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On the construction and identification of Boltzmann processes

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Abstract. Given the existence of a solution $\{f(t,x,z)\}_{t\geq 0}$ of the Boltzmann equation for hard spheres, we introduce a stochastic differential equation driven by a Poisson random measure that depends on the densities $\{f(t,x,z)\}_{t\geq 0}$. The marginal distributions of its solution solve a linearized Boltzmann equation in the weak form. Further, if the distributions admit a probability density, we establish, under suitable conditions, that the density at each t coincides with the solution of the Boltzmann equation $f(t,x,z)$. The stochastic process is therefore called the Boltzmann process.

Keywords. Boltzmann equation, Poisson random measures, stochastic differential equations, relative entropy.

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1 Introduction

The Boltzmann equation describes the time evolution of the density of molecules in a dilute (or rarified) gas for a given initial distribution. Each molecule (or particle) moves in a straight line without any external forces acting on it until it collides with another particle and gets deflected. The Boltzmann equation forms the basis for the kinetic theory of gases [6].
The Boltzmann equation has the general form

$$\frac{\partial f}{\partial t}(t,x,z) + z \cdot \nabla_x f(t,x,z) = \mathcal{C}(f,f)(t,x,z), \tag{1.1}$$

where f is a probability density function that depends on time $t \geq 0$, the space (location) variable $x \in \mathbb{R}^3$, and velocity, $z \in \mathbb{R}^3$. The function $\mathcal{C}$ is a certain quadratic form in f, called collision operator (or integral).
Set $\Xi := (0,\pi] \times [0,2\pi)$. Then $\mathcal{C}$ can be written in the general form

$$\mathcal{C}(f,f)(t,x,z) = \int_{\mathbb{R}^3 \times \Xi} \{f(t,x,z^\star)f(t,x,v^\star) - f(t,x,z)f(t,x,v)\}B(z,dv,d\theta)d\phi. \tag{1.2}$$

The dynamics of collisions are encoded in the collision kernel $B(z,dv,d\theta)$. Each $v \in \mathbb{R}^3$ in (1.2) denotes the velocity of an incoming particle which may hit, at the fixed location $x \in \mathbb{R}^3$, particles whose velocity is z. Let $z^\star \in \mathbb{R}^3$ and $v^\star \in \mathbb{R}^3$ denote the resulting outgoing (post-collision) velocities corresponding to the incoming (pre-collision) velocities z and v respectively. The angle $\theta \in (0,\pi]$ denotes the azimuthal or colatitude angle of the deflected velocity, $v^\star$, and $\phi \in [0,2\pi)$ measures the longitude of $v^\star$.
In the Boltzmann equation, the collisions are assumed to be elastic and hence, conservation of momentum and kinetic energy hold, i.e. considering particles of mass $m = 1$, the following equalities hold:

$$\begin{cases} z^\star + v^\star = z + v \\ |z^\star|^2 + |v^\star|^2 = |z|^2 + |v|^2 \end{cases} \tag{1.3}$$

In fact,

$$\begin{cases} z^\star = z + (\mathbf{n}, v - z)\mathbf{n} \\ v^\star = v - (\mathbf{n}, v - z)\mathbf{n} \end{cases} \tag{1.4}$$

where

$$\mathbf{n} = \frac{z^\star - z}{|z^\star - z|} \tag{1.5}$$

where $(\cdot,\cdot)$ denotes the scalar product, and $|\cdot|$, the Euclidean norm in $\mathbb{R}^3$.

Remark 1.1. *The Jacobian of the transformation (1.4) is 1 in magnitude, and $(z^\star)^\star = z$ since the collision dynamics are reversible.*

The outgoing velocity z^* is then uniquely determined in terms of the co-latitude angle $\theta \in (0, \pi]$ measured from the center, and longitude angle $\phi \in [0, 2\pi)$ of the deflection vector $\mathbf{n}$ in a sphere with north-pole z and south-pole v centered at $\frac{z+v}{2}$ (and with radius determined by the conserved kinetic energy) which are used in equation (1.1) and (1.2) (see e.g. the article by H. Tanaka [19] for futher details). It follows

$$(v - z, \mathbf{n}) = |v - z| \cos(\frac{\pi}{2} - \frac{\theta}{2}) = |v - z| \sin(\frac{\theta}{2}), \tag{1.6}$$

where θ is the angle between $v - z$ and $v^\star - z^\star$, and $\frac{\pi}{2} - \frac{\theta}{2}$ is the angle between $\mathbf{n}$ and $v - z$. In polar coordinates we obtain

$$(v - z, \mathbf{n}) d\mathbf{n} = |v - z| \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) d\theta d\phi \tag{1.7}$$

Figure 1.1 below visualizes the deflection vector and its angles mentioned above.

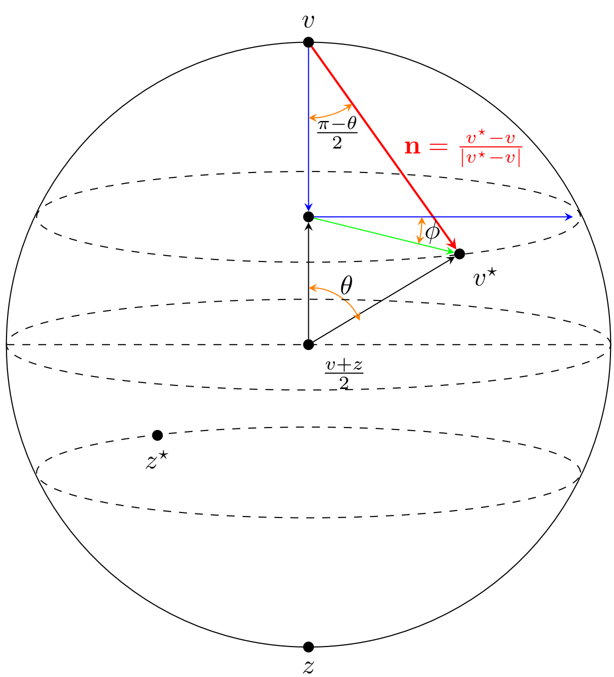

Figure 1.1: Parameterization of collisions

The collision measure $B(z, dv, d\theta)$ appearing in (1.2) has then the form

$$B(z, dv, d\theta) = |v - z| \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) d\theta dv. \tag{1.8}$$

(1.1) with (1.8) is then the "Boltzmann equation with hard spheres", proposed by Ludwig Eduard Boltzmann. In further modifications of the Boltzmann model the collision measure $B(z, dv, d\theta)$ is a σ-additive positive measure defined on the Borel σ-field $\mathcal{B}(\mathbb{R}^3) \times \mathcal{B}((0, \pi])$, depending (Lebesgue) measurably on $z \in \mathbb{R}^3$. The form of B depends on the version of Boltzmann equation one has in mind (see e.g. [20]). In general, one sets

$$B(z, dv, d\theta) = \sigma(|v - z|) dv Q(d\theta) \tag{1.9}$$

where σ, known as velocity cross-section, is a positive function on $\mathbb{R}^+$, and Q is a σ-finite measure on $\mathcal{B}((0, \pi])$. If Q is a finite measure, it is called the cut-off case.

