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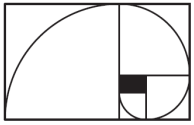

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A survey on Bruce-Roberts numbers

Bárbara K. Lima-Pereira, Bruna Oréfice-Okamoto, and João Nivaldo Tomazella

Abstract. This survey presents the main developments around the Bruce-Roberts number, an invariant for function germs on analytic varieties that generalizes the classical Milnor number. We summarize definitions, fundamental properties, known relations with the Milnor number, connections with the logarithmic characteristic variety, and relationships with polar multiplicities and the Euler obstruction. We also indicate several recent results and open problems, particularly concerning Cohen-Macaulayness of the logarithmic characteristic variety in the non-isolated hypersurface case.

Keywords. Bruce-Roberts number, relative Bruce-Roberts number, logarithmic characteristic varieties.

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This work is dedicated to the memory of Professor Maria Aparecida Soares Ruas, whose guidance and enthusiasm were essential to the development of this line of research.

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1 Introduction

When the third author began his PhD, his advisor, Prof. Dr. Maria Aparecida Soares Ruas, known as Cidinha, introduced him to the paper [12], in which a number appears for the first time somewhat discreetly and would later be referred to as the Bruce-Roberts number. From that moment on, Professor Ruas emphasized the importance of investigating this number and its possible relationships with other invariants in Singularity Theory, pointing out that much remained to be explored.

At that time, however, we did not give this concept the attention it deserved, as our main interest was focused on establishing conditions for the (in a certain sense) relative topological triviality of families of function germs, in terms of the integral closure of an appropriate ideal ([43, 42]).

After its introduction, the Bruce-Roberts number remained largely unexplored for some time. Nearly twenty years later, in independent works developed during the PhD of the second author and in the study presented in [21], the subject resurfaced with the aim of investigating the relationships between the Bruce-Roberts number and the Milnor number of a function (see [38]), as well as between the Bruce-Roberts number and the Euler obstruction.

Since then, several authors have devoted their research to the study of the Bruce-Roberts number and its applications (see [6, 7, 8, 21, 25, 32, 28, 31, 30, 29, 33, 39, 38, 40]), once again confirming Professor Cidinha's vision and intuition, as she always insisted that there was still much to be discovered in this field.

The Milnor number $\mu(f)$ is probably the most important invariant of a holomorphic function germ $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}$. It is defined algebraically as the colength of the Jacobian ideal Jf, generated by the partial derivatives of f in $\mathcal{O}_n$, the local ring of holomorphic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$. The finiteness of $\mu(f)$ is equivalent to that f has isolated singularity, which is also equivalent, by Mather's theorem, to that f is finitely determined with respect to $\mathcal{R}$, the group of coordinate changes in the source. From the topological viewpoint, Milnor showed in [36] that the Milnor fibre of f has the homotopy type of a wedge of $(n-1)$-spheres and that $\mu(f)$ is precisely the number of such spheres. Another important property is the conservation of the Milnor number, which implies that $\mu(f)$ is equal to the number of critical points of a Morsification of f.

An interesting and natural extension of this theory appears when we look at function germs defined over singular varieties. In [12], Bruce and Roberts studied function germs $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ considered over a germ of analytic variety $(X, 0)$ in $(\mathbb{C}^n, 0)$. They defined an invariant, which we call the Bruce-Roberts number of f with respect to X, and denote it by $\mu_{BR}(f, X)$. This number is a generalization of the Milnor number in the

sense that $\mu_{BR}(f, X) = \mu(f)$ when $X = \mathbb{C}^n$. Moreover, it also has the properties that $\mu_{BR}(f, X)$ is finite if and only if f has isolated singularity over $(X, 0)$ (in the stratified sense), if and only if f is finitely $\mathcal{R}_X$-determined, in which $\mathcal{R}_X$ is the subgroup of $\mathcal{R}$ of coordinate changes which preserve X, (see [12], p. 64).

This paper is organized as follows. Section 2 recalls the definition and main properties of the Bruce–Roberts number. Section 3 explores its relation to the Milnor number and other classical invariants, specifically when the analytic variety is a hypersurface or a complete intersection. Section 4 examines the logarithmic characteristic variety and its link to the Bruce–Roberts number; in particular, we investigate hypotheses that guarantee the logarithmic characteristic variety is Cohen–Macaulay. Section 5 presents applications, including connections with the Euler obstruction and invariance in families. Finally, Section 6 reviews recent developments and generalizations, such as the Bruce–Roberts' Tjurina number and extensions to 1-forms.

2 The Bruce-Roberts number

Let $\mathcal{O}_n$ be the ring of germs of analytic functions $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}$, and let $(X, 0) \subset (\mathbb{C}^n, 0)$ be the germ of an analytic variety. Two germs $f, g\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ are said to be $\mathcal{R}_X$-equivalent (respectively C^0-$\mathcal{R}_X$-equivalent) if there exists a germ of diffeomorphism (respectively homeomorphism) $\psi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\psi(X) = X$ and $f = g \circ \psi$.

We denote by Θ_n the $\mathcal{O}_n$-module of germs of holomorphic vector fields on $(\mathbb{C}^n, 0)$. Any element $\xi \in \Theta_n$ can be written in the form

$$\xi = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where $a_i \in \mathcal{O}_n$. Given $h \in \mathcal{O}_n$, the derivative of h in the direction of ξ is defined by

$$dh(\xi) = \xi(h) = \sum_{i=1}^{n} a_i \frac{\partial h}{\partial x_i}.$$

Let $X \subset (\mathbb{C}^n, 0)$ be a germ defined by an ideal $I_X \subset \mathcal{O}_n$. We denote by $\Theta_{X,0}$, or simply Θ_X, also written $\mathrm{Der}(-\log X)$, the $\mathcal{O}_n$-submodule of Θ_n consisting of germs of vector fields tangent to X at its smooth points; that is,

$$\Theta_X = \{\xi \in \Theta_n \mid dh(\xi) \in I_X \text{ for all } h \in I_X\}.$$

Definition 2.1. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ and $(X, 0) \subset (\mathbb{C}^n, 0)$ an analytic variety.*

The Bruce-Roberts number *of f with respect to X is defined by*

$$\mu_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)}.$$

The relative Bruce-Roberts number *is defined by*

$$\mu^{-}_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X) + I_X}.$$

Here $df(\Theta_X)$ denotes the image of Θ_X under the differential of f, $df : \Theta_n \to \mathcal{O}_n$.

These invariants were introduced in [12] and can be regarded as natural generalizations of the classical Milnor number to the context of function germs defined on analytic varieties.

The main properties of $\mu_{BR}(f, X)$ are the following (see [12]):

(a) $\mu_{BR}(f, X)$ is invariant under the action of the group $\mathcal{R}_X$ of diffeomorphisms $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ which preserve $(X, 0)$;

(b) $\mu_{BR}(f, X) < \infty$ if and only if f is finitely determined with respect to the $\mathcal{R}_X$-equivalence;

(c) $\mu_{BR}(f, X) < \infty$ if and only if f restricted to each logarithmic stratum (see Section 4) is a submersion in a punctured neighbourhood of the origin.

