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Grassmannian calculus for probability

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Abstract. The present overview and gentle introduction to Grassmannian calculus and some of its applications to probability collects and extends the notes of a mini-course given by the authors at the Brazilian School of Probability, August 5-9, 2024, in Salvador, Bahia, Brazil. The content is by no means comprehensive, and is a personal summary and interpretation of results and applications of this interesting area of research.

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Contents

1 Introduction

The use of Grassmannian calculus in probability theory remains a somewhat unfamiliar subject within the probabilistic community. While the powerful methods of Grassmann algebra have been successfully applied in theoretical physics, particularly in statistical mechanics and field theory, their potential contributions to probability theory have yet to be fully recognized and utilized. The aim of this survey is to bridge this gap by providing an accessible yet rigorous introduction to Grassmannian methods in probability, highlighting their relevance, applications, and potential for further development.

Role in Algebra

Grassmannian calculus, originally introduced in the 19th century by Hermann Grassmann [33], is a mathematical framework based on anticommuting variables, known as Grassmann variables. These variables, denoted as $\theta = (\theta_i)_i$, satisfy the fundamental relation $\theta_i\theta_j = -\theta_j\theta_i$, making them essential tools in areas where antisymmetric structures naturally arise. An example of such context are differential forms, since Grassmann calculus provides a natural language for integration on manifolds [48]. Furthermore, it serves as a foundation for Clifford algebras, Lie superalgebras, and cohomological theories (see for instance [7, Ch. III.7 and Ch. III.11] or [14]). These structures have significant implications in homological algebra, representation theory, and geometric topology.

Role in Probability

While perhaps not being mainstream, Grassmann calculus has been successfully applied in probability theory, too. One notable area of application is in the study of large random matrix ensembles, where Grassmann variables facilitate the derivation of integral representations and correlation functions [6]. Another recognized direction lies in stochastic processes with fermionic structure, where supersymmetric methods have been employed to analyze diffusion and percolation problems ([5, 38]). These contributions underscore the potential of Grassmannian methods in tackling problems that resist conventional probabilistic techniques. As stated by [39], Grassmann calculus has the unique property that “it permits the expression in formulas of the results of geometric constructions”.

In light of this remark, Grassmannian techniques have been successfully utilized in combinatorial probability, particularly in the study of loop models, spanning forests, and the enumeration of walks [8, 31, 26]. These applications leverage the inherent antisymmetric nature of Grassmann variables to elegantly encode counting problems that involve non-intersecting paths and constraints on connectivity. Other applications include for instance dimer models and the Ising model [25, 41, 46].

Role in Supersymmetry

In theoretical physics, Grassmann variables play a central role in supersymmetry (SUSY), an area that extends conventional quantum mechanics and quantum field theory by positing a symmetry between bosonic and fermionic degrees of freedom (see [52]). Supersymmetric theories have been applied to various fields of physics, from string theory to optics, and even a supersymmetric extension of the Standard Model was considered. For a review on the topic we refer the reader for example to [50]. Note that supersymmetric methods have also been adapted to study disorder systems, Anderson localization, and the behavior of spin glasses for example in [53, 23].

On the one hand, most researchers in probability theory are unfamiliar with Grassmannian calculus. On the other hand, mathematicians already acquainted with Grassmann algebra typically lack familiarity with probability theory. While we do not pretend to have covered all aspects of the field, this survey seeks to offer a treatment of the core concepts, techniques, and major achievements of Grassmannian calculus within probability, thereby making the subject more accessible to probabilists, and to provide sufficient background in probability theory ensuring that those well-versed in Grassmannian methods can appreciate the depth and breadth of its applications in this field.

Structure of the paper

These notes begin with providing mathematical preliminaries in Sec. 2 and introduce Grassmann calculus in Sec. 3. We are going to deal with Grassmann Gaussians in Sec. 4. We then proceed to show applications of Grassmann calculus: the first one are uniform spanning trees in Sec. 5. Another statistical mechanics model, the Abelian sandpile is presented in Sec. 6 followed by supersymmetry in Sec. 7. We will finally give two further applications of Grassmann calculus in probability theory: reinforced processes in Sec. 8 and random matrix theory in Sec. 9.

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2 Mathematical preliminaries

2.1 Notation

We denote $\mathbb{N} := \{1, 2, \ldots\}$. We write $[n] := \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. We will use boldfonts to denote vectors (for example, $\boldsymbol{x} = (x_1, \ldots, x_n)$). The set of $n \times n$ square matrices with entries in a field $\mathbb{F}$ is called $M_n(\mathbb{F})$. The cardinality of a set A is denoted as $\#A$. Given $a \in \mathbb{C}$, we denote by $\Re(a)$ and $\Im(a)$ the real and imaginary part of a, respectively.

Matrices

Definition 2.1 (Positive-(semi)definite matrix). A matrix $A \in M_n(\mathbb{C})$ is positive-(semi)definite if

$$\Re(\overline{\boldsymbol{x}}^T A\boldsymbol{x}) > 0$$

(resp. ≥ 0) for any $\boldsymbol{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, where $\overline{\boldsymbol{x}}^T$ denotes the transpose complex conjugate of $\boldsymbol{x}$. A similar definition applies to $A \in M_n(\mathbb{R})$ by replacing $\overline{\boldsymbol{x}}^T$ with $\boldsymbol{x}^T$, that is, the transpose of $\boldsymbol{x}$.