To write (1.1), (1.2) in its weak form (in the functional analytic sense), we need a result by Tanaka (cf. [18], Appendix):

Proposition 1.2. *Let $\psi(u, v, y, z) \in C_0(\mathbb{R}^{12})$, as a function of $u, v, y, z \in \mathbb{R}^3$. For each $\theta \in (0, \pi]$ fixed*

$$\int_{\mathbb{R}^6 \times [0,2\pi)} \psi(u, v, u^\star, v^\star) d\phi du dv = \int_{\mathbb{R}^6 \times [0,2\pi)} \psi(u^\star, v^\star, u, v) d\phi du dv \tag{1.10}$$

Consider the Boltzmann equation (1.1) with collision operator (1.2). Multiply (1.1) by a function ψ (of $(x, z) \in \mathbb{R}^6$) belonging to $C_0^1(\mathbb{R}^6)$, and integrate with respect to x and z, using integration by parts, we arrive at the weak formulation of the Boltzmann equation:

$$\begin{aligned} &\int_{\mathbb{R}^6} \psi(x, z) \frac{\partial f}{\partial t}(t, x, z) dx dz - \int_{\mathbb{R}^6} f(t, x, z)(z, \nabla_x \psi(x, z)) dx dz \\ &= \int_{\mathbb{R}^6} f(t, x, z) L_f \psi(x, z) dx dz \end{aligned} \tag{1.11}$$

for all $t \in \mathbb{R}^+$ with

$$L_f \psi(x, z) = \int_{\mathbb{R}^3 \times (0,\pi] \times [0,2\pi)} \{\psi(x, z^\star) - \psi(x, z)\} f(t, x, v) B(z, dv, d\theta) d\phi,$$

where B is as in (1.9).

We will introduce in Section 2 the McKean -Vlasov type Stochastic Differential Equation (SDE) (2.1) and prove that the distribution of its solution

satisfies the Boltzmann equation in its weak form. In other words, we construct in Theorem 2.1 the SDE which has its associated Kolmogorov equation as the Boltzmann equation (1.11). We call its solutions "Boltzmann processes". Under some suitable regularity conditions on the distributions, in Section 3 we get the existence of the solutions of (2.1) for the hard sphere case treated by Boltzmann, however in the non -cutoff case. To obtain this result we first assume the existence of a solution of the Boltzmann equation, using which the desired Markov process is constructed. This ingenious suggestion was given to us by Professor Presutti. An alternate way to overcome this obstacle was proposed and studied by us in relation to the Boltzmann-Enskog equation [2]. For hard and soft potentials, solvability and uniqueness of the Boltzmann-Enskog equation were carried out in subsequent works [12] and [13]. Indeed, there is a vast literature on various aspects of Boltzmann equation or Boltzmann type equations as well as their stochastic interpretation and we refer the reader to [7], [9], [11],[10], [16] [18, 19], [20], and the references therein. To our knowledge this is the first article which constructs the SDE which has its associated Kolmogorov equation as the Boltzmann equation itself. However, it is well known, that long time ago Tanaka [18], [19] constructed the SDE associated to the spatial homogeneous Boltzmann equation. The McKean Vlasov type SDE (2.1) associated to the Boltzmann equation (1.11) is constructed here in Section 2 for the non -cutoff case for hard sphere, as well as for Maxwell molecules, or soft and hard potentials. The existence of its solutions is obtained here for the non-cutoff case for hard spheres with $\sigma(|z-v|)= |z-v|$.
The solution of (2.1) is obtained by first proving in Theorem 3.2 the existence of the solution of (3.2), depending on the solution $\{f(t,x,z)\}_{t\in[0,T]}$ of the Boltzmann equation (1.1), and then assuming that its solution has time-marginal densities for all $t \in [0,T]]$. In Theorem 3.4 it is proven that the density marginals of (3.2) coincide with $\{f(t,x,z)\}_{t\in[0,T]}$. We therefore conclude in Theorem 3.5 that equation (3.2) coincides with the McKean Vlasov SDE (2.1) and the stochastic process that solves it is a Boltzmann process with density $\{f(t,x,z)\}_{t\in[0,T]})$, since the latter family of densities solve the Boltzmann equation (1.1).

Definition 1.3. A collection of densities $\{f(t,x,z\}_{t\in[0,T]}$, with $x,z \in \mathbb{R}^3$, is a strong (resp. weak) solution of the Boltzmann equation in $[0,T]$ if for any $t \in [0,T]$ it solves (1.1) (resp. (1.11)).

We denote by $\mathbb{D} := \mathbb{D}([0,T],\mathbb{R}^3)$ the space of all right continuous functions with left limits on $[0,T]$ taking values in $\mathbb{R}^3$, and equipped with the topology induced by the Skorohod metric.

Definition 1.4. A stochastic process $(X_s, Z_s)_{s\in[0,T]}$ with values in $\mathbb{D}\times\mathbb{D}$, and with their time-marginal probability densities denoted by $f(t,x,z)$ at each $t \in [0,T]$, is called a "Boltzmann process" if $f(t,x,z)$ solve equation (1.11) for all $t \in [0,T]$.

We remark that the infinitesimal generator of a Boltzmann process is given by $(z, \nabla_x) + L_f$. Costantini and Marra [8] analyzed hydrodynamical limits of a process given by a the drift term involving (z, ∇_x) and L_f in addition to a martingale.

We use the following notation $\mathbb{R}_+^0 := \{t \in \mathbb{R} : t \geq 0\}$. In this article we assume the following hypothesis.

Hypotheses A:

A1. The measure Q on $[0,\pi)$ is finite outside any neighbourhood of 0, and for all $\epsilon > 0$, it satisfies

$$\int_0^\epsilon \theta Q(d\theta) < \infty.$$

A2. The function $\sigma : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ (entering (1.9)) is given by $\sigma(z) := cz^\gamma$, with $c > 0$, $\gamma \in (-1,1]$ fixed.

The case of $\gamma = 1$ corresponds to hard-sphere interaction, and $\gamma = 0$ refers to Maxwellian molecules. When $\gamma \in (-1,0)$, we are in the realm of molecules with soft potentials while $\gamma \in (0,1)$ corresponds to molecules with hard potentials.