3 Milnor number and Bruce-Roberts number

For $f \in \mathcal{O}_n$, the Jacobian ideal Jf contains $df(\Theta_X)$ for any analytic variety $(X, 0)$. Hence, finiteness of the Bruce-Roberts number of f with respect to $(X, 0)$ implies that the Milnor number

$$\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_n / Jf$$

is finite. We remark that the converse is not true as we can see in the following example.

Example 3.1. Let $f : (\mathbb{C}^2, 0) \to \mathbb{C}$ be the function defined by $f(x, y) = xy$, and let X be the x-axis. Then,

$$\Theta_X = \langle (1, 0), (0, y) \rangle \text{ and } df(\Theta_X) = \langle y, xy \rangle.$$

Moreover, the Milnor number of f is $\mu(f) = 1$, while the Bruce–Roberts number $\mu_{BR}(f, X)$ is not finite (f is not a submersion on the stratum $\{y = 0\}$).

It is true that the Bruce-Roberts number is smaller than or equal to the Milnor number, hence, it is natural to investigate the relationship between these numbers.

When $(X,0)$ is an isolated hypersurface singularity and $\mu_{BR}(f,X)$ is finite, the fiber $(f^{-1}(0)\cap X,0)$ is an isolated complete intersection singularity (ICIS), (see [8, Proposition 2.8]).

For an ICIS, there exist the well-known Milnor and Tjurina numbers: Let $(Y,0)$ be the ICIS defined by a map germ $\phi=(\phi_1,\ldots,\phi_k):(\mathbb{C}^n,0)\to(\mathbb{C}^k,0)$; that is, $I_Y=\langle\phi_1,\ldots,\phi_k\rangle$, the Tjurina number of $(Y,0)$, which can be interpreted as the minimal number of parameters in a versal deformation of ϕ, is

$$\tau(Y,0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n^k}{\mathrm{Im}(J\phi)+I_Y\mathcal{O}_n^k}.$$

Here, $\mathrm{Im}(J\phi)$ denotes the $\mathcal{O}_n$-submodule of $\mathcal{O}_n^k$ generated by the columns of the Jacobian matrix of ϕ (see [20, Theorem 1.16, p. 238]).

Moreover, each fiber $\phi^{-1}(s)$ (for a generic s) has the homotopy type of a bouquet of spheres of dimension $n-k$. The Milnor number of $(Y,0)$, $\mu(Y,0)$, is the number of spheres in this bouquet(see [22, 23]). For its computation, one uses, the well-known Lê-Greuel formula recursively (see [11, 26]): if Y_1 is the ICIS defined by $(\phi_1,\ldots,\phi_{k-1})$, then

$$\mu(Y,0)+\mu(Y_1,0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle\phi_1,\ldots,\phi_{k-1},J\phi\rangle},$$

where $J\phi$ denotes the minors of maximal order of the Jacobian matrix of ϕ.

For $k=1$, that is, in the hypersurface case, we have

$$\tau(Y,0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle J\phi,\phi\rangle}\quad\text{and}\quad\mu(Y,0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle J\phi\rangle}=\mu(\phi).$$

The following result, proved in [28] and independently in [25], describes how these invariants are related to the Bruce-Roberts number.

Theorem 3.2 ([25, 28]). *Let $(X,0)\subset(\mathbb{C}^n,0)$ be an isolated hypersurface singularity defined by $\phi\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$, and let $f\in\mathcal{O}_n$ be finitely $\mathcal{R}_X$-determined. Then*

$$\mu_{BR}(f,X)=\mu(f)+\mu(f^{-1}(0)\cap X,0)+\mu(X,0)-\tau(X,0).$$

This result was obtained earlier in [38] for the case where the hypersurface $(X,0)$ is weighted homogeneous. In this case, the relation is

$$\mu_{BR}(f,X)=\mu(f)+\mu(f^{-1}(0)\cap X,0).$$

The proof for this particular case strongly uses the characterization of the logarithmic vector fields presented by Wahl [51] (see also [12, Proposition 7.2]).

To study how this result generalizes to other types of analytic varieties, we must be able to compute the Bruce-Roberts number in those settings. This task can be challenging, as it requires determining Θ_X for a given variety X in $(\mathbb{C}^n, 0)$, which involves calculating the corresponding syzygies. Consider the following example.

Example 3.3.

a) Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be the hypersurface defined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Then

$$\xi \in \Theta_X \iff d\phi(\xi) \in \langle \phi \rangle \iff \xi_1 \frac{\partial \phi}{\partial x_1} + \cdots + \xi_n \frac{\partial \phi}{\partial x_n} = \lambda \phi,$$

for some $\lambda \in \mathcal{O}_n$.

The goal is to determine all $\xi = (\xi_1, \ldots, \xi_n) \in \Theta_n$ such that there exists $\lambda \in \mathcal{O}_n$ with

$$\xi_1 \frac{\partial \phi}{\partial x_1} + \cdots + \xi_n \frac{\partial \phi}{\partial x_n} - \lambda \phi = 0.$$

To compute Θ_X in this case, we compute the syzygy module of the sequence

$$\left(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}, \phi \right)$$

and project onto the first n coordinates.

b) Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be the variety defined by $I_X = \langle \phi_1, \ldots, \phi_k \rangle$. Then

$$\begin{aligned} \xi \in \Theta_X &\iff d\phi_i(\xi) \in I_X \quad \forall i = 1, \ldots, k, \\ &\iff \xi_1 \frac{\partial \phi_i}{\partial x_1} + \cdots + \xi_n \frac{\partial \phi_i}{\partial x_n} = \lambda_{i1}\phi_1 + \cdots + \lambda_{ik}\phi_k \end{aligned}$$

for some $\lambda_{i1}, \ldots, \lambda_{ik} \in \mathcal{O}_n$. To compute Θ_X, we consider the modules

$$T_i = \text{syzygy} \left(\frac{\partial \phi_i}{\partial x_1}, \ldots, \frac{\partial \phi_i}{\partial x_n}, \phi_1, \ldots, \phi_k \right), \quad i = 1, \ldots, k,$$

and project onto the first n coordinates of T_i, obtaining T_i', $i = 1, \ldots, k$. Then

$$\Theta_X = \bigcap_{i=1}^{k} T_i'.$$

It is possible to compute the generators of Θ_X for certain varieties using computer algebra software; however, the feasibility of such computations depends on the complexity of the variety. In [37] and [48], algorithms are provided for obtaining a set of generators of Θ_X in the cases of an isolated hypersurface singularity and an ICIS, respectively.

In order to prove Theorem 3.2, we make use of the trivial vector fields of a hypersurface. We define them below in the more general setting of ICIS.

Definition 3.4. *Let $(X,0)$ be an ICIS defined by $\phi : (\mathbb{C}^n,0) \to (\mathbb{C}^k,0)$, with $I_X = \langle \phi_1, \ldots, \phi_k \rangle$. The module of trivial vector fields of X is defined by*

$$\Theta_X^T = \ker t\phi + I_X \Theta_n,$$

where $t\phi : \Theta_n \to \Theta(\phi)$ is the homomorphism of $\mathcal{O}_n$-modules given by $\xi \mapsto d\phi \circ \xi$, and $\Theta(\phi)$ denotes the $\mathcal{O}_n$-module of germs of vector fields along ϕ. We observe that $\Theta(\phi)$ can be identified with $\mathcal{O}_n^k$.