Definition 2.2 (Hermitian matrix). A matrix $A \in M_n(\mathbb{C})$ is Hermitian if $A = \overline{A}^T$, that is, for all $1 \leq i, j \leq n$ one has $A(i, j) = \overline{A(j, i)}$.

Let $n \in \mathbb{N}$ and $\mathbf{X} = (X_i)_{i=1}^n$ be a vector of real-valued random variables, each of which has all finite moments.

Cumulants

For a reference on this paragraph see for example [40].

Definition 2.3 (Joint cumulants of random vectors). The cumulant generating function $K(\mathbf{t})$ of $\mathbf{X}$ for $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ is defined as

$$K(\mathbf{t}) := \log\left(\mathbb{E}\left[\exp\left(\mathbf{t} \cdot \mathbf{X}\right)\right]\right) = \sum_{\mathbf{m} \in \mathbb{N}^n} \kappa_{\mathbf{m}}(\mathbf{X}) \prod_{j=1}^{n} \frac{t_j^{m_j}}{m_j!},$$

where $\mathbf{t}\cdot\mathbf{X}$ denotes the scalar product in $\mathbb{R}^n$, $\mathbf{m}=(m_1,\,\dots,\,m_n)\in\mathbb{N}^n$ is a multi-index with n components, and

$$\kappa_{\mathbf{m}}(\mathbf{X}) = \frac{\partial^{|m|}}{\partial t_1^{m_1}\cdots\partial t_n^{m_n}}K(\mathbf{t})\Big|_{t_1=\ldots=t_n=0}\,,$$

being $|m| = m_1+\cdots+m_n$.

The joint cumulant of the components of $\mathbf{X}$ can be defined as a Taylor coefficient of $K(\mathbf{t})$ for $\mathbf{m}=(1,\,\dots,\,1)$; in other words

$$\kappa(\mathbf{X}) = \frac{\partial^{n}}{\partial t_1\cdots\partial t_n}K(\mathbf{t})\Big|_{t_1=\ldots=t_n=0}\,.$$

In particular, for any $A\subseteq[n]$, the joint cumulant $\kappa(X_i\,:\,i\in A)$ of $\mathbf{X}$ can be computed as

$$\kappa(X_i\,:\,i\in A) = \sum_{\pi\in\Pi(A)}(|\pi|-1)!(-1)^{|\pi|-1}\prod_{B\in\pi}\mathbb{E}\left[\prod_{i\in B}X_i\right]\,,$$

with $\Pi(A)$ the set of partitions of the set A and $|\pi|$ the cardinality of π. Let us remark that, by some straightforward combinatorics, it follows from the previous definition that

$$\mathbb{E}\left[\prod_{i\in A}X_i\right] = \sum_{\pi\in\Pi(A)}\prod_{B\in\pi}\kappa(X_i\,:\,i\in B)\,. \tag{2.1}$$

If $A=\{i,\,j\}$, $i,\,j\in[n]$, then the joint cumulant $\kappa(X_i,\,X_j)$ is the covariance between X_i and X_j. In addition, for a real-valued random variable X, one has the equality

$$\kappa(\underbrace{X,\,\dots,\,X}_{n\text{ times}}) = \kappa_n(X),\quad n\in\mathbb{N},$$

which we call the *n-th cumulant of X*.

2.2 Real and complex Gaussians

We will now give a brief recap of Gaussian integration in complex variables. We will restrict ourselves to centered vectors but a more general theory can of course be studied, as for example in the reference [37] where the results we are presenting are proved.

Let $A\in M_n(\mathbb{C})$ be an Hermitian positive-definite matrix with complex entries.

Definition 2.4 (Complex Gaussian random vector). A random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_n) \in \mathbb{C}^n$ has a complex Gaussian distribution with mean zero and inverse covariance matrix A if it has a density equal to

$$|\det(A)| \exp\left(-\left(\overline{\mathbf{Z}}, A\mathbf{Z}\right)\right)$$

with respect to the measure

$$\prod_{\alpha=1}^{n} \frac{\mathrm{d}\,\Re(Z_\alpha)\,\mathrm{d}\,\Im(Z_\alpha)}{\pi}$$

which is a product of Lebesgue measures on the real resp. imaginary part of Z_α. We thus have

$$A^{-1} = \mathrm{E}\left[\mathbf{Z}\overline{\mathbf{Z}}^{\mathbf{T}}\right],$$

in other words for all $1 \le i,\ j \le n$ it holds that

$$\begin{aligned}
&\mathrm{E}[Z_i Z_j] = 0\\
&\mathrm{E}[Z_i \overline{Z}_j] = A^{-1}(i,\, j) = \overline{A^{-1}(j,\, i)}\\
&\mathrm{E}[|Z_i|^2] = A^{-1}(i,\, i).
\end{aligned}$$

Equivalently, for all $1 \le i \le n$ one has $Z_i = X_i + \mathrm{i}Y_i$, where $(\mathbf{X}, \mathbf{Y})$ is a $2n$-dimensional Gaussian vector with

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}, \frac{1}{2}\begin{pmatrix}\Re A^{-1} & -\Im A^{-1}\\ \Im A^{-1} & \Re A^{-1}\end{pmatrix}\right).$$

2.3 Cumulants of Gaussian vectors

A characterization of real univariate Gaussian random variable is that all its cumulants of order at least three vanish. More can be said about cumulants of Gaussian vectors, and we will now proceed to proving what the cumulants of the vector of their squares look like. Recall now the notation $\mathfrak{S}_n$ for the set of permutations of n elements, and $\#\sigma$ to denote the number of cycles in the cyclic decomposition of $\sigma \in \mathfrak{S}_n$. Both results are taken from [36] (let us also mention that the original statement is for Gaussian processes, but we will need only the vector-valued version in these notes).