There are many useful consequences of **A1**. Let us set

$$\alpha(z,v,\theta,\phi) := (\mathbf{n},(v-z))\mathbf{n} \tag{1.12}$$

Define $\hat{\alpha}(z,v,\theta,\phi) := \alpha(z,v,\theta,\phi)\sigma(|z-v|)$. Condition **A1** implies that there exists a constant C such that the following estimates hold.

$$\int_\Xi |\hat{\alpha}(z,v,\theta,\phi)|Q(d\theta)d\phi \leq C|z-v|^{1+\gamma}, \tag{1.13}$$

and hence

$$\int_\Xi |\hat{\alpha}(z,v,\theta,\phi)|Q(d\theta)d\phi \leq C(|z|^{1+\gamma} + |v|^{1+\gamma}). \tag{1.14}$$

Moreover, the following parameter transformation was introduced for each $z \neq v$ in [18] (see also [9], Section 3, or [11]).

$$\begin{aligned}\alpha(z,v,\theta,\phi) &= \frac{1-\cos(\theta)}{2}(v-z) + \frac{\sin(\theta)}{2}\Gamma(v-z,\phi) \\ &= \sin^2(\frac{\theta}{2})(v-z) + \frac{\sin(\theta)}{2}\Gamma(v-z,\phi)\end{aligned} \tag{1.15}$$

for all $\phi \in [0, 2\pi)$, where

$$\Gamma(v-z,\phi) = I(v-z)\cos(\phi) + J(v-z)\sin(\phi)$$

and $\frac{v-z}{|v-z|}$, $\frac{I(v-z)}{|v-z|}$, $\frac{J(v-z)}{|v-z|}$ form an orthogonal basis and hence are defined through Figure 1, where $\frac{v-z}{|v-z|}$ appears. It follows in particular that

$$\int_0^{2\pi} \Gamma(v-z,\phi)d\phi = 0. \tag{1.16}$$

In order to study solutions to the Boltzmann equation it is feasible to study continuity properties of $(u-v,n)n$ in u,v for fixed θ,ϕ. However, it was already pointed out by Tanaka that $(u,v) \longmapsto (u-v,n)n$ cannot be smooth. To overcome this problem Tanaka introduced in Lemma 3.1 of [18] another transformation of parameters, which describes a rotation around the longitude angle, is bijective and has Jacobian 1. As a consequence of this transformation ϕ_0, he proved the following Lemma 1.5.

Lemma 1.5. *(cf. [18], [11]) There exists a measurable function* $\phi_0 : \mathbb{R}^{12} \to (0, 2\pi]$ *such that*

$$|\Gamma(v-z,\phi) - \Gamma(v'-z',\phi+\phi_0(z,v,z',v'))| \le 3|z-v-(z'-v')| \tag{1.17}$$

and hence

$$|\alpha(z,v,\theta,\phi) - \alpha(z',v',\theta,\phi+\phi_0(z,v,z',v'))| \le 2\theta(|z-z'|+|v-v'|) \tag{1.18}$$

and

$$|\alpha(z,v,\theta,\phi)| \le 2\theta(|z|+|v|) \tag{1.19}$$

Moreover by using the estimate (1.17), the following inequality is obtained (cf. [11], Section 3):

$$\int_0^{\pi} \left| \int_0^{2\pi} \alpha(z,v,\theta,\phi) - \alpha(z',v',\theta,\phi)d\phi \right| Q(d\theta) \le C(|z-z'|+|v-v'|), \tag{1.20}$$

where by an abuse of notation we use the same symbol $C > 0$ in (1.14) and (1.20), even though the constants are different.

Let $\{f(t,x,z)\}_{t\in\mathbb{R}_+^0}$ be a collection of densities on $(\mathbb{R}^3\times\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3\times\mathbb{R}^3))$. Let us introduce the operator $\mathcal{C}_t(f,f)(\cdot)$ defined through the right side of equation (1.11)

$$\mathcal{C}_t(f,f)(\psi) := \int_{\mathbb{R}^6} f(t,x,z) L_f \psi(x,z) dx dz \tag{1.21}$$

It is easy to verify that

$$\mathcal{C}_t(f,f)(\psi) = 0 \quad \text{for} \quad \psi(x,z) = a + (b,z) + c|z|^2 \tag{1.22}$$

$\forall\, a, c \in , b \in \mathbb{R}^3$. (For a rigorous proof see Chapter II.7 [5] or [7], [4].)

The integral form of equation (1.11) corresponds to

$$\begin{aligned} \int_{\mathbb{R}^6} \psi(x,z) f(t,x,z) dxdz &= \int_{\mathbb{R}^6} \psi(x,z) f(0,x,z) dxdz \\ &+ \int_0^t \int_{\mathbb{R}^6} f(t,x,z)\{(z, \nabla_x \psi(x,z)) + L_f \psi(x,z)\} dxdzds, \end{aligned} \tag{1.23}$$

It is worthwhile to note that if a second collection of probability densities $\{g(t,x,z)\}_{t\in\mathbb{R}_+^0}$ on $(\mathbb{R}^3 \times \mathbb{R}^3, \mathcal{B}(\mathbb{R}^3 \times \mathbb{R}^3))$ is given, then

$$\mathcal{C}_t^S(f,g)(\psi) = 0 \quad \text{for} \quad \psi(x,z) = a + (b,z) + c|z|^2 \tag{1.24}$$

$\forall\, a, c \in , b \in \mathbb{R}^3$ with the operator $\mathcal{C}_t^S$ defined through

$$\mathcal{C}_t^S(f,g)(\cdot) := \mathcal{C}_t(f,g)(\cdot) + \mathcal{C}_t(g,f)(\cdot) \tag{1.25}$$

with

$$\mathcal{C}_t(f,g)(\psi) := \int_{\mathbb{R}^6} g(t,x,z) L_f \psi(x,z) dxdz. \tag{1.26}$$

The following Povzner type inequality is essentially contained in [16, Lemma 3.6]. (See also [7], Theorem 6.2.1 and Appendix B of Chapter 6 for $p \geq 2$ and references there.)

Lemma 1.6. *For all $\theta \in (0,\pi]$, $p \geq 2$ and $\gamma \in (0,1]$,*

$$\begin{aligned} &\int_0^{2\pi} \left(\langle z + \alpha(z,v,\theta,\phi)\rangle^{2p} + \langle v - \alpha(z,v,\theta,\phi)\rangle^{2p} - \langle z\rangle^{2p} - \langle v\rangle^{2p} \right) d\phi \\ &\leq -\frac{\sin^2(\theta)}{2} \left(\langle v\rangle^{2p} + \langle z\rangle^{2p} \right) + C_p \sin^2(\theta) \sum_{k=1}^{\lfloor \frac{p+1}{2} \rfloor} \left(\langle v\rangle^{2k} \langle z\rangle^{2p-2k} + \langle v\rangle^{2p-2k} \langle z\rangle^{2k} \right), \end{aligned}$$

where $\langle v\rangle := (1+|v|^2)^{\frac{1}{2}}$, $\lfloor x\rfloor \in \mathbb{Z}$ is defined by $\lfloor x\rfloor \leq x < \lfloor x\rfloor + 1$ and $C_p > 0$ is some constant.

Using conservation laws and Lemma 1.6, it can be proven that if $\{f(t,x,u)\}_{t\in[0,T]}$ is a weak solution of the Boltzmann equation in $[0,T]$, with initial finite second moment, i.e. $\int_{\mathbb{R}^6} |z|^2 f(0,x,z) dxdz < \infty$, then for all $p \geq 1$

$$\int_{\mathbb{R}^6} |z|^p f(t,x,z) dxdz < \infty \quad \forall t \in [0,T]. \tag{1.27}$$

For a proof we refer the reader to [16, Theorem 3.6].