In the next proposition, we present a usefull characterization of the module of trivial vector fields of an ICIS.

Proposition 3.5 ([30, Proposition 2.1]). *Let $(X,0)$ be the ICIS determined by the map $\phi = (\phi_1, \ldots, \phi_k) : (\mathbb{C}^n,0) \to (\mathbb{C}^k,0)$. Then*

$$\Theta_X^T = I_{k+1} \begin{pmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\ \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_k}{\partial x_1} & \cdots & \frac{\partial \phi_k}{\partial x_n} \end{pmatrix} + \left\langle \phi_i \frac{\partial}{\partial x_j},\ i = 1, \ldots, k,\ j = 1, \ldots, n \right\rangle,$$

where the first term on the right-hand side is the submodule of Θ_X generated by the $(k+1)$-minors of the matrix.

We note that Θ_X^T is not equal to Θ_X.

Example 3.6. Let $(X,0) \subset (\mathbb{C}^3,0)$ be the ICIS defined by $\phi : (\mathbb{C}^3,0) \to (\mathbb{C}^2,0)$ with $\phi(x,y,z) = (x^3 + x^2y^2 + y^7 + z^3, xyz)$. It is known that $(X,0)$ is not weighted homogeneous. In this case,

$$\Theta_X^T = I_3 \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_1}{\partial z} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_2}{\partial z} \end{pmatrix} + \left\langle \phi_i \frac{\partial}{\partial x}, \phi_i \frac{\partial}{\partial y}, \phi_i \frac{\partial}{\partial z},\ i = 1, 2 \right\rangle.$$

Let

$$\xi = 7xz\frac{\partial}{\partial x} + 3yz\frac{\partial}{\partial y} + 7z^2\frac{\partial}{\partial z}.$$

Then

$$d\phi_i(\xi) \in \langle \phi_1, \phi_2 \rangle, \quad i = 1, 2.$$

In this example, $\xi \in \Theta_X$ but $\xi \notin \Theta_X^T$, and ξ is not an Euler vector field, that is, $\xi(\phi_i) \neq \phi_i$, $i = 1, 2$.

As a step toward proving Theorem 3.2, we obtained the following characterization of the Tjurina number of an ICIS, which is of independent interest.

Proposition 3.7 ([30, Proposition 4.6]). *Let $(X, 0)$ be an ICIS and let $f \in \mathcal{O}_n$ be a finitely $\mathcal{R}_X$-determined function germ. Then*

$$\dim_{\mathbb{C}} \frac{\Theta_X}{\Theta_X^T} = \dim_{\mathbb{C}} \frac{df(\Theta_X)}{df(\Theta_X^T)} = \tau(X, 0).$$

In fact, the equality between the first and third numbers in the previous proposition appears also in Tajima's work (see [47]).

Since the invariants appearing in Theorem 3.2 are defined for an ICIS $(X, 0)$, it is natural to ask whether the same relation holds in this case. Unfortunately, this is not always the case, as shown by the following example.

Example 3.8. Let $(X, 0)$ be the ICIS determined by the map-germ $\phi : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ defined by $\phi(x, y, z) = (x^3 + x^2y^2 + y^7 + z^3, xyz)$, and $f \in \mathcal{O}_n$, such that $f(x, y, z) = x^2 + yz$. Using the software singular Singular, [14], we calculate the Milnor and Tjurina numbers, respectively, of the ICIS

$$\mu(X, 0) = 51, \ \tau(X, 0) = 47.$$

Since these numbers are different $(X, 0)$ is not weighted homogeneous. Moreover,

$$\mu_{BR}(f, X) = 35 \neq 34 = \mu(f) + \mu(X \cap f^{-1}(0), 0) + \mu(X, 0) - \tau(X, 0).$$

At present, the known relation among these invariants in the case of an ICIS is given by the following result.

Theorem 3.9 ([30, Theorem 4.7]). *Let $(X, 0)$ be an ICIS and let $f \in \mathcal{O}_n$ be a finitely $\mathcal{R}_X$-determined function germ. Then*

$$\begin{aligned}\mu_{BR}(f, X) = \mu(f) + \mu(X \cap f^{-1}(0), 0) + \mu(X, 0) - \tau(X, 0) \\ - \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{Jf + I_X} + \dim_{\mathbb{C}} \frac{I_X \cap Jf}{I_X Jf},\end{aligned}$$

where $I_X \subset \mathcal{O}_n$ denotes the ideal defining $(X, 0)$.

Example 3.10. With the setting of example 3.8, by using the software Singular, we see that, in fact

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{Jf + I_X} = 1 \text{ e } \dim_{\mathbb{C}} \frac{I_X \cap Jf}{I_X Jf} = 2,$$

which is in agreement with previous theorem.

As mentioned earlier, in the Bruce and Robert's paper, [12], another invariant of a function germ with respect to an analytic variety is introduced, namely the relative Bruce–Roberts number, $\mu^-_{BR}(f, X)$ (Definition 2.1). In [12, Proposition 7.7], it is proved that if the analytic variety under consideration is a weighted homogeneous ICIS, then the relative Bruce–Roberts number coincides with the Milnor number of the ICIS defined by the intersection of the variety with the zero set of the function. This proposition is generalized in the following result.

Theorem 3.11 ([30, Theorem 2.2]). *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an ICIS and let $f \in \mathcal{O}_n$ be such that $\mu_{BR}(f, X) < \infty$. Then $(X \cap f^{-1}(0), 0)$ defines an ICIS and*

$$\mu(X \cap f^{-1}(0), 0) = \mu^-_{BR}(f, X) - \mu(X, 0) + \tau(X, 0).$$

It is worth noting that Theorem 3.11 provides a very elegant proof of the well-known fact that if an ICIS is weighted homogeneous, then its Milnor number coincides with its Tjurina number.

4 The logarithmic characteristic variety and Bruce-Roberts number

In this section, we study the relationship between the Bruce–Roberts number and the logarithmic characteristic variety. The Bruce–Roberts number admits a particularly elegant geometric interpretation when the logarithmic characteristic variety is Cohen–Macaulay. This variety, originally defined by Saito in [45], appears in several works in Singularity Theory, and many significant consequences follow from the Cohen–Macaulay property. We now recall its definition.

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a reduced analytic subvariety. Take a representative X on some small open neighbourhood U of 0 in $\mathbb{C}^n$. For each $x \in U$, we denote by $\Theta_X(x)$ the linear subspace of T_xU generated by the vectors $\delta(x)$, with $\delta \in \Theta_{X,x}$. Here, $\Theta_{X,x}$ is defined in the same way as $\Theta_{X,0}$, namely, it consists of germs at x of vector fields tangent to (X, x).

