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Noether's problems

Vyacheslav Futorny and João Schwarz

Abstract. The purpose of these notes is to give an introduction to the classical (commutative) and the noncommutative Noether's problems and show their intrinsic connection. We also discuss the subrings of invariants by finite groups of families of noncommutative Galois algebras, including the Weyl algebras and the generalized Weyl algebras.

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Contents

Chapter 1

Introduction

We will assume that all algebras are considered over the filed k of characteristic zero.

Given a linear action of a finite group G on the ring of polynomials $\mathsf{k}[x_1, \ldots, x_n]$, the Classical Noether's Problem refers to the structure of the field of G-invariant rational functions in $x_1, \ldots, x_n$ asking whether this field is a purely transcendental extension of k. Translating into a geometric language the Classical Noether's Problem is equivalent to the question of rationality of the quotient variety $\mathbb{A}^n(\mathsf{k})/G$. We will discuss known cases with positive answer to the Classical Noether's Problem.

Passing to a noncommutative case we consider the algebra of differential operators on the polynomial ring $\mathsf{k}[x_1, \ldots, x_n]$ - the Weyl algebra $A_n(\mathsf{k})$, which is the simplest noncommutative deformation of the polynomial ring. As the Weyl algebra is an Ore domain, it admits the skew field of fractions. Extending a linear action of a group G to the Weyl algebra one may wonder about the structure of the skew subfield of G-invariants of $A_n(\mathsf{k})$. An analog of the Noether's Problem for the Weyl algebra A_n was first considered by Alev and Dumas in [3], who showed that for $n = 1$, $n = 2$ and an arbitrary finite group G the skew subfield of G-invariants of $A_n(\mathsf{k})$ is isomorphic to the skew field of $A_n(\mathsf{k})$. The same holds when the G-module is isomorphic to a direct sum of one dimensional representations. This led to the formulation of the Noncommutative Noether's Problem.

Though both, the Classical Problem and the Noncommutative Noether's Problem make sense for infinite groups, we will be interested only in the case of finite groups.

There has been a growing interest in the study of invariants of the Weyl algebras and, in particular, in the Noncommutative Noether's Problem. The cases in which this problem has a positive solution are of special interest in view of the rigidity of the Weyl algebras proved by Alev and Polo [7]: $A_n(\mathsf{k})^G$ is not isomorphic to $A_n(\mathsf{k})$ when k is algebraically closed, for any non trivial linear action of G. Moreover, by a recent result of Tikaradze, if $D^G \simeq A_n(\mathsf{k})$ for some domain D then $D \simeq A_n(\mathsf{k})$ and G is trivial. Nevertheless, weakening the isomorphism condition to birational equivalence (that is, their coordinate rings have isomorphic fields of fractions) gives nontrivial examples even for rigid algebras, in particular those coming from solving the Noncommutative Noether's Problem.

The Noncommutative Noether's Problem is also connected to the Gelfand-Kirillov Conjecture on the birational equivalence between the universal enveloping algebras and Weyl algebras. It was used to reprove the Gelfand-Kirillov Conjecture for gl_n and sl_n [54] and show it for all finite W-algebras of type A [53].

Checking the finite groups which appear in the examples with positive solutions for the Noncommutative Noether's Problem, one can see that the Classical Noether's Problem is also solved positively for the same groups. It leads to a natural conjecture that there might be an intrinsic connection between the cases with positive solution for these two problems.

Indeed we will show that the Classical Noether's Problem implies the Noncommutative Noether's Problem, that is, the rationality of the quotient variety $\mathbb{A}^n(\mathsf{k})/G$ for a linear finite group G implies that $A^n(\mathsf{k})^G$ and $A^n(\mathsf{k})$ are birationally equivalent. This was shown in [59]. Applying this result we immediately recover all previously known cases with positive solution for the Noncommutative Noether's Problem, and obtain new examples. In particular, we obtain the affirmative answer for all pseudo-reflections groups, for the alternating groups ($n = 3, 4, 5$) and for any finite group when $n = 3$ and k is algebraically closed.

In the case of the complex field this technique can be extended to the ring $D(X)$ of differential operators on any affine irreducible variety X equipped with an action of a finite group of automorphisms G. Generalizing the result above one shows that the subring of invariants $D(X)^G$ is birationally equivalent to $D(X)$ whenever X and the quotient variety X/G are birationally equivalent.

We give a comprehensive description of these facts with all necessary preliminaries, examples and applications. The case of pseudo-reflection groups is considered separately with a different approach, which allows us to find the Weyl generators of the skew field of fraction by a fairly simple algorithm.

Though the counterexamples are known to the Classical Noether's Problem, there are no known counterexamples to noncommutative version.

In many cases the subalgebras R^G with a ring R and a finite group G have a structure of Galois orders over certain commutative domains. This feature reflects a hidden skew group algebra structure of these algebras. The theory of Galois rings and orders was developed in [54], [55]. Examples include finite W-algebras of type A [53], in particular the universal enveloping algebra of gl_n. Having the Galois order structure on a given algebra allows to study effectively its representation theory, more explicitly the Gelfand-Tsetlin categories of modules over this algebra which have torsion for certain maximal commutative subalgebras [55].

We show that the invariants $A^n(\mathsf{k})^G$ of the Weyl algebra have the Galois order structure for many pseudo-reflection groups G. Similar results are obtained for the invariants of the algebra of differential operators on n-dimensional torus and quantizations. These Galois orders form a special family of linear Galois orders of shift type. Their quantizations belong to the family of quantum linear Galois orders which also include the universal enveloping algebra $U_q(gl_n)$, the quantum Heisenberg algebra, the subalgebras of G-invariants of the quantum affine space, and of the quantum torus.

All shift (respectively quantum) linear Galois orders satisfy the Gelfand-Kirillov conjecture (respectively the quantum Gelfand-Kirillov conjecture).

Another family of Galois orders consists of generalized Weyl algebras over integral domains with infinite order automorphisms [12] and their subrings of invariants with respect to pseudo-reflection groups.

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Chapter 2

Classic Noether's Problem

2.1 Invariant polynomials

Let $\Lambda_n = \mathsf{k}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables. If G is a finite group acting linearly on Λ_n, then denote by Λ_n^G the subring of G-invariant polynomials:

$$\Lambda_n^G = \{f \in \Lambda_n \mid g \cdot f = f, \ \forall g \in G\}.$$

Naturally, finite-dimensional representations of G are a source of linear actions on Λ_n. Let V be an n-dimensional G-module. Then the algebra $\mathsf{k}[V] \simeq \Lambda_n$ of polynomial functions on V admits a natural G-action. Denote by $\mathsf{k}[V]^G \simeq \Lambda_n^G$ the subalgebra of G-invariant polynomial functions on V.

Both algebras $\mathsf{k}[V]$ and $\mathsf{k}[V]^G$ are integral domains and hence admit the fields of fractions. The field of fractions of $\mathsf{k}[V]$ is isomorphic to the field of rational functions $\mathsf{k}(x_1, \ldots, x_n)$ and The field of fractions of $\mathsf{k}[V]^G$ is isomorphic to its invariant subfield $\mathsf{k}(x_1, \ldots, x_n)^G$.

Proposition 2.1. *The field extension*

$$\mathsf{k}(x_1, \ldots, x_n)^G \subset \mathsf{k}(x_1, \ldots, x_n)$$

is algebraic.

Proof. It is sufficient to show that every x_k is algebraic over $\mathsf{k}(x_1, \ldots, x_n)^G$. Let $z = x_k$ and consider the following polynomial

$$h_y(t) = \prod_{g \in G} (t - g \cdot y).$$

Extend the action of G on $\mathsf{k}(t, x_1, \ldots, x_n)$ with $g \cdot t = t$, for all $g \in G$. Then $h_z(t) \in \mathsf{k}(t, x_1, \ldots, x_n)^G$ and $h_y(y) = 0$. Write the polynomial $h_y(t)$ as a polynomial in t:

$$h_z(t) = h^0(x_1, \ldots, x_n) + h^0(x_1, \ldots, x_n)t + \ldots h^{|G|}(x_1, \ldots, x_n)t^{|G|}.$$

As $h_y(t)$ is G-invariant, then all coefficients $h^i(x_1, \ldots, x_n)$, $i = 1, \ldots, |G|$ are G-invariant. Hence, z is algebraic over $\mathsf{k}(x_1, \ldots, x_n)^G$ and the statement follows. □

As a consequence we immediately obtain that the transcendence degrees of $\mathsf{k}(x_1, \ldots, x_n)$ and $\mathsf{k}(x_1, \ldots, x_n)^G$ over k are the same ($\mathsf{k}(x_1, \ldots, x_n)$ is a pure transcendental extension of k of degree n). Theorem 2.4 below gives a stronger result.

Remark 2.2. The polynomial used in the proof of the proposition above suggests a natural way to construct invariant polynomials. Define the following operator on Λ_n: for $f \in \Lambda_n$ set

$$R(f) = \frac{1}{|G|} \prod_{g \in G} g \cdot f.$$

Then $R(f) \in \Lambda_n^G$ and $R : \Lambda_n \to \Lambda_n^G$ is the *Reynolds operator.*

Example 2.3.

- Let $n = 1$, $G \simeq \mathbb{Z}_2$ with the action $x \mapsto -x$. Then $Im(R) \simeq \mathsf{k}[x^2]$;
- Let $n = 2$, $G \simeq \mathbb{Z}_2$ with the action $x_1 \mapsto -x_1$, $x_2 \mapsto -x_2$. Then $Im(R)$ is generated by x_1^2, x_2^2 and $x_1 x_2$.

The algebra of invariant polynomial functions $\mathsf{k}[V]^G$ for an algebraic group G and a finite-dimensional G-module V can be quite complex [99] with rapid growth of complexity. In general, the subalgebra $\mathsf{k}[V]^G$ is not necessarily finitely generated which was the original question of Hilbert. A counter-example was constructed by Nagata in 1959. However, when G is reductive, the subring of G-invariants is finitely generated [34][1]. For later use, we prove this result for finite groups.

Theorem 2.4 (Noether). *Let R be a finitely generated k-algebra, G a finite group of k-automorphisms of R. Then R^G is a finitely generated k-algebra, the extension $R^G \subset R$ is integral and R is a finite R^G-module.*

[1] The culmination point of the XIXth century invariant theory was the Gordon's result that the algebra is finitely generated when $G = SL_2$. Then, in 1890, Hilbert surprised the mathematical world by generalizing this result for all groups SL_n in a non-constructive fashion. This led to the Gordon's famous exclamation "Das ist Theologie und nicht Mathematik"

Proof. For an arbitrary $a \in R$ consider the polynomial

$$h(t) = \prod_{g \in G} (t - g \cdot a).$$

The coefficients of $h(t)$ clearly belong to the subring R^G and $h(a) = 0$. Hence R is integral over R^G. Choose a finite set of generators $r_1, \ldots, r_m$ of R as a k-algebra and let $S \subset R^G$ be the k-subalgebra generated by the coefficients of monic polynomials in $R^G[t]$ that annihilate the generators $r_1, \ldots, r_m$. Then S is finitely generated, and hence Noetherian. Moreover, R is integral over S and finitely generated over it; hence it is a finite S-module. As S is Noetherian, R^G is also a finite S-module. This clearly implies that R^G is a finitely generated k-algebra. As R is integral over R^G, then R is finitely generated over R^G. Hence it is a finite R^G-module. □

In spite of the complexity of the invariant subring in general, sometimes the situation is simple, as in the classical case of the symmetric group S_n acting by permutations on the variables $x_1, \ldots, x_n$. The S_n-invariant polynomials are the symmetric polynomials and $\Lambda_n^{S_n} \simeq \mathsf{k}[e_1, \ldots, e_n]$, where e_i is the $i-th$ elementary symmetric polynomial. As the latter polynomials are algebraically independent, we get $\Lambda_n^{S_n} \simeq \Lambda_n$. Shephard and Todd [107] showed that this holds when G is a complex reflection group by classifying all such finite groups G. On the other hand, Chevalley ([31]) obtained a conceptual understanding of the situation, and proved that for a finite group G and for all fields of zero characteristic the isomorphism between Λ_n^G and Λ_n holds if and only if G is a pseudo-reflection group. We recall that a *pseudo-reflection* is a finite order linear automorphism whose space of fixed points has codimension 1. A pseudo-reflection group is a finite group generated by pseudo-reflections. Note that over $\mathbb{Q}$ the pseudo-reflection groups are the Weyl groups, and over $\mathbb{R}$ they are the Euclidean reflection groups, which are finite Coxeter groups. We recall that Coxeter groups are generated by a certain set S with relations of the form $s^2 = e$, $(ss')^{m(ss')} = e$, $m(ss') \in \{2, 3, \ldots, \infty\}$ and $m(ss') = m(s's)$ for $s, s' \in S$.

Despite similarities, complex reflection groups are not in general Coxeter groups, such as the general irreducible complex reflection groups $G(m, p, n)$ for example [107]. Note that Coxeter groups can be infinite.

One of the basic questions of the invariant theory: given a linear action of a group G on Λ_n, when is the subalgebra of invariant polynomial functions $\mathsf{k}[V]^G$ isomorphic to $\mathsf{k}[V]$?

Example 2.5. The symmetric group is a reflection group generated by the transpositions, which act as euclidean reflections. So the classical result about the symmetric polynomials is a particular case of the situation described by the Chevalley-Shephard-Todd theorem, which can be stated as follows.

Theorem 2.6 (Chevalley-Shephard-Todd Theorem). *Let G be a finite group which acts linearly on Λ_n and let $|G|$ be a coprime with char* k. *Then the following statements are equivalent:*

- $\Lambda_n^G \simeq \Lambda_n$;
- *G is a pseudo-reflection group;*
- *Λ_n is a finitely generated flat Λ_n^G-module;*
- *Λ_n^G is a regular ring.*

For the details of the proof we refer to [21, Theorem 7.2.1]. In case when the characteristic of the field divides the order of the group, the analogue of the Chevalley-Shephard-Todd theorem was studied in [27]: for the action of an irreducible finite group G, Λ_n^G is generated by algebraically independent elements if and only if G is generated by pseudo-reflections and the direct summand property holds. In a more general situation, necessary and sufficient conditions for the ring of invariants to be a complete intersection were obtained in [74]. All simple algebraic groups G, for which Λ_n^G is polynomial, were classified in [105].

Things get more interesting if one looks at the birational picture.

2.2 Birational equivalence

Two domains R_1 and R_2 are *birationally equivalent* if $Frac\ R_1 \simeq Frac\ R_2$. Let $\mathbb{A}^n(\mathsf{k})$ denote the n-dimensional affine space over k. An affine variety $X \subset \mathbb{A}^n(\mathsf{k})$ is *birational* to an affine variety $Y \subset \mathbb{A}^m(\mathsf{k})$ if their coordinate rings are birationally equivalent, which is equivalent to the existence to an invertible rational map between these varieties. In particular, for an affine variety X with an action of the group G the quotient variety X/G is birational to X if the field of functions on X is isomorphic to the subfield of G-invariants.

Let $K_n = \mathsf{k}(x_1, \ldots, x_n)$ be the field of rational functions in $x_1, \ldots, x_n$, that is K_n is isomorphic to the field of fractions of Λ_n. Consider a finite group G with a linear action on Λ_n. This action extends naturally to the action on the field of rational functions K_n:

$$x \cdot \frac{f}{g} = \frac{x \cdot f}{x \cdot g}$$

for $x \in G$ and $f, g \in \Lambda_n$. Denote by K_n^G the subfield of K_n consisting of invariant rational functions: $\frac{f}{g} \in K_n^G$ if and only if $x \cdot \frac{f}{g} = \frac{f}{g}$ for all $x \in G$. We will say that K_n^G is *the subfield of G-invariants of* K_n. Note that K_n^G is always finitely generated over k as a subfield of K_n.

For $G = S_n$ we clearly have $K_n^{S_n} \simeq K_n$. This leads to the following natural question about the subfields of invariants for the subgroups of S_n:

Problem 1. Let G be a finite group that acts transitively by permutations on the set of variables $x_1, \ldots, x_n$. When is the subfield of invariants K_n^G a purely transcendental extension of the base field?

This problem was first considered by Burnside ([29]), and systematically studied by Emmy Noether ([97]), who was interested in the inverse problem in the Galois theory: given a finite group G, determine whether the field of rational numbers $\mathbb{Q}$ admits a Galois extension L such that $Gal(L, \mathbb{Q}) = G$. Noether has shown that if the above problem had a positive solution for the action of the group G, then by the Hilbert's Irreducibility Theorem ([72, Chapter 3]), the inverse Galois problem had a positive solution as well. Noether's approach to this problem has not been as successful as initially imagined, but nonetheless this is still an intensive area of research (cf. [72]).

We will call the *the Classic Noether's Problem* (CNP for short) (sometimes in the literature also called the *Linear Noether's Problem*) the following problem.

Problem 2 (Classic Noether's Problem). Let G be a finite group acting linearly on K_n. When is the subfield K_n^G a purely transcendental extension of the base field? Or equivalently: when

$$K_n^G \simeq K_n?$$

In this form, the Noether's Problem is connected to many branches of algebra, such as PI-algebras ([41]) and moduli spaces ([38]) for example.

The Noether's Problem is related to the 14th Problem of Hilbert posed in 1900 which asks whether for a subfield K of K_n containing k the intersection $K \cap \Lambda_n$ is finitely generated. Again, a counter-example of Nagata shows that this is not the case in general, though the answer is affirmative if the transcendence degree of K is less or equal than 2.

The Classic Noether's Problem has a positive solution for $n = 1$:

Example 2.7 (Lüroth Theorem). If K is a subfield of a transcendental extension $\mathsf{k}(x)$ of transcendence degree 1, then K itself is a simple transcendental extension, that is $K \simeq \mathsf{k}(f)$ for some polynomial $f(x)$. Equivalently, an algebraic curve of genus zero can be parametrized by a rational parameter.

In particular, for invariant polynomials in one indeterminate we have [91]: Let $\Lambda = K[x]$, K a commutative field, $F = K(x)$, G a group of automorphisms of Λ such that $G(K) \subset K$. Then either:

- $\Lambda^G \subset K$ and $\Lambda^G = F^G = K^G$, or
- Λ^G is not a subset of K and $\Lambda^G = K^G[u]$, $F^G = K^G(u)$, where $u \notin K$ is an invariant polynomial of minimal possible degree.

From Example 2.7 we obtain

Corollary 2.8. $K_2^G \simeq K_2$ *for any group* G.

Proof. Let $\Lambda = \mathsf{k}[x,y]$ and $Frac\,\Lambda = \mathsf{k}(x,y)$. Set $K' = \mathsf{k}(\frac{y}{x})$ and $S = K'[x]$. Then $Frac\,S = Frac\,\Lambda = \mathsf{k}(x,y)$. Note that $G(K') \subset K'$. Indeed, suppose that $g \in G$ acts as follows: $gx = ax + by$ and $gy = a'x + b'y$, for some $a, b, a', b' \in \mathsf{k}$. Then we have

$$g(\frac{y}{x}) = \frac{a' + b'(\frac{y}{x})}{a + b(\frac{y}{x})} \in K'.$$

Since

$$[Frac\,S : Frac\,S^G] = |G|,$$

then $S^G \neq K'$, and $Frac\,S^G = K'^G(t)$ for some element $t \in S^G$ by Example 2.7. We appy the Lüroth theorem to the field K'^G and obtain that $K'^G \simeq \mathsf{k}(z)$ is a simple transcendental extension of k. We conclude that

$$Frac\,\Lambda^G \simeq Frac\,S^G \simeq K'^G(t) \simeq \mathsf{k}(z,t).$$

□

Example 2.9 (Miyata Theorem). Let G be a subgroup of the group of invertible upper triangular $n \times n$ matrices and $V = \mathsf{k}^n$. Then $Frac\,\mathsf{k}[V]^G$ is a purely transcendental extension of k.

As a consequence of the Miyata Theorem we immediately obtain the following example which was originally shown in [48].

Example 2.10. The Classic Noether's Problem has a positive solution if k is algebraically closed and G is finite abelian subgroup of $GL(V)$. More general, it has a positive solution if k is algebraically closed and G is an abelian group consisting of diagonalizable matrices.

Geometrically the Classic Noether's Problem asks whether $\mathbb{A}^n(\mathsf{k})/G$ is *birational* to $\mathbb{A}^m(\mathsf{k})$ for some m, or equivalently to the projective space $\mathbb{P}^m(\mathsf{k})$. Note that if this is the case, then the transcendence degree considerations imply that $m = n$. If the variety X is *birational* to $\mathbb{A}^n(\mathsf{k})$ then we say that X is *rational* (cf. [106]). Roughly speaking, the rationality of X means that modulo some subspaces of smaller dimensions it can be identified with the affine space. Clearly, the rationality of the variety $\mathbb{A}^n(\mathsf{k})/G$ is a weaker condition than the situation addressed by the Chevalley-Shephard-Todd theorem, where $\mathbb{A}^n(\mathsf{k})/G \simeq \mathbb{A}^n(\mathsf{k})$.

Remark 2.11. We note that the rationality of the quotient variety is hardly expected if G is not a connected group. Moreover, no non-rational examples is known if G is a connected group [102]. For example, rationality holds for algebraically closed k and connected solvable G by the Lie-Kolchin theorem and the Miyata's theorem.

The Lüroth theorem and Corollary 2.8 imply that the varieties $\mathbb{A}^1(\mathsf{k})/G$ and $\mathbb{A}^2(\mathsf{k})/G$ are rational for any group G.

Example 2.12 (Rational surfaces). Rational surface is a rational variety of dimension two, which is birational to the projective plane. The Castelnuovo rationality criterion states that any smooth algebraic surface X over algebraically closed field of characteristic zero, with no regular differential 1-forms and 2- forms and with zero bigenus P_2, is a rational surface. Zariski [117] proved that Castelnuovo's theorem also holds over fields of positive characteristic. Examples of rational surfaces include del Pezzo surfaces, Fermat cubic and Cayley cubic, Segre surface among sothers [9].

An n-dimensional variety of dimension X is unirational if there is a dominant map from the projective space $\mathbb{P}^n$ to X. The Castelnuovo's theorem implies that any unirational complex surface is rational. On the other hand, in positive characteristic this is not the case as there exist surfaces (e.g. Zariski surfaces) which are not rational. Most unirational complex varieties of dimension 3 or larger are not rational (cf. [70]).

Example 2.13 (Zariski-Castelnuovo Theorem). Let k be an algebraically closed field. The Castelnuovo rationality criterion in characteristic zero case and Zariski's extension for positive characteristic imply the rationality of plane involution: If K is a subfield of a a purely transcendental extension $\mathsf{k}(x,y)$ of k of transcendence degree 2 then K is also a purely transcendental extension of k of transcendence degree 2.

Corollary 2.14 ([29]). *If k is an algebraically closed field and G is a finite group, then the variety $\mathbb{A}^3(\mathsf{k})/G$ is rational.*

Proof. Let $\Lambda = \mathsf{k}[x,y,z]$ and $Frac\,\Lambda = \mathsf{k}(x,y,z)$. Set $K' = \mathsf{k}(\frac{y}{x},\frac{z}{x})$ and $S = K'[x]$. Then $Frac\,S = Frac\,\Lambda = \mathsf{k}(x,y,z)$ and $G(K') \subset K'$. Indeed, let $g \in G$ be such that $gx = ax+by+cz$ and $gy = a'x+b'y+c'z$, for some $a,b,c,a',b',c' \in \mathsf{k}$. Then

$$g(\frac{y}{x}) = \frac{a' + b'(\frac{y}{x}) + c'(\frac{z}{x})}{a + b(\frac{y}{x}) + c(\frac{z}{x})} \in K'.$$

Similarly $g(\frac{z}{x}) \in K'$. We apply Example 2.7: since $[Frac\,S : Frac\,S^G] = |G|$, we have $S^G \neq K'$, and hence $Frac\,S^G = K'^G(t)$ for some element $t \in S^G$. Since K'^G is an intermediate field between k and K' then $K'^G = \mathsf{k}(u,v)$, for some $u,v \in K'$ by the Zariski-Castelnuovo Theorem. Here u and v are algebraically independent elements. We conclude that

$$Frac\,\Lambda^G = Frac\,S^G = \mathsf{k}(u,v,t).$$

□

Example 2.15 (Stably-rational varieties). Stably-rational varieties is an important class of varieties. We say that a variety X is *stably-rational* if $X \times \mathbb{A}^m(\mathsf{k})$ (equivalenty $X \times \mathbb{P}^m(\mathsf{k})$) is rational for some $m \geq 0$. An important question is the Zariski problem: is every stably-rational variety rational? This is known to be true for dimensions 1 and 2. For higher dimensions (difficult) counter-examples have been found [17].

Following the discussion above we summarize the well known examples with the positive solution for the CNP for finite groups.

- For $n = 1$ and $n = 2$.

  Remark 2.16. When $n = 1$ there is no need to assume that the action of the group is linear. This follows from the Lüroth theorem. The same holds when $char \mathsf{k} = 0$ for $n = 2$, as in this case every finite group of automorphisms is linearizable [43].

- For $n = 3$, k is algebraically closed.

- For any n when G is a pseudo-reflection group.

  Indeed, let D be a commutative domain and G a finite group of automorphisms of D. Then one sees easily that $Frac\, D^G = (Frac\, D)^G$ (cf. Theorem 3.15 for a more general situation). Since in this case $\Lambda_n^G \simeq \Lambda_n$ then the statement follows.

- For $n = 2k$ and $G = S_k$ with the action of G on $\mathsf{k}(x_1, \ldots, x_k, y_1, \ldots, y_k)$ by simultaneous permutation of the x's and y's [86].

  It follows from [91, Remark 3], as $\mathsf{k}(x_1, \ldots, x_k)^{S_k}$ is rational.