Let $\{\mu_t(dx,dz)\}_{t\in\mathbb{R}^0_+}$ be a collection of probabilities on $(\mathbb{R}^3\times\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3\times\mathbb{R}^3))$. Let us define the operator

$$\mathcal{C}_t(f,\mu)(\psi) := \int_{\mathbb{R}^6} L_f\psi(x,z)\mu_t(dx,dz). \tag{1.28}$$

acting on all ψ for which the integral on the right side is finite.

Lemma 1.7.

$$\mathcal{C}_t(f,\mu)(|z|^2) = \int_{\mathbb{R}^9\times\Xi}(|v|^2-|z|^2)\sigma(|z-v|)\sin^2(\frac{\theta}{2})Q(d\theta)d\phi f(t,x,v)dv\mu_t(dx,dz). \tag{1.29}$$

Proof.

$$\begin{aligned}\mathcal{C}_t(f,\mu)(|z|^2) &= \int_{\mathbb{R}^6} L_f|z|^2\mu_t(dx,dz) \qquad (1.30)\\ &= \int_{\mathbb{R}^9\times\Xi}(|z^\star|^2-|z|^2)\sigma(|z-v|)Q(d\theta)d\phi f(t,x,v)dv\mu_t(dx,dz).\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathcal{C}_t(f,\mu)(|z|^2)\\ &= \int_{\mathbb{R}^9\times\Xi}(|z^\star|^2+|v^\star|^2-|z|^2-|v|^2)\sigma(|z-v|)Q(d\theta)d\phi f(t,x,v)dv\mu_t(dx,dz)\\ &\quad -\int_{\mathbb{R}^9\times\Xi}(|v^\star|^2-|v|^2)\sigma(|z-v|)Q(d\theta)d\phi f(t,x,v)dv\mu_t(dx,dz)\\ &\quad -\int_{\mathbb{R}^9\times\Xi}(|v^\star|^2-|v|^2)\sigma(|z-v|)Q(d\theta)d\phi f(t,x,v)dv\mu_t(dx,dz), \qquad (1.31)\end{aligned}$$

where in the last equality we used that the kinetic energy is conserved during the elastic collision, see (1.3).

Combining equation (1.30) and (1.31), we obtain

$$\begin{aligned}&\mathcal{C}_t(f,\mu)(|z|^2)\\ &= \frac{1}{2}\int_{\mathbb{R}^9\times\Xi}\left[|(z^\star|^2-|z|^2)-(|v^\star|^2-|v|^2)\right]\sigma(|z-v|)Q(d\theta)d\phi f(t,x,v)dv\mu_t(dx,dz)\\ &= \frac{1}{2}\int_{\mathbb{R}^9\times\Xi}\left[(|z|^2+2(z,\alpha)+|\alpha|^2-|z|^2)-(|v|^2-2(v,\alpha)+|\alpha|^2-|v|^2)\right]\\ &\qquad\qquad\times\sigma(|z-v|)Q(d\theta)d\phi f(t,x,v)dv\mu_t(dx,dz)\\ &= \int_{\mathbb{R}^9\times\Xi}(z+v,\alpha)\sigma(|z-v|)Q(d\theta)d\phi f(t,x,v)dv\mu_t(dx,dz), \qquad (1.32)\end{aligned}$$

Using the parametrization (1.15) for $\alpha = \alpha(z, v, \theta, \phi)$ and (1.16) we obtain

$$\begin{aligned}
&\mathcal{C}_t(f, \mu)(|z|^2) \\
&= + \int_{\mathbb{R}^9 \times \Xi} (z + v, v - z) \sin^2(\frac{\theta}{2}) \sigma(|z - v|) Q(d\theta) d\phi f(t, x, v) dv \mu_t(dx, dz) \\
&= - \int_{\mathbb{R}^9 \times \Xi} (|z|^2 - |v|^2) \sin^2(\frac{\theta}{2}) \sigma(|z - v|) Q(d\theta) d\phi f(t, x, v) dv \mu_t(dx, dz)
\end{aligned}$$

□

The plan of this article is the following: In Section 2, given a family of densities $\{f(t, x, z)\}_{t \in [0,T]}$ satisfying few regularity conditions stated in conditions $\mathbf{B_0}$ - $\mathbf{B_3}$, we construct the McKean -Vlasov type Stochastic Differential Equation (SDE) (2.1) and prove in Theorem 2.1 that, if its solution $(X_., Z_.)$ exists, its density time marginals $\{f(t, x, z)\}_{t \in [0,T]}$ satisfy the Boltzmann equation in its weak form (1.11). The SDE (2.1) is of McKean -Vlasov type, because the time marginals of the density of the solving process appear also in the compensator of the Poisson noise. In Section 3 we consider the hard sphere case without cutoff ($\gamma = 1$) and construct the SDE (3.2) of the same form, by depending in the same way as in (2.1) through the compensator on $\{f(t, x, z)\}_{t \in [0,T]}$, however without making any assumption on the distribution of its solution. This equation is hence not any more of McKean -Vlasov type, despite the r.h.s. is defined in the same way. In the Paragraphs 3.1 and 3.2 we make the additional assumption that $\{f(t, x, z)\}_{t \in [0,T]}$ is a collection of densities which solves the Boltzmann equation (1.1). In Paragraph 3.1 in Theorem 3.2 we get that under few regularity conditions $\mathbf{B_4}$ - $\mathbf{B_6}$ on $\{f(t, x, z)\}_{t \in [0,T]}$ (3.2) has a weak solution $(X_., Z_.)$ in $[0, T]$. In Paragraph 3.2 we prove in Theorem 3.4 that under the hypothesis that $(X_., Z_.)$ has a density (and few integrability conditions stated in condition **C1**) its density coincides with the solution of the Boltzmann equation $\{f(t, x, z)\}_{t \in [0,T]}$ appearing in the compensator. As a direct consequence we can conclude in Theorem 3.5 that the SDE (3.2) coincides with the McKean VLasov SDE (2.1) and the solving stochastic process is a Boltzmann process with density $\{f(t, x, z)\}_{t \in [0,T]}$. We have so constructed a Boltzmann process with density $\{f(t, x, z)\}_{t \in [0,T]}$ solving the Boltzmann equation (1.1). The Kolmogorov equation of this Boltzmann process is (1.1).

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2 The Boltzmann process

We use the following notation throughout the paper. $U_0 = \mathbb{D} \times [0,\pi) \times (0, 2\pi]$.

Let $\{f(t,x,v)\}_{t\in\mathbb{R}^0_+}$ be a collection of densities on $(\mathbb{R}^3 \times \mathbb{R}^3, \mathcal{B}(\mathbb{R}^3 \times \mathbb{R}^3))$. Then $m(t,v)$ denotes the marginal density of velocity v at time t, i.e. $m(t,v) := \int_{\mathbb{R}^3} f(t,x,v)dx$ so that $f(t,x|v)m(t,v) := f(t,x,v)$, upon disintegration of measures.