Lemma 4.1 ([12, Lemma1.5]). *There is a unique stratification $\{X_\alpha;\ \alpha \in I\}$ of U that satisfies the following properties:*

1. *Each stratum X_α is a smooth connected immersed submanifold of U and U is the disjoint union $\cup_{\alpha \in I} X_\alpha$.*

2. *If $x \in U$ lies in a stratum X_α, then the tangent space $T_x X_\alpha$ coincides with $\Theta_X(x)$.*

3. *If X_α and X_β are two distinct strata with X_α meeting the closure of X_β then X_α is contained in the frontier of X_β.*

Definition 4.2. *The stratification $\{X_\alpha : \alpha \in I\}$ of the previous lemma is called the* logarithmic stratification *of X and the strata X_α, the* logarithmic strata. *The germ $(X, 0)$ is* holonomic *if, for some neighborhood U of 0 in $\mathbb{C}^n$, the logarithmic stratification has only finitely many strata.*

Example 4.3. By [12, Porposition 1.7], if $(X, 0)$ is an ICIS in $\mathbb{C}^n$, then the logarithmic strata are the connected component of $\mathbb{C}^n \setminus X$, $X \setminus \{0\}$ and $\{0\}$.

Definition 4.4. *Define*

$$LC_U(X) = \{(x, \alpha) \in T^*_U\mathbb{C}^n \mid \alpha(\xi(x)) = 0 \text{ for all } \xi \in \Theta_X\},$$

where $T^*_U\mathbb{C}^n$ denotes the cotangent bundle of $\mathbb{C}^n$ restricted to U. The logarithmic characteristic variety *of X, denoted by $LC(X)$, is then defined as the germ of $LC_U(X)$ along $T^*_0\mathbb{C}^n$.*

To compute $LC(X)$, the following procedure is used. Denote the coordinates on $T^*\mathbb{C}^n$ by $x_1, \ldots, x_n, p_1, \ldots, p_n$ and consider the polynomial ring $\mathcal{O}_n[p] := \mathcal{O}_n[p_1, \ldots, p_n]$.

Given a vector field $\xi \in \Theta_n$, write

$$\xi = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}, \quad a_j \in \mathcal{O}_n,$$

and associate to ξ the linear form

$$\hat{\xi} := \langle \xi, p \rangle := \sum_{j=1}^{n} a_j p_j \in \mathcal{O}_n[p].$$

Let $\{\xi_1, \ldots, \xi_k\}$ be a set of generators of the $\mathcal{O}_n$-module Θ_X. Then the linear forms $\hat{\xi}_1, \ldots, \hat{\xi}_k$ generate an ideal $I_{LC(X)} \subset \mathcal{O}_n[p]$. The logarithmic characteristic variety $LC(X)$ is then regarded as the closed complex subspace of $T^*\mathbb{C}^n$ defined by this ideal.

Example 4.5. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be the hypersurface defined by

$$\phi(x, y, z) = x^3 + y^3 + z^3.$$

The germ $(X,0)$ is weighted homogeneous, and

$$\Theta_X = I_2\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 3y^2 & 3z^2 \end{pmatrix} + \left\langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right\rangle.$$

Therefore, the ideal $I_{LC(X)} \subset \mathcal{O}_n[p]$ is generated by the linear forms

$$I_2\begin{pmatrix} p_1 & p_2 & p_3 \\ 3x^2 & 3y^2 & 3z^2 \end{pmatrix} + \langle xp_1 + yp_2 + zp_3 \rangle.$$

Remark 4.1. By [12, Proposition 1.14], the germ $(X,0)$ is holonomic if and only if $\dim LC(X) = n$. In this case, let

$$X_0 = \mathbb{C}^n \setminus X, \ldots, X_r = \{0\}$$

denote the logarithmic strata of X. Then, as germs of sets,

$$LC(X) = \bigcup_{i=0}^{r} Y_i,$$

where $Y_i = \overline{N^*X_i}$ denotes the closure in $T^*\mathbb{C}^n$ of the conormal bundle N^*X_i of X_i. The sets Y_i $(i = 0, \ldots, r)$ are precisely the irreducible components of $LC(X)$. We denote by m_i the multiplicity of $LC(X)$ along Y_i.

We now consider the relationship between the logarithmic characteristic variety and the Bruce-Roberts number. If $LC(X)$ is Cohen-Macaulay, then $\mu_{BR}(f,X)$ coincides with the number of critical points of a Morsification of f on each logarithmic stratum X_i, counted with multiplicity.

Theorem 4.6 ([12, Corollary 5.8]). *If f has an isolated critical point at 0 and n_i is the number of critical points of a morsification of f on X_i, then*

$$\sum m_i n_i \leq \mu_{BR}(f,X) \tag{4.1}$$

with equality if and only if $LC(X)$ is Cohen-Macaulay at $(0, df(0))$.

In [12, Proposition 5.10], Bruce and Roberts showed that if $(X,0)$ has codimension greater than 1 in $(\mathbb{C}^n,0)$, then $LC(X)$ is not Cohen–Macaulay at any point of $X \times 0 \subset LC(X)$. In the codimension 1 case, they also proved that $LC(X)$ is Cohen–Macaulay at any point outside $X \times 0$.

Therefore, it only makes sense to ask whether $LC(X)$ is Cohen-Macaulay when $(X,0)$ is a hypersurface; in general, this remains an open problem. In [38], we use Wahl's characterization of $Der(-\log X)$ given in [51] to show that, if $(X,0)$ is a weighted homogeneous hypersurface with an isolated singularity, then $LC(X)$ is a determinantal variety, that is, it is

defined by the minors of a certain matrix and has the expected codimension. Hence, by the Eagon-Hochster results [16], one concludes that, in this case, $LC(X)$ is Cohen-Macaulay.

Afterwards, there were no further developments on this topic for almost a decade, until the PhD thesis of Lima-Pereira in 2020, where it was proved, using Theorems 3.2 and 4.6, that $LC(X)$ is Cohen-Macaulay for any hypersurface with isolated singularity. More recently, this result was extended to holonomic weighted homogeneous hypersurfaces whose singular set has dimension 1, see [32]. Altogether, we obtain the following result.

Theorem 4.7 ([32, Corollary 3.12]). *Let $(X, 0)$ be a holonomic hypersurface whose singular set has dimension at most 1, such that it is defined by a Euler homogeneous function*[1]*. Then, $LC(X)$ is Cohen-Macaulay.*

In the following example, we exhibit a non-holonomic weighted homogeneous hypersurface whose logarithmic characteristic variety is not Cohen-Macaulay. This shows that the holonomic hypothesis in the previous theorem cannot be removed. The remaining open problems concern non–weighted homogeneous hypersurfaces with non-isolated singular sets, as well as weighted homogeneous hypersurfaces whose singular set has dimension greater than 1.

Example 4.8. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be the hypersurface defined by

$$\phi(x, y, z) = xy(x - y)(x + y(z + 1)).$$

[12, Example 18-b] shows that $(X, 0)$ is not holonomic and by making the calculations with Singular,

$$\dim \mathcal{O}_{LC(X)} = 4 \neq 3 = \operatorname{depth} \mathcal{O}_{LC(X)}.$$

Therefore $LC(X)$ is not Cohen-Macaulay.

As mentioned earlier, the logarithmic characteristic varieties of germs of codimension greater than one are not Cohen-Macaulay. However, Bruce and Roberts [12] introduced another notion of logarithmic characteristic variety. With the notation of Remark 4.1, let $N^*X_0 = \mathbb{C}^n \times \{0\}$ denote the zero section, defined by the ideal $P = \langle p_1, \ldots, p_n \rangle$. Clearly, $I_{LC(X)} \subset P$. The *relative logarithmic characteristic variety* $LC(X)^-$ is the complex subspace of $T^*\mathbb{C}^n$ defined by the saturation of $I_{LC(X)}$ with respect to P, that is, $I_{LC(X)} : P^\infty$. As germs of sets,

$$LC(X)^- = \bigcup_{i=1}^{r} Y_i = \bigcup_{i=1}^{r} \overline{N^*X_i} = LC(X) \cap \pi^{-1}(X),$$

where $\pi \colon T^*\mathbb{C}^n \to \mathbb{C}^n$ denotes the canonical projection.