- For any n, when the representation of G is isomorphic to a direct sum of one dimensional representations [48].
  Follows immediately from the Miyata's theorem.

- For any representation of a finite p-group over an arbitrary field of characteristic p, where p is a prime number [91].

- For $n = 3, 4, 5$ and $G = A_n$ is an alternating group [85]. For $n > 5$ this is still an open problem (cf. [72]).

The first counter-examples to the CNP are due to Swan and Voskresenskii [112], [115] for the action of the cyclic group of order 47 over the purely transcendental extension of the field of rational numbers of the same degree. Later Saltman found counter-examples also for algebraically closed fields. A detailed reference for these results are [43] and [72]. The smallest group that can give a counter-example to CNP is the cyclic group of order 8. Necessary and sufficient conditions for the CNP to have a positive solution for finite abelian groups acting by permutations are found in [82]. A survey of recent results on the CNP and related questions is [67].

Chapter 3

Algebras of differential operators

3.1 Differential operators

Let R be a commutative unital k-algebra, $char\,\mathsf{k} = 0$. The ring of differential operators $D(R)$ on R was introduced by Grothendieck in [63]. This is an associative k-subalgebra of $\mathrm{End}_{\mathsf{k}}(R)$ defined inductively as follows. First we embed R in $\mathrm{End}_{\mathsf{k}}(R)$: every element $r \in R$ defines the scalar multiplications $l_r \in \mathrm{End}_{\mathsf{k}}(R)$: $a \mapsto ra$ for all $a \in R$. Moreover, we identify R with $\mathrm{End}_R(R)$ and set $D(R)_0 = R$. Then for any $n > 1$, set

$$D(R)_n \;=\; \{\, d \in \mathrm{End}_{\mathsf{k}}(R) \;:\; d\,b - b\,d \in D(R)_{n-1} \ \text{ for all } \ b \in R\}$$

and $D(R) := \cup_{n=0}^{\infty} D(R)_n$. In particular, $R \subset D(R)$. Clearly, each $D(R)_n$ is a k-vector space. Moreover, $D(R)_i \subset D(R)_{i+1}$ for all $i \geq 0$ and

$$D(R)_i D(R)_j \subset D(R)_{i+j}$$

for all i, j. Therefore $D(R)$ comes with a natural filtration

$$\mathcal{D} = \{D(R)_i\}_{i \geq 0}$$

with commutative associated graded algebra. If $d \in D(R)_n$ and $d \notin D(R)_{n-1}$ then d is a differential operator of *order* n. Hence, the filtration above is given by orders of differential operators.

Example 3.1. If X is a smooth affine variety and $R = \mathcal{O}(X)$ is the coordinate ring of X, then the associated graded algebra of $D(R)$ is $\mathcal{O}(T^*X)$.

Define a Lie bracket on $\mathrm{End}_{\mathsf{k}}(R)$ by

$$[f, g] = fg - gf$$

for $f, g \in \mathrm{End}_{\mathsf{k}}(R)$.

For $d_1, d_2 \in D(R)_1$ and $a \in R$ we have that $[d_1, d_2](a) = d_1 d_2(a) - d_2 d_1(a) \in R$ and $[d_1, d_2] \in D(R)_1$. Hence, the subspace $D(R)_1$ is a Lie subalgebra of $\mathrm{End}_{\mathsf{k}}(R)$ with respect to the bracket $[\,,\,]$.

If d is a differential operator of order n and $a_1, \ldots, a_{n+1} \in R$ then we have the following reduction formula [94]:

$$d(a_1 \ldots a_{n+1}) = \sum_{k=0}^{n} (-1)^{n+k} \sum_{i(1)<\ldots<i(k)} a_1 \ldots \hat{a}_{i(1)} \ldots \hat{a}_{i(k)} \ldots a_{n+1} d(a_{i(1)} \ldots a_{i(k)}).$$

Applying this formula for a differential operator $d \in D(R)_1$ such that $d(1) = 0$, we obtain $d(ab) = ad(b) + d(a)b$, that is any such differential operator of order 1 satisfies the Leibniz rule.

A k-*derivation* of R is an element d of the space $\mathrm{End}_{\mathsf{k}}(R)$ which satisfies the Leibniz rule: $d(ab) = ad(b) + d(a)b$ for all $a, b \in R$. The set $Der_{\mathsf{k}} R \subset \mathrm{End}_{\mathsf{k}}(R)$ of all such derivations is a k-vector space and an R-module:

$$(r \cdot d)(a) = rd(a) \in Der_{\mathsf{k}} R$$

for all $r, a \in R$. Moreover, $Der_{\mathsf{k}} R$ is a Lie subalgebra o $\mathrm{End}_{\mathsf{k}}(R)$ with the Lie bracket defined above. It is known as *the derivation algebra* of R. Clearly, $Der_{\mathsf{k}} R \subset D(R)_1$. Moreover, we have the following decomposition.

Proposition 3.2. *$D(R)_1 \simeq Der_{\mathsf{k}} R \oplus R$.*

Proof. Let $d \in D(R)_1$ and $z = d - d(1)$. Then for all $r, s \in R$ we have

$$0 = [s, [z, r]](1) = s(z(r) - rz(1)) - z(rs) + rz(s).$$

As $z(1) = 0$, we get $z(rs) = sz(r) + rz(s)$, and hence $z \in Der_{\mathsf{k}} R$. As $Der_{\mathsf{k}} R \cap R = 0$, we conclude that $D(R)_1 = Der_{\mathsf{k}} R \oplus R$. □

Hence, the ring $D(R)$ always contains the subring generated by $Der_{\mathsf{k}} R$ and R. Moreover, we have the following result which shows that under certain conditions $D(R)$ is generated by $Der_{\mathsf{k}} R$ and R.

Proposition 3.3 ([89, Section 15]). *If R is affine and regular, then $D(R)$ is generated by $Der_{\mathsf{k}} R$ and R.*

The converse statement is known as the Nakai Conjecture [95]. This is still an open problem.

Rings of differential operators play an important role in many areas, in particular in ring theory ([89]) and in representation theory ([68]). The most important example is the n-th Weyl algebra $A_n(\mathsf{k}) \simeq D(\mathsf{k}[x_1, \ldots, x_n])$, which will be discussed in the next section.

The next theorem summarizes the principal properties of $D(R)$ for affine regular domains, see [89, Chapter 15].

Theorem 3.4. *Let R be a regular affine domain and $n = tdeg\, R$. Then*

- *$D(R)$ is finitely generated.*
- *$D(R)$ is a simple Noetherian domain and R is a simple $D(R)$-module.*
- *The left and right Krull dimension of $D(R)$ equals n.*
- *The Gelfand-Kirillov dimension of $D(R)$ equals $2n$;*
- *The global dimension of $D(R)$ equals n.*

Suppose now that the commutative ring R is *reduced*, that is without nonzero nilpotent elements. In addition suppose that R is finitely generated k-algebra over a field k of characteristic zero and of Krull dimension ≤ 1. Then $D(R)$ is a noetherian ring and it's left and right Krull dimension coincides with the Krull dimension of R [94, Theorem 3.1].

Example 3.5. Let $R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$, the coordinate ring of the normal cubic cone. Then $D(R)$ is neither left nor right noetherian [22]. This shows the necessity of the conditions above.

Example 3.6. If X is any curve, even singular, then $D(\mathcal{O}(X))$ is always finitely generated and left and right Noetherian [110].

Let R be any ring with identity and $S \subset R$ a multiplicatively closed subset. If S is a left Ore set then R admits a left (respectively right) ring of fractions with respect to S, that is a localization by S on the left (respectively on the right). If R is a domain such that $R^* = R \setminus \{0\}$ is both a left and a right Ore set then R is called *Ore domain.*

Proposition 3.7. *Let R be an affine domain and a k-algebra. Then $D(R)$ is an Ore domain.*

Proof. Let $K := Frac\, R$ be the fraction field of R. We can realize $D(R)$ as a subset of $D(K)$ in the following way ([89, 15.5.5(iii)]):

$$D(R) = \{d \in D(K) | d(R) \subset R\}.$$

Now, since K is finite field extension of k, $D(K)$ is a non-commutative domain with finite Gelfand-Kirillov dimension. Since $D(R)$ is a subring of $D(K)$, the same properties hold for it. Hence, $D(R)$ does not contain a subalgebra isomorphic to the free associative algebra in two variables. Then $D(R)$ is an Ore domain (left and right) by the result of Jategaonkar [80, Proposition 4.13]. □

We finish the discussion of rings of differential operators by pointing out that our definition makes perfect sense for smooth varieties in prime characteristic $p \neq 0$ (cf. [109]).

If R is an Ore domain (not necessarily commutative) then localizing by R^* we obtain the ring of fractions $Frac(R)$ (if R is commutative then

$Frac(R) = Frac\, R$ is the field of fractions of R). We will call $Frac(R)$ the *skew field of fractions* of R. Hence,

$$Frac : R \mapsto Frac(R)$$

defines a functor from the category of Ore domains with injective homomorphisms to the category of skew fields. If $\phi : R_1 \to R_2$ is a morphism of domains such that $Frac(\phi)$ is an isomorphism of skew fields then we say that R_1 and R_2 are birational equivalent. As we saw above, for a finitely generated commutative k-algebra R with no zero divisors, the ring $D(R)$ is an Ore domain (both left and right), and hence $D(R)$ admits the skew field of fractions $Frac(D(R))$.

3.2 Weyl algebras

Through this section assume $char\, \mathsf{k} = 0$.

The Weyl algebra $A_n(\mathsf{k})$ is isomorphic to the the ring of differential operators on the polynomial algebra Λ_n. It can also be described as the unital associative algebra generated over k by the elements $x_1, \dots, x_n$, $\partial_1, \dots, \partial_n$ subject to the relations

$$\partial_i x_j - x_j \partial_i = \delta_{ij},\ \ x_i x_j = x_j x_i,\ \ \partial_i \partial_j = \partial_j \partial_i,$$

for $1 \leq i, j \leq n$. Hence, the Weyl algebras are the simplest noncommutative deformations of polynomial algebras.

Naturally, this definition makes sense in prime characteristic, but in this case the Weyl algebra behaves quite differently from the characteristic 0 case [101].

Historically, the Weyl algebras appeared from the matrix formalism of quantum mechanics. The modern age of Weyl algebras started from the discovery of their connection with Lie theory: let $\mathfrak{n}$ be a nilpotent finite dimensional Lie algebra over an algebraically closed field of zero characteristic. Then all primitive quotients of its enveloping algebra are isomorphic to a Weyl algebra ([37]). The name "Weyl algebra" was introduced in 1968 by Dixmier, following a suggestion of Segal (cf. [36]). The Weyl algebras are one of the most studied noncommutative rings ([89]).

There is another approach in defining a Weyl algebra. It can be introduced as an iterated Ore extension $\mathsf{k}[x_1, \dots, x_n][y_1; \partial_1] \dots [y_n; \partial_n]$.

Let's summarize the main properties of the Weyl algebra. First some notation. Given a n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we denote

$$|\alpha| = \sum_i \alpha_i,\ \ \alpha! = \prod_i \alpha_i,\ \ x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

Similarly we define ∂^α.

Proposition 3.8.

- *$A_n(\mathsf{k})$ has as basis $\{x^\alpha \partial^\beta\}, \alpha, \beta \in \mathbb{N}^n$*
- *the Weyl algebra has a finite dimensional filtration $\mathcal{B} = \{B_i\}_{i \geq 0}$, $B_i = \langle x^\alpha y^\beta \rangle_{\mathsf{k}}, |\alpha| + |\beta| \leq i$, the* Bernstein filtration*, such that the associated graded algebra is Λ_{2n}.*
- *The center and units of $A_n(\mathsf{k})$ restrict to the scalars.*
- *Every one sided ideal of $A_n(\mathsf{k})$ is generated by at most 2 elements (cf.[24]).*

Remark 3.9. As a ring of differential operators, $A_n(\mathsf{k})$ also admits a filtration by the order of differential operators. The graded associated algebra is isomorphic to Λ_{2n}. Note that in this case the filtration is not finite dimensional; hence the importance of the Bernstein filtration.

As an illustration of the relevance of the Weyl algebra, we will discuss its connection with one of the main problems of affine algebraic geometry: the Jacobian Conjecture.

Let $F : \mathsf{k}^n \to \mathsf{k}^m$ be a map. We say that it is a *polynomial map* if there exists $F_1, \ldots, F_m \in \Lambda_n$ such that $F = (F_1, \ldots, F_m)$. A polynomial map that has an inverse which is also a polynomial map is called a polynomial isomorphism.

Let $F : \mathsf{k}^n \to \mathsf{k}^n$ be a polynomial map. Consider its Jacobian $J(F)$. If F is a polynomial isomorphism, then $J(F)$ is an invertible at every point; hence the determinant $\Delta(F)$ is a polynomial function invertible at every point, and hence a constant.

Suppose that a base field is the field of real or complex numbers. Then, if $\Delta(F)(p) \neq 0$ in some point p, then the function is locally invertible by the inverse function theorem. However, even in the case of a polynomial F, it is not necessarily invertible.

Inspired by the above phenomena in the analytic situation, Keller stated in 1939 the following conjecture [76]:

Conjecture 1 (Jacobian Conjecture). *Let $F : \mathsf{k}^n \to \mathsf{k}^n$ be a polynomial map. If $\Delta(F) = 1$ in the whole k^n, then F has a polynomial inverse.*

It is easy to see that the Conjecture is true when $n = 1$. Indeed, let $F : \mathsf{k} \to \mathsf{k}$ be a polynomial map. The condition on the Jacobian Conjecture implies that dF/dx is a constant. Hence F is linear and has polynomial inverse.

However, it is still an open problem whether the Conjecture holds even for $n = 2$. Some partial results have been obtained. For instance, it is false in positive characteristic. For a survey of recent results, see [11].

Let us now discuss how the Weyl algebra enters the picture. As we saw, $A_n(\mathsf{k})$ is a simple algebra. Hence every nonzero endomorphism of $A_n(\mathsf{k})$ is injective. Dixmier [36] stated a number of problems for the Weyl algebras. One of them is the following conjecture.

Conjecture 2 (Dixmier Conjecture). *Every nonzero endomorphism of the Weyl algebra $A_n(\mathsf{k})$ is surjective. Hence, every such endomorphism is an automorphism.*

Very important by itself, the Dixmier Conjecture implies the Jacobian Conjecture. For a proof we refer to [33]. Recently it has been discovered that both Conjectures are essentially equivalent; for a short proof see [14].

Representation theory of Weyl algebras has also important applications in analysis. Let f be a real polynomial in n indeterminates and let $\Omega \subset \mathbb{R}^n$ be an open region such that f is nonnegative on Ω and zero on its boundary. For any complex number λ with positive real part consider the function

$$f_\Omega(\lambda) : \mathbb{R}^n \to \mathbb{C}$$

given by $f_\Omega(\lambda)(x) = f(x)^\lambda$ if $x \in \Omega$, and $f(x) = 0$ otherwise. At the ICM-1954 I. M. Gelfand asked whether this function f_Ω can be extended to a meromorphic function on the whole complex plane. The problem was initially solved by Bernstein and Gelfand [23] and by Atiyah [8] using the Hironaka theory of resolutions of singularities. Later, A.Joseph obtained an elementary proof of this result using only elementary properties of Weyl algebra modules (see [80, Chapter 8]).

3.3 Weyl fields

Since $A_n(\mathsf{k})$ is a simple noetherian Ore domain, it admits the skew field of fractions $F_n(\mathsf{k}) := Frac(A_n(\mathsf{k}))$, which is usually called the *Weyl field*. In this section we are going to discuss the properties of the Weyl fields. First we need some preliminaries.

Let R be any left and right Noetherian domain. Let $A = R[x; \alpha, \delta]$ be an Ore extension, where α is an automorphism of R, δ is an α-derivation of R, that is δ is a homomorphism of abelian groups such that

$$\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$$

for all $a, b \in R$, and

$$xr = \alpha(r)x + \delta(r).$$

By the Hilbert basis theorem for Ore extensions (cf. [89]), A is also a left and right Noetherian domain, and hence it admits the skew field of fractions $\mathrm{Frac}(A)$. Set $K = \mathrm{Frac}(R)$. Then α extends uniquely to an automorphism of K:

$$\alpha(xy^{-1}) = \alpha(x)\alpha(y)^{-1},$$

for $x, y \in R$. Also, δ extends to a unique α-derivation of K by

$$\delta(s^{-1}) = -\alpha(s)^{-1}\delta(s)s^{-1},$$

for $0 \neq s \in R$. We can, then, form the Ore extension $K[x;\alpha,\delta]$ and we have:

Proposition 3.10. $\mathrm{Frac}(A) = \mathrm{Frac}(K[x;\alpha,\delta])$.

Proof. The inclusion $\mathrm{Frac}(A) \subset \mathrm{Frac}(K[x;\alpha,\delta])$ is clear. On the other hand, any element of $K[x;\alpha,\delta]$, after taking common denominators, may be written in the form $s^{-1}a$, with $a \in A, s \in R$. So, if $x, y \in K[x;\alpha,\delta]$, then

$$xy^{-1} = s^{-1}ab^{-1}t$$

for some $s, t \in R$, $a, b \in A$. Hence

$$\mathrm{Frac}(K[x;\alpha,\delta]) \subset \mathrm{Frac}(A).$$

□

We will denote $\mathrm{Frac}(K[x;\alpha,\delta])$ by $K(x;\alpha,\delta)$, and write simply $K(x;\alpha)$ if $\delta = 0$ and $K(x;\delta)$ if $\alpha = id$.

Let K be a division ring, α an automorphism of K and δ an α-derivation. The skew field $K(x;\alpha,\delta)$ can be regarded as a subfield of the ring of skew Laurent series $K((x^{-1},\alpha^{-1},-\delta\alpha^{-1}))$. Its elements are Laurent series of the form

$$\sum_{i\geq m}^{\infty} a_i x^{-i},$$

where $i \in \mathbb{Z}$, $a_i \in K$, $a_m \neq 0$, with the multiplication given by

$$x^{-1}a = \sum_{n\geq 1} \alpha^{-1}(-\delta\alpha^{-1})^{n-1}(a)x^{-n} = \alpha^{-1}(a) - x^{-1}\delta\alpha^{-1}(a)x^{-1}.$$

In fact, multiplying on the left and on the right by x one obtains the same relations as in $K[x;\alpha,\delta]$. Hence, $K[x;\alpha,\delta]$ is a subring of $K((x^{-1},\alpha^{-1},-\delta\alpha^{-1}))$ and hence $K(x;\alpha,\delta)$ is a subfield. When $\alpha = id$ we write $K((x^{-1},\delta))$. These are the rings of *pseudo-differential operators.*

Lemma 3.11. *Let K be a division ring with center $Z(K)$. Let σ be an automorphism of K such that σ^n is not inner (is not a conjugation by a fixed element) for every $n \geq 1$. Then the center Z of $K(x;\sigma)$ is the subfield $Z(K) \cap K^\sigma$, where $K^\sigma = \{k \in K | \sigma(k) = k\}$.*

Proof. In the embedding of $K(x;\sigma)$ in $K((x^{-1},\sigma^{-1}))$ every $f \in K(x;\sigma)$ can be written in the form $\sum_{j\geq m} a_j x^{-j}$, where $j \in \mathbb{Z}$, $a_j \in K$. As $\delta = 0$, the multiplication is given by

$$x^j a = \sigma^j(a)x^j, a \in K, j \in \mathbb{Z}.$$

If $f \in Z$ then $xf = fx$ and $af = fa$ for all $a \in K$. As $xa_j = \sigma(a_j)x$, the first equality implies that $a_j \in K^\sigma$, while the second equality implies $aa_j = a_j\sigma^{-j}(a)$ for all $j \geq m$. As σ^j is not inner, we necessarily have $a_j = 0$ for all $j \neq 0$. □

Now we are going to specify this discussion to the Weyl fields. Consider first $A_1(\mathsf{k})$ and set $w = \partial x$. We have $wx = xw + x$ and hence, the subalgebra of $A_1(\mathsf{k})$ generated by x and w is $\mathsf{k}[x][w; d]$, in which d is the derivation $x\partial_x$. We have $\partial w = (w+1)\partial$, and hence the subalgebra of $A_1(\mathsf{k})$ generated by ∂ and w is $\mathsf{k}[w][\partial; \theta]$, in which θ is the automorphism of $\mathsf{k}[w]$ defined by $w \mapsto w + 1$.

It is clear that the skew fields of fractions of these two subalgebras equal the whole skew field of fractions of $A_1(\mathsf{k})$. Hence

$$F_1(\mathsf{k}) \simeq \mathsf{k}(x)(w; d) \simeq \mathsf{k}(w)(\partial; \theta).$$

Inductively for $A_n(\mathsf{k})$, calling $w_i = \partial_i x_i$, we have the following proposition.

Proposition 3.12.

- $F_n(\mathsf{k}) \simeq \mathsf{k}(x_1, \ldots, x_n)(w_1; d_1) \ldots (w_n; d_n)$ *with* $d_i = x_i\partial_{x_i}$.
- $F_n(\mathsf{k}) \simeq \mathsf{k}(w_1, \ldots, w_n)(\partial_1; \theta_1) \ldots (\partial_n; \theta_n)$, *where* $\theta_i(w_j) = w_j + \delta_{ij}$ *and fixes all* ∂_j *with* $j < i$.

Since for any i, no power of θ_i is an inner automorphism, we can apply this proposition and the previous lemma to obtain the following statement.

Corollary 3.13. *The center of* $F_n(\mathsf{k})$ *equals* k.

We also have

Theorem 3.14. *Let* L *and* L' *be two purely transcendental extensions of* k *with finite transcendence degrees. Then* $F_n(L) \simeq F_m(L')$ *if and only if* $n = m$ *and* $tdeg\, L = tdeg\, L'$.

Proof. Since the centers of $F_n(L)$ and $F_m(L')$ are L and L' respectively by the corollary above, we must have $tdeg\, L = tdeg\, L'$. The Gelfand-Kirillov transcendence degree of $F_n(\mathsf{k})$ (which coincides with the Gelfand-Kirillov dimension of $A_n(\mathsf{k})$) is $2n$. This implies immediately that $m = n$. □

3.4 Invariant differential operators

Suppose that the algebra R is equipped with an action of a finite group G. Then this action can be extended to the ring $D(R)$ of differential operators on R by conjugation: if $d \in D(R)$ then $(g * d) \cdot f = (g \circ d \circ g^{-1}) \cdot f$ for any $f \in R, g \in G$. The elements of $D(R)$ invariant under the action of G are called *G-invariant differential operators* on R. Study of invariant differential operators is an active area of research (cf. [104]).

We recall an important result of C. Faith [46].

Theorem 3.15. *Let R be a left and a right Ore domain and let $Frac(R)$ denotes its skew field of fractions. Let G be a finite group of automorphisms of R. Then R^G is also a left and a right Ore domain and $Frac(R^G) = Frac(R)^G$.*

Proof. Note first that if I and J are two non trivial left ideals of R then $I \cap J \neq \{0\}$. Indeed, if $i \in I$ and $j \in J$ are two nonero elements, then by the left Ore condition, $Ri \cap Rj \neq \{0\}$. Inductively, the intersection of any finite number of non trivial left ideals is a nontrivial left ideal of R. Let $0 \neq x \in Frac(R)^G$. It has the form $t^{-1}b$. Let $N = \bigcap_{g \in G} g(Rt)$. It is a non trivial left ideal that satisfies $G(N) \subset N$, and hence, by Isaac and Bergson's Theorem ([92, Corollary 1.5]) there exists $v = dt \in N$, non null and G-invariant. Then

$$x = t^{-1}b = t^{-1}d^{-1}db = v^{-1}u,$$

where $u = db$. As x, v are G-invariant, so is u. Hence, we have (*) every element from $Frac(R)^G$ different from zero has the form $v^{-1}u$, with $0 \neq u, v \in R^G$. Let now $a, b \in R^G$ be non null elements; $ab^{-1} \in Frac(R)^G$ is of the form $v^{-1}u$ for some $0 \neq u, v \in R^G$. This implies that $va = ub$, which is exactly the left Ore condition for R^G. The right Ore condition is proved similarly. Finally, by (*) its clear that $Frac(R)^G = Frac(R^G)$. □

Chapter 4

Noncommutative Noether's Problem

4.1 Rigid Algebras

Recall the following natural problem of the invariant theory:

- Let A be an associative algebra and H and G be two finite groups of automorphisms of A. Suppose $A^G \simeq A^H$. Is it true that $H \simeq G$?

This is known to be true for the first Weyl algebra $A_1(\mathbb{C})$ [4]. If one of the groups is trivial then we have, in the terminology of [7], the *Galois embedding problem*:

- Let A be an associative algebra and G a finite group of automorphisms of A. When is it possible to have

$$A^G \simeq A?$$

The most famous example of the Galois embedding problem is the positive solution given by the Chevalley-Shephard-Todd theorem: $\Lambda_n^G \simeq \Lambda_n$ when G is a pseudo-reflection group. It is also not hard to find examples of noncommutative algebras A for which $A^G \simeq A$ for non-trivial group G.

Example 4.1 ([78]). Let R be an algebra with an automorphism σ of order $n-1$ ($\sigma^n = \sigma$) over the complex numbers. Let β be an n-th primitive root of unity. Consider the noncommutative algebra $A = R[z, \sigma]$ together with an automorphism g such that $g|_R = id_R$ and $gz = \beta z$. Let $G = \langle g \rangle$. Then we have

$$A^G \simeq R[z^n, \sigma^n] \simeq A.$$

However, as noted by the authors in [78], such situation for noncommutative algebras seems to be unusual. In fact, one expects in general the following property to hold for a noncommutative algebra A:

- *(Rigidity)* For a finite group of automorphisms G of A, the isomorphism $A^G \simeq A$ is only possible when $G = \{e\}$.