Proposition 2.5 (Cumulants of real Gaussian vector). *Let S be a finite set, $\#S = n \in \mathbb{N}$. If $\varphi = (\varphi_x)_{x\in S}$ is a centered real Gaussian vector with covariance matrix $C/2 \in M_n(\mathbb{R})$, then for all $k \ge 1$, $x_1, \ldots, x_k \in S$ one has*

$$\kappa\left(\varphi_{x_1}^2,\,\ldots,\,\varphi_{x_k}^2\right) = \frac{1}{2}\,\mathrm{cyp}(C)(x_1,\,\ldots,\,x_k) := \frac{1}{2}\sum_{\sigma\in\mathfrak{S}_n\colon \#\sigma=1}\prod_{\alpha=1}^{k} C(x_\alpha,\,x_{\sigma(\alpha)}).$$

Proof. The desired result follows from Malyshev's formula [40, Equation (3.2.8)] for generalized cumulants as a sum of products of ordinary cumulants. Note that the sum is restricted to cyclic permutations only. All Gaussian cumulants are zero except those of order two, so the result is a sum of products of covariances in the form $C(x_{i_1},\,x_{j_1})\cdots C(x_{i_k},\,x_{j_k})$. Since each value $1,...,k$ occurs once as a first index and once as a second index, $(j_1,...,j_k)$ is a permutation of $(i_1,...,i_k)$. For each cyclic permutation, there are 2^{k-1} distinct partitions of the $2k$ indices that satisfy the connectivity condition, all giving rise to the same contribution $C(x_1,\,x_{\sigma(1)})\cdots C(x_k,\,x_{\sigma(k)})/2^k$. As a consequence the joint cumulant is one-half the sum of the cyclic products. □

Proposition 2.6 (Cumulants of complex Gaussian vector). *Let S be as in Proposition 2.5. If $\mathbf{Z} = (Z_x)_{x\in S}$ is a centered complex Gaussian process with covariance matrix $C \in M_n(\mathbb{C})$, then for all $k \geq 1$, $x_1,\,\ldots,\,x_k \in S$ one has*

$$\kappa\left(|Z_{x_1}|^2,\,\ldots,\,|Z_{x_k}|^2\right) = \mathrm{cyp}(C)(x_1,\,\ldots,\,x_k) = \sum_{\sigma\in\mathfrak{S}_n\colon \#\sigma=1}\prod_{\alpha=1}^{k} C(x_\alpha,\,x_{\sigma(\alpha)}).$$

Proof. The desired formula follows after arguing as in the proof of Proposition 2.5 and using the covariance matrix C instead of $C/2$. □

3 Grassmannian calculus

For this Section we are indebted to [1].

Let $\mathbb{F}$ be a field which contains the field $\mathbb{Q}$ of rational numbers.

Definition 3.1 (Associative algebra). An associative algebra $\mathcal{A}$ over a field $\mathbb{F}$ is a vector space over $\mathbb{F}$ with an operation $\wedge$ satisfying the following conditions: for all $x,\,y,\,z \in \mathcal{A}$, $a \in \mathbb{F}$ one has

Left distributivity $x \wedge (y+z) = x\wedge y + x \wedge z$.

Right distributivity $(x+y)\wedge z = x \wedge y + y \wedge z$.

Compatibility $a(x\wedge y) = (ax)\wedge y = x \wedge (ay)$.

Associativity $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

Example 3.2. An example of an associative algebra $(\mathcal{A}, \wedge)$ is given by the $n \times n$ matrices $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ with the usual matrix multiplication.

For the rest of the notes, we will omit the symbol $\wedge$ unless there is a risk of confusion with other operations. In our applications, $\mathbb{F}$ will typically be the set of real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, and we will not work with any other field of numbers.

Definition 3.3 (Generators of $\mathcal{A}$). The set $\{\xi_i : i \in I\}$ forms a set of generators for $\mathcal{A}$ if, for all $x \in \mathcal{A}$, there exists $n \in \mathbb{N}$ such that

$$x = \mathrm{pol}_{\mathbb{F}}(\xi_{i_1}, \ldots, \xi_{i_n}), \quad i_j \in I, 1 \le j \le n.$$

In other words every element of the algebra can be written as a finite polynomial in the $\xi's$.

We are now ready to define a Grassmann algebra.

Definition 3.4. A Grassmann algebra $\mathcal{G}_{\mathcal{A}}$ is an associative algebra over $\mathbb{R}$ (or $\mathbb{C}$) generated by a set of generators $\{\xi_i : i \in I\}$ that satisfy the following anticommutation relations:

$$\xi_i \xi_j = -\xi_j \xi_i. \tag{3.1}$$

Corollary 3.5 (Nilpotency of the ξ's). *Since the base field of the algebra contains $\mathbb{Q}$, and in particular contains $1/2$, it follows that*

$$\xi_i^2 = 0, \quad i \in I.$$

Example 3.6. An example of a Grassmannian algebra is the algebra of differential forms on an n-dimensional manifold with the operation $\wedge$ defined as the standard wedge product. To show this, let V be a n-dimensional manifold with coordinates $x_1, \ldots, x_n$. A differential form on V can then be written as