Hypotheses B: We assume that $t \to f(t,x,v)$ is differentiable for each $x, v \in \mathbb{R}^3$ fixed, and satisfies

$\mathbf{B_0}$. $|\frac{\partial f}{\partial t}|$ is bounded on any compact subset of $\mathbb{R}^0_+ \times \mathbb{R}^6$.

$\mathbf{B_1}$. $\frac{\partial f}{\partial t}(t,\cdot) \in L^1(\mathbb{R}^6), \quad \forall t \in \mathbb{R}^0_+,$

$\mathbf{B_2}$. $\sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} |u|^{1+\gamma} f(s,x,u)du \in C([0,T]) \quad \forall\, T > 0.$

$\mathbf{B_3}$. $\sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} |u|^{1+\gamma} \frac{\partial}{\partial t} f(t,x,u)du \in L^1([0,T]) \quad \forall\, T > 0.$

Theorem 2.1. *Let $\{f(t,x,v)\}_{t\in\mathbb{R}^0_+}$ be a collection of densities which satisfies hypothesis* **B**. *Suppose that the hypothesis* **A** *holds. Let X_0 and Z_0 be $\mathbb{R}^3$- valued random variables. Suppose that for any fixed $T > 0$ there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P})$, an adapted process $(X_t, Z_t)_{t\in[0,T]}$ with values on $\mathbb{D}\times\mathbb{D}$, which has time marginals with density $f(t,x,v)$, and such that it satisfies a.s. the following stochastic equation for $t \in [0,T]$:*

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t Z_s ds \\ Z_t = Z_0 + \displaystyle\int_0^t \int_{U_0\times\mathbb{R}^0_+} \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\ \sigma(|Z_s - v_s|) f(s, X_s | v_s)]}(r) dN, \end{cases} \tag{2.1}$$

where $dN := N(dv, d\theta, d\phi, dr, ds)$ is a Poisson random measure with compensator $m(s,v)dvQ(d\theta)d\phi dsdr$. Then $(X_t, Z_t)_{t\in[0,T]}$ is a Boltzmann process.

Proof. From (1.14) it follows for each $T > 0$

$$\int_0^T \mathbb{E}[\int_{U_0\times\mathbb{R}_+^0} |\alpha(Z_s, v_s, \theta, \phi)| 1_{[0,\ \sigma(|Z_s - v_s|) f(s, X_s|v_s)]}(r) m(s,v) dv Q(d\theta) d\phi dr] ds$$
$$= \int_0^T \mathbb{E}[\int_{U_0} |\hat{\alpha}(Z_s, v_s, \theta, \phi)| f(s, X_s, v) dv Q(d\theta) d\phi] ds$$
$$\leq C \int_0^T \int_{\mathbb{R}^9} (|z|^{1+\gamma} + |v|^{1+\gamma}) f(s,x,z) f(s,x,v) dx dz dv ds,$$
$$\leq 2C \int_0^T \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^6} \left(|z|^{1+\gamma} f(s,x,z) dz\right) f(s,x,v) dv dx ds < \infty.$$

for some constant $C > 0$. In the above estimates we have used that the function $f(t)$ is the probability density of the process (X_t, Z_t), as well the assumption $\mathbf{A_2}$ and $\mathbf{B_2}$. It follows that we can apply the Itô formula to $(X_s, Z_s)_{s\in\mathbb{R}^+}$ [17]. In fact let t, $\Delta t > 0$, $\psi \in C_0^2(\mathbb{R}^3 \times \mathbb{R}^3)$, then

$$\psi(X_{t+\Delta t}, Z_{t+\Delta t})$$
$$= \psi(X_t, Z_t) + \int_t^{t+\Delta t} (Z_s, \nabla_x \psi(X_s, Z_s)) ds$$
$$+ \int_t^{t+\Delta t} \int_{U_0\times\mathbb{R}_0^+} \Big\{ \psi(X_s, Z_s + \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\ \sigma(|Z_s - v_s|) f(s, X_s|v_s)]}(r)) - \psi(X_s, Z_s) \Big\} dN$$

It follows that

$$\mathbb{E}[\psi(X_{t+\Delta t}, Z_{t+\Delta t}) - \psi(X_t, Z_t)]$$
$$= \mathbb{E}\left[\int_t^{t+\Delta t} (Z_s, \nabla_x \psi(X_s, Z_s)) ds\right]$$
$$+ \mathbb{E}\Bigg[\int_t^{t+\Delta t} \int_{U_0} \{\psi(X_s, Z_s + \alpha(Z_s, v_s, \theta, \phi)) - \psi(X_s, Z_s)\} \times \sigma(|Z_s - v_s|) f(s, X_s, v_s) dv Q(d\theta) d\phi ds\Bigg]$$

Upon dividing by Δt on both sides, we obtain

$$
\begin{aligned}
&\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \int_{\mathbb{R}^6} \psi(x,u)\{f(t+\Delta t,x,u) - f(t,x,u)\} dxdu \\
&= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_{\mathbb{R}^6} (u, \nabla_x \psi(x,u)) f(s,x,u) dxduds + \\
&\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_{\mathbb{R}^6 \times \mathbb{R}^3 \times [0,\pi) \times (0,2\pi]} \{\psi(x, u+\alpha(u,v,\theta,\phi)) - \psi(x,u)\} \\
&\times \sigma(|u-v|) f(s,x,v) f(s,x,u) dv Q(d\theta) d\phi dxduds \qquad (2.2)
\end{aligned}
$$

Letting $\Delta t \to 0$ in every term of (2.2) we obtain (1.11). Indeed, for e.g., let us consider the second term on the right side of (2.2). To find this limit, we first prove the continuity of the function

$$
\begin{aligned}
g(s) := \int_{\mathbb{R}^6 \times \mathbb{R}^3 \times [0,\pi) \times (0,2\pi]} &\{\psi(x, u+\alpha(u,v,\theta,\phi)) - \psi(x,u)\} \\
&\times \sigma(|u-v|) f(s,x,v) f(s,x,u) dv Q(d\theta) d\phi dxdu.
\end{aligned}
$$