This phenomenon we are going to address in this section.

Recall that a polynomial algebra is a relatively free object in the variety of commutative algebras. Naturally, one of the first attempts in generalizing the phenomena of Chevalley-Shephard-Todd theorem was made for free and relatively free algebras. We recall the necessary definitions.

Definition 4.2.

- Denote by $L_n(\mathsf{k}) = \mathsf{k}\langle x_1, \ldots, x_n\rangle$ the free associative algebra in n indeterminate and by $L(\mathsf{k}) = \mathsf{k}\langle x_1, x_2, \ldots, \rangle$ the free associative algebra in countably many variables. An ideal T of $L_n(\mathsf{k})$ or $L(\mathsf{k})$ is called a *T-ideal* if it is closed with respect to all algebra endomorphisms.

- An algebra A satisfies a polynomial identity if there is an element $f(x_1, \ldots, x_n) \in L(\mathsf{k})$ such that $f(a_1, \ldots, a_n) = 0$ for all n-tuples of elements of A. In this case A is called a PI-algebra.

- Let T be a T-ideal of $L_n(\mathsf{k})$ and let $\mathfrak{A}$ be the class of all algebras satisfying all polynomial identities in T. Then $\mathfrak{A}$ is called the variety associated to T.

- Given a T-ideal in $L_n(\mathsf{k})$, the *relatively free algebra of rank* n, is the quotient $L_n(\mathsf{k})/T$. It is a free object in the variety of algebras associated to T, in $\{y_1, \ldots, y_n\}$, which are the images of the canonical generators of $L_n(\mathsf{k})$ in the quotient.

We have a natural linear action of the group $GL_n(\mathsf{k})$ on the free associative algebra $L_n(\mathsf{k})$ by linear substitution: if $t = (t_{ij}) \in GL_n(\mathsf{k})$, then

$$t \cdot x_j = \sum_{i=1}^{n} t_{ij} x_i,$$

$j = 1, \ldots, n$, and $t \cdot f(x_1, \ldots, x_n) = f(t \cdot x_1, \ldots, t \cdot x_n)$. We have the following result for the invariant subalgebras of free associative algebras [81], [77]:

Theorem 4.3. *Let G be a finite group of linear algebra automorphisms of $L_n(\mathsf{k})$. Then the subalgebra of invariants $L_n(\mathsf{k})^G$ is also a free algebra.*

Free algebras can be embedded into free algebras of smaller rank. Given that, what is the rank of the subalgebra of G-invariants? The rank is finite (that is, $L_n(\mathsf{k})^G$ is finitely generated) if and only if G is generated by scalar multiplications [35, Theorem 1.2, iii)], in which case the group G is cyclic, generated by a scalar matrix αI, and α is a primitive $|G|$th root of unity. We also have an isomorphism

$$L_n(\mathsf{k})^G \simeq L_{n|G|}(\mathsf{k}).$$

In particular, we have the following result which shows that the free associative algebra is rigid [35]:

Theorem 4.4. *Let G be a finite group of linear automorphisms of $L_n(\mathsf{k})$. If $L_n(\mathsf{k})^G \simeq L_n(\mathsf{k})$ then G is trivial.*

Let us now consider independent commuting indeterminates

$$\{x_{ij}^k, 1 \leq i, j \leq m,\ k \in \mathbb{N}\}.$$

For each $k \in \mathbb{N}$, the matrix $X_k = (x_{ij}^k)$ is called a *generic matrix* over k. The k-*algebra of generic matrices* is generated by all generic matrices over k. These are the key objects in the theory of polynomial identities and invariants of matrix rings. A well known result in PI theory shows that the algebra of generic matrices is isomorphic to the quotient of a free associative algebra in countably many variables by the T-ideal of all polynomial identities satisfied by the algebra of all $m \times m$ matrices over k.

For any n and m consider the finitely generated algebra of generic matrices $GM_m(n)$ which is isomorphic to the quotient $\mathsf{k}\langle x_1, \ldots, x_n\rangle/I$, where I consists of all those polynomials that vanish identically on all $m \times m$ matrices over k. The algebra $GM_m(n)$ is universal in the following sense: given arbitrary $m \times m$ matrices $M_1, \ldots, M_n$ over k (or, more generally, over any ring S) we have the evaluation homomorphism

$$GM_m(n) \to M_m(S),$$

where $x_i \mapsto M_i,\ i = 1, \ldots, n$.

The algebra $GM_m(n)$ has the following simple concrete realization [89, Proposition 13.1.20]. Let

$$O = \mathsf{k}[a_{ij}^k],\ \ i, j = 1, \ldots, m,\ \ , k = 1, \ldots, n.$$

Consider the algebra $M_m(O)$ of $m \times m$ matrices with entries from O and denote by X_k the matrix (a_{ij}^k) for each $k = 1, \ldots, n$.

Theorem 4.5. *The linear map from $GM_m(n)$ to $M_m(O)$ which sends x_i to X_i, $i = 1, \ldots, n$, defines an isomorphism of $GM_m(n)$ with the subalgebra of $M_m(O)$ generated by $X_1, \ldots, X_n$.*

Despite the fact that the ring of generic matrices $GM_m(n)$ is closely related to the matrix algebra, a remarkable fact is that $GM_m(n)$ is an Ore domain [41]. One of the main problems in the PI-algebra theory is whether the center of the skew field of fractions of the ring of generic matrices is rational (cf. [19], [20]). Moreover, it can be shown that this problem is equivalent to the rationality of the invariants of a purely transcendental extension of the base field by the action of the symmetric group [41].

The ring of generic matrices is another example of a rigid ring. The rigidity is a consequence of the following result of Guralnick [64].

Theorem 4.6. *Given any finite group acting linearly on the ring of generic matrices, its invariant subring is not even relatively free.*

With this result, using the structure theory of PI-algebras, one can show that essentially all relatively free algebras are rigid (cf. [49, Section 7]).

Remark 4.7. Invariants of relatively free algebras were studied by Domokos in [39]. It was shown that if $A \simeq \mathsf{k}\langle x_1, \dots, x_n\rangle / I$ is a relatively free k-algebra with $n \geq 2$ and G is a finite group acting linearly on A, then A^G is a relatively free if and only if G is a pseudo-reflection group and T contains the polynomial $[[x_2, x_1], x_1]$. See [40], section 6.3 for additional results and references.

So this attempt to generalize Chevalley-Shephard-Todd theorem falls into the phenomena of rigidity. In 1995, Alev and Polo [7] studied the cases of the Weyl algebra and enveloping algebras of semisimple Lie algebras. Their result is as follows:

Theorem 4.8.

- *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. If for some finite group of automorphisms G of $U(\mathfrak{g})$ we have*

$$U(\mathfrak{g})^G \simeq U(\mathfrak{g}')$$

  for another finite dimensional semisimple Lie algebra $\mathfrak{g}'$, then $G = id$ and $\mathfrak{g} \simeq \mathfrak{g}'$.

- *There is no non-trivial finite group of automorphisms G such that $A_n(\mathsf{k})^G \simeq A_n(\mathsf{k})$.*

In fact, for the Weyl algebra a stronger result is known. Solving a 30 years old conjecture of Smith, Tikaradze [113] showed the following.

Theorem 4.9. *If D is a noncommutative domain and G is a finite group of $\mathbb{C}$-linear automorphisms of D, then $D^G \simeq A_n(\mathbb{C})$ if and only if $G = id$ and $D \simeq A_n(\mathbb{C})$.*

Kirkman, Kuzmanovich and Zhang [78] considered versions of the Chevalley-Shephard-Todd theorem for another generalization of the polynomial algebras, the *Artin-Schelter regular algebras* (cf. [78, Definition 1.5]) and addressed the question of their rigidity. To present the main result we recall the notion of the homogenization of the universal enveloping enveloping algebra and the Rees algebra. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with basis $b_1, \ldots, b_n$. Then the *homogenization* $H(\mathfrak{g})$ of the universal enveloping algebra is the algebra generated by the vector space $\mathfrak{g} + \mathsf{k}z$ with relations

$$zb = bz, \;\; bb' - b'b = [b, b']z, \;\; b, b' \in \mathfrak{g}.$$

The *Rees ring* of the Weyl algebra with respect to the standard filtration is the algebra with generators $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_n$ and h, and relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0, \;\; [\partial_i, x_j] = \delta_{ij}h^2, \;\; i, j = 1, \ldots, n$$

and h is central.

Theorem 4.10.

- *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be finite dimensional Lie algebras without 1-dimensional Lie ideals. If G is a finite group of graded automorphisms of the homogenization $H(\mathfrak{g})$, such that $H(\mathfrak{g})^G \simeq H(\mathfrak{g}')$, then $G = id$ and $\mathfrak{g} \simeq \mathfrak{g}'$.*

- *Let A be the Rees ring of the Weyl algebra, with respect to the standard filtration. Then for any graded group of automorphisms G, the isomorphism $A^G \simeq A$ implies that G is trivial.*

The following question was posed in [78]. Let A be a rigid Ore domain with a skew field of fractions $Frac(A)$. Is it possible to have an isomorphism

$$Frac(A)^G \simeq Frac(A)$$

for some group of automorphisms G?

As we shall see, things become much more interesting on the level of skew fields ring of fractions. In particular this question is related to the Noncommutative Noether's Problem.

4.2 Gelfand-Kirillov Conjecture

In his address to the ICM-1966 in Moscow, A.A.Kirillov proposed a project to classify the enveloping algebras $U(\mathfrak{g})$ of finite dimensional algebraic Lie algebras $\mathfrak{g}$ over an algebraically closed field of zero characteristic up to birational equivalence, that is to classify isomorphism classes of their division rings of fractions by finding canonical representatives in each isomorphism class. The following conjecture was formulated by I.M.Gelfand and A.A.Kirillov in [61]:

Gelfand-Kirillov Conjecture: Let $\mathfrak{g}$ be a finite dimensional algebraic Lie algebra over an algebraically closed field k of characteristic 0. Then the skew field of fractions $Frac(U(\mathfrak{g}))$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic to a certain Weyl field $F_m(L)$, where L is a purely transcendental extension of the base field.

This is a very bold statement: it says that up to birational equivalence $Frac(U(\mathfrak{g}))$ is completely determined by two positive integers: m and $tdeg\, L$ cf. Theorem 3.14. The Gelfand-Kirillov Conjecture became an influential problem in Lie theory (cf. [37, Problem 3]), and here we outline the main contributions.

The conjecture was shown to be valid for gl_n, sl_n and for nilpotent $\mathfrak{g}$ by Gelfand and Kirillov ([61]). Then it was verified for solvable Lie algebras ([26], [88], [73]). The conjecture was verified in [96] for the semi-direct product of sl_n, sp_{2n} and so_n with their standard modules. A weaker version of the conjecture was introduced by Gelfand and Kirillov in 1969: they proved that for every simple $\mathfrak{g}$ there exists a finite field extension F of the center Z of $Frac(U(\mathfrak{g}))$ (which is isomorphic to a field of rational functions), such that $Frac(U(\mathfrak{g})) \otimes_Z F$ is a Weyl field [62]. The same is conjectured to hold for all algebraic Lie algebras ([5]). At the other extreme, with obvious modifications, the Gelfand-Kirillov Conjecture was verified for maximal primitive quotients of the enveloping algebras of simple finite dimensional Lie algebras [32]. It was also shown to be true for all algebraic Lie algebras of dimension at most 8 [6].

However, the Gelfand–Kirillov Conjecture is false in general [5], and the counter-example can be found already in dimension 9. Despite considerable efforts, nothing was known about the case of simple Lie algebras outside of type A until the work of A.Premet [100], who showed that the conjecture is false for types B, D,E, F. It remains a very difficult open problem for types C and G.

Despite being false in general, the Gelfand–Kirillov Conjecture became influential in studies of birational equivalence of Ore domains (cf. [54], [53], [45]). It also played an important role in the theory of quantum groups where a quantized version of the conjecture was thoroughly studied (cf. [28, I.2.11, II.10.4]), with the quantum affine spaces in place of the Weyl algebra. It was also studied in positive characteristic [25].

4.3 Invariants of the Weyl algebra

Before we discuss the invariants of the Weyl fields in the next subsection, we first consider the invariant subalgebras of the Weyl algebra.

The first and the most natural question of the invariant theory of the Weyl algebra is the analogue of the Noether's Theorem (cf. Theorem 2.4):

Let G be a finite group of automorphisms of $A_n(\mathsf{k})$. Is the invariant subalgebra $A_n(\mathsf{k})^G$ finitely generated?

The answer to this question is positive. A noncommutative generalization of the Noether's Theorem is due to S. Montgomery and L. Small [93]. The proof of this result is quite long and we need a preliminary discussion of skew group rings.

Definition 4.11. Let R be a ring and G a group of automorphisms of R. The skew group ring $R * G$ is, as left R-module, freely generated by the elements of G, and product extended by linearity from

$$rgsh = rg^{-1}(s)gh,\ r, s \in R, g, h \in G.$$

Clearly, R is identified with a subring $R * id$ of $R * G$. If G is finite then the skew group ring $R * G$ is left or right Noetherian whenever R is.

Lemma 4.12. *Let R be a ring, G a finite group acting by ring automorphisms, and $S = R * G$. Suppose $|G|$ is invertible in R, and set $e = \frac{1}{|G|}\sum_{g\in G} g$. Then*

1. $e^2 = e$.

2. $eS = eR$.

3. $eSe = eR^G \simeq R^G$

Proof. The statement (1) is clear. Let us prove (2). Clearly $eR \subset eS$. By the definition of multiplication in S we have $rg = gg(r)$ for $r \in R, g \in G$. Consider an arbitrary element $s \in S$. Then it can be written in the form

$$s = \sum_{g\in G} r_g g \in S,$$

and we have

$$es = \sum_{g\in G} er_g g = \sum_{g\in G} egg(r_g).$$

As $eg = e$ it follows that

$$eg = \sum_{g\in G} eg(r_g) = e\sum_{g\in G} g(r_g) \in eR.$$

Hence, the statement (2) follows. Let us prove (3). By (2), we have $eSe = eRe$. For any $r \in R$ we obtain

$$ere = \frac{e}{|G|}\sum_{g\in G} rg = \frac{e}{|G|}\sum_{g\in G} gg(r) = \frac{1}{|G|}\sum_{g\in G} egg(r) =$$

$$\frac{1}{|G|}\sum_{g\in G} eg(r) = \frac{e}{|G|}\sum_{g\in G} g(r) = \frac{e}{|G|}\tau(r) = e\tau(\frac{r}{|G|}),$$

where $\tau : R \to R^G$ is defined by

$$r \mapsto \sum_{g\in G} g(r), r \in R.$$

This shows that $eSe = e\tau(R)$. Since $|G|$ is invertible in R it follows that every $r \in R^G$ can be written in the form $r = \tau(\frac{1}{|G|}r)$. We conclude that $R^G \subset \tau(R)$ and then $R^G = \tau(R)$. Hence $eSe = eR^G$. As $er = re$ for every $r \in R^G$, the map $r \mapsto er$ defines an isomorphism of rings $R^G \to eR^G$. □

Theorem 4.13 (Montgomery-Small [93]). *Let R be a finitely generated (not necessarily commutative) unital algebra. Let G be a finite group of algebra automorphisms of R with $|G|$ invertible in k. Then, if R is a left (or right) Noetherian, R^G is a finitely generated algebra.*

Proof. Let $S = R * G$. Then S is left (or right) Noetherian following the same property of R. It is easy to see that S is a finitely generated algebra: if $\{q_1, \ldots, q_m\}$ is a generating set of R as an algebra, and $G = \{g_1, \ldots, g_d\}$, then the union of these two subsets generate S (identifying R with $R * id$ and G with $1 * G$). Consider again

$$e = \frac{1}{|G|}\sum_{g\in G} g.$$

In particular, eSe is a subring of S (with unit e) and eS is a left eSe-module. Clearly, SeS is a two sided ideal of S. Let us show that eS is a finitely generated left eSe-module. As S is a left Noetherian ring, SeS is a finitely generated left ideal. Then $SeS = \sum_i Sx_i$. Write

$$x_i = \sum_j v_{ij} e w_{ij},$$

where $v_{ij}, w_{ij} \in S \,\forall i, j$. Let $r \in S$ be an arbitrary element. Then $er = eee(R) \in e(SeS)$, and, hence

$$er = e(\sum_i s_i x_i) = \sum e s_i v_{ij} e w_{ij} = \sum e s_i v_{ij} e^2 w_{ij}.$$

This way, the finite set $\{ew_{ij}\}$ generates eS as a left eSe-module.
Write now

$$eS = \sum_{i=1}^{n} eSex_i, x_i \in S,$$

and let $t_1, \ldots, t_m$ be a finite set of algebra generators of S. Now we write

$$et_j = \sum_{i=1}^{n} ey_{ij}ex_i, \;\; ex_kt_j = \sum_{i=1}^{n} ez_{ijk}ex_i,$$

with $y_{ij}, z_{ijk} \in S$ for all $1 \leq j \leq m$, $1 \leq k \leq n$. Consider the finite set $E = \{ex_ie, ey_{ij}e, ez_{ijk}e\}$, $1 \leq i, k, \leq n$, $1 \leq j \leq m$. Then we have

$$et_1t_2e = (\sum_{i=1}^{n} ey_{i1}ex_i)t_2e = \sum_{i=1}^{n} ey_{i1}e(ex_it_2)e$$

$$= \sum_{i=1}^{n} ey_{i1}e(\sum_{l=1}^{n} ez_{l2i}ex_l)e = \sum_{i=1}^{n} ey_{i1}e(\sum_{l=1}^{n} ez_{l2i}eex_ie).$$

This way we can show inductively that each monomial $et_{j_1} \ldots t_{jk}e$ with $1 \leq j_1, \ldots, j_k \leq m$ can be written as a finite sum of products of elements of E. Every element of eSe is a linear combination of such elements, and hence we conclude that eSe is finitely generated as an algebra by the set E. By the above lemma, R^G is also finitely generated. □

As an immediate consequence of the theorem above we obtain

Corollary 4.14. *For every finite group of automorphisms G of the Weyl algebra $A_n(\mathsf{k})$, the invariant subalgebra $A_n(\mathsf{k})^G$ is a finitely generated algebra.*

Now we are going to discuss the structural properties of $A_n(\mathsf{k})^G$. We start with the following property of the skew product.

If a ring R is equipped with the action of some group G, then the elements of G act as automorphisms on R. Recall that an automorphism of a ring given by the conjugation action of a fixed element of G is called inner, while an automorphism which is not inner is called an outer automorphism.

Proposition 4.15. *Given a simple ring R and a finite subgroup G of outer automorphisms of R, the skew group ring $R * G$ is also a simple ring.*

Proof. Let I be a non-zero ideal of $R * G$. Let

$$x = \sum_{g \in G} x_g g \in R * G.$$

We define the length of x to be the cardinality of the set $\{x_g \neq 0\}$. Let $y \in I$ be a non-zero element of the minimal length. Multiplying on the left by a suitable element of G we can assume that $y_{id} \neq 0$. Besides, as R is simple, we have $Ry_{id}R = R$. Hence there exist $j, k \in R$ such that $jy_{id}k = 1$. Multiplying y on the left by j and on the right by k we can assume $y_{id} = 1$. So, let

$$y = id + \sum_{id \neq g \in G} y_g g.$$

If $y = id$ then $I = R * G$ and we are done. Otherwise, let $id \neq h \in G$ with $y_h \neq 0$. For any $r \in R$ consider the element $ry - yr \in I$ which is equal to

$$\sum_{id \neq g \in G} (ry_g - y_g g^{-1}(r))g.$$

The sum has a smaller length than y and, hence, equals zero, by our assumption on the length. Then we have (*)

$$ry_h = y_h h^{-1}(r)$$

for all $r \in R$. We get that $y_h R = Ry_h$ is a two sided ideal of R, and hence $y_h R = R$ from the simplicity of R. Last equality implies that y_h is a unit. Applying (*) we obtain

$$h^{-1}(r) = (y_h)^{-1} r y_h,$$

which implies that h is an inner automorphism. But this contradicts the hypotheses, since $h \neq id$. Hence $y = id$, and $I = R * G$. □

Let us also recall the following result from the Morita theory.

Proposition 4.16. *Two rings T and S are Morita equivalent if and only if there exists $n \geq 1$ and an idempotent e of $M_n(S)$ such that*

$$T \simeq eM_n(S)e, \quad M_n(S)eM_n(S) = M_n(S).$$

Here $M_n(S)$ denotes the ring of $n \times n$ matrices with entries from S.

Proof. Cf. [89, Proposition 3.5.6]. □

Now we can establish the simplicity of the invariant subalgebra.

Theorem 4.17. *Let R be a simple ring and G a finite group of outer automorphisms of R. Then the invariant subalgebra R^G is simple. Moreover, if R is left or right Noetherian, then so is R^G.*

Proof. Since the property to be simple is Morita invariant, it is sufficient to show that R^G and $S = R*G$ are Morita equivalent. We use the criterion from the above proposition for $n = 1$ (that is, $M_n(S)$ is just S). Let

$$e = \frac{1}{|G|} \sum_{g \in G} g.$$

Then $e^2 = e$ and $eSe \simeq R^G$ (cf. Lemma 4.12). We also have that $SeS = S$ as S is a simple ring. Therefore, S and R^G are Morita equivalent by Proposition 4.16. This implies the simplicity of R^G. The property to be left or right Noetherian is also Morita invariant. Moreover, if R is a left or right Noetherian then so is $R * G$. We conclude that R^G is left or right Noetherian respectively. □

We apply these results for the Weyl algebras.

Corollary 4.18. *For any finite group G of automorphisms of $A_n(\mathsf{k})$, the invariant subalgebra $A_n(\mathsf{k})^G$ is a simple left and right Noetherian domain.*

Proof. Recall that the units of the Weyl algebra are the scalars, and hence all automorphisms are outer. The statement follows from the theorem above. □

More can be said about the invariant subalgebra $A_n(\mathsf{k})^G$ when the action of the group G on the Weyl algebra is linear. Recall that the Weyl algebra $A_n(\mathsf{k})$ has two standard filtrations: the Bernstein filtration (defined by the total degree of elements with respect to the generators of $A_n(\mathsf{k})$) and the filtration defined by orders of differential operators. An essential feature of a linear action is that it preserves these two filtrations of the Weyl algebra $A_n(\mathsf{k})$. Hence, such group action induces the action on the respective associated graded algebras. It is not hard to see that

$$gr\, A_n(\mathsf{k})^G \simeq (gr\, A_n(\mathsf{k}))^G.$$

For an effective computation of the generators of the invariant subalgebras of the Weyl algebra, the last isomorphism is crucial, as it translates the problem to the one of the classical invariant theory. A detailed account of the algorithms for finding generators and relations for the invariant subalgebra of the Weyl algebra together with references, can be found in [114]. If we are only interested in generators of $A_n(\mathsf{k})^G$ we have the following surprising result of T.Levasseur and T.Stafford [83].

Theorem 4.19. *Let G be a finite group of linear automorphisms of $A_n(\mathsf{k})$. Then $A_n(\mathsf{k})$ contains two subalgebras $\mathsf{k}[x_1, \ldots, x_n]$ and $\mathsf{k}[\partial_1, \ldots, \partial_n]$ isomorphic to the polynomial algebra in n indeterminates, where the latter one consists of differential operators with constant coefficients. Then the invariant subalgebra $A_n(\mathsf{k})^G$ is generated by $\mathsf{k}[x_1, \ldots, x_n]^G$ and $\mathsf{k}[\partial_1, \ldots, \partial_n]^G$.*

4.4 Invariant Weyl fields

Given the preeminence of Weyl fields in questions of noncommutative birational equivalence, J.Alev and F.Dumas considered the following noncommutative analogue of Noether's problem in 1998 [2]:

- Let $F_n(\mathsf{k})$ be a Weyl field and G be a finite group of automorphisms of $F_n(\mathsf{k})$. When do we have an isomorphism

$$F_n(\mathsf{k})^G \simeq F_n(\mathsf{k})?$$

The question was inspired by the following results.

- If G is *any* finite subgroup of automorphisms of $A_1(\mathbb{C})$, then ([1])

$$F_1(\mathbb{C})^G \simeq F_1(\mathbb{C})$$

with the induced actions.

- If G is a finite group of automorphisms of $F_1(\mathbb{C})$ that fixes one of the commutative subfields $\mathbb{C}[x], \mathbb{C}[\partial]$ or $\mathbb{C}[\partial x]$, then ([2]):

$$F_1(\mathbb{C})^G \simeq F_1(\mathbb{C})$$

- If G is a finite abelian group of linear automorphisms of $A_n(\mathsf{k})$ with algebraically closed field k, then $F_n(\mathsf{k})^G \simeq F_n(\mathsf{k})$ [2].

This problem was revisited in [3] but only for automorphisms of the Weyl fields induced by the automorphisms of the Weyl algebra. Let G be a finite group which acts linearly on Λ_n. Then this action extends to the linear action of G on $A_n(\mathsf{k})$ as it was described above, and also to the action on $F_n(\mathsf{k})$. *For convenience we set* $F_0(\mathsf{k}) := \mathsf{k}$.

The following analog of the Noether's Problem for the Weyl algebra A_n was first considered by Alev and Dumas in [3]:

For which G and n, the skew field $F_n(\mathsf{k})^G$ is isomorphic to the Weyl field $F_m(L)$ for some $m \geq 1$ and some purely transcendental extension L of k?

Acknowledging the importance of the Weyl algebras and following [3], we call this analog of the Noether's Problem, the *Noncommutative Noether's Problem* (NNP for short).