[1] A function $\phi \in \mathcal{O}_n$ is said to be Euler homogeneous if there exists a vector field $\varepsilon \in \Theta_n$ such that $\varepsilon(\phi) = \phi$)

Just as $LC(X)$ is related to $\mu_{BR}(f,X)$, the space $LC(X)^-$ is related to the relative Bruce-Roberts number $\mu^-_{BR}(f,X)$. More precisely, $LC(X,0)^-$ is Cohen-Macaulay at $(0,p)$ if and only if

$$\mu^-_{BR}(f,X) = \sum_{i=1}^{r} n_i m_i, \tag{4.2}$$

where $X_1,\dots,X_r$ are the logarithmic strata excluding $X_0 = \mathbb{C}^n \setminus X$, n_i denotes the number of critical points of a Morsification f_t on the stratum X_i, and m_i is the multiplicity of the corresponding irreducible component Y_i of $LC(X,0)$; see [12, Proposition 5.11].

Bruce and Roberts proved in [12, Proposition 7.3] that if $(X,0)$ is a weighted homogeneous ICIS, then $LC(X)^-$ is Cohen-Macaulay. In [30, Theorem 3.1], using Theorem 3.11, this result is extended to general ICIS, leading to the following theorem.

Theorem 4.9 ([30, Theorem 3.1]). *If $(X,0)$ is an ICIS, then $LC(X)^-$ is Cohen–Macaulay.*

Equations (4.1) and (4.2) show that, like the Milnor number, the numbers $\mu_{BR}(f,X)$ and $\mu^-_{BR}(f,X)$ can be computed in terms of the number of stratified critical points of a Morsification of f with respect to the logarithmic stratification of X. Moreover, the Cohen–Macaulayness of $LC(X)$ and $LC(X)^-$ implies the conservation of both numbers under arbitrary deformation of f. Several authors have recently published papers on these topics (see [2, 8, 21, 38]).

5 Applications

The results in Sections 3 and 4 have many applications developed by several authors. We present some of them here.

5.1 Bruce-Roberts number and Euler obstruction

Let $(X,0)$ be an isolated hypersurface singularity defined by a germ $\phi : (\mathbb{C}^n,0) \to (\mathbb{C},0)$. Let $f \in \mathcal{O}_n$ be finitely $\mathcal{R}_X$-determined. From Theorem 3.2, by replacing f with a generic linear projection $p\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ and applying the Lê-Greuel formula, we obtain

$$\mu_{BR}(p,X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle\phi\rangle + J(p,\phi)} - \tau(X,0).$$

According to Gaffney [18], the $(n-1)^{\text{th}}$ polar multiplicity is defined by

$$m_{n-1}(X,0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle\phi\rangle + J(p,\phi)},$$

and therefore,

$$m_{n-1}(X,0) = \mu_{BR}(p,X) + \tau(X,0).$$

Furthermore, Pérez and Saia [24] proved that

$$1 + (-1)^{n-1}\mu(X,0) = \sum_{i=0}^{n-1}(-1)^i m_i(X,0),$$

where $m_i(X,0)$ denotes the i^{th} polar multiplicity. From the Lê–Tessier formula, we also have

$$\mathrm{Eu}(X,0) = \sum_{i=0}^{n-2}(-1)^{n-i-2}m_i(X,0),$$

where $\mathrm{Eu}(X,0)$ denotes the Euler obstruction of $(X,0)$ (see [35]). Combining these expressions, we obtain

$$m_{n-1}(X,0) - \mu(X,0) = \mathrm{Eu}(X,0) + (-1)^{n-1}.$$

This leads to the following result.

Theorem 5.1 ([28, Corollary 4.5]). *Let $p\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a generic linear projection, and let $(X,0) \subset (\mathbb{C}^n,0)$ be a hypersurface with an isolated singularity. Then:*

1. $m_{n-1}(X,0) = \mu_{BR}(p,X) + \tau(X,0)$;
2. $\mathrm{Eu}(X,0) = \mu_{BR}(p,X) + \tau(X,0) - \mu(X,0) + (-1)^n$.

It follows from the above result that the Bruce-Roberts number of a generic linear projection on an isolated hypersurface singularity does not depend on the choice of the projection.

In [40], the aforementioned relations between the Bruce-Roberts numbers and other invariants are combined with the results of [41] to show that

$$\mu_{BR}(f,X) = \mu(f) + \mu(X,0) + (-1)^{d-1}\big(B_{f,X}(0) - 1\big) - \tau(X,0),$$

where $B_{f,X}$ denotes the Brasselet number ([15, Definition 3.18]) of a function $f : (X,0) \to \mathbb{C}$.

5.2 Bruce-Roberts numbers in families

Let $(X,0) \subset (\mathbb{C}^n,0)$ be a germ of analytic variety and $f \in \mathcal{O}_n$. A 1-parameter deformation of f is a holomorphic map

$$F : U \times V \to \mathbb{C},$$

with U and V open sets in $\mathbb{C}^n$ and $\mathbb{C}$, respectively, such that $F(x,0) = f(x)$, for all $x \in U$. We denote $f_t : (\mathbb{C}^n, 0) \to \mathbb{C}$ the germ of the function defined by $f_t(x) = F(x,t)$.

We say that F is μ_{BR}-constant, respectively μ-constant, if $\mu_{BR}(f_t, X) = \mu_{BR}(f, X)$, respectively $\mu(f_t) = \mu(f)$, for $t \in V$ small enough.

The constancy of the Milnor number in a family of holomorphic functions is controlled via the integral closure of the Jacobian ideal (see [19]). Similar results have been obtained for the Bruce-Roberts number in [2], although in that work the additional hypothesis that $LC(X,0)$ is Cohen-Macaulay is required. It follows from our Theorem 4.7 that the results in [2] hold for any isolated hypersurface singularity.

Recall that the *polar curve* of F is defined by

$$C := \{(x,t) \in \mathbb{C}^n \times \mathbb{C} \mid dF(\xi_i)(x,t) = 0 \text{ for all } i = 1, \ldots, p\},$$

where $\xi_1, \ldots, \xi_p$ are generators of Θ_X. The following corollary is an immediate consequence of [2, Proposition 3.5] together with Theorem 4.7.

Corollary 1. *With the above notation, if $(X,0)$ is a hypersurface germ whose singular set has dimension at most 1, then the following statements are equivalent:*

(i) F is a μ_{BR}-constant deformation of f;

(ii) The polar curve of F with respect to $\{t = 0\}$ does not split, that is, $C = \{0\} \times \mathbb{C}$.

Moreover, if $\frac{\partial F}{\partial t} \in \overline{df(\Theta_X)}$, then F is a μ_{BR}-constant deformation of f, where $\overline{I}$ denotes the integral closure of an ideal I.