Since

$$
\{\psi(x, u+\alpha(u,v,\theta,\phi)) - \psi(x,u)\} \simeq \nabla_u \psi(x,u) \alpha(u,v,\theta,\phi)
$$

with $(x,z) \in K$ compact set, and

$$
|\alpha(u,v,\theta,\phi)| \sigma(|u-v|) \leq |u-v|^{1+\gamma} |\sin(\frac{\theta}{2})|,
$$

by denoting with F a compact set in $\mathbb{R}^3$ which includes all projections x of $(x,z) \in K$, it follows that

$$
\begin{aligned}
|g(s) - g(s_0)| \leq C \int_{F \times \mathbb{R}^6} &|f(s,x,u) f(s,x,v) - f(s_0,x,u) f(s_0,x,v)| \\
&\times (|u|^{\gamma+1} + |v|^{\gamma+1}) dxdudv, \qquad (2.3)
\end{aligned}
$$

with

$$
C := \|\nabla_u \psi\|_\infty 2\pi \int_0^\pi \theta Q(d\theta)
$$

We split the integral on the right side of (2.3) into two terms, one with $|u|^{\gamma+1}$ (resp. $|v|^{\gamma+1}$), and get

$$
\begin{aligned}
&\int_{F \times \mathbb{R}^6} |f(s,x,u) f(s,x,v) - f(s_0,x,u) f(s_0,x,v)| |u|^{\gamma+1} dxdudv \\
=& \int_{F \times \mathbb{R}^6} |f(s,x,u) f(s,x,v) - f(s,x,u) f(s_0,x,v)| |u|^{\gamma+1} dxdudv \\
&+ \int_{F \times \mathbb{R}^6} |f(s,x,u) f(s_0,x,v) - f(s_0,x,u) f(s_0,x,v)| |u|^{\gamma+1} dxdudv \\
=& J_1(s) + J_2(s) \qquad (2.4)
\end{aligned}
$$

where $J_1(s)$ (resp. $J_2(s)$) is the first (resp. second) term on the right side of (2.4).

$$J_1(s) = \int_{F\times\mathbb{R}^6} |u|^{\gamma+1} f(s,x,u)| \int_{s_0}^{s} \frac{\partial f}{\partial r}(r,x,v)dr|dxdudv$$
$$\leq \left(\sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} |u|^{\gamma+1} f(s,x,u)du \right) \int_{s_0}^{s} \int_{\mathbb{R}^6} |\frac{\partial f}{\partial r}(r,x,v)|dxdvdr$$

By $\mathbf{B_1}$ and $\mathbf{B_2}$ $\lim_{s\to s_0} J_1(s) = 0$.
Let us consider $J_2(s)$.

$$J_2(s) = \int_{F\times\mathbb{R}^6} |u|^{\gamma+1} f(s_0,x,v)| \int_{s_0}^{s} \frac{\partial f}{\partial r}(r,x,u)dr|dxdudv$$
$$\leq \int_{s_0}^{s} \int_{F\times\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |u|^{\gamma+1} |\frac{\partial f}{\partial r}(r,x,u)|du \right) f(s_0,x,v)dxdvdr$$

Since $\sup_{x\in\mathbb{R}^3} \int_{R^3} |u|^{\gamma+1}|\frac{\partial f}{\partial r}(r,x,u)|du$ is integrable in $[s_0, s]$ by $\mathbf{B_3}$, we obtain $\lim_{s\to s_0} J_2(s) = 0$.
Likewise, and without any changes in the arguments it follows

$$\lim_{s\to s_0} C \int_{F\times\mathbb{R}^6} |f(s,x,u)f(s,x,v) - f(s_0,x,u)f(s_0,x,v)||v|^{\gamma+1}dxdudv = 0$$

Hence $\lim_{s\to s_0} g(s) = g(s_0)$, so that g is a continuous function and

$$\lim_{\Delta t\downarrow 0} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} g(s)ds = g(t).$$

Note that in the above arguments we have taken $s > s_0$ for simplicity. One may also take $s_0 > s$. □

Theorem 2.1 motivates the following Definition.

Definition 2.2. Let $\{f(t,x,v)\}_{t\in\mathbb{R}_+^0}$ be a collection of densities satisfying Hypothesis **B**. Suppose that for any fixed $T > 0$, there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P})$ and an adapted process $(X_t, Z_t)_{t\in[0,T]}$ with values on $\mathbb{D}\times\mathbb{D}$ such that

- **i)** $(X_t, Z_t)_{t\in[0,T]}$ has time marginals with density $f(t,x,u)$, for $t \in [0,T]$,
- **ii)** $(X_t, Z_t)_{t\in[0,T]}$ is a solution of the McKean -Vlasov SDE (2.1).

Then we say that "the McKean -Vlasov equation (2.1) with density functions $\{f(t,x,v)\}_{t\in\mathbb{R}_+^0}$ is associated to the the Boltzmann equation (1.1)". If the above property holds for $T \in [0,S]$ with $S > 0$, then the McKean -Vlasov SDE (2.1) with density functions $\{f(t,x,v)\}_{t\in[0,S]}$ is associated to the Boltzmann equation (1.1) up to time S.

Remark 2.3. *Let us assume hypothesis* **A**. *From Theorem 2.1 it follows that any stochastic process* $(X_t, Z_t)_{t\in[0,T]}$ *solving a McKean -Vlasov equation* (2.1) *associated to the Boltzmann equation* (1.11) *is (according to Definition 1.4) a Boltzmann process. The Boltzmann equation* (1.11) *is hence the Kolmogorov equation associated to the McKean -Vlasov equation* (2.1).

3 Existence of the Boltzmann process

In Theorem 2.1 we proved that any process $(X_t, Z_t)_{t\in[0,T]}$ solving the McKean -Vlasov equation (2.1) associated to (1.11) in $[0, T]$ is a Boltzmann process. In this section we analyze the following: given a strong solution $\{f(t, x, z)\}_{t\in[0,T]}$ of the Boltzmann equation (1.1), we find sufficient conditions for the existence of a solution of the McKean -Vlasov equation (2.1) with density $\{f(t, x, z)\}_{t\in[0,T]}$. The solution process $(X_t, Z_t)_{t\in[0,T]}$ is then a Boltzmann process.

We present an overview on the construction of Boltzmann processes. We briefly outline the construction of the process $(X_t, Z_t)_{t\in[0,T]}$ under suitable conditions before stating the main result on Boltzmann processes. The proofs of the ensuing results on the existence of a solution to a certain linearized stochastic system will appear in a separate paper [1].

3.1 Construction of a solution of a SDE defined through a collection of densities solving (1.1)

In this paragraph, we assume that $\{f(t, x, z)\}_{t\in[0,T]}$ is a collection of densities which solves the Boltzmann equation (1.1) and satisfies the following conditions:

- $\mathbf{B_4}$. $\sup_{s\in[0,T],x\in\mathbb{R}^3} \int_{\mathbb{R}^3} f(s, x, v)dv \le C_T < \infty$.
- $\mathbf{B_5}$. For every $K > 0$, there exists a corresponding constant $C_T^K > 0$ such that
$$\sup_{s\in[0,T],|x|\le K} \int_{\mathbb{R}^3} \max(1, |v|^{1+\gamma})|\nabla_x f(s, x, v)|dv \le C_T^K < \infty.$$
- $\mathbf{B_6}$. $\sup_{s\in[0,T],x\in\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^{\gamma+2} f(s, x, v)dv \le c_T < \infty$.