In what follows we are going to use the noncommutative analogues of the notion of transcendence degree. Historically, the most used one is the Gelfand-Kirillov transcendence degree [61]. However, this invariant is very difficult to compute in practice, and it has some theoretical deficiencies: for instance, let $D \subset Q$ be division algebras. Is it true (as expected) the the Gelfand-Kirillov transcendence degree of Q is bigger than the one of D? The answer to such basic question is yet unknown. J. J. Zhang introduced in [118] another version of transcendence degree, the *lower transcendence degree*, henceforth denoted by LD, which coincides with the Gelfand-Kirillov transcendence degree in all known cases and remedies the theoretical issues of the latter one. It is related to the developments in noncommutative algebraic geometry [118, Section 9].

The lower transcendence degree is a powerful tool which can be used for example to establish the following result.

Proposition 4.20. *If the isomorphism $F_n(\mathsf{k})^G \simeq F_m(L)$ holds, then $m = n$ and $L = \mathsf{k}$.*

Proof. Suppose $F_n(\mathsf{k})^G \simeq F_m(L)$. Let $t = tdeg\, L$. It is well known that no commutative subfield of $F_n(\mathsf{k})$ has a transcendence degree larger than n [89, 6.6.18]. Suppose that the transcendence degree of the field $F_m(L)$ is $m + t$, Then

$$(*)\, m + t \leq n.$$

Since G is finite, we have

$$[F_n(\mathsf{k}) : F_n(\mathsf{k})^G] \leq |G|,$$

by the noncommutative Artin Lemma ([92, Lemma 2.18]). Now, since $[F_n(\mathsf{k}) : F_n(\mathsf{k})^G] < \infty$ and $LD\, F_n(\mathsf{k}) = 2n$, we get

$$LD\, F_n(\mathsf{k})^G = LD\, F_n(\mathsf{k}) = 2n.$$

But $LD\, F_m(L) = 2m + t$. This together with the inequality $(*)$ implies that $m = n$ and $t = 0$. □

Hence, for a finite group G, we can restate the NNP as follows: for which n and G there exists an isomorphism

$$F_n(\mathsf{k})^G \simeq F_n(\mathsf{k})?$$

The following are examples when the NNP has a positive solution.

Example 4.21.

- For $n = 1$ and $n = 2$ and an arbitrary finite group G [3]. When G is a finite subgroup of $SL_2(\mathbb{C})$ an explicit isomorphism $F_2(\mathbb{C})^G \simeq F_2(\mathbb{C})$ was constructed in [3].

- For any n and any group G whose natural representation decomposes into a direct sum of one dimensional representations [3]. In particular, it holds for all $n \geq 1$ if G is abelian and k is algebraically closed.

Invariant subfields with respect to infinite groups actions were also considered previously. In this case $m + trdeg_{\mathsf{k}}\, L \leq n$. If the action of a group G is triangular then one can not guarantee that the skew subfield of

G-invariants is isomorphic to a Weyl field [3, Remark 1.2.3, 1.3.2]. However, if the action of G decomposes as a direct sum of one dimensional G-modules, then indeed

$$F_n(\mathsf{k})^G \simeq F_m(L),$$

and $m + trdeg_{\mathsf{k}} L = n$. Also, all values $0, 1, \ldots, n$ can appear as $trdeg_{\mathsf{k}} L$ for actions of infinite subgroups of the torus $\mathbb{T}^n$ [3], including the extreme case n, when the skew field of invariants is commutative.

In what follows we will only be interested in the invariants of finite groups.

Chapter 5

NNP for complex reflection groups

Let V be a finite dimensional complex vector space. A *complex reflection* $r : V \to V$ is a diagonalizable non-identity linear invertible operator of finite order, which fixes pointwise a hyperplane. This implies that all but one eigenvalues of r equal 1, and the last one is a complex root of unity different from 1.

A complex reflection group of V is a group generated by complex reflections of V. A complex reflection group W generated by reflections of V is irreducible if V is irreducible W-module, that is the only W-invariant subspaces of V are 0 and V itself.

Any complex reflection group acting on some vector space V is isomorphic to a product of irreducible complex reflection groups, acting on the corresponding direct summands of V. As we discussed in Section 2, the irreducible complex reflection groups were classified by Shephard and Todd. These groups either form a 3-parameter family $G(m, p, n)$, where m, p, n are positive integers parameters (with p dividing m), or they belong to one of 34 exceptional cases. The group $G(m, p, n)$ is a semidirect product of an abelian group of order $\frac{m^n}{p}$ by the symmetric group S_n. It has a natural representation in $\mathbb{C}^n$, where S_n acts by the permutation of the coordinates. This G-module is irreducible unless $G = S_n$ $(m = 1)$ or $G = G(2, 2, 2)$ is the Klein group.

In this section we will show that the NNP has positive solution for every n and any complex reflection group following [44], the case $W = S_n$ was considered in [56].

We prove the following theorem:

Theorem 5.1. *The NNP has a positive solution for any complex reflection group.*

Before we will need some preliminaries.

5.1 Invariants of localized rings

As before Λ denote the polynomial algebra over k with n variables. Let W be an arbitrary finite group acting by linear automorphisms on Λ.

Consider a W-invariant element $\Delta \in \Lambda$, localization Λ_Δ with the induced action of W and the W-invariants Λ_Δ^W in Λ_Δ. We have an embedding $\Lambda_\Delta^W \to \Lambda_\Delta$ and the induced map

$$D(\Lambda_\Delta) \to D(\Lambda_\Delta^W).$$

By the restriction of domain we obtain the map

$$\phi_\Delta : D(\Lambda_\Delta)^W \to D(\Lambda_\Delta^W).$$

Proposition 5.2. *Let Δ be a W-invariant element in Λ. Then the map ϕ_Δ is injective.*

Proof. Note that $D(\Lambda_\Delta)$ is a simple ring and W acts by outer automorphisms. Then $D(\Lambda_\Delta)^W$ is a simple ring, by [92, Corollary 2.6]. Since ϕ_Δ is not trivial, it is injective. □

Suppose that R is an Ore domain and S is an Ore subset. Denote by R_S the localization of R by S and let $Frac(R)$ be the skew field of fractions of R. The following isomorphism is clear:

$$Frac(R) \simeq Frac(R_S).$$

In particular, this can be applied to the algebra Λ. Moreover, the next statement shows that the operation of localization by Δ and the operation of taking the W-invariants commute on Λ_Δ and $D(\Lambda_\Delta)$.

We have the following useful isomorphisms between the invariants of localized rings [44, Proposition 1]. Note that these isomorphisms are independent of the field k.

Theorem 5.3. *Let Δ be a W-invariant element of Λ, S a multiplicatively closed set in Λ. Denote by Λ_S and Λ_Δ the localizations of Λ by S and by the multiplicative set generated by Δ respectively. Then we have the following isomorphisms.*

1. $D(\Lambda_S) \simeq D(\Lambda)_S$;
2. $(\Lambda_\Delta)^W \simeq (\Lambda^W)_\Delta$;
3. $D(\Lambda_\Delta)^W \simeq (D(\Lambda)^W)_\Delta$;

4. $Frac(A_n(\mathbb{C}))^W = Frac(A_n(\mathbb{C})^W)$.

Proof. The first statement is Proposition 7.3. Let us prove the second statement. Since Δ is a W-invariant polynomial, then an element h belongs to $(\Lambda_\Delta)^W$ if and only if $\Delta^k h \in \Lambda^W$ for some $k \geq 0$, which is also equivalent to the condition $h \in (\Lambda^W)_\Delta$. Hence, (2) follows.

For the third statement note that $D(\Lambda_S) \simeq D(\Lambda)_S$ for any multiplicatively closed subset S of Λ by Proposition 7.3. Again, $d \in D(\Lambda_\Delta)^W$ if and only if $\Delta^k d \in D(\Lambda)^W$ for some $k \geq 0$, which implies (3).

Finally, the last statement follows from Theorem 3.15. □

Our goal will be to find an adequate Δ for which the homomorphism $\phi_\Delta : D(\Lambda_\Delta)^W \to D(\Lambda_\Delta^W)$ is surjective.

5.2 Proof of Theorem 5.1

What is the idea of the proof? We would like to emulate the positive solution of the CNP for complex reflection groups. If $A_n(\mathbb{C})^W$ would be isomorphic to $A_n(\mathbb{C})$, then we would just take the skew field of fractions of both algebras. However, this is false, since the Weyl algebra is rigid. Nonetheless, with a small modification this idea works. Namely, there exists a W-invariant polynomial f such that

$$A_n(\mathbb{C})_f^W \simeq A_n^*(\mathbb{C}),$$

where $A_n(\mathbb{C})_f$denotes the localization of $A_n(\mathbb{C})$ by the multiplicative set generated by f, and the second algebra is a localization of the Weyl algebra.

Let W be a complex reflection group acting by reflection representation on an n-dimensional complex vector space V. Denote by $\mathcal{A} = \{H_s\}$ the set of reflecting hyperplanes. The group W acts on $\mathcal{A}$ by permutation. For any $H \in \mathcal{A}$ the point-wise stabilizer of H in W is a finite cyclic group of order n_H. Let α_H be a linear form for which H is the zero set. Introduce

$$\delta = \prod_{H \in \mathcal{A}} \alpha_H,\ J = \prod_{H \in \mathcal{A}} \alpha_H^{n_H - 1}.$$

Its easy to show that

$$w \cdot J = det(w) J$$

for every $w \in W$ ([111, Exercise 4.3.5]). Let N be the order of W. Then $\Delta = J^N$ is a W-invariant polynomial.

The polynomial f can be chosen to be Δ. We fix a basis $\{v_1, \ldots, v_n\}$ of V and let $\{x_1, \ldots, x_n\}$ be the dual basis of V^*. Then

$$\Lambda = \mathbb{C}[V] \simeq \mathbb{C}[x_1, \ldots, x_n].$$

Let Λ_δ be the localization of Λ by the multiplicative set $\{1, \delta, \delta^2, \dots, \}$. Then, we have

$$\Lambda_\delta \simeq \Lambda_J \simeq \Lambda_\Delta,$$

since they are just localizations by

$$\{\alpha_H^L,\ \ k \geq 0,\ \ H \in \mathcal{A}\}.$$

In fact, $\Lambda_\delta = \mathbb{C}[V^{reg}]$, where

$$V^{reg} = V \setminus \bigcup_{H \in \mathcal{A}} H.$$

Recall the following theorem on the isotropy subgroups of complex reflection groups.

Theorem 5.4 (Steinberg). *Let W be a complex reflection group on a vector space V. Given any subset X of V, the isotropy group of X is generated by the complex reflections of W that it contains.*

Proof. [75, Corollary 26-1]. □

Equipped with this result we analyze the restrictions of W on V^{reg} and $\mathbb{C}[V^{reg}]$.

Lemma 5.5. *The action of W restricts to a free action on V^{reg} and $\mathbb{C}[V^{reg}]$.*

Proof. Assume that for some $w \in W$ and $v \in V^{reg}$, the element $w \cdot v$ belongs to a hyperplane fixed by a reflection $s \in W$. Then $w^{-1}sw$ belongs to the isotropy group of v, which is also a reflection group by the Steinberg's Theorem. Since $v \in V^{reg}$, we conclude that $s = id$, which is a contradiction. Hence $w \cdot v \in V^{reg}$ and the action is clearly free. □

For an affine G-variety X with an action of the group G denote by X/G the geometric quotient of X by G equipped with the quotient topology together with a surjective morphism $X \to X/G$ whose fibers are exactly the G-orbits in X. We recall the following result of Cannings and Holland [30]:

Theorem 5.6. *Let X be an affine irreducible algebraic variety over $\mathbb{C}$ with an action of a finite group G on it.*

1) *There exits a maximal open dense G-invariant subset $V \subset X$, on which the induced action of G is free. Let*

$$\pi : X \to X/G$$

be the canonical projection and $V' = \pi(V)$. Then V' is open dense in X/G and, since V is a complete pre-image, $V = \pi^{-1}(V')$, the map π restricts to the quotient map:

$$\pi|_V : V \to V'.$$

2) If G acts freely, we have the following isomorphism

$$D(X)^G \simeq D(X/G).$$

The action of the group W on V extends naturally to the action on Λ and on the localization Λ_Δ. Consider the ring $D(\Lambda_\Delta)$ of differential operators on Λ_Δ. Then we have the following result.

Corollary 5.7. *$D(\Lambda_\Delta)^W \simeq D((\Lambda_\Delta)^W)$.*

Proof. By Lemma 5.5, the action of W on V^{reg} is free. But V^{reg} can be identified with $Specm\,\Lambda_\Delta$. Note that $\mathcal{O}(Specm\,\Lambda_\Delta/W) = \Lambda_\Delta^W$. The statement follows from Theorem 5.6. □

As a consequence we have the following isomorphisms:

$$(\Lambda_\Delta)^W \simeq (\Lambda^W)_\Delta \simeq \Lambda_\Delta,$$

where the first isomorphism follows from Theorem 5.3, (2), while the second isomorphism from the Chevalley-Shephard-Todd Theorem. Applying Corollary 5.7 we obtain

$$D(\Lambda_\Delta)^W \simeq D((\Lambda_\Delta)^W) \simeq D(\Lambda_\Delta) \simeq D(\Lambda)_\Delta.$$

Now Theorem 5.3, (3) implies

$$(D(\Lambda)^W)_\Delta \simeq D(\Lambda_\Delta)^W \simeq D(\Lambda)_\Delta.$$

Therefore we get the isomorphisms of the skew fields of fractions:

$$Frac(A_n(\mathbb{C})) \simeq Frac(A_n(\mathbb{C})_\Delta) \simeq Frac((A_n(\mathbb{C})_\Delta)^W)$$

$$\simeq Frac(A_n(\mathbb{C})_\Delta)^W \simeq Frac(A_n(\mathbb{C})^W) \simeq Frac(A_n(\mathbb{C}))^W.$$

This proves Theorem 5.1.

The NNP is important for many applications. As an example we will consider the analog of the Gelfand-Kirillov Conjecture for spherical subalgebras of rational Cherednik algebras.

5.3 Rational Cherednik algebras

Let W be a finite complex reflection group with a finite dimensional complex representation in the space V. Consider the set S of complex reflections, and for each $s \in S$ choose an eigenvector $\alpha_s \in V^*$ for the non trivial eigenvalue of s. Similarly choose dual elements $\alpha_s^\vee \in V$. Let $(\,,\,)$ be the non-degenerate pairing $V^* \times V \to \mathbb{C}$. We normalize these vectors in such way that $(\alpha_s, \alpha_s^\vee) = 2$ for all s with respect to $(\, , \,)$. Let $c : S \to \mathbb{C}$ be a conjugation invariant function

$$c(w^{-1}sw) = c(s), \ \ w \in W, \ \ s \in S.$$

For a vector space U denote by $U^{\otimes k}$ the tensor product of k copies of U, and by $T(U)$ the *tensor algebra* of U:

$$T(U) = \mathbb{C} \oplus U \oplus (U^{\otimes 2}) \oplus (U^{\otimes 3}) \dots,$$

with multiplication determined by the tensor product. Now we define rational Cherednik algebra associated with W, c and V.

Definition 5.8. The rational Cherednik algebra $H_c(V, W)$ is generated by

$$T(V \oplus V^*) \rtimes \mathbb{C}W$$

with the relations

$$[x, x'] = [y, y'] = 0;$$

$$[y, x] = x(y) - \sum_{s \in S} c(s)(\alpha_s, y)(x, \alpha_s^\vee)s,$$

$x, x' \in V^*, y, y' \in V$.

Denote by V^{reg} the regular part of V obtained by the removing the reflecting hyperplanes from V.

Definition 5.9. Let X be an affine variety. Then we define the ring of differential operators $D(X)$ on X as the ring of differential operators on the coordinate ring $\mathcal{O}(X)$, that is $D(X) := D(\mathcal{O}(X))$.

We have the following Dunkl embedding of rational Cherednik algebra into the semidirect product of the ring of differential operators $D(V^{reg})$ with the group algebra of W:

Theorem 5.10 (Dunkl embedding [45]). *There is an embedding*

$$H_c(V, W) \to D(V^{reg}) \rtimes \mathbb{C}W.$$

Let δ be the product of linear forms vanishing on the reflection hyperplanes, and $H_c(V, W)_\delta$ be the localization of $H_c(V, W)$ by δ. Then we have an isomorphism

$$H_c(V, W)_\delta \simeq D(V^{reg}) \rtimes \mathbb{C}W.$$

Consider the idempotent

$$e = \frac{1}{|W|} \sum_{w \in W} w$$

as an element of $H_c(V, W)$.

Definition 5.11. The subalgebra $U_c(V,W) := eH_c(V,W)e$ with the unit e is called the *spherical subalgebra* of the rational Cherednik algebra.

Corollary 5.12. *There is an embedding of $U_c(V,W)$ into $D(V^{reg})^W$. Moreover, both rings are birationally equivalent (have isomorphic skew fields of fractions).*

Proof. By the Dunkl embedding, we have

$$U_c(V,W)_\delta \simeq eD(V^{reg}) \rtimes \mathbb{C}We.$$

But the later ring is isomorphic to $D(V^{reg})^W$. Hence,

$$U_c(V,W)_\delta \simeq D(V^{reg})^W.$$

This immediately implies that both algebras have isomorphic skew fields of fractions. □

Applying Corollary 5.12 and using the fact that the NNP solves positively for complex reflection groups, we obtain the following result.

Theorem 5.13 ([44]). *The skew field of fractions of $U_c(V,W)$ is isomorphic to the Weyl field $F_n(\mathbb{C})$, where $n = dim\, V$.*

5.4 NNP for complex torus

In this section we show that the NNP holds for actions of the classical Weyl groups on the complex algebraic torus. Let $\mathbb{T}^n$ be the n-dimensional complex algebraic torus and $D(\mathbb{T}^n)$ is the algebra of differential operators on $\mathbb{T}^n$ (algebra of differential operators on the ring of functions $\mathcal{O}(\mathbb{T}^n)$ on $\mathbb{T}^n$). The following theorem was established in [44].

Theorem 5.14. *There are natural actions of the classical Weyl groups W on the complex algebraic torus $\mathbb{T}^n$ such that*

$$Frac(D(\mathbb{T}^n)^W) \simeq F_n(\mathbb{C}).$$

Hence the NNP holds in this case.

Let's now move to the details of this proof, that is somewhat involved. The ring of functions on $\mathbb{T}^n$ is isomorphic to the ring of Laurent polynomials:

$$\mathcal{O}(\mathbb{T}^n) = \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}].$$

We have n involutions $\epsilon_1, \ldots, \epsilon_n$ of $\mathcal{O}(\mathbb{T}^n)$ that act as follows:

$$\epsilon_i(x_i) = -x_i^{-1},\ \epsilon_i(x_j) = x_j,\ j \neq i.$$

We extend these actions on $D(\mathbb{T}^n)$ as follows:

$$\epsilon_i(\partial_i) = x_i^2\partial_i,\ \epsilon_i(\partial_j) = \partial_j,\ j \neq i.$$

Since $D(\mathbb{T}^n)$ is isomorphic to the localization of the Weyl algebra A_n by the multiplicative set generated by $x_1, \ldots, x_n$, the formulas above determine completely the actions on $D(\mathbb{T}^n)$. In this fashion we obtain an action of $\mathbb{Z}_2^n$ on $D(\mathbb{T}^n)$ considering $\epsilon_1, \ldots, \epsilon_n$ as a basis. We also have the usual permutation action of the symmetric group S_n. Using these actions of $\mathbb{Z}_2^n$ and S_n on $D(\mathbb{T}^n)$ we can define the actions of all classical Weyl groups on the ring of differential operators on the torus.

We will denote by B_n and D_n the corresponding Weyl groups of type B and D respectively. Note that the Weyl group in type C with n generators is isomorphic to the Weyl group B_n.

In the following lemma we describe the invariant differential operators on the torus with respect to the B_n-action.

Lemma 5.15.

1. *The subalgebra of B_n-invariants of $D(\mathbb{T}^n)$ is a polynomial algebra generated by*

$$e_i(x_1 - x_1^{-1}, \ldots, x_n - x_n^{-1}),\ \ i = 1, \ldots, n,$$

where e_i is the i-th elementary symmetric polynomial.

2. *Let Z be the subvariety of $\mathbb{T}^n$ defined by the following equation*

$$\prod_{1\leq i\leq j\leq n} (x_i^2 - x_j^{-2}) \prod_{i\leq i<j\leq n} (x_i^2 - x_j^2) = 0,$$

and $U = \mathbb{T}^n \setminus Z$. Then U is a B_n-invariant subvariety and the action of B_n is free on U. In particular, the projection

$$\pi : U \to U/B_n$$

is étale (i.e. flat and unramified).

Proof. Let us prove the first statement. Consider the lexicographical order on Laurent monomials. Let $\pi = (k_1, \ldots, k_n)$ be a sequence of integers with the property that

$$k_1 \geq k_2 \geq \ldots \geq k_n \geq 0$$

and $x_1^{k_1} \ldots x_n^{k_n}$ be the corresponding monomial. Denote by

$$m_\pi = x_1^{k_1} \ldots x_n^{k_n} + \ldots$$

a B_n-invariant polynomial with a minimal number of monomials. The polynomials m_π form a basis of the subalgebra of B_n-invariants. We call π the degree of m_π. If $\pi = (1, 0, \ldots, 0)$ then

$$m_\pi = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i^{-1} = e_1(x_1 - x_1^{-1}, \ldots, x_n - x_n^{-1}).$$

Consider now the case of an arbitrary $\pi = (k_1, \ldots, k_n)$. The leading monomial $x_1^{k_1} \ldots x_n^{k_n}$ of m_π coincides with the leading monomial of

$$M_\pi = s_1^{k_1 - k_2} \ldots s_{n-1}^{k_{n-1} - k_n} s_n^{k_n},$$

where $s_i = e_i(x_1 - x_1^{-1}, \ldots, x_n - x_n^{-1})$, $i = 1, \ldots, n$. Then $m_\pi - M_\pi$ has a smaller leading monomial and we can proceed by induction on the degree of m_π.

For the proof of (2) we denote

$$\Delta = \prod_{1 \le i,j \le n} (x_i^2 - x_j^{-2}) \prod_{i \le i < j \le n} (x_i^2 - x_j^2)(x_i^{-2} - x_j^{-2}) \prod_{i=1}^{n} (x_i - x_i^{-1}).$$

Then one can easily see that Δ is B_n-invariant (with respect to the action defined above). Denote by $V(\Delta)$ the algebraic subset of $\mathbb{T}^n$ corresponding to Δ. Then the action of B_n is free on $\mathbb{T}^n \setminus V(\Delta)$. Since U coincides with $\mathbb{T}^n \setminus V(\Delta)$, the statement follows. □

Next we describe the invariant differential operators on the torus with respect to the D_n-action.

Lemma 5.16.

1. *The subalgebra of D_n-invariants of $D(\mathbb{T}^n)$ is generated by the elementary symmetric polynomials*

$$e_i = e_i(x_1 - x_1^{-1}, \ldots, x_n - x_n^{-1}), \;\; i = 1, \ldots, n-1$$

and

$$\Delta_n^{\pm} = \frac{1}{2}\Big(\prod_{i=1}^{n}(x_i + x_i^{-1}) \pm \prod_{i=1}^{n}(x_i - x_i^{-1})\Big).$$

Moreover, there exists a polynomial $P \in \mathbb{C}[e_1, \ldots, e_{n-1}, \Delta_n^+]$ such that Δ_n^- belongs to the localization $\mathbb{C}[e_1, \ldots, e_{n-1}, \Delta_n^+]_P$.

2. *Let $Z \subset \mathbb{T}^n$ be the variety defined by the equation $\Delta = 0$ and $U = \mathbb{T}^n \setminus Z$. Then U is an affine D_n-invariant subvariety of $\mathbb{T}^n$ on which the action of D_n is free. Moreover, the projection $\pi : U \to U/D_n$ is étale.*

Proof. (1) We proceed with the proof analogously to the preceding lemma. Let us order the Laurent monomials lexicographically. Let $\pi = (k_1, \ldots, k_n)$ be a sequence of integers with $k_1 \geq k_2 \geq \ldots \geq |k_n| \geq 0$. Note that k_n can be negative. Set

$$\lambda_\pi = |\{g \in D_n | g(x_1^{k_1} \ldots x_n^{k_n}) = x_1^{k_1} \ldots x_n^{k_n}\}|.$$

Associated with π define the following D_n-invariant Laurent polynomial

$$m_\pi = \lambda_\pi^{-1} \sum_{g \in D_n} g(x_1^{k_1} \ldots x_n^{k_n}).$$

The polynomials m_π form a basis of the space of D_n-invariant Laurent polynomials. The leading monomial of m_π is $x_1^{k_1} \ldots x_n^{k_n}$. We define its degree to be π. The same leading monomial has the element

$$M_\pi = s_1^{k_1 - k_2} \ldots s_{n-1}^{k_{n-1} - k_n} (\Delta_n^{sign(k_n)})^{|k_n|}.$$

The difference $m_\pi - M_\pi$ has a smaller leading term. Hence, we can proceed by induction on the degree of m_π to conclude that

$$e_1, e_2, \ldots, e_n, \Delta_n^{\pm}$$

is a generating set of $D(\mathbb{T}^n)^{D_n}$. Next we show how to choose the required polynomial P.

Note that both $e_n = \Delta_n^+ + \Delta_n^-$ and $D = \Delta_n^+ \Delta_n^-$ are D_n-invariant, and the degree of the leading monomial in e_n is $(1, 1, \ldots, 1, 1)$. Moreover, D can be expressed as a polynomial in $e_1, \ldots, e_n$, the leading monomial in D has degree $(2, 2, \ldots, 2, 0)$. Hence, e_n cannot enter in the expression of D with degree greater than 1. It is easy to see that the polynomial part of D consists of squares. Thus $D \notin \mathbb{C}[e_1, \ldots, e_{n-1}]$ since it has the same leading monomial as e_{n-1}^2 and the second in the lexicographical order monomial in s_{n-1}^2 has degree $(2, 2, \ldots, 1, 1)$. We conclude that

$$\Delta_n^+ \Delta_n^- = p_1(e_1, \ldots, e_{n-1}) + e_n p_0(e_1, \ldots, e_{n-1}),$$

that is

$$\Delta_n^- = \frac{\Delta_n^+ p_0 + p_1}{\Delta_n^+ - p_0}.$$

Then $P = \Delta_n^+ - p_0$ is a required polynomial. This completes the proof of the first statement.

