrow \phi & & \psi \downarrow \cong \\ \partial V \times [0,1) & \xrightarrow{f \times Id} & \partial W \times [0,1), \end{array}$$

   where U_X and U_Y are the collared neighborhood of X and Y in V and W respectively, and ϕ and ψ are PL diffeomorphisms such that $\phi(x) = (x, 0)$, $x \in \partial V$ and $\psi(y) = (y, 0)$, $y \in \partial W$.

3. $F_{|\partial V} = f : \partial V \to \partial W$.

Given maps $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$, one may define $(f \sqcup g, X_1 \sqcup X_2, Y_1 \sqcup Y_2)$ by mapping the disjoint union of X_1 and X_2 into the disjoint union of Y_1 and Y_2 by the map such that $(f \sqcup g)|_{X_1} = f$ and $(f \sqcup g)|_{X_2} = g$.

Definition 7.2 ([5]). Two maps $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are cobordant if the triple $(f \sqcup g, X_1 \sqcup X_2, Y_1 \sqcup Y_2)$ is null-cobordant.

Let $f : X \longrightarrow Y$ be a map, with X a compact locally orientable Witt space of pure dimension m and Y a closed n-dimensional smooth manifold. Then, we can define the map $f_! : IH_i^{\bar{p}}(X) \to IH_i^{\bar{p}}(Y)$ in such a way that the following diagram commutes

$$\begin{array}{ccc} H_i(X) & \xrightarrow{f_*} & H_i(Y) \\ \uparrow{\scriptstyle \omega_X} & & {\scriptstyle \omega_Y}\uparrow{\scriptstyle \simeq} \\ IH_i^{\bar{p}}(X) & \xrightarrow{f_!} & IH_i^{\bar{p}}(Y) \end{array}$$

i.e. $f_! = (\omega_Y)^{-1} \circ f_* \circ \omega_X$, where the map ω_Y is an isomorphism since Y is smooth.

Definition 7.3. For any partition $\ell = (\ell_1, \ldots, \ell_s)$ and numbers u_1, u_2 satisfying $u_i \leq [m/2]$ for all i and

$$(\ell_1 + \ell_2 + \ldots \ell_s) + u_1 + u_2 = n, \tag{7.1}$$

let us denote the intersection product by $w_\ell(Y) = w_{\ell_1}(Y) \cdots w_{\ell_s}(Y)$. The Stiefel-Whitney-Wu numbers of any triple (f, X, Y) corresponding to the numbers $\ell_1, \ldots, \ell_s$, u_1, u_2 are defined by

$$\varepsilon(w_\ell(Y) \bullet f_!(v_{m-u_1}(X)) \bullet f_!(v_{m-u_2}(X))).$$

Theorem 7.4 ([5]). *Let $f : X \longrightarrow Y$ be a be a normally nonsingular (or placid) map where X is compact locally orientable Witt space of pure dimension m and Y a closed n-dimensional smooth manifold. If (f, X, Y) is null-cobordant, with $(f, X, Y) = \partial(F, V, W)$ and W is a smooth manifold, then for any partition ℓ and numbers u_1, u_2 satisfying $u_i \leq [m/2]$ for all i and (7.1), the Stiefel-Whitney-Wu numbers*

$$\varepsilon(w_\ell(Y) \bullet f_!(v_{m-u_1}(X)) \bullet f_!(v_{m-u_2}(X))).$$

vanish.

Let $f : X \to Y$ be a map of pseudomanifolds. In general, there is no existence neither unicity of induced map in intersection homology, *i.e.* of a Gysin map $f_!$ making the following diagram (7.2) commutative. However, if f is proper and normally nonsingular, then there is an unique Gysin map

$$f_! : IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(Y)$$

such that the following diagram commutes [14, §5.4.3]:

$$
\begin{array}{ccc}
H_i(X) & \xrightarrow{f_*} & H_i(Y) \\
\uparrow{\scriptstyle \omega_X} & & \uparrow{\scriptstyle \omega_Y} \\
IH_i^{\bar{p}}(X) & \xrightarrow{f_!} & IH_i^{\bar{p}}(Y).
\end{array}
\tag{7.2}
$$

The same result holds for placid maps as well (see [14] and [2, Proposition 3.2]).

Theorem 7.5 ([5]). *Let $f : X \longrightarrow Y$ be a normally nonsingular (or placid) map, with X and Y compact locally orientable Witt spaces of pure dimension m and n respectively. If (f, X, Y) is null-cobordant, then for any u with $0 \leq u \leq n$, the following Wu numbers vanish:*

$$(\varepsilon(v_{n-u}(Y) \bullet f_!(v_u(X))) = 0.$$

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