$$F = F_0 + \cdots + F_n,$$

where $F_0 \in C^\infty(V)$ is a 0-form, i.e., an ordinary function from V to $\mathbb{R}$, and F_i is an i-form for any $i \in [n]$, i.e.,

$$F_i(x) = f(x) \,\mathrm{d}\, x_{i_1} \wedge \cdots \wedge \mathrm{d}\, x_{i_n},$$

with $x \in V$ and $f \in C^\infty(V)$. The form F_p is the degree-p part of F and a form F has degree p if $F = F_p$. The set $\{\mathrm{d}\, x_i : i = 1, ..., n\}$ forms a set of generators for this algebra. To conclude the example and show that this is indeed a Grassmann algebra, note that $\mathrm{d}\, x_i \wedge \mathrm{d}\, x_j = -\,\mathrm{d}\, x_j \wedge \mathrm{d}\, x_i$ for any $i, j \in [n]$.

Corollary 3.5 entails an important property: all elements of a Grassmann algebra are affine polynomials in the ξ's, as we shall see now.

Definition 3.7 (Even and odd, parity). A non-zero monomial $f \in \mathcal{G}_{\mathcal{A}}$ is called even if it contains an even number of generators, and it is called odd otherwise. We define the parity of such a monomial as

$$p(f) := \begin{cases} 0 & \text{if } f \text{ is even,} \\ 1 & \text{if } f \text{ is odd.} \end{cases}$$

We can extend the definition to even resp. polynomials whose monomials are all even resp. odd.

Lemma 3.8 (Graded commutation relations). *For all non-zero monomials f, $g \in \mathcal{G}_{\mathcal{A}}$ one has*

$$fg = (-1)^{p(f)p(g)} gf.$$

Proof. To swap the order of the terms fg, one has to bring each generator appearing in g to the front of f, yielding a $(-1)^{p(g)}$ for each generator. In addition, each generator in g has to perform a number of swaps equal to the number of factors of f, gaining an additional $(-1)^{p(f)}$. □

From Lemma 3.8 we readily obtain two important consequences. The first is that even polynomials commute, while odd ones anticommute. This gives rise to the nomenclature "bosons" for even terms in the algebra and "fermions" for the odd ones. Secondly, it allows us to prove immediately Pauli exclusion principle, which we phrase as follows.

Proposition 3.9 (Pauli exclusion principle). *If f is an odd element of $\mathcal{G}_{\mathcal{A}}$ then $f^2 = 0$.*

As an example, one can compute

$$(\xi_1 + \xi_2 + \xi_1\xi_2\xi_3)^2 = \xi_1\xi_2 + \xi_2\xi_1 = 0$$

noticing that all other terms yield a factor of ξ_i^2 for some $1 \leq i \leq 3$. Even though there are several terms whose square is zero (indeed, one can show that all polynomials whose monomials share at least one generator enjoy this property) Pauli exclusion principle becomes significant, from a physical viewpoint, when f is homogenous of degree 1.

Example 3.10. Not all polynomials squared are zero. Indeed,

$$(\xi_1\xi_2 + \xi_3\xi_4)^2 = 2\prod_{i=1}^{4} \xi_i \neq 0,$$

where we have used $(\xi_1\xi_2)^2 = (\xi_3\xi_4)^2 = 0$.

3.1 Functions of Grassmann variables

Definition 3.11. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function given by $f(x) = \sum_{k=0}^{\infty} a_k x^k$, and let $F \in \mathcal{G}_{\mathcal{A}}$ be an element of the Grassmann algebra. We define the composition of an analytic function with an element of the Grassmann algebra, $f(F)$, by

$$f(F) := \sum_{k=0}^{\infty} a_k F^k.$$

As a consequence of this definition, and the nilpotency of the generators, one can observe for example that for any generator ξ

$$\exp(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{k!} = 1 + \xi.$$

Proposition 3.12. *If $f, g \in \mathcal{G}_{\mathcal{A}}$ are even polynomials, then*

$$\exp(f + g) = \exp(f)\exp(g). \tag{3.2}$$

Proof. The proof follows because f and g are even polynomials, so when variables are commutative the binomial formula still holds. □

Remark 3.13. Note that Proposition 3.12 does not necessarily hold for odd polynomials. As an example, since $(\xi_1 + \xi_2)^2 = \xi_1\xi_2 + \xi_2\xi_1 = 0$, we have

$$\exp(\xi_1 + \xi_2) = 1 + \xi_1 + \xi_2,$$

whereas

$$\exp(\xi_1)\exp(\xi_2) = 1 + \xi_1 + \xi_2 + \xi_1\xi_2.$$

3.2 Differentiation and integration

From now on, we will simplify the structure of the Grassmann algebra even more by giving it only a finite number of generators $\{\xi_i : 1 \le i \le n\}$.