To show the statement (2) we consider the same polynomial Δ as in the proof of Lemma 5.15. □

Proof of Theorem 5.14. Let $\mathbb{T}^n = Spec\, \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$, $\Lambda = \mathbb{C}[x_1, \ldots, x_n]$ and $f = x_1 \ldots x_n$. Denote by Γ the localization of Λ by the multiplicative set generated by f.

The proof of the statement for the symmetric group S_n is similar to the proof of Theorem 5.1. Next we consider the case of the group B_n. By Lemma 5.15 the action of B_n restricts to a free action on the localization Γ_Δ of Γ by Δ. Applying Theorem 5.6 we get the isomorphisms

$$D(U)^{B_n} \simeq D(U/B_n), \quad D(\Gamma_\Delta)^{B_n} \simeq D(\Gamma_\Delta^{B_n}).$$

By Theorem 5.3 we conclude that

$$D(\Gamma)_\Delta^{B_n} \simeq D(\Gamma_\Delta^{B_n}).$$

Since $\Gamma^{B_n} \simeq \Lambda$ we have

$$D(\Gamma)_\Delta^{B_n} \simeq D(\Lambda_\Delta) \simeq D(\Lambda)_\Delta.$$

Forming the skew field of fractions we conclude that

$$Frac(D(\mathbb{T}^n)^{B_n}) \simeq Frac(D(\mathbb{T}^n)).$$

Next consider the case of the group D_n. Repeating the same steps as above we get

$$D(\Gamma)_\Delta^{D_n} \simeq D(\Gamma_\Delta^{D_n}).$$

Applying Lemma 5.16 we have $\Gamma^{D_n} \simeq \Lambda_P$ for a certain P. Therefore

$$D(\Gamma)_\Delta^{D_n} \simeq D(\Gamma_{\Delta P}) \simeq D(\Gamma)_{\Delta P}.$$

Forming the skew field of fractions we conclude that

$$Frac(D(\mathbb{T}^n)^{D_n}) \simeq Frac(D(\mathbb{T}^n)).$$

This completes the proof of the theorem. □

Chapter 6

NNP for pseudo-reflection groups

In this section we prove that the Noncommutative Noether's Problem has a positive solution for all pseudo-reflection groups over any field of zero characteristic generalizing the case of complex reflection groups. This was shown in [59]. The method of the proof allows us to exhibit explicitly the Weyl generators of the invariant skew subfield of $F_n(\mathsf{k})$ using a simple algorithm.

A pseudo-reflection group is a group generated by pseudo-reflections (cf. Section 2). If the field is $\mathbb{C}$ then these are the complex reflection groups.

6.1 Proof of the NNP for irreducible pseudo-reflection groups

We proceed by considering first irreducible pseudo-reflection groups. Recall that a pseudo-reflection group W is called *irreducible* if its natural representation is irreducible.

We will make use of the following notion of the *field of definition* of G - the smallest subfield where a representation of the group G is defined. More precisely,

Definition 6.1. Let $\rho : G \to GL_n(\mathsf{k})$ be a linear representation of a finite group G. Let $\mathsf{k}' \subset \mathsf{k}$ be a subfield. Suppose there exists a homomorphism $\rho' : G \to GL_n(\mathsf{k}')$ such that ρ can be obtained from ρ' by the extension of scalars. We say that ρ has k' as the field of definition if k' is the smallest subfield with this property.

Given a linear representation $\rho : G \to GL_n(\mathsf{k})$ denote by χ the corresponding character function. Let $\mathbb{Q}(\chi)$ be the field extension of $\mathbb{Q}$ by $Im\,\chi$.

By [75, Appendix B], we have

Proposition 6.2. *Let W be an irreducible pseudo-reflection group and $\rho : W \to GL_n(\mathsf{k})$ a representation of W. Then ρ has $\mathbb{Q}(\chi)$ as the field of definition.*

We shall also need the following fact from the invariant theory of pseudo-reflection groups. Let M be the $n \times n$ matrix whose ij's entry is $\partial_{x_j} e_i$, where

$$\Lambda^W = \mathsf{k}[x_1, \dots, x_n]^W \simeq \mathsf{k}[e_1, \dots, e_n].$$

Let J' be the determinant of M.

Consider the set $\mathcal{S}$ of all pseudo-reflections in W. Each $s \in \mathcal{S}$ fixes a hyperplane H_s. Let L_s be a linear form whose kernel is H_s for each $s \in \mathcal{S}$. Set

$$J = \prod_{s \in \mathcal{S}} L_s.$$

It has the following properties:

Proposition 6.3 ([75, 20-2, Proposition A and B, 21-1, Proposition A and B]). *$J \neq 0$ and $w.J = det(w)J$ for every $w \in W$. Moreover, J is a multiple of J'.*

As in the case of complex reflection groups set $\Delta = J^{|W|}$ ([44, Section 3]).

Let E_i, $i = 1, \dots, n$ be the column vector, where we have 1 in the ith position and 0 in all others. Let

$$F_i = \begin{pmatrix} f_{i1} \\ \vdots \\ f_{in} \end{pmatrix}$$

be a solution of the linear system

$$(*)\, MF_i = E_i.$$

By the Kramer's rule, $f_{ij} \in \Lambda_J$, $1 \leq i, j \leq n$, where Λ_J is the localization of Λ by J.

For each $i = 1, \dots, n$ set

$$d_i = \sum_{k=1}^{n} f_{ik} \partial_k.$$

Then $d_i \in D(\Lambda_\Delta) = D(\Lambda)_\Delta$ and we have

$$d_i(e_j) = \delta_{ij}, \;\; i, j = 1, \ldots, n.$$

We will show that all differential operators d_i, $i = 1, \ldots, n$ are W-invariant. By Theorem 6.2 we can assume that e_i's, and hence d_i's, have coefficients in $\mathbb{Q}(\chi)$. Observe the following: let $\mathsf{k}' \subset \mathsf{k}$ be a subfield fixed by W and d a differential operator with coefficients in k', then the question of W-invariance of d is the same, weather we consider the base field k or k'. As $\mathbb{Q}(\chi)$ is fixed by W, to show that the d_i's are invariant differential operators on Λ_Δ^W, we can replace k by $\mathbb{Q}(\chi)$. Now our field of definition is a subfield of $\mathbb{C}$. Repeating the above argument we can assume that $\mathsf{k} = \mathbb{C}$.

Recall the following result of Knop:

Theorem 6.4 ([79, Theorem 3.1]). *Let X be a complex affine irreducible normal variety. Then*

$$D(X)^W = \{d \in D(X) | d(\mathcal{O}(X)^W) \subseteq \mathcal{O}(X)^W\}.$$

Denote by Δ' the result of expressing Δ as a polynomial in e_i, $i = 1, \ldots, n$. By the Chevalley-Shephard-Todd theorem, $\Lambda_\Delta^W \simeq \mathsf{k}[e_1, \ldots, e_n]_{\Delta'}$. Taking into account the action of operators d_i's and Theorem 6.4 we obtain the desired invariance of d_i's under the action of W. Indeed, for each $i = 1, \ldots, n$ the operator d_i sends every e_j to an element of $\mathsf{k}[e_1, \ldots, e_n]_{\Delta'}$ and the same holds for Δ'. Since these elements generate $\mathsf{k}[e_1, \ldots, e_n]_{\Delta'}$ the statement follows.

Recall that we have an injective map $\phi_\Delta : D(\Lambda_\Delta)^W \to D(\Lambda_\Delta^W)$. Moreover, as in the case of complex reflection groups we can show the following.

Proposition 6.5. *The map $\phi_\Delta : D(\Lambda_\Delta)^W \to D(\Lambda_\Delta^W)$ is surjective.*

Proof. It is sufficient to show that the images of

$$\phi_\Delta(d_i), \;\; \phi_\Delta(e_i), \;\; i = 1, \ldots, n$$

are the Weyl generators of $D(\Lambda_\Delta^W)$. Let $A := \Lambda_\Delta^W$. The A-module of Kähler differentials $\Omega_{\mathsf{k}}(A)$ is freely generated over A with basis $d_{e_1}, \ldots, d_{e_n}$. Then, by [89, 15.1.12], the A-module of derivations $\mathrm{Der}_{\mathsf{k}}(A)$ is freely generated by the unique extensions of ∂_{e_i}, $i = 1, \ldots, n$ from $\mathsf{k}[e_1, \ldots, e_n]$ to A. Clearly, $\phi_\Delta(d_i) = \partial_{e_i}$, $i = 1, \ldots, n$, and hence ϕ_Δ is surjective. □

Combining Proposition 5.2 and Proposition 6.5 we conclude

$$D(\Lambda_\Delta)^W \simeq D(\Lambda_\Delta^W).$$

Applying Theorem 5.3 we finally have

Corollary 6.6. *Let k be an arbitrary field of zero characteristic and W an irreducible pseudo-reflection group. Then the NNP holds for W.*

6.2 Proof of the NNP for general pseudo-reflection groups

In this subsection we consider general pseudo-reflection groups and prove the following theorem.

Theorem 6.7. *The NNP holds for all pseudo-reflection groups over fields of zero characteristic.*

Let V be a finite dimensional vector space. If g is a linear automorphism of V then we set

$$Fix\, g = \{v \in V | gv = v\} = \mathrm{Ker}\,(Id - g), \;\; [V, g] = Im(Id - g).$$

If g is a pseudo-reflection, $g \neq id$, then $Fix\, g$ is a hyperplane and $[V, g]$ is one dimensional. If $a \in V$ generates $[V, g]$ then for every $v \in V$ there exists $\psi(v) \in \mathsf{k}$ such that $v - gv = \psi(v)a$. Then ψ is a linear functional on V and $\mathrm{Ker}\,\psi = Fix\, g$.

In the following we collect basic properties of pseudo-reflections.

Lemma 6.8.

(1) *Let g be a pseudo-reflection of order $m > 1$, $H = Fix\, g$, L_H any linear functional such that $H = \mathrm{Ker}\, L_H$. Let a be a generator of $[V, g]$. Then there exists an m-th primitive root of unity μ such that*

$$gv = v - (1 - \mu)\frac{L_H(v)}{L_H(a)}a,$$

for all $v \in V$.

(2) *Let r and s be pseudo-reflections ($r, s \neq id$), $H = Fix\, r$, $J = Fix\, s$, x a generator of $[V, r]$, and y a generator of $[V, s]$. If $x \in J$ and $y \in H$ then $rs = sr$.*

(3) *A subspace $V' \subset V$ is invariant by a pseudo-reflection $g \neq id$ if and only if $V' \subseteq Fix\, g$ or $[V, g] \subseteq V'$.*

Proof. Given a pseudo-reflection g consider a linear functional ψ such that $gv = v - \psi(v)a$ and $H = \mathrm{Ker}\,\psi$, as above. Hence $ga = \mu a$ for a primitive m-th root of unity μ, and hence $\psi(a) = 1 - \mu$. We have $\psi = \lambda L_H$, where $0 \neq \lambda \in \mathsf{k}$. We get

$$\lambda = \frac{1 - \mu}{L_H(a)}.$$

This implies the first statement.

Now applying (1), we find $\mu, \nu \in \mathsf{k}$ such that $\forall\, v \in V$

$$rs(v) = v - (1-\mu)\frac{L_H(v)}{L_H(a)}a - (1-\nu)\frac{L_J(v)}{L_J(b)}b + (1-\mu)(1-\nu)\frac{L_H(b)L_J(v)}{(L_H(a)L_J(b))}a.$$

If $y \in H$ then $L_H(y) = 0$ and the last term is 0. Analogously, the last term in the expression of $sr(v)$ is 0 and other terms in both expressions are equal. Therefore $rs = sr$.

Finally, if $V' \subseteq Fix\, g$ or $[V, g] \subseteq V'$, then V' is invariant by the statement (1). Conversely, if V' is g-invariant and is not contained in $Fix\, g$, then $[V', g] \neq 0$, and hence

$$[V, g] = [V', g] \subseteq V'.$$

□

The following proposition generalizes the case of complex reflection groups and shows how to decompose a pseudo-reflection group into the product of irreducible one.

Proposition 6.9. *Let W be a finite group of pseudo-reflections on V. Consider a decomposition*

$$V = V_1 \oplus \ldots \oplus V_m$$

of the $\mathsf{k}W$-module V into irreducible submodules and set W_i to be the restriction of W to V_i, $i = 1, \ldots, m$. Then W_i is either a pseudo-reflection group or trivial, and

$$W \simeq W_1 \times \ldots \times W_m.$$

Proof. By Lemma 6.8, (3), if g is a non-identity pseudo-reflection then $[V, g] \subseteq V_j$ for some j. Let W_i be the subgroup of W generated by the pseudo-reflections g such that $[V, g] \subset V_i$ (if there is no such pseudo-reflections then $W_i = Id$). The subgroup W_i acts trivially on all V_j, $j \neq i$, and by By Lemma 6.8, (2), the subgroups W_i and W_j commute. Therefore, W is the direct product of the subgroups W_i's, and each W_i is irreducible pseudo-reflection group on V_i, or trivial. □

Consider now the Weyl algebra $A_n(\mathsf{k})$ with a linear action of a pseudo-reflection group W extended from a linear action on n-dimensional vector space V. By Proposition 6.9 we have $W \simeq W_1 \times \ldots \times W_m$. Suppose that $n = n_1 + \ldots + n_m + k$. Then

$$A_n(\mathsf{k}) = A_{n_1}(\mathsf{k}) \otimes \ldots \otimes A_{n_m}(\mathsf{k}) \otimes A_k(\mathsf{k}).$$

For each $i = 1, \ldots, m$, the subgroup W_i acts on $A_{n_i}(\mathsf{k})$ and fixes all $A_{n_j}(\mathsf{k})$ with $j \neq i$. Also, the whole group W fixes $A_k(\mathsf{k})$. Then we have

$$A_n(\mathsf{k})^W \simeq A_{n_1}(\mathsf{k})^{W_1} \otimes \ldots \otimes A_{n_m}(\mathsf{k})^{W_m} \otimes A_k(\mathsf{k}).$$

Applying Corollary 6.6 we immediately obtain Theorem 6.7.

6.3 Computation of invariant subfields

Theorem 6.7 guarantees the existence of an isomorphism between the Weyl field $F_n(\mathsf{k})$ and its invariant subfield $F_n(\mathsf{k})^W$ for any irreducible pseudo-reflection group W acting on n-dimensional vector space. But finding such an isomorphism explicitly might be quite difficult. We will present here an algorithmic procedure which allows exhibit explicitly the Weyl generators in $F_n(\mathsf{k})^W$ which realize such an isomorphism with $F_n(\mathsf{k})$ [59].

Let W be an irreducible pseudo-reflection group acting on $\mathsf{k}[x_1, \dots, x_n]$. Then, as we saw, $\mathsf{k}[x_1, \dots, x_n]^W$ has the algebraically independent generators $e_1, \dots, e_n$. The Weyl generators of $F_n(\mathsf{k})^W$ are $e_1, \dots, e_n$ and the operators $d_i, i = 1, \dots, n$ introduced immediately after the linear equation (*) (cf. Proposition 6.5).

The algorithm of finding the Weyl generators $F_n(\mathsf{k})^W$ consists of the following three steps:

- The first step is the classical problem of finding generators of the ring of invariant polynomials under a finite group action (cf. [34]). We also observe that for finite Coxeter groups explicit invariant bases are well known ([69] and references for 3.12).
- The second step is to compute the matrix M in (*), which involves n^2 operations with partial derivatives.
- Finally, the third step consists in solving the system (*).

We illustrate this algorithm with the following examples.

Example 6.10. Assume $n = 3$, $W = S_3$ and

$$J = (x_1 - x_2)(x_2 - x_3)(x_3 - x_2).$$

The group S_3 acts by permutations of variables. The polynomial J is not W-invariant but $\Delta = J^6$ is. Note that localizations of $\Lambda = \mathsf{k}[x_1, x_2, x_3]$ by the multiplicative sets generated by J and Δ are isomorphic. Consider the following elements of $F_3(\mathsf{k})^{S_3}$:

$$X_1 = x_1 + x_2 + x_3, \;\; X_2 = x_1x_2 + x_2x_3 + x_1x_3, \;\; X_3 = x_1x_2x_3;$$

$$Y_1 = \frac{x_1^2(x_2 - x_3)}{J}\partial_1 + \frac{x_2^2(x_3 - x_1)}{J}\partial_2 + \frac{x_3^2(x_1 - x_2)}{J}\partial_3;$$

$$Y_2 = \frac{x_1(x_3 - x_2)}{J}\partial_1 + \frac{x_2(x_1 - x_3)}{J}\partial_2 + \frac{x_3(x_2 - x_1)}{J}\partial_3;$$

$$Y_3 = \frac{(x_2 - x_3)}{J}\partial_1 + \frac{(x_3 - x_1)}{J}\partial_2 + \frac{(x_1 - x_2)}{J}\partial_3.$$

Then,

$$Y_i X_j - X_j Y_i = \delta_{ij}$$

for $i, j = 1, 2, 3$.

These elements are the Weyl generators of $F_3(\mathsf{k})^{S_3}$. In fact, by Proposition 6.5, we have an isomorphism

$$\phi_\Delta : D(\Lambda_\Delta)^{S_3} \to D(\Lambda_\Delta^{S_3}).$$

The Weyl generators of $D(\Lambda_\Delta^{S_3})$ are the elementary symmetric polynomials e_1, e_2, e_3, and $\partial_{e_1}, \partial_{e_2}, \partial_{e_3}$. Using the explicit form of ϕ_Δ we see that these Weyl generators of $D(\Lambda_\Delta^{S_3})$ are precisely the images of X_1, X_2, X_3 and Y_1, Y_2, Y_3 respectively.

Example 6.11. Assume now that $n = 2$ and $W = B_2$. Let

$$J = 8x_1x_2(x_2^2 - x_1^2).$$

Then we have the following Weyl generators of $F_2(\mathsf{k})^{B_2}$:

$$X_1 = x_1^2 + x_2^2, \;\; X_2 = x_1^4 + x_2^4;$$

$$Y_1 = \frac{4x_2^3\partial_1 - 4x_1^3\partial_2}{J};$$

$$Y_2 = \frac{-2x_2\partial_1 + 2x_1\partial_2}{J}.$$

Example 6.12. Assume that $n = 2$ and $W = I_2(8)$, the dihedral group of order 16.

Let

$$J = x_1^7x_2 - 7x_1^5x_2^3 + 7x_1^3x_2^5 - x_1x_2^7.$$

We have the following Weyl generators of $F_2(\mathsf{k})^{I_2(8)}$:

$$X_1 = x_1^2 + x_2^2, \;\; X_2 = (1/4)x_1^6x_2^2 - (1/2)x_1^4x_2^4 + (1/4)x_1^2x_2^6;$$

$$Y_1 = \frac{((1/2)x_1^6x_2 - 2x_1^4x_2^3 + (3/2)x_1^2x_2^5)\partial_1 - ((1/2)x_1x_2^6 - 2x_1^3x_2^4 + (3/2)x_1^5x_2^2)\partial_2}{J};$$

$$Y_2 = \frac{-2x_2\partial_1 + 2x_1\partial_2}{J}.$$

Chapter 7

CNP implies NNP

By the Chevalley-Shephard-Todd theorem the CNP holds for all pseudo-reflection groups over any field of zero characteristic. We also saw in the previous section that the NNP holds for all pseudo-reflection groups over any field of characteristic zero. This suggests that there might be a deeper connection between these two Noether's problems. In fact, in this section we prove the following general result [59]:

Theorem 7.1. *The CNP implies the NNP for an arbitrary linear action of a finite group and an arbitrary field of characteristic zero.*

Theorem 7.1 means exactly the following: let G be a finite group acting linearly on a finite dimensional vector space V such that $\mathsf{k}(V)^G \simeq \mathsf{k}(V)$. Then

$$Frac(D(V)^G) \simeq Frac(D(V)).$$

This was conjectured by the second author in [105].

A linear action of a finite group G on an n-dimensional k-vector space V can be naturally extended to an action on the ring $\mathcal{O}(V^*) = \mathsf{k}[x_1, \ldots, x_n]$. In its turn, this action extends to the action on the Weyl algebra $A_n(\mathsf{k}) \simeq D(\mathcal{O}(V^*))$. Recall that a positive solution for the Classical Noether's Problem for this action means that

$$\mathrm{Frac}(\mathcal{O}(V^*)^G) \cong \mathsf{k}(x_1, \ldots, x_n).$$

Denote by $\mathcal{B}$ the subalgebra of $A_n(\mathsf{k})^G$ generated by the the following invariant subrings:

$$\mathcal{O}(V^*)^G = \mathsf{k}[x_1, \ldots, x_n]^G, \ \ \mathcal{O}(V)^G = \mathsf{k}[\partial_1, \ldots, \partial_n]^G.$$

Then $\mathcal{B} = A_n(\mathsf{k})^G$ by Theorem 4.19. We will closely follow the argument in the proof of this fact.

Set $S = \mathcal{O}(V^*)^G \setminus \{0\}$. Since S is ad-nilpotent on $A_n(\mathsf{k})$, and hence on $\mathcal{B} = A_n(\mathsf{k})^G$, then S is an Ore subset of both $A_n(\mathsf{k})$ and $\mathcal{B}$ ([80, Theorem 4.9]).

Denote $F := \mathrm{Frac}(\mathcal{O}(V^*)^G)$. We have the following lemma [83]:

Lemma 7.2 ([83, Lemma 8]). *Let L be a finite field extension of k, with $trdeg_k\, L = l$. Consider the ring of differential operators $D(L)$ on L. Let A be a subalgebra of $D(L)$ containing L, with a filtration induced from that of $D(L)$ (defined by the order of differential operators). If the associated graded algebra of A contains as a subalgebra a finitely generated graded L-algebra B such that $Kdim\, B = l$, then $A = D(L)$.*

We also need the following generalization of a well known fact for regular commutative rings (cf. [89, 15.1.25]):

Proposition 7.3 ([94, Proposition 1.8]). *Let R be a finitely generated commutative domain, not necessarily regular, S any multiplicatively closed subset of R. Then both left and right localizations of $D(R)$ by S exist and they are isomorphic to $D(R_S)$.*

We have

Lemma 7.4. $\mathrm{Frac}(A_n(\mathsf{k})^G) \cong \mathrm{Frac}(D(F))$.

Proof. We have $\mathcal{B} \subset A_n(\mathsf{k})^G$ by definition. On the other hand,

$$A_n(\mathsf{k})^G \subset D(\mathcal{O}(V^*)^G)$$

by restriction of domain. We have

$$D(\mathcal{O}(V^*)^G)_S = D(\mathcal{O}(V^*)^G_S) = D(F)$$

by Proposition 7.3. After localization by S we obtain:

$$\mathcal{B}_S \subset A_n(\mathsf{k})^G_S \subset D(\mathcal{O}(V^*)^G)_S = D(F).$$

Consider the filtration on $\mathcal{B}_S$ induced from $D(F)$. Since

$$\mathcal{O}(V^*)^G \subset \mathcal{B},$$

we have that $gr\, \mathcal{B}_S$ contains

$$F \otimes \mathcal{O}(V^*)^G$$

as a graded F-subalgebra. Since $\mathcal{O}(V^*)$ is finite over $\mathcal{O}(V^*)^G$ (as a module), then it has the Krull dimension n ([89]). Applying Lemma 7.2 we get $\mathcal{B}_S = D(F)$. We conclude that

$$\mathrm{Frac}(D(F)) \subset \mathrm{Frac}(A_n(\mathsf{k})^G) \subset \mathrm{Frac}(D(F)),$$

which implies the desired equality. $\square$

Remark 7.5. Note that in fact in the above proof we do not need the equality $\mathcal{B} = A_n(\mathsf{k})^G$ but only the embedding $\mathcal{B} \subset A_n(\mathsf{k})^G$.

As a consequence of Lemma 7.4 we can prove Theorem 7.1. Under the condition of the theorem we have that $F \cong \mathsf{k}(x_1, \ldots, x_n)$. Then $D(F)$ is isomorphic to the localization of A_n by

$$\mathsf{k}[x_1, \ldots, x_n] \setminus \{0\}$$

and the statement follows by taking the skew fields of fractions.

Theorem 7.1 allows us to transfer all known cases with positive solution for the CNP to the NNP. In particular, we have the following immediate application of Theorem 7.1.

Corollary 7.6. *The Noncommutative Noether's Problem holds in the following cases for any field of characteristic zero:*

- *For all linear representations of all pseudo-reflection groups;*
- *For alternating groups $\mathcal{A}_n$ with usual permutation action for $n = 3, 4, 5$;*
- *For any group when $n = 3$ and k is algebraically closed.*

To get a feeling of the invariants of the alternating groups we consider the groups $\mathcal{A}_3$ and $\mathcal{A}_4$ explicitly.

Example 7.7. We start with the group $\mathcal{A}_3$. Let V be a 3-dimensional vector space with a basis e_1, e_2, e_3. Consider the linear representation of $\mathcal{A}_3$ on V by permutations of the basis elements. The induced action by algebra automorphisms on $D(S(V^*))$ gives the action of $\mathcal{A}_3$ on the Weyl algebra $A_3(\mathsf{k})$.