On any fixed filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P})$ satisfying the usual conditions, let $S_T := S_T^1(\mathbb{R}^d)$ denote the linear space of all adapted càdlàg processes $(X_t)_{t\in[0,T]}$ with values on $\mathbb{R}^d$ equipped with norm
$$\|X\|_{S_T^1} := \mathbb{E}[\sup_{s\in[0,T]} |X_s|]. \tag{3.1}$$

Under hypotheses $\mathbf{B_4}$ - $\mathbf{B_6}$, and adopting the notation

$$f(s,x,v) = f(s,x \,|\, v)m(s,v)$$

upon disintegration of measures, we first prove the existence of a weak solution to the stochastic differential equation

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t Z_s ds \\ Z_t = Z_0 + \displaystyle\int_0^t \int_{U_0 \times \mathbb{R}_+^0} \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\ \sigma(|Z_s - v_s|) f(s, X_s | v_s)]}(r) dN, \end{cases} \tag{3.2}$$

for $t \in [0,T]$ where $dN := N(dv, d\theta, d\phi, dr, ds)$ with its compensator given by $m(s,v)dvQ(d\theta)d\phi ds dr$ with values in $S_T^1 := S_T^1(\mathbb{R}^3 \times \mathbb{R}^3)$.

Here we do not assume that (3.2) is of McKean -Vlasov type.

First, we recall the definition of weak solutions in the context of stochastic analysis [15].

Definition 3.1. A "weak solution" of equation (3.2) in the time interval $[0,T]$ is a triplet $((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P}), N(dv, d\theta, d\phi, dr, ds),\ (X_t, Z_t)_{t\in[0,T]})$ for which the following properties hold:

- $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P})$ is a stochastic basis;
- $N(dv, d\theta, d\phi, dr, ds)$ is an adapted Poisson random measure with compensator $m(s,v)dvQ(d\theta)d\phi ds dr$;
- $(X_\cdot, Z_\cdot) := (X_t, Z_t)_{t\in[0,T]})$ is an adapted càdlàg stochastic process with values in $\mathbb{R}^d \times \mathbb{R}^d$ which satisfies (3.2) P -a.s.

The existence of solutions to the stochastic system (3.2) is stated in the following theorem, proven in [1].

Theorem 3.2. *Let* $\gamma = 1$ *and Hypothesis* $\mathbf{A}$ *be satisfied. Let* $T > 0$ *and* $\{f(t,x,v)\}_{t\in[0,T]}$ *be a collection of densities which satisfy* $f(t,x,u) \in C([0,T] \times \mathbb{R}^6)$ *and Hypotheses* $\mathbf{B}$. *Let the initial distribution of* (X_0, Z_0) *admit finite second moment. There exists a weak solution*

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P}), N(dv, d\theta, d\phi, dr, ds), (X_t, Z_t)_{t\in[0,T]})$$

of (3.2) *such that* $(X_\cdot, Z_\cdot) \in S_T^1$. *Moreover,*

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t|^2] + \sup_{t\in[0,T]} \mathbb{E}[|Z_t|^2] < \infty \tag{3.3}$$

The proof of the theorem uses truncation of the functions α and σ in order to first obtain the existence of a solution to (3.2) up to a stopping time. This is wrought by constructing a suitable Picard scheme. Next, any such solution is shown to be extendable to the whole of $[0,T]$ by patching of measures. It is worthwhile to remark that our methods lead to a weak solution (in the sense of stochastic analysis), and we do not claim uniqueness of solutions.

We remark that the estimate (3.3) is proven by symmetry arguments similar to those appearing in the proof of Lemma 1.2. The form of the stochastic system (3.2) with the process taking values in $\mathbb{R}^6$ at each $t \in [0,T]$, one obtains that the solution lies in $\mathbb{D} \times \mathbb{D}$.

3.2 Construction of Boltzmann processes with densities satisfying (1.1)

We recall the concept of relative entropy which plays a key role in the proof of the following theorem. Recall that for any two probability measures μ, ν on a common measurable space $(X, \mathcal{X})$, the relative entropy of ν with respect to μ, denoted $R(\nu \,||\, \mu)$, is defined by

$$R(\nu \,||\, \mu) = \int_X \left(\log \frac{d\nu}{d\mu}\right) d\nu$$

if ν is absolutely continuous with respect to μ. Otherwise, we set $R(\nu \,||\, \mu) = \infty$. The following Lemma is well known.

Lemma 3.3. *Let μ, ν be two probability measures on a measurable space $(X, \mathcal{X})$. Then $R(\nu \,||\, \mu) \geq 0$ and $R(\nu \,||\, \mu) = 0$ if and only if $\mu = \nu$.*

We assume that $\{f(t,x,z)\}_{t\in[0,T]}$ is a collection of densities which solves the Boltzmann equation (1.1) and satisfies hypotheses $\mathbf{B}$ as well as the following condition:

C1. The densities $f(t,x,z)$ and $g(t,x,z)$ are in $C^{1,2}([0,T] \times \mathbb{R}^6)$ and are strictly positive-valued functions with $g \log g, g \log f \in L^1(\mathbb{R}^6)$ for each $t \in [0,T]$ and $\lim_{|x|\to\infty} g(t,x,z) = 0$ and $g(0,x,z) = f(0,x,z)$ a.s.

Theorem 3.4. *Let $(X_t, Z_t)_{t\in[0,T]}$ be a stochastic process that solves the stochastic system (3.2). Suppose that $(X_t, Z_t)_{t\in[0,T]}$ has time marginals with density $g(t,x,z)$, for each $t \in [0,T]$. Suppose that $\{f(t,x,z)\}_{t\in[0,T]}$ and $\{g(t,x,z)\}_{t\in[0,T]}$ satisfy hypotheses $\mathbf{B_0}$–$\mathbf{B_6}$ and $\mathbf{C1}$. Then $g(t,x,z) = f(t,x,z) \quad$ a.e. for all $t \in [0,T]$.*

Proof. We will write $R(g \,||\, f)$ for the relative entropy of the measure with probability density g with respect to the measure with probability density

f. The theorem is proved by establishing the following equality.

$$R_t(g|f) := \int_{\mathbb{R}^6} \log\left(\frac{g(t.x.z)}{f(t,x,z)}\right) g(t,x,z)dxdz = 0 \quad \forall t \in [0,T] \tag{3.4}$$

We first apply the Itô formula [14] to $\log(g(t,X_t,Z_t))$, where $(X_t,Z_t)_{t\in[0,T]}$ solves (3.2) and take expectation to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^6} \log(g(t,x,z))g(t,x,z)dxdz - \int_{\mathbb{R}^6} \log(g(0,x,z))g(0,x,z)dxdz \\
&= \int_0^t \int_{\mathbb{R}^6\times\mathbb{R}^3\times\Xi} \{\log(g(s,x,z^\star)) - \log(g(s,x,z))\} \\
&\qquad \times \sigma(|z-v|)f(s,x,v)g(s,x,z)Q(d\theta)d\phi dvdxdzds. \qquad (3.5)
\end{aligned}$$

Indeed, in arriving at (3.5), we have used the following two calculations:

(i) $\displaystyle\int_0^t \int_{\mathbb{R}^6} \frac{\partial}{\partial s}\log(g(s,x,z))g(s,x,z)dxdzds$

$$= \int_0^t \int_{\mathbb{R}^6} \frac{\partial}{\partial s} g(s,x,z)dxdzds = \int_{\mathbb{R}^6} (g(t,x,z) - g(0,x,z))dxdz = 0$$

since g is a probability density.