Definition 3.14 (Right derivative). The right derivative is the linear map $\mathcal{G}_{\mathcal{A}} \to \mathcal{G}_{\mathcal{A}}$ which acts on monomials $\xi_{i_1} \cdots \xi_{i_p}$, $1 \le i_1 < \ldots < i_p \le n$, as

$$\begin{cases} \frac{\partial}{\partial_{\xi_j}} \xi_{i_1} \cdots \xi_{i_p} = (-1)^{\alpha-1} \xi_{i_1} \cdots \xi_{i_{\alpha-1}} \xi_{i_{\alpha+1}} \cdots \xi_{i_p} & \text{if there is } 1 \le \alpha \le p \\ & \text{such that } i_\alpha = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

In other words, if $x = x_1 + \xi_i x_2$ where $x_1, x_2 \in \mathcal{G}_{\mathcal{A}}$ are Grassmann terms in which ξ_i does not occur, one has

$$\frac{\partial}{\partial \xi_i} x = x_2.$$

It follows that the derivative is also nilpotent, in the sense that

$$\left(\frac{\partial}{\partial \xi_i}\right)^2 = 0$$

for all i. Moreover, the derivative satisfies the *Leibniz rule*

$$\frac{\partial}{\partial \xi_i}(fg) = \left(\frac{\partial}{\partial \xi_i} f\right) g + (Pf)\left(\frac{\partial}{\partial \xi_i} g\right). \tag{3.4}$$

where $f, g \in \mathcal{G}_{\mathcal{A}}$ and P is the *parity operator* [51, Section 2.2] defined by

$$P(\xi) = -\xi.$$

This operator essentially multiplies a monomial of k variables by $(-1)^k$ (whence Lemma 3.8 can be rephrased as $fg = P(gf)$).

Proposition 3.15 (Taylor expansion for Grassmann algebras). *Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function with expansion $f(x) = \sum_{m \geq 0} a_m x^m$, $a_m \in \mathbb{R}$. Let $F, G \in \mathcal{G}_{\mathcal{A}}$ such that F is generated by $\xi_1, ..., \xi_n$ and G is an odd polynomial. Then*

$$f(F+G) = \sum_{k=0}^{N} \delta_G^k(f(F)), \tag{3.5}$$

where N is the nilpotency index of G, i.e., the biggest number such that $G^N \neq 0$, and

$$\delta_G^k(f(F)) = \sum_{m \geq k} a_m \sum_{\substack{j_0 + \dots + j_k = m-k \\ j_0, \dots, j_k \geq 0}} F^{j_0} G F^{j_1} \cdots G F^{j_k},$$

with k representing the number of occurrences of G.

Proof. First, note that we can write

$$(F+G)^m = \sum_{k=0}^{m} \sum_{\substack{j_0 + \dots + j_k = m-k \\ j_0, \dots, j_k \geq 0}} F^{j_0} G F^{j_1} \cdots G F^{j_k}.$$

This formula can be easily checked by an induction argument. Define the operator $\delta_G : \mathcal{G}_{\mathcal{A}} \to \mathcal{G}_{\mathcal{A}}$ by

$$\delta_G(F^m) = \sum_{j=0}^{m-1} F^j G F^{m-1-j}, \quad m \geq 1,$$

$\delta_G(\alpha F_1 + \beta F_2) = \alpha\delta_G(F_1) + \beta\delta_G(F_2)$ for all $F_1,\, F_2 \in \mathcal{G}_{\mathcal{A}}$ and $\alpha, \beta \in R$, and $\delta_G(1) = 0$. This operator essentially replaces the derivative of $F^m G$ with respect to F in this non-commutative setup. By using the Leibniz rule (3.4) it is possible to show that the k-th composition can be written as

$$\delta_G^k(F^m) = \sum_{\substack{j_0+\ldots+j_k=m-k \\ j_0,\ldots,j_k\geq 0}} F^{j_0} G F^{j_1} \cdots G F^{j_k}, \quad m \geq k,$$

with the notational convention that

$$\delta_G^0(F^m) = F^m.$$

Thus, by using the nilpotency of G and the fact that $\delta_G^k(F^m) = 0$ for $m < k$, we can write

$$\begin{aligned} f(F+G) &= \sum_{m\geq 0} a_m \sum_{k=0}^{m} \delta_G^k(F^m) = \sum_{k\geq 0}\sum_{m\geq k} a_m \delta_G^k(F^m) \\ &= \sum_{k\geq 0} \delta_G^k \left(\sum_{m\geq k} a_m F^m \right) = \sum_{k=0}^{N} \delta_G^k(f(F)). \end{aligned}$$

□

Example 3.16. As an example, for $f(\cdot) = \exp(\cdot)$, $F = \xi_1\xi_2$, $G = \xi_3$ one obtains from (3.5)

$$\exp(\xi_1\xi_2 + \xi_3) = \exp(\xi_1\xi_2) + \exp(\xi_1\xi_2)\xi_3 = 1 + \xi_1\xi_2 + (1 + \xi_1\xi_2)\xi_3.$$

This is an instance of the more general fact that, if F and G commute and G is a polynomial of degree at most one, (3.5) simplifies to

$$f(F+G) = f(F) + f'(F)G.$$

Note that the oddness of G is crucial for (3.5) to make sense. Indeed, in that case this substitution can be expressed in terms of a *finite* Taylor series.