The 1-dimensional subspace W generated by the element $e_1 + e_2 + e_3$ is $\mathcal{A}_3$-invariant, with trivial action of $\mathcal{A}_3$. Hence, there exists a 2-dimensional invariant complement U, that is $V \simeq W \oplus U$ is a direct sum of $\mathcal{A}_3$-invariant subspaces. Then we have

$$A_3(\mathsf{k})^{\mathcal{A}_3} \simeq A_1(\mathsf{k}) \otimes A_2(\mathsf{k})^{\mathcal{A}_3}.$$

Since the NNP holds for any linear action on the second Weyl algebra ([3]), taking the skew field of fractions of the above gives the desired result for $\mathcal{A}_3$.

Example 7.8. Next we consider the group $\mathcal{A}_4$. Let V be a 4-dimensional vector space with a fixed basis e_1, e_2, e_3, e_4, and consider the action of $\mathcal{A}_4$ on V by permutations of these basis elements. Choose a new basis of V as follows:

$$f_0 = e_1 + e_2 + e_3 + e_4;$$

$$f_1 = e_1 + e_2 - e_3 - e_4;$$
$$f_2 = e_1 - e_2 + e_3 - e_4;$$
$$f_3 = e_1 - e_2 - e_3 + e_4.$$

Let W be the subspace of V generated by f_0 and U the subspace of V generated by f_1, f_2 and f_3. Clearly, both subspaces are $\mathcal{A}_4$-invariant and $V = W \oplus U$.

The group $\mathcal{A}_4$ consists of the following elements

$$id,\ o_1 = (1\,2)(3\,4),\ o_2 = (1\,3)(2\,4),\ o_3 = (1\,4)(2\,3),\ o_5 = (1\,2\,3),\ o_6 = (1\,3\,2).$$

The first 4 elements generate the Klein subgroup $\mathcal{K}_4$ (normal in $\mathcal{A}_4$), which acts as follows:

$$o_1(f_1) = f_1,\ o_1(f_2) = -f_2,\ o_1(f_3) = -f_3;$$
$$o_2(f_1) = -f_1,\ o_2(f_2) = f_2,\ o_2(f_3) = -f_3;$$
$$o_3(f_1) = -f_1,\ o_3(f_2) = -f_2,\ o_3(f_3) = f_3.$$

And the action of the remaining elements of $\mathcal{A}_4$:

$$o_5(f_1) = -f_3,\ o_5(f_2) = f_1,\ o_5(f_3) = -f_2;$$
$$o_6(f_1) = f_2,\ o_6(f_2) = -f_3,\ o_6(f_3) = -f_1.$$

CLAIM 1: The NNP holds for the restricted action of $\mathcal{K}_4$ on $A_3(\mathsf{k})$.

Let x_i, $y_i = \partial_i$, $i = 1, 2, 3$ be the generators of $A_3(\mathsf{k})$. Set $w_i = y_i x_i$, $i = 1, 2, 3$. Then we have $w_i^2 = y_i^2 x_i^2 - w_i$ for all i, and $A_3(\mathsf{k})^{\mathcal{K}_4}$ is generated by

$$x_i^2,\ w_i,\ y_i^2,\ i = 1, 2, 3.$$

Hence $\mathrm{Frac}(A_3(\mathsf{k})^{\mathcal{K}_4})$ is generated by

$$X_1 = y_1^2,\ Y_1 = 1/2 w_1 y_1^{-2},\ X_2 = y_2^2,\ Y_2 = 1/2 w_2 y_2^{-2},$$
$$X_3 = y_3^2, \quad Y_3 = 1/2 w_3 y_3^{-2}.$$

These elements satisfy the Weyl relations and thus the NNP holds for $\mathcal{K}_4$.

CLAIM 2: The NNP holds for the restricted action of $\mathcal{A}_4$ on $A_3(\mathsf{k})$.

Note that

$$\mathrm{Frac}(A_3(\mathsf{k}))^{\mathcal{A}_4} \simeq (\mathrm{Frac}(A_3(\mathsf{k})^{\mathcal{K}_4}))^{\mathcal{A}_3},$$

since $\mathcal{K}_4$ is normal in $\mathcal{A}_4$ and the quotient is isomorphic to $\mathcal{A}_3$. By Claim 1,

$$\mathrm{Frac}(A_3(\mathsf{k}))^{\mathcal{K}_4} \simeq F_3(\mathsf{k}).$$

Moreover, taking into account the action of the elements $o_5, o_6 \in \mathcal{A}_4$ on the Weyl generators, we see that the induced action of $\mathcal{A}_3$ on $F_3(\mathsf{k})$ is just the usual permutation action. Applying the solution of the NNP for $\mathcal{A}_3$ we obtain the positive solution of the NNP for the action of $\mathcal{A}_4$ on $A_3(\mathsf{k})$.

Since we have an isomorphism

$$A_4(\mathsf{k})^{\mathcal{A}_4} \simeq A_1(\mathsf{k}) \otimes A_3(\mathsf{k})^{\mathcal{A}_4},$$

the Claim 2 implies the positive solution of the NNP for the action of $\mathcal{A}_4$ on $A_4(\mathsf{k})$.

Also Theorem 7.1 allows us to give a shorter proof of the Proposition 4.20:

Corollary 7.9. *If $F_n(\mathsf{k})^G$ is isomorphic to $F_m(L)$ for some m and some purely transcendental extension L of k of transcendence degree t, then $m = n$ and $t = 0$.*

Proof. We have that $F_m(L) \simeq \mathrm{Frac}(D(F))$. Now use [43, Lemma 3.2.2]. The center of $F_m(L)$ has the transcendence degree t over k. By the primitive element theorem, $F = \mathsf{k}(y_1, \ldots, y_n)(f)$, for certain algebraically independent $y_1, \ldots, y_n$, and by [89, 15.2.4], the second skew-field has center of transcendence degree 0. So $t = 0$ and $m = n$. □

Chapter 8

Generalization to affine irreducible varieties

In this section we generalize Theorem 7.1 for the rings of differential operators on any affine irreducible variety in the case of the complex field. So, throughout this section, the field k will be the field of complex numbers.

Let X be a complex affine irreducible variety with the coordinate ring $\mathcal{O}(X)$, and G a finite group of automorphisms of X. Denote by $D(X)$ the algebra of differential operators on X, which is defined as the ring of differential operators $D(\mathcal{O}(X))$. The action of G on X extends naturally to the action of G on $D(X)$. Since X is irreducible, then $D(X)$ is an Ore domain by Proposition 3.7.

We have the following generalization of Theorem 7.1.

Theorem 8.1. *If the quotient variety X/G is birational to an irreducible affine variety Y then $D(X)^G$ is birationally equivalent to $D(Y)$.*

We start with the following lemma.

Lemma 8.2. *Let X be a complex irreducible affine variety with a free action by automorphisms of a finite group G. If X/G is birationally equivalent to an affine irreducible variety Y then*

$$\mathrm{Frac}(D(X)^G) \cong \mathrm{Frac}(D(Y)).$$

Proof. Note that we do not require the variety X to be smooth, unlike in a similar statement in [89].

Let S be the set of regular elements in $\mathcal{O}(X)^G$. Since X/G is birational to Y, then we have

$$\mathrm{Frac}(\mathcal{O}(X)^G) = \mathrm{Frac}(\mathcal{O}(X)^G_S) \cong \mathrm{Frac}(\mathcal{O}(Y)).$$

Since

$$\mathrm{Frac}(D(X)^G) \cong \mathrm{Frac}(D(X/G))$$

by Theorem 5.6, then applying Proposition 7.3, we obtain

$$\mathrm{Frac}(D(X)^G) \cong \mathrm{Frac}(D(X/G)_S) \cong \mathrm{Frac}(D(\mathcal{O}(X)_S^G)) \cong \mathrm{Frac}D(\mathrm{Frac}(\mathcal{O}(Y))),$$

and hence

$$\mathrm{Frac}(D(X)^G) \cong \mathrm{Frac}(D(Y)).$$

□

We proceed with the proof of Theorem 8.1. By Theorem 5.6, 1), there exists an open dense subset $V \subset X$ on which the action of G is free and such that the quotient map

$$\pi : X \to X/G$$

restricts to the map $\pi : V \to V'$, where V' is open dense in X/G. By the Noether's Theorem (Theorem 2.4), the map π is finite, hence affine ([65], Exercise 5.17). Let $\tilde{V'}$ be a principal open subset of V'. Since π is affine, then

$$\tilde{V} = \pi^{-1}(\tilde{V'})$$

is affine. Since $\tilde{V}$ is a union of orbits, G restricts to a free action on it. We now have a quotient map $\pi : \tilde{V} \to \tilde{V'}$ with $\tilde{V}$ affine. Also, since $\tilde{V'} \subset X/G$, then $\tilde{V'}$ is birational to Y. Applying Lemma 8.2 we obtain

Lemma 8.3. $\mathrm{Frac}(D(\tilde{V})^G) \cong \mathrm{Frac}(D(Y))$.

For $f \in \mathcal{O}(X)$ denote $Spec\,\mathcal{O}(X)_f$ the principal open subset. These sets constitute a basis of the Zariski topology, and hence there exists a principal open subset

$$Spec\,\mathcal{O}(X)_h \subset \tilde{V}$$

for $h \in \mathcal{O}(X)$. Set

$$f = \prod_{g \in G} g \cdot h.$$

Then f is G-invariant. Thus we have

Lemma 8.4. *There exists a principal open set* $Spec\,\mathcal{O}(X)_f \subset \tilde{V}$ *with* G*-invariant* f.

Now we generalize the argument given in the proof for unitary reflection groups in [44]. By Lemma 8.4 there exists a principal open set

$$Spec\,\mathcal{O}(X)_f \subset \tilde{V}$$

with G-invariant f. Then we have the following inclusions of varieties:

$$\operatorname{Spec}\mathcal{O}(X)_f \subset \tilde{V} \subset X.$$

Let $D(.)$ be the sheaf of differential operators functor which associates the ring of differential operators to a given variety. Functor $D(.)$ is contravariant and we have chain of inclusions

$$D(\mathcal{O}(X)) \subset D(\tilde{V}) \subset D(\mathcal{O}(X)_f) = D(\mathcal{O}(X))_f$$

(cf. [84, Proposition 2.4.18]). Taking the field of fractions, and then the G-invariants, we have the following chain:

$$\mathrm{Frac}D(\mathcal{O}(X))^G \subset \mathrm{Frac}(D(W))^G \subset \mathrm{Frac}(D(\mathcal{O}(X))_f)^G$$

$$= \mathrm{Frac}(D(\mathcal{O}(X))_f^G) = \mathrm{Frac}(D(\mathcal{O}(X))^G).$$

Then applying Lemma 8.3 we have

$$\mathrm{Frac}(D(X)^G) \simeq \mathrm{Frac}(D(W)^G) \cong \mathrm{Frac}(D(Y)),$$

which implies the statement of Theorem 8.1.

Remark 8.5. Observe that the above proof of Theorem 8.1 requires no assumptions on smoothness and normality of the varieties.

From Theorem 8.1 we immediately deduce the following corollary.

Corollary 8.6. *Let X be a complex affine irreducible variety of dimension n and G a finite group of automorphisms of X.*
a) If X/G is birational to X then $D(X)^G$ and $D(X)$ are birationally equivalent.
b) If the quotient variety X/G is rational then

$$(\mathrm{Frac}(D(X)))^G \simeq F_n(\mathbb{C}).$$

Proof. Statement a) follows from Theorem 8.1 for $Y = X$, while b) is obtained from Theorem 8.1 by taking $Y = \mathbb{A}^n(\mathbb{C})$. □

As an application, we can easily recover the NNP for the torus (Theorem 5.14). Indeed, applying Lemmas 5.15 and 5.16, we see that the quotients $\mathbb{T}^n/B_n$ and $\mathbb{T}^n/D_n$ are rational. Hence, we get

$$Frac(D(\mathbb{T}^n)^{B_n}) \simeq Frac(D(\mathbb{T}^n)^{D_n}) \simeq F_n(\mathbb{C}).$$

Example 8.7. The following gives an example of the situation described in Corollary 8.6 a).

Consider an elliptic curve E and define the map $\tau : E \to E$ which sends $P \mapsto P+P$ (multiplication by 2). The map τ is an isogeny, and hence it is surjective with finite kernel, $E[2]$ ([90] I.7). Since E is an abelian variety, we can view $E[2]$ as a finite group of automorphisms of E with the action given by translations: $Q \in E[2]$ maps $P \in E$ to $Q + P$. With this we have

$$E/E[2] \simeq E.$$

Removing a finite number of points from E and from the corresponding inverse image, we obtain the desired birational equivalence of affine varieties.

In the example above we have

$$Frac(D(E)^{E[2]}) \simeq Frac(D(E)),$$

but the skew field of fractions of the ring of differential operators on a curve of positive genus cannot be a Weyl field [18].

Chapter 9

Galois algebras

The Weyl algebras have a hidden skew group structure which allows to view them as members of the family of so-called *Galois rings.* The theory of Galois rings and Galois orders was developed in [54], [55] and had a strong impact on the representation theory of various classes of algebras, first of all the universal enveloping algebra of gl_n [98], restricted Yangians of type A and, more general, finite W-algebras of type A [52], [53]. The main motivation for the development of this theory was a study of representations of infinite dimensional associative algebras via representations of their commutative subalgebras. A general framework for this approach was introduced in [42], where the theory of general Harish-Chandra categories was developed. Theory of Galois rings is a refinement of the theory of Harish-Chandra categories in the case when we have a pair $\Gamma \supset U$ with Γ commutative, and with certain embedding of U in a skew monoid ring. This theory has led to breakthrough in representation theory for many algebras, in particular the universal enveloping algebra $U(gl_n)$, its quantization, finite W-algebras of type A (see [50] for a detailed discussion). A somewhat different formalism was recently developed in [66]. A different approach to study so-called *flag Galois orders* was proposed in [116]. It was also noted in [116] that there exists a remarkable connection between quantized Coulomb branches and Galois orders, in particular with OGZ algebras of type A ([87]) and spherical subalgebras of cyclotomic Cherednik algebras ([45]).

Natural examples of Galois rings come from the generalized Weyl algebras of rank 1 ([12]) over integral domains with infinite order automorphisms and their tensor products. This includes in particular such algebras as the Weyl algebra A_n, the quantum plane, the q-deformed Heisenberg algebra, quantized Weyl algebras, $U(sl_2)$ and its quantization, and the Witten-Woronowicz algebra.

In this section we discuss the subrings of invariants of a family of linear

Galois rings, which contains in particular the Weyl algebras. We show that the Gelfand-Kirillov conjecture holds for all members of this family. Moreover, realizing these invariant subrings again as Galois rings allows to apply the general theory to study their representation theory uniformly and effectively. We start with the discussion of invariant cross products.

9.1 Invariant cross products

Let R be a ring, $\mathfrak{M}$ a monoid acting on R by ring automorphisms. We will denote the action of $m \in \mathfrak{M}$ on $r \in R$ by r^m. Consider the *cross product*, that is the skew monoid ring $R * \mathfrak{M}$, where

$$(rm)(r'm') = (r(r')^m)(mm'), \;\; r, r' \in R, \;\; m, m' \in \mathcal{M}.$$

Any element of $R * \mathfrak{M}$ can be written in the form

$$x = \sum_{m \in \mathfrak{M}} x_m m.$$

Define *supp* x as the set of those $m \in \mathfrak{M}$ for which x_m is not zero.

Let G be a finite group acting on $\mathfrak{M}$ by conjugation. We can define an action of G on $R * \mathfrak{M}$ as

$$g(rm) = g(r)g(m), \;\; g \in G, \;\; r \in R, \;\; m \in \mathfrak{M}.$$

We denote

$$\mathfrak{K} = (R * \mathfrak{M})^G$$

the ring of invariants by the action of G.

From now on we will consider the case when $R = L$ is a field, and assume that L is a finite Galois extension of a field K such that $G = Gal(L, K)$. The monoid $\mathfrak{M}$ will be assumed to have the following property: if $m, m' \in \mathfrak{M}$ and their restrictions to K coincide, then $m = m'$. In this case we say that $\mathfrak{M}$ is a *separating* monoid of automorphisms.

Example 9.1. A *shift operator algebra* is the following cross product

$$\mathbb{S} := \mathsf{k}(t_1, \ldots, t_n) * \mathbb{Z}^n,$$

where $L = \mathsf{k}(t_1, \ldots, t_n)$ is the field of rational functions, $\mathfrak{M} = \mathbb{Z}^n$ is a free abelian group with the canonical generators $\sigma_1, \ldots, \sigma_n$. The action of $\mathbb{Z}^n$ on L is given by the following *shifts*:

$$\sigma_i(t_j) = t_i - \delta_{ij}, \;\; i, j = 1, \ldots, n.$$

There are natural actions of the classical Weyl groups S_n, $B_n = C_n$ and D_n on $\mathbb{S}$ as follows. The action of the symmetric group S_n on L is defined by permutations of the variables t_i, while its action on $\mathbb{Z}^n$ is by conjugation:

$$\pi(\sigma_i) = \sigma_{\pi(i)}, \ \ \pi \in S_n, \ \ i = 1, \dots, n.$$

Consider the group $\mathbb{Z}_2^n$ which is the direct product of n copies of $\mathbb{Z}_2$. Choosing a basis $\epsilon_1, \dots, \epsilon_n$ of $\mathbb{Z}_2^n$, we have its action on $\mathbb{Z}^n$ by conjugation:

$$\epsilon_i(\delta_i) = \delta_i^{-1}, \ \ \epsilon_i(\delta_j) = \delta_j,$$

for all $i \neq j$. The action of $\mathbb{Z}_2^n$ on $\mathbb{C}(t_1, \dots, t_n)$ is:

$$\epsilon_i(t_i) = 2 - t_i, \ \ \epsilon_i(t_j) = t_j, \ \ j \neq i.$$

With this information we obtain the actions of the classical Weyl groups on $\mathbb{S}$.

9.2 Galois rings and orders

We will assume all rings are k-algebras and k is algebraically closed of characteristic zero.

Definition 9.2 ([54]). Let Γ be a commutative finitely generated domain and U an associative algebra finitely generated over Γ. We say that U is a Γ-Galois ring if the following conditions are satisfied:

- We have a finitely generated field extension L of the field of fractions $K = Frac\,\Gamma$ with the Galois group G;
- $\mathfrak{M} \subset Aut_{\mathbb{C}} L$ is a separating monoid of automorphisms of L;
- The group G acts on $\mathfrak{M}$ by conjugation;
- There is an embedding $U \to \mathfrak{K} = (L{*}\mathfrak{M})^G$ such that $KU = UK = \mathfrak{K}$.

Remark 9.3. Γ is not required to be central in U.

Example 9.4. Let Γ be a commutative domain, finitely generated as a k-algebra, $\sigma \in Aut_{\mathsf{k}}\Gamma$ an automorphism of infinite order and $A = \Gamma[x;\sigma]$, the skew polynomial Ore extension, with

$$xd = \sigma(d)x, \ \ d \in \Gamma.$$

Then

$$\Gamma[x;\sigma] \simeq \Gamma * \mathfrak{M},$$

where

$$\mathfrak{M} = \{\sigma^n \mid n = 0, 1, \dots\} \simeq \mathbb{N}.$$

The isomorphism fixes Γ and maps x to the generator $\overline{1}$ of the monoid $\mathbb{N}$ ($\overline{1}$ acts on Γ as σ).

Let $L = K$ be the field of fractions of Γ and $G = \{e\}$. Then A is a Galois Γ-ring in $K * \mathfrak{M}$. The localization of A by x is isomorphic to $\Gamma * \mathbb{Z}$.

Further examples of Galois algebras include the generalized Weyl algebras over integral domains with infinite order automorphisms and their tensor products (cf. Section 13), such as the Weyl algebra A_n, the quantum plane, the q-deformed Heisenberg algebra, the quantized Weyl algebras, the Witten–Woronowicz algebra, and many others.

Example 9.5. Let $A_n = A_n(\mathsf{k}) = \mathsf{k}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ be the n-th Weyl algebra over the field k with standard generators ∂_i, x_i, $1 \le i, j \le n$. For each $i = 1, \dots, n$ denote $t_i = \partial_i x_i$ and consider $\sigma_i \in \mathrm{Aut}\ \mathsf{k}[t_1, \dots, t_n]$ such that $\sigma_i(t_j) = t_j - \delta_{ij}$ for all $j = 1, \dots, n$. Let $\mathbb{Z}^n$ be the free abelian group generated by $\sigma_1, \dots, \sigma_n$.

The polynomial algebra $\Gamma = \mathsf{k}[t_1, \dots, t_n]$ is a maximal commutative subalgebra of A_n, and A_n is a free right (left) module over Γ.

There exists a natural embedding

$$A_n \to \mathsf{k}[t_1, \dots, t_n] * \mathbb{Z}^n,$$

such that

$$x_i \mapsto \sigma_i, \ \ \partial_i \mapsto t_i\sigma_i^{-1}, \ \ i = 1, \dots, n.$$

Set $S = \Gamma \setminus \{0\}$. We have

$$A_n[S^{-1}] \simeq \mathsf{k}(t_1, \dots, t_n) * \mathbb{Z}^n.$$

Hence A_n is a Galois Γ-ring in $\mathsf{k}(t_1, \dots, t_n) * \mathbb{Z}^n$.

Example 9.6. Consider the Lie algebra $\mathfrak{gl}_n$ of $n \times n$ complex matrices with the standard basis of elementary matrices $\{e_{i,j}, \ 1 \le i, j \le n\}$. For each $k \leqslant n$ denote by $\mathfrak{gl}_k$ the Lie subalgebra of $\mathfrak{gl}_n$ spanned by $\{e_{ij} \mid i, j = 1, \dots, k\}$. Then we have the following embeddings of Lie subalgebras

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n$$

and the corresponding embeddings of the universal enveloping algebras

$$U_1 \subset U_2 \subset \ldots \subset U_n,$$

where $U_k = U(\mathfrak{gl}_k)$, $1 \le k \le n$. Let Z_k be the center of U_k for each k. This is the polynomial algebra generated by k elements. Denote by Γ the subalgebra of $U = U_n = U(\mathfrak{gl}_n)$ generated by the centers Z_k, $k = 1, \dots, n$. This is the *Gelfand-Tsetlin subalgebra* [119], isomorphic to a polynomial algebra in $\frac{n(n+1)}{2}$ variables.

Introduce the polynomial algebra Λ in the variables $\{\lambda_{ij} \mid 1 \leqslant j \leqslant i \leqslant n\}$ together with an embedding

$$\tau : \Gamma \longrightarrow \Lambda$$

whose image is generated by the following elements of Λ:

$$\sum_{i=1}^{k}(\lambda_{ki}+k-1)^s\prod_{j\neq i}\left(1-\frac{1}{\lambda_{ki}-\lambda_{kj}}\right),$$

for all $1 \leq s \leq k \leq n$.

Let $G = S_1 \times \ldots \times S_n$ be the product of symmetric group. Then G has a natural action on Λ, where for each k the component S_k permutes the variables $\lambda_{k1}, \ldots, \lambda_{kk}$. It turns out that the image of τ consists of G-invariants and $\Lambda^G \simeq \Gamma$.

Denote by K the field of fractions of Γ and by L the field of fractions of Λ. Note that Λ is the integral closure of Γ in L. Then $L^G = K$ and G is the Galois group of the field extension $K \subset L$.

Set $\mathfrak{M}$ to be the free abelian group $\mathbb{Z}^{\frac{n(n-1)}{2}}$. The group $\mathfrak{M}$ acts by shifts on L and there is an embedding of U in $(L * \mathfrak{M})^G$ [54]. Moreover, U is a Galois Γ-ring.

The last example was extended in [53], [52] for shifted Yangians and finite W-algebras of type A, and certain invariant rings of differential operators on the torus [54] .

We recall briefly the structure theory of Galois rings following [54]. There is the following useful characterization of Galois rings:

Proposition 9.7 ([54], Proposition 4.1). *Assume that a* Γ*-ring*

$$U \subset \mathfrak{K} = (L * \mathfrak{M})^G$$

is generated by $u_1, \ldots, u_k$. *If*

$$\bigcup_{i=1}^{k} supp\, u_i$$

generates $\mathfrak{M}$ *as a monoid then* U *is a Galois ring. In particular, if* $LU = L * \mathfrak{M}$ *then* U *is a Galois ring.*

The main properties of Galois rings are collected in the following propositions.

We recall the following statement.

Proposition 9.8 ([54, Proposition 4.2]). *Let* U *be a Galois algebra over* Γ *embedded in* $\mathfrak{K} = (L * \mathfrak{M})^G$ *and* $S = \Gamma \setminus \{0\}$. *Then* S *is a left and right denominator set in* U *and the localization of* U *by* S *both on the left and on the right are isomorphic to* $\mathfrak{K}$. *In particular, if* $\mathfrak{K}$ *is an Ore domain, then so is* U.

Among Galois rings we have an important family of *Galois orders*.

Definition 9.9.

- A Galois Γ-ring is called a right (left) Galois order over Γ if for every right (left) finite dimensional K- vector space $W \subset \mathfrak{K}$, $W \cap \Gamma$ is a finitely generated right (left) Γ-module. If U is both left and right Galois order, then U is called a Galois order.

- Let U be an associative algebra and Γ a commutative subalgebra of U. We say that Γ *is a Harish-Chandra subalgebra* if for every $u \in U$, $\Gamma u \Gamma$ is finitely generated left and right Γ-module [42].

We have the following characterization of Galois orders.

Proposition 9.10 ([54]). *Let U is a Galois Γ-ring.*

- *If Γ is Noetherian and U is a left (right) projective Γ-module then U is a left (right) Galois order.*

- *If Γ is a finitely generated domain over k and U a Galois order over Γ then Γ is a Harish-Chandra algebra in U.*

Example 9.11. The Weyl algebra $A_n(\mathsf{k})$ is a Galois order over $\mathsf{k}[x_1, \ldots, x_n]$.