Other interesting results concerning μ_{BR}-constant families of functions were obtained by Grulha in [21]. If X is an isolated hypersurface singularity, then from Theorem 4.7 and [12, Corollary 5.8], it follows that

$$\mu_{BR}(f, X) = \sum_{i=0}^{r} n_i m_i.$$

According to [12, p. 80], $m_i = 1$ for $i = 1, \ldots, r-1$, and from [28, Corollary 5.5] and [46, Proposition 2.3], we have

$$\mu_{BR}(f, X) = \mu(f) + (-1)^{n-1} \operatorname{Eu}_f(X,0) + m_{n-1}(X,0) - \tau(X,0),$$

where $\operatorname{Eu}_f(X,0)$ denotes the Euler obstruction of f_t on X at 0 (see [10, 46]). Therefore, from this expression and Theorem 3.2, we deduce that

Theorem 5.2 ([21, Theorem 5.18]). *Let $X \subset \mathbb{C}^n$ be a hypersurface with an isolated singularity and let $F\colon (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}$ be a family of functions with isolated singularities. Then:*

(i) If $\mu_{BR}(X,f_t)$ is constant along the family, then $\mu(f_t)$, $\mu(X\cap f_t^{-1}(0),0)$, and $\mathrm{Eu}_{f_t}(X,0)$ are also constant.

(ii) If $\mu(f_t)$ is constant along the family, then the constancy of either $\mathrm{Eu}_{f_t}(X,0)$ or $\mu(X\cap f_t^{-1}(0),0))$ implies that $\mu_{BR}(X,f_t)$ is constant along the family.

Moreover, if $(X,0)$ is an ICIS, it follows from [12, Proposition 5.11] and Theorem 3.11 that

$$\begin{aligned}\mu^-_{BR}(f,X) &= \sum_{i=1}^{r} n_i m_i = \sum_{i=1}^{r-1} n_i m_i + n_r m_r = \sum_{i=1}^{r-1} n_i + m_r \\ &= (-1)^{n-1}\,\mathrm{Eu}_f(X,0) + m_d(X,0) - \tau(X,0)\end{aligned}$$

$$\begin{aligned}\mu(X\cap f^{-1}(0),0) &= \mu^-_{BR}(f,X) - \mu(X,0) + \tau(X,0) \\ &= (-1)^{n-1}\,\mathrm{Eu}_f(X,0) + m_d(X,0) - \mu(X,0).\end{aligned}$$

Corollary 2. *Let $X\subset\mathbb{C}^n$ be an ICIS, and let $F\colon(\mathbb{C}^n\times\mathbb{C}^r,0)\to\mathbb{C}$ be a family of functions with isolated singularities. Then the constancy of any one of the numbers $\mu^-_{BR}(X,f_t)$, $\mu(X\cap f_t^{-1}(0),0)$, or $\mathrm{Eu}_{f_t}(X,0)$ implies the constancy of the other two.*

The following is a classical result of Greuel [19], which presents several equivalent conditions for the μ-constancy of a family of functions.

Theorem 5.3 ([19, p.161]). *For any deformation $F:(\mathbb{C}^n\times\mathbb{C},0)\to(\mathbb{C},0)$ of a function germ f with isolated singularity at the origin, the following statements are equivalent:*

1. *F is a μ-constant deformation of f;*
2. *for every holomorphic curve $\gamma:(\mathbb{C},0)\to(\mathbb{C}^n\times\mathbb{C},0)$,*

$$\nu\left(\frac{\partial F}{\partial t}\circ\gamma\right) > \inf_{i=1,\dots,n}\left\{\nu\left(\frac{\partial F}{\partial x_i}\circ\gamma\right)\right\},$$

   where ν denotes the usual valuation of a complex curve;

3. *same statement as in (ii) with "$>$" replaced by "$\geq$";*
4. *$\partial F/\partial t\in\overline{J_F}$, where $\in\overline{J_F}$ denotes the integral closure of the Jacobian ideal*

$$J_F = \langle \partial F/\partial x_1,\dots,\partial F/\partial x_n\rangle \,in\, \mathcal{O}_{n+1};$$

5. *$\partial F/\partial t\in\sqrt{J_F}$, where $\sqrt{J_F}$ denotes the radical of J_F;*

6. *the polar curve of F with respect to $\{t=0\}$ does not split, i.e.,*

$$C=\left\{(x,t)\in\mathbb{C}^n\times\mathbb{C}\;\middle|\;\frac{\partial F}{\partial x_i}=0,\ i=1,\dots,n\right\}=\{0\}\times\mathbb{C}\quad \textit{near } (0,0).$$

In [2], the corresponding version for the Bruce–Roberts number is analyzed, and the following statements are considered.

(1_r) F is a μ_{BR}-constant deformation of f;

(2_r) for every holomorphic curve $\gamma:(\mathbb{C},0)\to(\mathbb{C}^n\times\mathbb{C},0)$,

$$\nu\left(\frac{\partial F}{\partial t}\circ\gamma\right)>\inf\left\{\nu\big(dF(\xi_i)\circ\gamma\big)\ :\ i=1,\dots,p\right\};$$

(3_r) same statement as in (2r) with "$>$" replaced by "$\geq$";

(4_r) $\dfrac{\partial F}{\partial t}\in\overline{J_F(\Theta_X)}$;

(5_r) $\dfrac{\partial F}{\partial t}\in\sqrt{J_F(\Theta_X)}$;

(6_r) the polar curve of F in V with respect to $\{t=0\}$ does not split, i.e.,

$$C=\big\{(x,t)\in\mathbb{C}^n\times\mathbb{C}\;\big|\;dF(\xi_i(x))=0,\ i=1,\dots,p\big\}=\{0\}\times\mathbb{C}\ \text{ near } (0,0).$$

The relations among these conditions are established in [2]. In particular, from [2, Theorem 3.3] we have

$$(2_r)\implies(3_r)\iff(4_r)\implies(5_r).$$

Moreover, [2, Examples 3.1 and 3.2] show that

$$(1_r)\not\Longrightarrow(4_r)\quad\text{and}\quad(5_r)\not\Longrightarrow(4_r).$$

In addition, [2, Proposition 3.5] proves that

$$(1_r)\implies(6_r),$$

and that, if C is Cohen–Macaulay, then

$$(6_r)\implies(1_r).$$

Finally, for a hypersurface $(X,0)$ whose logarithmic characteristic variety $LC(X)$ is Cohen–Macaulay, [2, Theorem 3.7] asserts the equivalence

$$(1_r)\iff(6_r),$$

while, if $(X,0)$ is a hypersurface with an isolated singularity, [2, Theorem 3.11] establishes that

$$(4_r) \implies (1_r).$$

Another classical result concerning the μ-constancy of families of function germs is due to Varchenko [49]. It states that if f is a weighted homogeneous function germ with an isolated singularity and f_t is a deformation of f obtained by adding terms of weighted degree greater than or equal to that of f, that is, a non-negative deformation, then the family f_t is μ-constant. Motivated by this result, the relationship between the Bruce–Roberts number and non-negative deformations was studied in [39].