Definition 3.17 (Grassmann-Berezin integration). The Grassmann-Berezin integral is defined as

$$\int \mathcal{D}(\boldsymbol{\xi})F = \int \mathrm{d}\,\xi_1\,\mathrm{d}\,\xi_2 \cdots \mathrm{d}\,\xi_n F := \partial_{\xi_1}\partial_{\xi_2}\cdots\partial_{\xi_n} F, \quad F \in \mathcal{G}_{\mathcal{A}}.$$

Surprisingly the integral is defined in the same way as the derivative (actually, the two coincide!). Even though surprising, this definition retains many useful properties of the integral, for example linearity, or the fact that an integral does not depend on the variables which are integrated out. It is however crucial to keep track of the order of integration because of (3.3). For example

$$\int \mathrm{d}\,\xi_1\,\mathrm{d}\,\xi_2(\xi_2\xi_1) = 1$$

but

$$\int \mathrm{d}\,\xi_2\,\mathrm{d}\,\xi_1(\xi_2\xi_1) = \frac{\partial}{\partial\xi_2}\left(\frac{\partial}{\partial\xi_1}(-\xi_1\xi_2)\right) = -\frac{\partial}{\partial\xi_2}\xi_2 = -1.$$

For this reason, to fix notation we will now start working with a special Grassmann algebra: the one in which $n = 2m$ and the set of generators is divided into two sets of variables $\{\xi_i,\,\overline{\xi}_i\,:\,1 \le i \le m\}$. The bar is reminiscent of complex conjugation, but we consider it only as a special notation to distinguish the two sets of variables. In this setup, we choose

$$\int \mathcal{D}(\boldsymbol{\xi},\,\overline{\boldsymbol{\xi}})F := \int \left(\prod_{i=1}^{n} \mathrm{d}\,\xi_i\,\mathrm{d}\,\overline{\xi}_i\right) F, \quad F \in \mathcal{G}_{\mathcal{A}}. \tag{3.6}$$

It is important to observe that integration satisfies

$$\int \mathrm{d}\,\xi\,\mathrm{d}\,\overline{\xi}\,\left(a_1 + a_2\xi + a_3\overline{\xi} + a_4\overline{\xi}\xi\right) = a_4, \quad a_i \in \mathbb{R},\, 1 \le i \le 4 \tag{3.7}$$

where ξ, $\overline{\xi}$ are arbitrary generators. In other words, only polynomial expressions in which all generator appear give a non-zero contribution to integrals.

There are many properties of the Grassmann integral that work exactly in the same way as for standard integrals: invariance under translation, change-of-variables formulas and Fubini's theorem. We will prove them now for completeness, beginning with translation invariance. The fact that it holds is intuitively clear because the integral is "morally" a derivative, and thus does not see shifts. Let's see the proof more precisely.

Proposition 3.18 (Invariance under translation for Grassmann-Berezin integration). *Let $I = \{i_1, ..., i_p\}$ be an ordered sequence of indices in $[n]$ and let $\chi_1, ..., \chi_n$ be odd elements of $\mathcal{G}_{\mathcal{A}}$ satisfying $\chi_j = 0$ for any $j \notin I$. Then*

$$\int \mathrm{d}\,\xi_{i_p} \cdots \mathrm{d}\,\xi_{i_1} f(\xi + \chi) = \int \mathrm{d}\,\xi_{i_p} \cdots \mathrm{d}\,\xi_{i_1} f(\xi), \tag{3.8}$$

where $f(\xi + \chi)$ denotes the substitution defined in (3.5).

Proof. To prove the statement, it is enough to show that (3.8) can be written as

$$\frac{\partial}{\partial\xi_{i_p}} \cdots \frac{\partial}{\partial\xi_{i_1}} f(\xi + \chi) = \frac{\partial}{\partial\xi_{i_p}} \cdots \frac{\partial}{\partial\xi_{i_1}} f(\xi).$$

To prove this, it suffices to consider the cases in which $f(\xi) = \xi_{i_1} \cdots \xi_{i_p}$ for some sequence of indices $\{j_1, ..., j_m\} \subseteq [n]$. For any $i \in I$ and $j \in [n]$ we have that

$$\frac{\partial}{\partial \xi_i}(\xi_j + \chi_j) = \frac{\partial}{\partial \xi_i}\xi_j = \delta_{ij}.$$

Using this relation together with the Leibniz rule, we note that $\frac{\partial}{\partial \xi_i}(\xi_{j_1} + \chi_{j_1}) \cdots (\xi_{j_m} + \chi_{j_m})$ equals the same object in which χ_i has been replaced by zero. By iterating this for $\frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_p}}$, we set $\chi_i = 0$ for any $i \in I$. However, by hypothesis these are the only nonzero χ_i. This concludes the proof. □

Changing variables in Grassmann-Berezin integration is, unlike translation invariance, a rule that defies its standard counterpart. The rule is as follows.

Proposition 3.19 (Linear change of variables for Grassmann-Berezin integration). *Let $A \in M_n(\mathbb{R})$ and let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Define new Grassmannian variables by $\chi_i = \sum_{j=1}^{n} A_{ij}\xi_j$, $i = 1, ..., n$. Then*

$$\int \mathrm{d}\,\xi_n \cdots \mathrm{d}\,\xi_1 f(\xi_1, ..., \xi_n) = \det(A) \int \mathrm{d}\,\chi_n \cdots \mathrm{d}\,\chi_1 f(\chi_1, ..., \chi_n). \tag{3.9}$$

Proof. Let $\mathfrak{S}_n$ denote the set of permutations of n elements and $\mathrm{sgn}(\sigma)$ the sign of the permutation σ. The statement follows after noting that

$$\begin{aligned}\int \mathrm{d}\,\xi_n \cdots \mathrm{d}\,\xi_1 \chi_1 \cdots \chi_n &= \int \mathrm{d}\,\xi_n \cdots \mathrm{d}\,\xi_1 \sum_{j_1=1}^{n} A_{1j_1}\xi_{j_1} \cdots \sum_{j_n=1}^{n} A_{nj_n}\xi_{j_n} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)} \\ &= \det(A).\end{aligned}$$

□

Note that for ordinary multivariate integrals the factor $\det(A)$ should be on the other side compared to (3.9): if $\mathbf{u} = A\mathbf{x}$ then

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n = \int_{\mathbb{R}^n} f\left(A^{-1}\mathbf{u}\right) \frac{1}{|\det A|}\,\mathrm{d}\,\mathbf{u}.$$

Finally we can state the analog of Fubini's theorem, which essentially mimics its standard counterpart up to a possible sign change.