Consider the action of the symmetric group S_n on $A_n(\mathsf{k})$. We will show that $A_n(\mathsf{k})^{S_n}$ is a Galois order over

$$D = \mathsf{k}[x_1, \ldots, x_n]^{S_n}.$$

Recall that $A_n(\mathsf{k})^{S_n}$ is generated by $\mathsf{k}[x_1, \ldots, x_n]^{S_n}$ and $\mathsf{k}[\partial_1, \ldots, \partial_n]^{S_n}$ by Theorem 4.19. Denote $K = Frac\,\Gamma$ and $L = \mathsf{k}(x_1, \ldots, x_n)$. The generators $\sigma_1, \ldots, \sigma_n$ of $\mathbb{Z}^n$ act on L as before: $\delta_i(t_j) = t_j - \delta_{ij}$. Consider an action of S_n on $\mathbb{Z}^n$ by conjugation, and set $\mathfrak{K} = (L * \mathbb{Z}^n)^{S_n}$. The algebra $A_n(\mathsf{k})^{S_n}$ is simple. Hence we have an embedding

$$A_n(\boldsymbol{k})^{S_n} \to \mathfrak{K}$$

induced by the homomorphism $A_n(\mathsf{k}) \to L * \mathbb{Z}^n$ described in the previous example.

Consider the elements $x_1 + \ldots + x_n$ and $\partial_1 + \ldots + \partial_n$. Their images in $\mathfrak{K}$ have supports that generate $\mathbb{Z}^n$ as a monoid. So, $A_n(\mathsf{k})^{S_n}$ is a Galois ring over D by Proposition 9.7. Moreover, the canonical embedding of D-modules

$$A_n(\mathsf{k})^{S_n} \to A_n(\mathsf{k})$$

splits, the inverse is the symmetrization map

$$\frac{1}{n!} \sum_{\pi \in S_n} \pi.$$

Since $A_n(\mathsf{k})$ is free over $\mathsf{k}[t_1, \dots, t_n]$, and the latter algebra is free over the invariant subalgebra D we have that $A_n(\mathsf{k})^{S_n}$ is a left and right projective D-module. Applying Proposition 9.10 we conclude that D is a Harish-Chandra subalgebra of $A_n(\mathsf{k})^{S_n}$ and $A_n(\mathsf{k})^{S_n}$ is a Galois order over D.

Example 9.12. The Galois ring structure of the universal enveloping algebra $U(\mathfrak{gl}_n)$ over the Gelfand-Tsetlin subalgebra Γ implies the embedding of $U(\mathfrak{gl}_n)$ into the tensor product

$$\mathcal{A}_1^{S_1} \otimes \mathcal{A}_2^{S_2} \otimes \ldots \otimes \mathcal{A}_{n-1}^{S_{n-1}} \otimes \mathsf{k}[t_1, \dots, t_n]^{S_n},$$

where $\mathcal{A}_k$ is a certain localization of the k-th Weyl algebra A_k. The NNP implies

$$(F_k)^{S_k} \simeq Frac(A_k^{S_k}) \simeq F_k,$$

and we have

$$Frac(U(\mathfrak{gl}_n)) \simeq F_1 \otimes \ldots \otimes F_{n-1} \otimes \mathsf{k}(y_1, \dots, y_k) \simeq F_{\frac{(n(n-1)}{2}} \otimes \mathsf{k}(y_1, \dots, y_k)$$

Hence, $U(\mathfrak{gl}_n)$ is birationally equivalent to the Weyl algebra A_m over $\mathsf{k}(y_1, \dots, y_n)$ for $m = n(n-1)/2$ (cf. [53, Proposition 5.2]).

Chapter 10

Linear Galois orders

In this section we assume that the field k is the field of characteristic zero.

Linear Galois orders form an important class of Galois algebras that was introduced in [44].

10.1 Galois orders of shift and quantum types

Let V be an n-dimensional vector space and $L \simeq \mathbb{C}(t_1, \dots, t_n)$ the field of fractions of the symmetric algebra $S(V^*)$. Let G be a pseudo-reflection group acting on V by reflections. This action can be extended to an action of G on L. Set $K = L^G$. The field K is a purely transcendental extension of $\mathbb{C}$ by the Chevalley-Shephard-Todd theorem. Let Γ be a polynomial subalgebra such that $\mathrm{Frac}\,\Gamma = K$.

Fix a submonoid $\mathfrak{M} \subset Aut_{\mathsf{k}} L$ and assume that G normalizes $\mathfrak{M}$.

Definition 10.1. A Galois algebra U over Γ in $(L * \mathfrak{M})^G$ is called a *linear Galois algebra*. Moreover,

1. if $\mathfrak{M} \simeq \mathbb{Z}^k$ for some $0 < k < n$, and the canonical generators $e_1, \dots, e_k$ of $\mathfrak{M}$ act by the shifts

$$e_i(t_j) = t_j - \delta_{ij}, \;\; i, j = 1, \dots, k$$

and

$$e_i(t_j) = t_j, \;\; i = 1, \dots, k, \;\; j = k+1, \dots, n,$$

then we say that U is linear Galois algebra of a *shift* type.

2. if $\mathfrak{M} \simeq \mathbb{Z}^k$ or $\mathbb{N}^k$ for some $0 < k < n$, and

$$e_i(t_j) = q^{\delta_{ij}} t_j, \;\; i, j = 1, \dots, k,$$

$0 \neq q \in \mathsf{k}$ is not a root of unity, and

$$e_i(t_j) = t_j, \;\; i = 1, \dots, k, \;\; j = k+1, \dots, n,$$

then we say that U is linear Galois algebra of a a *quantum* type (cf. [57, Section 5.4]).

Example 10.2.

- Recall that the n-th Weyl algebra $A_n(\mathsf{k})$ is a Galois order over $\mathsf{k}[t_1, \dots, t_n]$ embedded in $\mathsf{k}(t_1, \dots, t_n) * \mathbb{Z}^n$, where $t_i = \partial_i x_i$, $i = 1, \dots, n$, $\mathbb{Z}^n$ is generated by $\sigma_1, \dots, \sigma_n$ and the embedding is given by
$$x_i \mapsto \sigma_i, \;\; \partial_i \mapsto t_i \sigma_i^{-1}.$$
Then $A_n(\mathsf{k})$ is a linear Galois order of a shift type.
- The localization $A_n(\mathsf{k})_{x_1 \dots x_n}$ of $A_n(\mathsf{k})$ by the multiplicative set generated by $x_1, \dots, x_n$ is a linear Galois order ([56, Section 4.4]). It is isomorphic to the ring of differential operators on n-dimensional torus.

Next we give examples of linear Galois order of quantum type.

Example 10.3. Let $q \in \mathsf{k}$ is not a root of unity. We denote by $\mathsf{k}_q[x, y]$ the *quantum plane* over k:

$$\mathsf{k}_q[x, y] := \mathsf{k}\langle x, y |\, yx = qxy\rangle.$$

Let $\bar{q} = (q_1, \dots, q_n) \in (\mathsf{k} \backslash \{0\})^n$. The *quantum affine space* $\mathcal{O}_{\bar{q}}(\mathsf{k}^{2n})$ is the tensor product of quantum planes:

$$k_{q_1}[x_1, y_1] \otimes \dots \otimes \mathsf{k}_{q_n}[x_n, y_n].$$

We have the following isomorphism

$$\mathcal{O}_{\bar{q}}(\mathsf{k}^{2n}) \simeq \mathsf{k}[x_1, \dots, x_n] * \mathbb{N}^n$$

given by

$$x_i \mapsto x_i, \;\; y_i \mapsto \epsilon_i, \;\; i = 1, \dots, n,$$

which implies that $O_q(\mathsf{k}^{2n}) \subset \mathsf{k}(x_1, \dots, x_n) * \mathbb{N}^n$ is a linear Galois order of quantum type over $\Gamma = \mathsf{k}[x_1, \dots, x_n]$. If $q_1 = \dots = q_n = q$ then we simply write $\mathcal{O}_q(\mathsf{k}^{2n})$ for the quantum affine space $\mathcal{O}_{\bar{q}}(\mathsf{k}^{2n})$.

Example 10.4. Denote by $A_1^q(\mathsf{k})$ the first quantum Weyl algebra:

$$A_1^q(\mathsf{k}) = \mathsf{k}\langle x, y | yx - qxy = 1\rangle.$$

For any positive integer n and any $\bar{q} = (q_1, \ldots, q_n) \in (\mathsf{k}\backslash\{0\})^n$ define the n-th *quantum Weyl algebra*

$$A_n^{\bar{q}}(\mathsf{k}) = A_1^{q_1}(\mathsf{k}) \otimes \ldots \otimes A_1^{q_n}(\mathsf{k}).$$

In particular, for $q_1 = \ldots = q_n = q$ we set

$$A_n^q(\mathsf{k}) = A_1^q(\mathsf{k})^{\otimes n}.$$

The quantum Weyl algebra $A_n^q(\mathsf{k}) \subset \mathsf{k}(h_1, \ldots, h_n) * \mathbb{Z}^n$ is a linear Galois order of quantum type over $\Gamma = \mathsf{k}[h_1, \ldots, h_n]$, where $\mathbb{Z}^n$ has a basis $\epsilon_1, \ldots, \epsilon_n$ which acts on Γ as follows:

$$\epsilon_i(h_i) = q_i^{-1}(h_i - 1),\ \epsilon_i(h_j) = h_j,\ i, j = 1, \ldots, n.$$

The embedding is given by:

$$y_i x_i \mapsto h_i, x_i \mapsto \epsilon_i,\ y_i \mapsto h_i \epsilon_i^{-1},$$

$i = 1, \ldots, n$.

The following isomorphism is well-known:

Proposition 10.5 ([28]). *Let n be a positive integer and $(q_1, \ldots, q_n) \in (\mathsf{k} \setminus \{0, 1\})^n$. Then the skew fields of fractions of the tensor product of quantum Weyl algebras*

$$A_1^{q_1}(\mathsf{k}) \otimes_{\mathsf{k}} \cdots \otimes_{\mathsf{k}} A_1^{q_n}(\mathsf{k})$$

and of the tensor product of quantum planes

$$\mathsf{k}_{q_1}[x, y] \otimes_{\mathsf{k}} \cdots \otimes_{\mathsf{k}} \mathsf{k}_{q_n}[x, y]$$

are isomorphic.

Hence, the quantum Weyl algebra $A_n^{\bar{q}}(\mathsf{k})$ is birationally equivalent to the quantum affine space $\mathcal{O}_{\bar{q}}(\mathsf{k}^{2n})$. In particular, $A_n^q(\mathsf{k})$ is birationally equivalent to $O_q(\mathsf{k}^{2n})$.

Example 10.6. The quantum torus $\mathcal{O}_q(k^{*2n})$ is defined as a localization of $\mathcal{O}_q(k^{2n})$ by $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then $\mathcal{O}_q(k^{*2n})$ is naturally a linear Galois order of quantum type.

Observe that in both, shift and quantum cases, the algebra $L * \mathfrak{M}$ is a domain, since it is isomorphic to an iterated Ore extension of L. Hence $\mathfrak{K} = (L * \mathfrak{M})^G$ is an Ore domain by Theorem 3.15. We immediately obtain from Proposition 9.8 the following statement.

Corollary 10.7. *Every linear Galois algebra $U \subset (L * \mathfrak{M})^G$ of a shift or a quantum type is an Ore domain.*

We conclude this subsection with an example of a linear Galois algebra which is not of shift type and quantum type.

Example 10.8. Let $R = \mathbb{C}[x_1, x_2, x_3]$ and $\tilde{R} = R_1 \otimes R_2 \otimes R_3$, where each R_i is a copy of R, $i = 1, 2, 3$. Denote by σ the Nagata automorphism, which is known to be wild [108]. Let

$$\sigma_1 = \sigma \otimes 1 \otimes 1,\ \sigma_2 = 1 \otimes \sigma \otimes 1,\ \sigma_3 = 1 \otimes 1 \otimes \sigma.$$

Consider $\tilde{R} * \mathbb{Z}^3$, the skew group ring of $\tilde{R}$ with the group generated by $\sigma_i, i = 1, 2, 3$. The symmetric group S_3 acts on this ring by permuting the R_i factors and by conjugating $\mathbb{Z}^3$. Then the subring of invariants $U = (\tilde{R} * \mathbb{Z}^3)^{S_3}$ is as required a linear Galois algebra.

10.2 Gelfand-Kirillov Conjecture for shift type linear Galois algebras

Suppose $L * \mathfrak{M}$ is an Ore domain. Then $(L * \mathfrak{M})^G$ is an Ore domain and the skew field of fractions $\mathrm{Frac}((L * \mathfrak{M})^G)$ is isomorphic to the G-invariant subfield $(\mathrm{Frac}(L * \mathfrak{M}))^G$ with induced actions of G.

Let V be an n-dimensional k-vector space, $S(V)$ the symmetric algebra of V and L the field of fractions of $S(V)$.

If $G < GL_n(\mathsf{k})$ is a finite group then it acts linearly on L. Suppose G normalizes a submonoid $\mathfrak{M} \subset Aut_{\mathsf{k}}\, L$.

The following result was first shown in [44, Theorem 6] for complex reflection groups.

Theorem 10.9. *Suppose $L = \mathsf{k}(t_1, \ldots, t_n; z_1, \ldots z_m)$, for some integers n, m, and $\mathfrak{M} \simeq \mathbb{Z}^n$ acts by shifts on L, that is*

$$\varepsilon_i(t_j) = t_j + \delta_{ij},\ \ \varepsilon_i(z_k) = z_k,\ i, j = 1, \ldots, n,\ k = 1, \ldots, m,$$

for generators $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathfrak{M}$. Then we have

(1) $(L * \mathfrak{M})^G$ is birationally equivalent to $A_n(\mathsf{k}) \otimes \mathsf{k}[z_1, \ldots, z_m]$ for any pseudo-reflection group G;

(2) If $L^G \simeq L$ for a given group G then $(L*\mathfrak{M})^G$ is birationally equivalent to $A_n(\mathsf{k}) \otimes \mathsf{k}[z_1, \ldots, z_m]$.

Proof. As we already saw there is an embedding of the Weyl algebra $A_n(\mathsf{k})$ in $\mathsf{k}[t_1, \ldots, t_n]*\mathbb{Z}^n$, and their skew fields of fractions are isomorphic. Hence, item (1) follows from Theorem 6.7. If $L^G \simeq L$ then the CNP holds and (2) follows from Theorem 7.1. □

From Theorem 10.9 we get the following generalization of the Gelfand-Kirillov Conjecture for linear Galois algebras of shift type.

Corollary 10.10 ([44, Theorem 6], [71, Theorem 8.4]). *Let U be a linear Galois algebra in*

$$(\mathsf{k}(t_1, \ldots, t_k, z_1, \ldots, z_s) * \mathfrak{M})^G$$

over a commutative domain Γ, whose field of fractions is isomorphic to

$$\mathsf{k}(t_1, \ldots, t_k, z_1, \ldots, z_s)^G.$$

Suppose that $\mathfrak{M} \simeq \mathbb{Z}^k$ acts by shifts on $t_1, \ldots, t_k$ and fixes $z_1, \ldots, z_s$, and that $\mathsf{k}(t_1, \ldots, t_k, z_1, \ldots, z_s)^G$ is rational. Then the Gelfand-Kirillov Conjecture holds for U:

$$\operatorname{Frac} U \simeq \operatorname{Frac} A_n(\mathsf{k}(z_1, \ldots, z_s)).$$

In particular, this is the case if G is a pseudo-reflection group.

The last corollary has the following immediate generalization relevant to the case of $U(\mathfrak{gl}_n)$ and W-algebras of type A (cf. [44, Theorem 6]).

Corollary 10.11. *Assume that:*

- *$L = \mathsf{k}(t_{ij}; i = 1, \ldots, N; j = 1, \ldots, n_i; z_1, \ldots, z_m)$ for some positive integers N, $n_1, \ldots, n_N$ and m;*
- *$G = G_1 \times \ldots \times G_N$, where each G_i is a pseudo-reflection group which acts only on the variables $t_{i1}, \ldots, t_{in_i}$ and fixes the others;*
- *$\mathfrak{M} \simeq \mathbb{Z}^n$ acts by shifts on the corresponding variables $t_{11}, \ldots, t_{Nn_N}$, where*
$$n = \sum_{i=1}^{N} n_i;$$
- *For each i, the group G_i acts by conjugation on the corresponding part of $\mathbb{Z}^n$.*

Let U be a Galois algebra in $(L * \mathfrak{M})^G$ over a commutative subalgebra Γ, whose field of fractions K is isomorphic to L^G. Then

$$\operatorname{Frac}(U) \simeq \operatorname{Frac}(A_n(\mathsf{k}(z_1, \ldots, z_m))),$$

where $\mathsf{k}(z_1, \ldots, z_m)$ is the field of rational functions in variables $z_1, \ldots, z_m$.

Remark 10.12. In particular, the above result reproves the Gelfand-Kirillov Conjecture for finite W-algebras of type A from [53].

Chapter 11

Quantum Gelfand-Kirillov Conjecture

11.1 Quantum Weyl fields

A *quantum Weyl fields* is the skew field of fractions of a quantum Weyl algebra

$$A_n^{\overline{q}}(\mathsf{k}) \simeq A_1^{q_1}(\mathsf{k}) \otimes_{\mathsf{k}} \cdots \otimes_{\mathsf{k}} A_1^{q_n}(\mathsf{k}).$$

An algebra A is said to satisfy the *quantum Gelfand-Kirillov conjecture* if the skew field of fractions $\mathrm{Frac}(A)$ is isomorphic to a quantum Weyl field over a purely transcendental extension of k.

Recall that the quantum Weyl algebra $A_n^{\overline{q}}(\mathsf{k})$ and the quantum affine space $O_{\overline{q}}(\mathsf{k}^{2n})$ are birationally equivalent. Hence, the quantum Gelfand-Kirillov conjecture compares the skew field of fractions of a given algebra with skew fields of fractions of quantum affine spaces.

Next we consider examples of computation of quantum Weyl fields.

Example 11.1. The *Woronowicz deformation* isomorphic to the second Witten deformation can be realized as a generalized Weyl algebra $D(a, \sigma)$ ([16]), where $D = \mathbb{C}[h, z]$, $a = z + \alpha h + \beta$ with

$$\sigma(h) = s^4 h, \sigma(z) = s^2 z, \alpha = -1/s(1 - s^2), \beta = s/(1 - s^4).$$

$$s \in \mathsf{k}, s \neq 0, \pm 1, \pm i.$$

The Woronowicz deformation is birationally equivalent to the cross product $D * \mathbb{Z}$ by Proposition 13.5, where the action of $\mathbb{Z}$ on D is defined as follows: if e is a generator of the group $\mathbb{Z}$, then $e(h) = s^4 h$, $e(z) = s^2 z$.

Example 11.2. The *first Witten deformation* is a generalized Weyl algebra $D(a, \sigma)$, where

$$D = \mathbb{C}[c, h],\ a = c - \frac{h(h-1)}{p + p^{-1}},\ \sigma(c) = c,\ \sigma(h) = p^2(h-1)$$

with $p \in \mathsf{k}, p \neq 0, \pm 1, \pm i$. It is birationally equivalent to $\mathbb{C}[c] \otimes (\mathbb{C}[h] * \mathbb{Z})$ by Proposition 13.5. On the other hand, $\mathbb{C}[h] * \mathbb{Z}$ is birationally equivalent to the quantum Weyl algebra with the parameter p^2. We conclude that the field of fractions of the first Witten deformation is isomorphic to $\mathrm{Frac}\,(\mathbb{C}[c] \otimes \mathbb{C}_{p^2}[x, y])$.

Example 11.3. Consider the quantum group $A = \mathcal{O}_{q^2}(so(3, \mathbb{C}))$. Then A can be realized as a generalized Weyl algebra $\mathbb{C}[h, c](\sigma, a)$, where $a = c + h^2/q(1+q^2))$ and $\sigma(c) = c$, $\sigma(h) = q^2 h$ [15]. The algebra A is birationally equivalent to $\mathbb{C}[c, h] * \mathbb{Z}$ by Proposition 13.5, where the generator e of $\mathbb{Z}$ acts as σ on $\mathbb{C}[c, h]$. Since c is σ-invariant, then $\mathbb{C}[c, h] * \mathbb{Z}$ is birationally equivalent to $\mathbb{C}[c] \otimes (\mathbb{C}[h] * \mathbb{Z})$, and A satisfies the quantum Gelfand-Kirillov conjecture:

$$\mathrm{Frac}\,\mathcal{O}_{q^2}(so(3, \mathbb{C})) \cong \mathrm{Frac}\,(\mathbb{C}(c) \otimes \mathbb{C}_{q^2}[x, y]).$$

Example 11.4. Let X be the quantum 2-sphere ([16]) with the algebra of functions

$$A(S_q^2) \simeq \mathbb{C}\langle x, y, h\rangle / I,$$

where

$$I = (xh - qhx,\ yh - q^{-1}hy,\ q^{\frac{1}{2}}yx + (c-h)(d+h),\ q^{-\frac{1}{2}}xy + (c-qh)(d+qh).$$

It is isomorphic to a generalized Weyl algebra $\mathbb{C}[h](a, \sigma)$, where

$$a = -q^{-\frac{1}{2}}xy(c-h)(d+h))$$

and $\sigma(h) = qh$.

Since $\mathbb{C}[h](a, \sigma)$ is birationally equivalent to $\mathbb{C}[h] * \mathbb{Z}$, where $\mathbb{Z}$ is generated by e and $e(h) = qh$, then $A(S_q^2)$ is birationally equivalent to the quantum plane with parameter q. Hence, $A(S_q^2)$ satisfies the quantum Gelfand-Kirillov conjecture:

$$\mathrm{Frac}\,A(S_q^2) \cong \mathrm{Frac}\,\mathsf{k}_q[x, y].$$

11.2 q-difference Noether's Problem

The quantum Gelfand-Kirillov conjecture is strongly connected with the *q-difference Noether problem* for reflection groups, which was introduced in [51].

Let W_n be the Weyl group of type A_{n-1}, B_n, C_n, or D_n. We have a natural action of S_n on $\mathcal{O}_q(\mathsf{k}^{2n})$ as follows

$$\pi(x_i) = x_{\pi(i)},\ \ \pi(y_i) = y_{\pi(i)},\ \ \pi \in S_n,\ \ i = 1, \dots, n.$$

Also, we have the following action of the group $(\mathbb{Z}/2\mathbb{Z})^n$ on $\mathcal{O}_q(\mathsf{k}^{2n})$:

$$g(x_i) = (-1)^{g_i} x_i,\ \ g(y_i) = (-1)^{g_i} y_i,\ \ g \in (\mathbb{Z}/2\mathbb{Z})^n,\ \ i = 1, \dots, n.$$

Recall the Weyl group of type A_{n-1} is isomorphic to S_n and the Weyl group of type B_n (equivalently, of type C_n) is isomorphic to $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, then these groups act naturally on $\mathcal{O}_q(\mathsf{k}^{2n})$. Also, let

$$\mathcal{E}_n = \{\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{Z}/2\mathbb{Z})^n \mid \beta_1 + \cdots + \beta_n = 0\} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}.$$

The Weyl group of type D_n is isomorphic to $S_n \ltimes \mathcal{E}_n$, which defines its action on $\mathcal{O}_q(\mathsf{k}^{2n})$. Hence we have a naturally action of the group W_n on $\mathcal{O}_q(\mathsf{k}^{2n})$ by k-algebra automorphisms.

Let $\mathcal{F}_{q,n}$ (respectively $\mathcal{F}_{\bar{q},n}$) denote the skew field of fractions of $\mathcal{O}_q(\mathsf{k}^{2n})$ (respectively $\mathcal{O}_{\bar{q}}(\mathsf{k}^{2n})$). The action of W_n on $\mathcal{O}_q(\mathsf{k}^{2n})$ induces an action of W_n on $\mathcal{F}_{q,n}$. We denote by $\mathcal{F}_{q,n}^{W_n}$ the skew subfield of invariants of $\mathcal{F}_{q,n}$ under W_n.

The q-difference Noether problem for W_n asks whether the invariant quantum Weyl subfield $\mathcal{F}_{q,n}^{W_n}$ is isomorphic to a quantum Weyl field $\mathcal{F}_{\bar{q},n}$ for some $\bar{q} = (q_1, \dots, q_n)$.

The positive solution of the q-difference Noether problem was obtained in [51]:

Theorem 11.5. *The q-difference Noether problem for the group W_n has a positive solution, namely*

$$\mathcal{F}_{q,n}^{W_n} \simeq \mathcal{F}_{\bar{q},n},$$

where

$$\bar{q} = \begin{cases} (q, q, \dots, q), & \text{if } W_n = S_n \text{ (type } A_{n-1}), \\ (q^2, q^2, \dots, q^2), & \text{if } W_n = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n \text{ (type } B_n \text{ or } C_n), \\ (q, q^2, q^2, \dots, q^2), & \text{if } W_n = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1} \text{ (type } D_n). \end{cases}$$

As a consequence we get an isomorphism of k-algebras

$$\big(\mathrm{Frac}(A_n^q(\mathsf{k}))\big)^{W_n} \simeq \mathrm{Frac}(A_n^q(\mathsf{k})).$$

Using Theorem 11.5 (and the Galois orders technique) one shows that the quantum Gelfand-Kirillov conjecture holds for the quantum universal enveloping algebra $U_q(\mathfrak{gl}_n)$ ([51]):

Theorem 11.6. *The quantum Gelfand-Kirillov conjecture holds for* $U_q(\mathfrak{gl}_n(\mathbb{C}))$ *for* $q \in \mathbb{C}$ *not a root of unity: there exists a* $\mathbb{C}$*-algebra isomorphism*

$$\mathrm{Frac}\big(U_q(\mathfrak{gl}_n)\big) \simeq \mathrm{Frac}\Big(\mathcal{O}_q(\mathsf{k}^2)^{\otimes_{\mathsf{k}}(n-1)} \otimes_{\mathsf{k}} \mathcal{O}_{q^2}(\mathsf{k}^2)^{\otimes_{\mathsf{k}}(n-1)(n-2)/2}\Big),$$

where $\mathsf{k} = \mathbb{C}(z_1, \dots, z_n)$ *if the field of rational functions in* n *variables.*

Remark 11.7. The quantum Gelfand-Kirillov conjecture for $\mathfrak{gl}_n$ (for a generic q) follows from the positive solution of the q-difference Noether problem for the Weyl group of type D_n.