Theorem 5.4 ([39, Theorem 3.6]). *Let $(X,0) \subset (\mathbb{C}^n,0)$ be a weighted homogeneous hypersurface with isolated singularity, let $f:(\mathbb{C}^n,0)\to\mathbb{C}$ be a weighted homogeneous (with the same weights as X) $\mathcal{R}_X$-finitely determined function and let f_t be a deformation of f. The following statements are equivalent:*

1. *the family f_t is $C^0-\mathcal{R}_X$-trivial;*
2. *f_t is a non-negative deformation;*
3. *$\mu_{BR}(f_t,X)$ is constant.*

We observe that the results above provide sufficient conditions for a family of functions to have a constant Bruce–Roberts number, assuming the analytic variety is fixed.

In [13], Proposition 4.4 establishes an interesting result for toric varieties, showing that under certain conditions the Bruce–Roberts number remains invariant under simultaneous deformations of both the toric variety and the function.

5.3 Chern obstruction

In [33], the authors established a relation between the Chern obstruction and the relative Bruce–Roberts number under the assumption that the relative logarithmic characteristic variety is Cohen–Macaulay. The key ingredient in this relation is the following result.

Proposition 5.5 ([33, Proposition 5.2]). *Let $(X,0)\subset(\mathbb{C}^n,0)$ be a $d-$dimensional reduced analytic variety, $G=(g_1,\ldots,g_p):(X,0)\to(\mathbb{C}^p,0)$ a map-germ, where $d>p$, and $\mathcal{V}$ a Whitney stratification of $X\cap G^{-1}(0)$. Suppose that $f:(X,0)\to(\mathbb{C},0)$ has (at most) a stratified isolated singularity at the origin in $X\cap G^{-1}(0)$. Then the Chern obstruction $Ch_{X\cap G^{-1}(0),0}\{df\}$ is equal to the number of non-degenerated Morse critical points of a Morsification of f appearing in the regular part of $X\cap G^{-1}(0)$.*

In this way, a connection between the Chern obstruction and the relative Bruce–Roberts number is obtained. In particular, when the relative logarithmic characteristic variety is Cohen–Macaulay, the relative Bruce–Roberts number counts Morse critical points.

Theorem 5.6 ([33, Theorem 5.5]). *Let $(X,0) \subset (\mathbb{C}^n,0)$ be a reduced analytic variety and $F=(f,g_1,\ldots,g_p):(X,0)\to(\mathbb{C}^{p+1},0)$ be a map such that $f:(X,0)\to(\mathbb{C},0)$ has (at most) a stratified isolated singularity at the origin in $X\cap G^{-1}(0)$. Suppose that $X\cap G^{-1}(0)$ is a holonomic variety and that the relative logarithmic characteristic variety $LC(X\cap G^{-1}(0))^-$ of $X\cap G^{-1}(0)$ is Cohen-Macaulay. Then,*

$$Ch_{X\cap G^{-1}(0),0}\{df\}=\mu^-_{BR}(f,X\cap G^{-1}(0))-\sum_{\alpha=s+1}^{k+1} n_\alpha m_\alpha.$$

As a consequence of the previous theorem and Theorem 4.9, we obtain a particular relation when the variety is an ICIS.

Corollary 3 ([33, Theorem 5.6]). *Let $(X,0)\subset(\mathbb{C}^n,0)$ be a $(d+1)-$dimensional ICIS and $F=(f,g_1,\ldots,g_p):(X,0)\to(\mathbb{C}^{p+1},0)$ a map, such that $X\cap G^{-1}(0)$ and $X\cap F^{-1}(0)$ are also ICIS. Then, for a generic linear form l,*

$$Ch_{X\cap G^{-1}(0),0}\{df\}=\mu^-_{BR}(f,X\cap G^{-1}(0))-\mu^-_{BR}(l,X\cap G^{-1}(0)).$$

Although the assumption that the relative logarithmic characteristic variety is Cohen–Macaulay is rather restrictive, we expect that this condition is satisfied for any holonomic variety. Furthermore, this can be checked computationally, (see [33, Examples 5.10, 5.11 and 5.12]).

6 State of the Art

In recent years, several authors [3, 4, 5, 31, 40] have investigated relationships between the Bruce-Roberts numbers and other invariants in Singularity Theory, as well as extensions of these invariants to the setting of 1-forms. In particular, connections with the GSV-index have been explored, and links with foliations have been established, see [4].

6.1 The Bruce-Roberts Tjurina number

Another invariant which is related to a map germ is its Tjurina number, which is related to the number of parameters which are necessary in a versal deformation of the germ.

It is well known that the Tjurina number of an ICIS is smaller then or equal to its Milnor number and that the equality is equivalent to the fact that the ICIS can be defined by a weighted homogeneous map (see [44, 50]).

In [1, 7, 6, 8], it is defined a generalization of the Tjurina's number: the *Bruce-Roberts Tjurina number* of f with respect to $(X, 0)$ is

$$\tau_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f \rangle + df(\Theta_X)},$$

when this number is finite.

It is clear that the Bruce-Roberts number is greater than or equal to the Bruce-Roberts Tjurina number, hence, $\mu_{BR}/\tau_X \geq 1$. In the classical case, that is, $f : (\mathbb{C}^n, 0) \to \mathbb{C}$, we have $1 \leq \mu/\tau \leq n$, see [34]. The following result provides an estimate for the relative case.

Proposition 6.1 ([8, Corollary 3.3]).

$$\frac{\mu_{BR}(f, X)}{\tau_X(f)} \leq r_f\big(df(\Theta_X)\big),$$

where $r_f\big(df(\Theta_X)\big)$ *is the smallest integer* $r \in \mathbb{Z}_{\geq 1}$ *such that* $f^r \in df(\Theta_X)$.

In [6], it is defined a kind of relative quasihomogeneity: f and ϕ are relativelly quasihomoegenous if they are weighted homogeneous with the same weights. There is also an example which shows that the equality between the Bruce-Roberts number and the Bruce-Roberts Tjurina number is not equivalent to the relative quasihomogeneity. In the next result, the authors present sufficient conditions for this equivalence.

Theorem 6.2 ([6, Theorem 1.3]). *Let* (f, X) *be a pair with* $\mu_{BR}(f, X) < \infty$. *Suppose also that*

1. $f \in \mathcal{M}^3$, *where* $\mathcal{M} = \{f \in \mathcal{O}_n \mid f(0) = 0\}$ *is the (unique) maximal ideal of* $\mathcal{O}_n$, *or*

2. X *is a hypersurface with at most an isolated singularity at the origin.*

Then, the equality $\mu_{BR}(f, X) = \tau_X(f)$ *is equivalent to the relative quasihomogeneity of the pair* (f, X).

In Cidinha's last paper about this issue ([7]), they related the Bruce-Roberts and Bruce-Roberts Tjurina numbers to dimensions of special modules generalizing the relationships for the invariants presented in the first section.