Proposition 3.20 (Fubini theorem for Grassmann-Berezin integration). *Let $I = \{i_1, ..., i_p\}$ be an ordered sequence of indices in $[n]$,* *and let* $I^c = \{j_1, ..., j_{n-p}\}$. *Then, for any elements* $f \in \mathcal{G}_{\mathcal{A}}$ *generated by* $\xi_{i_1}, ..., \xi_{i_p}$ *and* $g \in \mathcal{G}_{\mathcal{A}}$ *generated by* $\xi_{j_1}, ..., \xi_{j_{n-p}}$, *we have*

$$\int \mathrm{d}\,\xi_{i_p} \cdots \mathrm{d}\,\xi_{i_1}\, \mathrm{d}\,\xi_{j_{n-p}} \cdots \mathrm{d}\,\xi_{j_1} f g = (-1)^{p(n-p)} \left(\int \mathrm{d}\,\xi_{i_p} \cdots \mathrm{d}\,\xi_{i_1} f \right) \left(\int \mathrm{d}\,\xi_{j_{n-p}} \cdots \mathrm{d}\,\xi_{j_1} g \right).$$

Proof. Expanding f and g in monomials, we see that the unique terms that contribute to the integrals are $\xi_{i_1} \cdots \xi_{i_p}$ and $\xi_{j_1} \cdots \xi_{j_{n-p}}$ in f and g, respectively. We can then assume without loss of generality that $f = \xi_{i_1} \cdots \xi_{i_p}$ and $g = \xi_{j_1} \cdots \xi_{j_{n-p}}$. By using the derivative rule (3.3) and the Leibniz rule (3.4), we get

$$\int \mathrm{d}\,\xi_{j_{n-p}} \cdots \mathrm{d}\,\xi_{j_1} f g = (-1)^{p(n-p)} f \left(\int \mathrm{d}\,\xi_{j_{n-p}} \cdots \mathrm{d}\,\xi_{j_1} g \right).$$

The result follow after integrating both sides with respect to $\mathrm{d}\,\xi_{i_p} \cdots \mathrm{d}\,\xi_{i_1}$. □

4 Grassmannian calculus and Gaussian random variables

This Subsection is devoted to studying "Gaussian integrals" in the Grassmannian setting. In particular, let $A \in M_n(\mathbb{R})$. By using the notation $(\bar{\boldsymbol{\xi}}, A\boldsymbol{\xi}) := \sum_{i,\,j=1}^n \bar{\xi}_i A(i,\,j)\xi_j$ we wish to compute the analog of a Gaussian measure, that is, objects of the form

$$\int \mathcal{D}(\boldsymbol{\xi},\,\bar{\boldsymbol{\xi}}) \mathrm{e}^{(\bar{\boldsymbol{\xi}},\,A\boldsymbol{\xi})}.$$

We will prove that

Proposition 4.1.

$$\int \mathcal{D}(\boldsymbol{\xi},\,\bar{\boldsymbol{\xi}}) \mathrm{e}^{(\bar{\boldsymbol{\xi}},\,A\boldsymbol{\xi})} = \int \mathcal{D}(\boldsymbol{\xi},\,\bar{\boldsymbol{\xi}}) \mathrm{e}^{-(\boldsymbol{\xi},\,A\bar{\boldsymbol{\xi}})} = \det(A).$$

Before we enter into the proof, let us note that there are no assumptions needed on A, unlike the real case in which

$$\int_{\mathbb{R}^n} \prod_{i=1}^n \frac{\mathrm{d}\,x_i}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}(\boldsymbol{x},\,A\boldsymbol{x})} = (\det A)^{-\frac{n}{2}}$$

requires A to be symmetric and positive-definite.

Proof of Proposition 4.1. Since the polynomial $(\overline{\boldsymbol{\xi}},\, A\boldsymbol{\xi})$ is even, by Proposition 3.12

$$\exp\left(\overline{\boldsymbol{\xi}},\, A\boldsymbol{\xi}\right) = \prod_{i=1}^{n} \exp\left(\overline{\xi}_i \sum_{j=1}^{n} A_{ij}\xi_j\right) = \prod_{i=1}^{n}\left(1 + \overline{\xi}_i \sum_{j=1}^{n} A_{ij}\xi_j\right). \tag{4.1}$$

When expanding (4.1), since we are integrating with respect to $\mathcal{D}(\boldsymbol{\xi},\, \overline{\boldsymbol{\xi}})$, only the term containing $\overline{\xi}_i$ for all $i = 1, ..., n$ survives integration, which gives rise to

$$\prod_{i=1}^{n} \overline{\xi}_i \left(\sum_{j=1}^{n} A_{ij}\xi_j\right) = \prod_{i=1}^{n} \overline{\xi}_i \left(\sum_{j_i=1}^{n} A_{ij_i}\xi_{j_i}\right).$$