Remark 11.8. The original proof of the quantum Gelfand-Kirillov conjecture for $\mathfrak{sl}_n$ was done by Fauquant-Millet [47] using a different approach. The advantage of the Galois orders technique is that it allows to exhibit explicitly the skew fields of fractions.

Another immediate corollary of Theorem 11.5 is the following generalization of the Gelfand-Kirillov Conjecture for linear Galois algebras of quantum type.

Theorem 11.9. *Let* U *be a linear Galois algebra of quantum type in*

$$(\mathbb{C}(x_1, \dots, x_n; z_1, \dots, z_m) * \mathfrak{M}^n)^G,$$

where $\mathfrak{M}$ *is either* $\mathbb{Z}$ *or* $\mathbb{N}$*. Then the quantum Gelfand-Kirillov conjecture holds for* U *and there exist* $r_1, \dots, r_n \in \mathbb{Z}$ *such that*

$$\mathrm{Frac}\, U \cong \mathrm{Frac}\,(\mathcal{O}_{\overline{q}}(k^{2n}) \otimes \mathbb{C}[z_1, \dots, z_n]),$$

where $\overline{q} = (q^{r_1}, \dots, q^{r_n})$.

Chapter 12

Quantum Galois orders of invariants

In this section we consider the invariant subrings of quantum Galois orders with respect to the cyclic group of order m, G_m, the product $G_m^{\otimes n}$ of n copies of G_m and the groups $G(m,p,n)$, $n,m \geq 1$, $p\,|\,m$.

12.1 Invariants of of quantum affine spaces

The group $G_m^{\otimes n}$ has the following natural action on $O_q(\mathsf{k}^{2n})$:

$$g(x_i) = g_i x_i,\;\; g(y_i) = y_i,\;\; i = 1,\dots,n,$$

for each $g = (g_1,\dots,g_n) \in G$ [43].

Taking x_i to x_i^m and y_i to y_i for all $i = 1,\dots,n$, we obtain an isomorphism for the invariant quantum subspace:

$$O_q(\mathsf{k}^{2n})^{G_m^{\otimes n}} \simeq O_{q^m}(\mathsf{k}^{2n}).$$

But $O_{q^m}(\mathsf{k}^{2n})$ is isomorphic to $\mathsf{k}[x_1,\dots,x_n] * \mathbb{N}^n$, where the generators $\epsilon_1,\dots,\epsilon_n$ of the free monoid $\mathbb{N}^n$ act as follows:

$$\epsilon_i(x_i) = q^m x_i,\;\; \epsilon_i(x_j) = x_j,\;\; j \neq i,\;\; i,j = 1,\dots,n.$$

Hence, $O_q(\mathsf{k}^{2n})^{G_m^{\otimes n}}$ is a Galois order over $\mathsf{k}[x_1,\dots,x_n]$.

Now let $G = G(m,p,n)$ with the following action on $O_q(\mathsf{k}^{2n})$: for $h = (g,\pi) \in G$, $g = (g_1,\dots,g_n) \in G_m^{\otimes n}$, $\pi \in S_n$ we have

$$h(x_i) = g_i x_{\pi(i)},\;\; h(y_i) = y_{\pi(i)},\;\; i = 1,\dots,n.$$

We also have an action of the group G on $\mathsf{k}[x_1, \dots, x_n] * \mathbb{N}^n$, where the action on each x_i is the same as above, and the action on $\mathbb{N}^n$ is by conjugations: $h(\epsilon_i) = \epsilon_{\pi(i)}$ for all i. Hence we have a canonical isomorphism:

$$O_q(\mathsf{k}^{2n})^G \simeq (\mathsf{k}[x_1, \dots, x_n] * \mathbb{N}^n)^G.$$

This isomorphism together with the fact that Γ is a polynomial algebra imply the following theorem [57, Theorem 5].

Theorem 12.1. *For $G = G(m, p, n)$ the invariant quantum subspace $O_q(\mathsf{k}^{2n})^G$ is a Galois order over $\Gamma = \mathsf{k}[x_1, \dots, x_n]^G$. Moreover, $O_q(\mathsf{k}^{2n})^G$ is free as left (right) Γ-modules.*

The invariant subalgebra $O_q(\mathsf{k}^{*2n})^G$ of the quantum torus $O_q(\mathsf{k}^{*2n})$ is the localization of $O_q(\mathsf{k}^{2n})^G$ by $x_1, \dots, x_n$ and $y_1, \dots, y_n$. This leads to an isomorphism

$$O_q(\mathsf{k}^{*2n})^G \simeq (\mathsf{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] * \mathbb{Z}^n)^G,$$

and we immediately get the following result [57, Theorem 6].

Theorem 12.2. *For $G = G(m, p, n)$ the invariant subring $O_q(\mathsf{k}^{*2n})^G$ of the quantum torus is is a Galois order over $\Gamma = \mathsf{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^G$ in $(\mathsf{k}(x_1, \dots, x_n) * \mathbb{Z}^n)^G$. Moreover, $O_q(\mathsf{k}^{*2n})^G$ is free as a left (right) Γ-module.*

12.2 Invariants of quantum Weyl algebras

In this subsection we consider certain invariant subrings of the quantized Weyl algebras $A_n^q(\mathsf{k})$.

Consider first the subring of invariants $A_n^q(\mathsf{k})^{S_n}$, where S_n acts by simultaneous permutations of the variables y_i and x_i, $i = 1, \dots, n$. Using the structure of the quantum Weyl algebra $A_n^q(\mathsf{k})$ as a Galois order over $\Gamma = \mathsf{k}[h_1, \dots, h_n]$ above, we obtain an embedding

$$A_n^q(\mathsf{k})^{S_n} \to (\mathsf{k}(h_1, \dots, h_n) * \mathbb{Z}^n)^{S_n},$$

where S_n permutes $h_1, \dots, h_n$ and acts on $\mathbb{Z}^n$ by conjugation. Using Proposition 9.7 we can show

Theorem 12.3 ([57]). *$A_n^q(\mathsf{k})^{S_n}$ is a Galois order over $\Gamma = \mathsf{k}[h_1, \dots, h_n]^{S_n}$. Moreover, $A_n^q(\mathsf{k})^{S_n}$ is free as a left (right) Γ-module.*

Alev and Dumas showed that every finite group G of automorphisms of the quantum Weyl algebra $A_1^q(\mathbb{C})$ is a cyclic group of some order m, whose generator acts by:

$$x \mapsto \alpha x, \;\; y \mapsto \alpha^{-1} y$$

for some m-th primitive root of unity α. Localization of $A_1^q(\mathbb{C})$ by x is isomorphic to $\mathbb{C}(x)[z,\sigma]$, with $z=(q-1)xy+1$ and $\sigma(x)=qx$. We have an embedding

$$A_1^q(\mathbb{C})^G \to \mathbb{C}(x^m)[z;\sigma] \cong \mathsf{k}(x^m) * \mathbb{N},$$

where $\sigma(x^m)=q^n x^m$. Therefore, we conclude

Theorem 12.4. *For any finite group G the algebra $A_1^q(\mathbb{C})^G$ is a Galois order over $\Gamma=\mathbb{C}[x^m]$. Moreover, $A_1^q(\mathbb{C})^G$ is free as a left (right) Γ-module.*

We summarize known examples of invariant Galois orders in the following theorem.

Theorem 12.5. *The following algebras are Galois orders over appropriate commutative subalgebras:*

- $A_n(\mathsf{k})^G$, *where G belongs to $\{G_m^{\otimes n}, \mathcal{A}_n, G(m,p,n)\}$;*
- $\mathcal{D}(\mathbb{T}^n)^W$, *where $W \in \{S_n, B_n = C_n, D_n\}$;*
- $\mathcal{O}_q(k^{2n})^G$ *and* $\mathcal{O}_q(k^{*2n})^G$ *for* $G=G(m,p,n)$;
- $\mathbb{C}_q[x,y]^G$ *and* $A_1^q(\mathbb{C})^G$ *for any finite group G;*
- $A_n^q(\mathsf{k})^{S_n}$.

Chapter 13

Generalized Weyl algebras

In this section we consider the invariants of *generalized Weyl algebras* which were introduced by Bavula [12]. Many important algebras in non-commutative geometry and representation theory can be realized as generalized Weyl algebras. This includes such algebras as the first Weyl algebra and its quantization; the quantum plane; the quantum sphere; the universal enveloping algebra of $sl_2(\mathsf{k})$ and its quantization; the Heisenberg algebra and its quantizations; quantum 2×2 matrices; Witten's and Woronowic's deformations; Noetherian down-up algebras and many others. Development of this theory led to many important applications (cf. [12], [15], [13]). The algebras considered in the previous sections can be put in a more general framework of generalized Weyl algebras and results on their invariants can be extended to this wider class of algebras.

Let D be a ring, $\sigma = (\sigma_1, \ldots, \sigma_n)$ an n-tuple of commuting automorphisms of D:

$$\sigma_i \sigma_j = \sigma_j \sigma_i,\ i, j = 1, \ldots, n.$$

Let $a = (a_1, \ldots, a_n)$ be an n-tuple of nonzero elements of the center of D, such that $\sigma_i(a_j) = a_j, j \neq i$. The generalized Weyl algebra $D(a, \sigma)$ [12] of rank n is generated over D by X_i^+, X_i^-, $i = 1, \ldots, n$ subject to the relations:

$$X_i^+ d = \sigma_i(d) X_i^+;\ X_i^- d = \sigma_i^{-1}(d) X_i^-,\ d \in D, i = 1, \ldots, n,$$

$$X_i^- X_i^+ = a_i;\ X_i^+ X_i^- = \sigma_i(a_i),\ i = 1, \ldots, n\,,$$

$$[X_i^-, X_j^+] = [X_i^-, X_j^-] = [X_i^+, X_j^+] = 0\,, i \neq j.$$

We will assume that D is an affine commutative Noetherian domain. In this case the $D(a, \sigma)$ is also a Noetherian Ore domain [12].

Consider the skew group ring $D * \mathbb{Z}^n$, where the free abelian group $\mathbb{Z}^n$ has a basis $e_1, \ldots, e_n$ and its action on D is defined as follows: ye_i acts as σ_i^y, for all i and $y \in \mathbb{Z}$.

The tensor product over k of two generalized Weyl algebras

$$D(a,\sigma) \otimes D(a',\sigma') \simeq (D \otimes D')(a * a', \sigma * \sigma'),$$

is a generalized Weyl algebra, where $*$ is the tensor product of automorphisms, and the concatenation of a, a'.

Example 13.1. Both $\mathsf{k}_q[x,y]$ and $A_1^q(\mathsf{k})$ are generalized Weyl algebras (cf. [12]). Indeed, $\mathsf{k}_q[x,y]$ is a generalized Weyl algebra of rank one with $D = \mathsf{k}[h]$, $a = h$, $\sigma(h) = qh$, and $A_1^q(\mathsf{k})$ is a generalized Weyl algebra $D(a,\sigma)$ of rank one with $D = \mathsf{k}[h]$, $a = h$, $\sigma(h) = q^{-1}(h-1)$. The isomorphism is given as follows:

$$yx \mapsto h, \;\; x \mapsto X, \;\; y \mapsto Y.$$

Hence $A_n(\mathsf{k})$, $A_n(\mathsf{k})^q$, $\mathcal{O}_q(k^{2n})$ and $\mathcal{O}_q(k^{*2n})$ are generalized Weyl algebras.

Example 13.2. Assume $\mathsf{k} = \mathbb{C}$ and consider the quantum group $U_q(sl_2)$ with $q \neq \pm 1$. The Quantum Gelfand-Kirillov Conjecture for $U_q(sl_2)$ was proved in [3]. We will show how to compute the skew field of fractions of $U_q(sl_2)$ using its realization as a generalized Weyl algebra which was shown in [57]. In fact, we have an isomorphism

$$U_q(sl_2) \simeq \mathbb{C}[c,h,h^{-1}](a,\sigma),$$

where σ fixes c and sends h to qh, and

$$a = c + \frac{\frac{h^2}{q^2-1} - \frac{h^{-2}}{q^{-2}-1}}{q - q^{-1}}.$$

The skew field of fractions $\operatorname{Frac} U_q(sl_2)$ is isomorphic to the skew field of fractions of $\mathbb{C}[c,h,h^{-1}]*\mathbb{Z}$ by Proposition 13.5, and the latter is birationally equivalent to

$$(\mathbb{C}[c] \otimes \mathbb{C}[h,h^{-1}]) * \mathbb{Z} \cong \mathbb{C}[c] \otimes \mathbb{C}_q[x^{\pm}, y^{\pm}].$$

Then

$$\operatorname{Frac} U_q(sl_2) \cong \operatorname{Frac}(\mathbb{C}_q[x,y] \otimes \mathbb{C}[c]).$$

Proposition 13.3. *If a_i is a unit in D for each $i = 1, \ldots, n$, then we have an isomorphism*

$$D(a,\sigma) \simeq D * \mathbb{Z}^n.$$

Proof. Indeed, the isomorphism is defined by the map $\phi : D(a,\sigma) \to D * \mathbb{Z}^n$, such that $\phi(X_i^+) = e_i$ and $\phi(X_i^-) = a_i e_i^{-1}$. □

For all integers $y_1, \ldots, y_n$ set $y_1\sigma_1 + \ldots + y_n\sigma_n := \sigma_1^{y_1} \ldots \sigma_n^{y_n}$. The following result was shown in [58, Theorem 14]:

Theorem 13.4. *If $\sigma_1, \ldots, \sigma_n$ are linearly independent over $\mathbb{Z}$, then $D(a,\sigma)$ is a Galois order over D in the skew group ring* (Frac D) $* \mathbb{Z}^n$*, and D is a Harish-Chandra subalgebra.*

As a consequence we obtain

Proposition 13.5. *The algebras $D(a,\sigma)$ and $D*\mathbb{Z}^n$ are birationally equivalent.*

Recall the following definition [13].

Definition 13.6.

- Let $R = \mathsf{k}[h](a,\sigma)$ be a generalized Weyl algebra of rank 1 with $\sigma(h) = h - 1$ such that there is no irreducible polynomial $p \in \mathsf{k}[h]$ for which both p and $\sigma^i(p)$ are multiples of a for any $i \geq 0$. These generalized Weyl algebras of *simple classical type.*

- Let $R = \mathsf{k}[h^{\pm}](a,\sigma)$ be a generalized Weyl algebra of rank 1 with $a \in \mathsf{k}[h]$, $\sigma(h) = qh$, $0, 1 \neq q \in \mathsf{k}$ is not a root of unity. Assume that there is no irreducible polynomial $p \in \mathsf{k}[h]$ for which both p and $\sigma^i(p)$ are multiples of a for any $i \geq 0$, then R is of *simple quantum type.*

A tensor product of generalized Weyl algebras of both quantum and classical types is called a generalized Weyl algebra of *mixed type.* If all generalized Weyl algebras of the tensor product are of the same type, then we call we call the tensor product a *generalized Weyl algebra of pure type.*

Let $D(a,\sigma)$ be a generalized Weyl algebra and G a finite group of automorphisms of D. Suppose G normalizes the set $\{\sigma_i\}$ and $g(a_i) = a_{g(i)}$ for all $i = 1, \ldots, n$. Then from the defining relations of generalized Weyl algebras we easily get an extension of G to a group of automorphisms on $D(a,\sigma)$. Indeed, if $g\sigma_i g^{-1} = \sigma_j$ for $g \in G$, then set $g(i) := j$ and define

$$g \cdot X_i = X_{g(i)}, \;\; g \cdot Y_i = Y_{g(i)}.$$

This defines the G action on $D(a,\sigma)$. Denote by $D(a,\sigma)^G$ the G-invariants with respect to this action. Invariant of generalized Weyl algebras with respect to finite group actions were studied in [60]. It was shown that $D(a,\sigma)^G$ inherits nice properties of $D(a,\sigma)$ since the embedding in Theorem 13.4 is G-equivariant. In particular, we get

Theorem 13.7. *If $D(a,\sigma) \subset Frac(D)*\mathbb{Z}^n$ is a Galois order then $D(a,\sigma)^G$ is a Galois order in $(Frac(D)*\mathbb{Z}^n)^G$ with Harish-Chandra subalgebra D^G. Moreover, if D is a projective module over D^G, then $D(a,\sigma)^G$ is a free D^G-module.*

Proof. Indeed, $(D * \mathbb{Z}^n)^G$ is a Galois order in $(Frac(D) * \mathbb{Z}^n)^G$ by [66, Lemms 2.10(iii)]. Hence, there exists $X \subset (D*\mathbb{Z}^n)^G$ such that $\bigcup_{x\in X} supp\, x$ generates $\mathbb{Z}^n$ as a monoid. By the form of the embedding in Theorem 13.4, we can find a set Y with the same property in $D(a,\sigma)^G$, so that it is a Galois order in the same invariant skew group ring.

Since $D(a,\sigma)$ is free over D, then $D(a,\sigma)^G$ is a free D^G-module if D is a projective D^G-module by [10, Corollary 4.5]. □

We will consider the invariants of the generalized Weyl algebras with respect to the following groups. Let G_m denote the cyclic group of order m. For $n \geq 1$ let $G_m^{\otimes n}$ be the product of n copies of G_m. If $p|m$ denote by $A(m,p,n)$ the subgroup of $G_m^{\otimes n}$ consisting of the elements $(h_1,\ldots,h_n)$ such that

$$(\prod_{i=1}^{n} h_i)^{m/p} = id.$$

For integers $m \geq 1$, $n \geq 1$ and $p|m$ consider the three parameter family of groups

$$G(m,p,n) = A(m,p,n) \rtimes S_n,$$

where S_n acts on $A(m,p,n)$ by permutations. These are all irreducible non-exceptional complex reflection groups of Shephard and Todd. In particular, this family contains all classical Weyl groups.

Let $R = D(\tilde{a},\tilde{\sigma})$ be a generalized Weyl algebra of rank n such that $\tilde{a} = (a_1,\ldots,a_n)$, $\tilde{\sigma} = (\sigma_1,\ldots,\sigma_n)$, each σ_i is of infinite order and

$$D \simeq \otimes_{i=1}^{n} D_i,\;\; R \simeq \otimes_{i=1}^{n} R_i,$$

where R_i is a generalized Weyl algebra of rank 1 for each i. The identification with the tensor product is given by: $a_i \mapsto 1 \otimes \ldots \otimes a_i \otimes \ldots 1$ with a_i is in the i-th position, and $\sigma_i = 1 \otimes \ldots \otimes \sigma_i \otimes \ldots 1$ with σ_i in the i-th position.

Suppose that R is a generalized Weyl algebra of pure type. Then $D_i \simeq \mathsf{k}[h_i]$ or $D \simeq \mathsf{k}[h_i^{\pm}]$ for all $i = 1,\ldots,n$.

We consider the following group actions on R:

- The symmetric group S_n action on R by algebra automorphisms:

$$\pi \cdot h_i = h_{\pi(i)},\;\; \pi \cdot X_i = X_{\pi}(i),\;\; \pi \cdot Y_i = Y_{\pi(i)}.$$

- Let $G_m = \langle g \rangle \subset k$ be the cyclic group of order m. The action of G_m on $D(a, \sigma)$:

$$g \cdot X = \xi X, \;\; g \cdot Y = \xi^{-1} Y, \;\; g \cdot d = d,$$

for all $d \in D$, where ξ is a primitive m-root of unity generating G_m. The invariant subalgebra $D(a, \sigma)^{G_m}$ is isomorphic to the generalized Weyl algebra $D(a_m, \sigma^m)$ with the generators X^m and Y^m, where

$$a_m = a\sigma^{-1}(a) \dots \sigma^{-(m-1)} a.$$

- The action of G_m above induces the diagonal action of the group $G = G_m^{\otimes n}$ on R. Then

$$R^G \simeq D(\tilde{a}_m, \tilde{\sigma^m}),$$

where the latter algebra is generated over D by X_i^m and Y_i^m, $i = 1, \dots, n$.

- The group $G(m, p, n)$ acts on R as follows: if $\xi = (g, \pi) \in A(m, p, n) \rtimes S_n$ and $g = (g_1, \dots, g_n)$, then

$$\xi \cdot h_i = h_{\pi(i)}, \; \xi \cdot X_i = g_i X_{\pi(i)}, \; \xi \cdot Y_i = g_i^{-1} Y_{\pi(i)}.$$

We have the following statement.

Theorem 13.8 ([60, Theorem 5.1]). *Let $R = D(\tilde{a}, \tilde{\sigma})$ be a generalized Weyl algebra of rank n and $G = G(m, p, n)$.*

- *The algebra R is a Galois order over D.*

- *Let $\mathbb{Z}^n$ be the free abelian group with the canonical basis $e_i, i = 1, \dots, n$, which acts on D by automorphisms as follows: $e_i(h_j) = \sigma_i(h_j)$, for $i, j = 1, \dots, n$. Let $\mathfrak{M} \subset \mathbb{Z}^n$ be a subgroup generated by*

$$e_1^m, \;\dots,\; e_n^m, \; (e_1 e_2 \dots e_n)^{m/p}.$$

Then the subring of invariants R^G is a Galois order in $(\mathrm{Frac}\, D * \mathfrak{M})^G$ over the Harish-Chandra subalgebra D^G.

- *The algebra R^G is free over D^G.*

- $R^{G(m,p,n)} = \bigoplus_{k=0}^{p-1} (X_1 \dots X_n)^{km/p} R^{G(m,1,n)}$.

Example 13.9. Consider the standard basis e, f, h of sl_2, where $[h, e] = e$, $[h, f] = -f$, $[e, f] = 2h$. The universal enveloping algebra $U(sl_2)$ can be realized as a generalized Weyl algebra $\mathsf{k}[H,C](a,\sigma)$, where $a = C - H(H+1))$, with the isomorphism given by

$$e \mapsto X,\ f \mapsto Y,\ h \mapsto H,\ h(h+1) + fe \mapsto C.$$

Define an action of the cyclic group G_m of order m on $U(sl_2)$ as follows. Let $g \in G_m$ be an element of order m. Then the action of g:

$$g \cdot h = h,\ g \cdot e = \xi e,\ f \mapsto \xi^{-1} f,$$

where ξ is a fixed m-th primitive root of unity.

The algebra $\mathsf{k}[H,C](a,\sigma)$ (and hence $U(sl_2)$) is birationally equivalent to $\mathsf{k}[H,C] * \mathbb{Z}$, where $\mathbb{Z}$ acts by σ. The action of G_m naturally extends to $\mathsf{k}[H,C] * \mathbb{Z}$, where the generator g acts on $\mathbb{Z}$ by sending $\overline{y}$ to ξ^y, $y \in \mathbb{Z}$. Therefore we have an embedding of $U(sl_2)^{G_m}$:

$$U(sl_2)^{G_m} \subset (\mathsf{k}[H,C] * \mathbb{Z})^{G_m}.$$

Since C is fixed by σ and also by the action of G_m, we have

$$\mathrm{Frac}(\mathsf{k}[H,C] * \mathbb{Z})^{G_m} \cong \mathrm{Frac}(\mathsf{k}[C] \otimes (\mathsf{k}[H] * \mathbb{Z})^{G_m}).$$

On the other hand, $k[H] * \mathbb{Z}$ is isomorphic to the localization $A_1(\mathsf{k})_x = A_1(\mathsf{k})_{x^m}$ of the first Weyl algebra. We have an action of G_m on $A_1(\mathsf{k})$ as follows:

$$g \cdot x = \xi^{-1} x,\ \ g \cdot \partial = \xi \partial.$$

We have the following ismorphsms:

$$\mathrm{Frac}(\mathsf{k}[H]*\mathbb{Z})^{G_m} \cong \mathrm{Frac}(A_1(\mathsf{k})_{x^m})^{G_m} \cong \mathrm{Frac}(A_1(\mathsf{k})^{G_m}_{x^m}) \cong \mathrm{Frac}(A_1(\mathsf{k})^{G_m}).$$

From here we get that $U(sl_2)^{G_m}$ is birationally equivalent to $\mathsf{k}[C] \otimes A_1(\mathsf{k})^{G_m}$. As $A_1(\mathsf{k})^{G_m} \simeq A_1(\mathsf{k})$, we finally have the isomorphisms of skew fields of fractions

$$\mathrm{Frac}(U(sl_2)^{G_m}) \cong \mathrm{Frac}(\mathsf{k}[C] \otimes A_1(\mathsf{k})) \cong \mathrm{Frac}\, U(sl_2),$$

where the second isomorphism follows from the Gelfand-Kirillov conjecture.

Observe that $U(sl_2)$ is rigid and hence $U(sl_2)^{G_m}$ is not isomorphic to $U(sl_2)$. Nevertheless,

$$\mathrm{Frac}(U(sl_2)^{G_m}) \simeq \mathrm{Frac}(U(sl_2)).$$

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Vyacheslav Futorny
Instituto de Matemática e Estatística,
Universidade de São Paulo,
São Paulo SP, Brasil
vfutorny@gmail.com
https://www.ime.usp.br/~futorny/

João Schwarz
Instituto de Matemática e Estatística,
Universidade de São Paulo,
São Paulo SP, Brasil
jfschwarz.0791@gmail.com