6.2 The Bruce-Roberts numbers of 1-forms

In [4, 5, 3], the authors extend the definition of the Bruce-Roberts numbers to the setting of 1-forms. Let $(X,0) \subset (\mathbb{C}^n,0)$ be an analytic variety, and let ω be a germ of a holomorphic 1-form at the origin. The Bruce-Roberts numbers of the 1-form ω with respect to $(X,0)$ are defined by

$$\mu_{BR}(\omega, X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X)} \quad \text{and} \quad \mu^-_{BR}(\omega, X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X) + I_X},$$

where I_X denotes the ideal of germs vanishing on $(X,0)$, and $\omega(\Theta_X)$ is defined in the natural way. Namely, if the 1-form ω is given by

$$\omega = \sum_{i=1}^{n} \omega_i \, dx_i,$$

and

$$\eta = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \in \Theta_X,$$

then

$$\omega(\eta) = \sum_{i=1}^{n} a_i \, \omega_i.$$

We also denote by $\langle \omega \rangle = \langle \omega_1, \ldots, \omega_n \rangle$ the ideal of $\mathcal{O}_n$ generated by the components of ω.

Using these invariants, the results of [28, 29, 30] can be extended to holomorphic 1-forms with isolated singularities. Let $(X,0) \subset (\mathbb{C}^n,0)$ be an isolated hypersurface singularity and let ω be a holomorphic 1-form with an isolated singularity at the origin. Then

$$\begin{aligned} \mu_{BR}(\omega, X) &= \mu(\omega) + \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) - \tau(X, 0), \\ \mu^-_{BR}(\omega, X) &= \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) - \tau(X, 0), \end{aligned}$$

where $\mu(\omega) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \omega \rangle}$ denotes the Milnor number for 1- forms, see [4] for details.

The previous equalities were extended to the case of isolated complete intersection singularities in [31]. Let $(X,0) \subset (\mathbb{C}^n,0)$ be an ICIS and let ω be a holomorphic 1-form with an isolated singularity at the origin. Then

$$\begin{aligned} \mu_{BR}(\omega, X) = \mu_0(\omega) + \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) - \tau(X, 0) \\ - \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \omega \rangle + I_X} + \dim_{\mathbb{C}} \frac{I_X \cap \langle \omega \rangle}{I_X \langle \omega \rangle}, \end{aligned}$$

and the relation defining the relative Bruce-Roberts number remains unchanged.

In order to establish these equalities, we use the algebraic characterization of the GSV-index given in [17, Theorem 2]. More precisely,

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega, X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_X + I_{k+1}\begin{pmatrix}\omega \\ t\phi\end{pmatrix}},$$

where

$$\begin{pmatrix}\omega \\ t\phi\end{pmatrix} = \begin{pmatrix} \omega_1 & \cdots & \omega_n \\ \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_k}{\partial x_1} & \cdots & \frac{\partial \phi_k}{\partial x_n} \end{pmatrix}.$$

In Theorem 3.9, by taking $\omega = df$ and applying the Lê-Greuel formula,

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = \mu(X \cap f^{-1}(0), 0) + \mu(X, 0),$$

we recover a particular case of the previous formula.

The following example illustrates the previous results in the case of a non-exact 1-form.

Example 6.3. Let $(X, 0)$ be the ICIS determined by the map-germ $\phi : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ defined by $\phi(x, y, z) = (x^3 + x^2y^2 + y^7 + z^3, xyz)$, and let $\omega = x^2\frac{d}{dx} + y^2\frac{d}{dy} + (x + z)\frac{d}{dz}$. Using the software singular Singular, [14], we calculate

$$\mu_{BR}(\omega, X) = 36, \;\; \mu(\omega) = 4, \;\; \mathrm{Ind}_{\mathrm{GSV}}(\omega, X) = 75, \;\; \tau(X, 0) = 47$$

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle\omega\rangle + \langle\phi_1, \phi_2\rangle} = 4 \text{ e } \dim_{\mathbb{C}} \frac{I_X \cap \langle\omega\rangle}{I_X\langle\omega\rangle} = 8.$$

To compute the last dimension, we can use the homolog library in Singular. An alternative is to use the fact that there is an isomorphism

$$\frac{I_X \cap \langle\omega\rangle}{I_X\langle\omega\rangle} \cong \mathrm{Tor}_1^{\mathcal{O}_n}\left(\frac{\mathcal{O}_n}{\langle\omega\rangle}, \frac{\mathcal{O}_n}{I_X}\right),$$

therefore,

$$\dim_{\mathbb{C}} \frac{I_X \cap \langle\omega\rangle}{I_X\langle\omega\rangle} = 2\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_X + \langle\omega\rangle},$$

for further details, see [27, Proposition 5.39 and Corollary 5.47].

Motivated by this isomorphism, the following conjecture was proposed.

Conjecture 6.4. Let $(X,0) \subset (\mathbb{C}^n,0)$ be an ICIS determined $\phi : (\mathbb{C}^n,0) \to (\mathbb{C}^k,0)$ and ω a 1-form such that $\mu_{BR}(\omega, X) < \infty$ then

$$\dim_{\mathbb{C}} \operatorname{Tor}_i^{\mathcal{O}_n}\left(\frac{\mathcal{O}_n}{\langle\omega\rangle}, \frac{\mathcal{O}_n}{I_X}\right) = \binom{k}{i} \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_X + \langle\omega\rangle}.$$

When $k = 1$, the conjecture is immediate. The case $k = 2$ was proved in [30]. Recent progress on this conjecture is presented in [9], where the authors reformulate it by adding the assumption $n > k$.

Conjecture 6.5 ([9], Conjecture 5.2). Let $(R, \mathcal{M})$ be a regular local ring of dimension n, let I be an ideal generated by an R-regular sequence of length $k \leq n-1$, and let J be an ideal generated by a regular sequence of length n. Then, for any $0 \leq i \leq k$, we have

$$\ell\big(\operatorname{Tor}_i^R(R/I, R/J)\big) = \binom{k}{i} \ell\left(\frac{R}{I+J}\right),$$

where $\ell(-)$ denotes the length as an R-module.

Still about Bruce-Roberts and 1-forms, in [5], the notion of the Bruce-Roberts Tjurina number to holomorphic 1-forms was defined and a relative GSV-index was presented. The authors further established relationships between these new invariants and also obtained an inequality of the type stated in Proposition 6.1 for the Bruce-Roberts number of 1-forms and the Bruce-Roberts Tjurina number of 1-forms.

7 Conclusion

This survey presents central results concerning the Bruce–Roberts number, with an emphasis on its relationships with other fundamental invariants in Singularity Theory, including the Milnor and Tjurina numbers, the Euler obstruction, the GSV-index, and logarithmic characteristic varieties. The results discussed illustrate the structural richness of these invariants and highlight several open problems, particularly in non-homogeneous settings or in the presence of singular loci of positive dimension.

Much of the development in this area has been influenced by the contributions of Maria Aparecida Soares Ruas, whose work played a significant role in shaping the study of the Bruce–Roberts number. The perspectives outlined here suggest further avenues for investigation and indicate that these invariants remain a fertile topic for future research.

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Bárbara K. de Lima-Pereira
Instituto de Ciência, Tecnologia e Inovação (ICTIN)
Universidade Federal de Lavras
Campus São Sebastião do Paraíso
São Sebastião do Paraíso, MG, Brasil
barbarapereira@ufla.br

Bruna Oréfice-Okamoto
Departamento de Matemática
Universidade Federal de São Carlos
São Carlos, SP, Brasil
brunaorefice@ufscar.br

João Nivaldo Tomazella
Departamento de Matemática
Universidade Federal de São Carlos
Rod. Washington Luís, km 235 - SP-310, CEP 13565-905
São Carlos, SP, Brasil
jntomazella@ufscar.br