(ii) $\displaystyle\int_{\mathbb{R}^6} z\cdot\nabla_x \log(g(s,x,z))g(s,x,z)dxdz = \int_{\mathbb{R}^6} z\cdot\nabla_x g(s,x,z)dxdz = 0$

where the last equality is obtained by integrating and using the condition that $\lim_{|x|\to\infty} g(t,x,z) = 0$. Likewise, one obtains upon taking expectation and recalling that $\{f(t,x,z)\}_{t\in[0,T]}$ is a collection of densities which solves the Boltzmann equation (1.1),

$$
\begin{aligned}
&\int_{\mathbb{R}^6} \log\left(f(t,x,z)\right) g(t,x,z)dxdz - \int_{\mathbb{R}^6} \log\left(f(0,x,z)\right) g(0,x,z)dxdz \\
&= \int_0^t \int_{\mathbb{R}^6} \frac{\mathcal{C}(f,f)(s,x,z)}{f(s,x,z)} g(s,x,z)dxdzds \\
&\quad + \int_0^t \int_{\mathbb{R}^6\times\mathbb{R}^3\times\Xi} \{\log\left(f(s,x,z^\star)\right) - \log\left(f(s,x,z)\right)\} \\
&\qquad\qquad \times \sigma(|z-v|) f(s,x,v) g(s,x,z) Q(d\theta)d\phi dvdxdzds \\
&= \int_0^t \int_{\mathbb{R}^9\times\Xi} \{\frac{g(s,x,z^\star)}{f(s,x,z^\star)} - \frac{g(s,x,z)}{f(s,x,z)}\} \\
&\qquad\qquad \times \sigma(|z-v|) f(s,x,v) f(s,x,z) Q(d\theta)d\phi dvdxdzds\,, \\
&\quad + \int_0^t \int_{\mathbb{R}^6\times\mathbb{R}^3\times\Xi} \{\log\left(f(s,x,z^\star)\right) - \log\left(f(s,x,z)\right)\} \\
&\qquad\qquad \times \sigma(|z-v|) f(s,x,v) g(s,x,z) Q(d\theta)d\phi dvdxdzds. \qquad (3.6)
\end{aligned}
$$

It is worthwhile to note that the last equality in the above display results upon rewriting

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^6} \frac{\mathcal{C}(f,f)(s,x,z)}{f(s,x,z)} g(s,x,z)dxdzds \\
&= \int_0^t \int_{\mathbb{R}^9\times\Xi} \{f(s,x,z^\star)f(s,x,v^\star) - f(s,x,z)f(s,x,v)\} \\
&\qquad\qquad \times \sigma(|z-v|) \frac{g(s,x,z)}{f(s,x,z)} Q(d\theta)d\phi dvdxdzds \\
&= \int_0^t \int_{\mathbb{R}^9\times\Xi} \{\frac{g(s,x,z^\star)}{f(s,x,z^\star)} - \frac{g(s,x,z)}{f(s,x,z)}\} \\
&\qquad\qquad \times \sigma(|z-v|) f(s,x,v) f(s,x,z) Q(d\theta)d\phi dvdxdzds\,,
\end{aligned}
$$

by using Proposition 1.2.

Combining equations (3.5) with (3.6) we obtain that

$$
\begin{aligned}
&R_t(g|f) \\
&= \int_{\mathbb{R}^6} \log\left(g(t,x,z)\right) g(t,x,z)dxdz - \int_{\mathbb{R}^6} \log\left(f(t,x,z)\right) g(t,x,z)dxdz \\
&= \int_0^t \int_{\mathbb{R}^9\times\Xi} \left[\frac{g(s,x,z)}{f(s,x,z)}\{1+\log\left(\frac{g(s,x,z^\star)/f(s,x,z^\star)}{g(s,x,z)/f(s,x,z)}\right)\} - \frac{g(s,x,z^\star)}{f(s,x,z^\star)}\right] \\
&\qquad\qquad \times \sigma(|z-v|) f(s,x,v) f(s,x,z) Q(d\theta)d\phi dvdxdzds \qquad (3.7)
\end{aligned}
$$

Transforming (3.7) in the equivalent equation below, and recalling that for $x \geq 0$ we have $1 + \log(x) - x \leq 0$, we easily see that

$$
\begin{aligned}
&R_t(g|f) \\
&= \int_0^t \int_{\mathbb{R}^9 \times \Xi} \left[1 + \log\left(\frac{g(s,x,z^\star)/f(s,x,z^\star)}{g(s,x,z)/f(s,x,z)} \right) - \frac{g(s,x,z^\star)/f(s,x,z^\star)}{g(s,x,z)/f(s,x,z)} \right] \\
&\qquad\qquad \times \frac{g(s,x,z)}{f(s,x,z)} \sigma(|z-v|) f(s,x,v) f(s,x,z) Q(d\theta) d\phi dv dx dz ds \\
&\leq 0.
\end{aligned}
$$

However, by Lemma 3.3, $R_t(g|f) \geq 0$, and hence, $R_t(g|f) = 0$. □

Under the assumptions of Theorem 3.4 it follows that $(X_t, Z_t)_{t\in[0,T]}$ in Theorem 3.2 solves the McKean-Vlasov equation (2.1) associated to the Boltzmann equation (1.1) and is a Boltzmann process with densities $\{f(t,x,z)\}_{t\in[0,T]}$ up to time T. It is worthwhile to recall that $\gamma = 1$ corresponds to the case of hard-sphere interaction.

The main result of this work is given below:

Theorem 3.5. *Let Hypotheses* **A** *be satisfied and* $\gamma = 1$. *Assume that* $\{f(t,x,z)\}_{t\in[0,T]}$ *is a collection of densities which solves the Boltzmann equation* (1.1), *and satisfies the hypotheses* $\mathbf{B_0}$–$\mathbf{B_6}$. *Let the random vector* (X_0, Z_0) *have finite second moment. Suppose that the weak solution of the stochastic system* (3.2) *has its distribution that admits a probability density at each time* $t \in [0,T]$ *given by* $g(t,x,z)$. *If condition* **C1** *is satisfied by* $\{f(t,x,z)\}_{t\in[0,T]}$ *and* $\{g(t,x,z)\}_{t\in[0,T]}$, *then the McKean-Vlasov equation* (2.1) (*that involves* $\{f(t,x,z)\}_{t\in[0,T]}$) *has a weak solution in* $[0,T]$ *with values in* $\mathbb{D}\times\mathbb{D}$, *and its Kolmogorov equation coincides with equation* (1.1).

Proof. The result follows from Theorem 3.2 and Theorem 3.4. In fact, equation (3.2) coincides with the McKean Vlasov SDE (2.1) and its solution is a Boltzmann process with time-marginal densities $\{\{f(t,x,z)\}_{t\in[0,T]}\}$ since the latter solve the Boltzmann equation (1.1). □

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