Similarly, the terms that give non-vanishing contribution after integrating must contain $\xi_1 \cdots \xi_n$ up to permutation. They have the form

$$\sum_{\sigma\in\mathfrak{S}_n} A_{1j_{\sigma(1)}} \cdots A_{nj_{\sigma(n)}} \overline{\xi}_n \xi_{j_{\sigma(n)}} \cdots \overline{\xi}_1 \xi_{j_{\sigma(1)}}. \tag{4.2}$$

Putting $\overline{\xi}_n \xi_{j_{\sigma(n)}} \cdots \overline{\xi}_1 \xi_{j_{\sigma(1)}}$ into a standard order, i.e., $\overline{\xi}_n \xi_n \cdots \overline{\xi}_1 \xi_1$, yields the sign of the permutation σ, so that (4.2) becomes

$$\sum_{\sigma\in\mathfrak{S}_n} \prod_{i=1}^{n} A_{ij_{\sigma(i)}} \mathrm{sgn}(\sigma) \overline{\xi}_n \xi_n \cdots \overline{\xi}_1 \xi_1.$$

Since

$$\int \mathcal{D}(\boldsymbol{\xi},\, \overline{\boldsymbol{\xi}}) \overline{\xi}_n \xi_n \cdots \overline{\xi}_1 \xi_1 = 1,$$

we get the claim. □

Next we state Wick's theorem which, like in the real case, allows one to compute multipoint functions for a Gaussian vector. In the statement, we are thinking of $\boldsymbol{\xi}$ resp. $\overline{\boldsymbol{\xi}}$ as $(n \times 1)$-dimensional vectors. The version of the Theorem we need is the following, but more can be found in [10, Theorem A.16], together with their proofs. We will use, for a matrix $A \in M_n(\mathbb{R})$, the notation

$$A_{i^c,\, j^c}, \quad i,\, j \in [n]$$

for the submatrix obtained removing row i and column j from A.

Theorem 4.2 (Wick's theorem). *Let A be an $n \times n$, B an $r \times n$ and C an $n \times r$ matrix respectively with coefficients in $\mathbb{R}$. For any sequences of indices $I = \{i_1, \ldots, i_r\}$ and $J = \{j_1, \ldots, j_r\}$ in $[n]$ of the same length r, if the matrix A is invertible we have that*

1. *if we make the choice of ordering* $i_1 < \cdots < i_r$ *and* $j_1 < \cdots < j_r$*, then*

$$\int \mathcal{D}(\boldsymbol{\xi}, \overline{\boldsymbol{\xi}}) \prod_{\alpha=1}^{r} \overline{\xi}_{i_\alpha} \xi_{j_\alpha} \exp\left((\overline{\boldsymbol{\xi}}, A\boldsymbol{\xi})\right) = (-1)^{\sum_\alpha i_\alpha + \sum_\alpha j_\alpha} \det\left(A_{i^c, j^c}\right).$$

2. $$\int \mathcal{D}(\boldsymbol{\xi}, \overline{\boldsymbol{\xi}}) \prod_{\alpha=1}^{r} \left(\overline{\boldsymbol{\xi}}^T C\right)_\alpha (B\boldsymbol{\xi})_\alpha \exp\left((\overline{\boldsymbol{\xi}}, A\boldsymbol{\xi})\right) = \det(A) \det\left(BA^{-1}C\right).$$

If $\#I \neq \#J$*, the integral is* 0.

5 Grassmannian calculus and uniform spanning trees

At this stage we would like to show one main area of application for Grassmannian variables: studying spanning trees. Indeed we will show that Grassmann variables can completely describe the edge probabilities of so-called uniform spanning tree, whose definition we will recall now.

For the rest of this Section, we will let $G = (V, E, w)$ be a finite connected graph with vertex set V and edge set E. We will assume that each edge $e = (u, v)$, $u, v \in V$, has an edge weight $w_e = w_{uv} > 0$. In particular, edges are unoriented.

Definition 5.1 (Spanning tree). A finite subgraph $T \subseteq G$ is called a spanning tree of G if

- it has no cycles, i. e. there is no non-empty subset of the edge set of H that forms a path[1] such that the first node of the path corresponds to the last;
- it is connected, and
- it is spanning, i. e. every $v \in V$ has at least one edge of H incident to it.

It is easy to see that any connected and finite graph possesses a finite number T_G of spanning trees (and that this number is non-zero). Therefore, it is legit to define a uniform probability on the set of spanning trees.

Definition 5.2 (UST). The uniform spanning tree is the probability measure on the set of all spanning trees of G with probability mass function

$$\mathbf{P}(t) = \frac{1}{T_G}$$

for t a spanning tree of G.

[1] A graph path is a sequence $\{v_1, v_2, \ldots, v_k\}$ of distinct vertices such that (x_i, x_{i+1}) are graph edges for all $1 \leq i \leq k-1$.

In order to understand the UST measure, one has to get hold of the constant T_G, more specifically needs to count the number of spanning trees. To this end, we will need the following object: the Laplacian matrix, which we have briefly encountered on the square lattice in the Introduction.

Definition 5.3 (Laplacian matrix). The Laplacian (matrix) $\Delta = \Delta(G)$ is the matrix indexed over V defined as

$$\Delta(u,\, v) := \begin{cases} w_{uv} & \text{if } e = (u,\, v) \in E, \\ -\deg_G(u) := -\sum_{v\in V} w_{uv} & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the definition that 0 is an eigenvalue for Δ with eigenvector

$$\underbrace{(1,\, 1,\, \ldots,\, 1)}_{|V| \text{ times}}.$$

One can in fact prove [12, Chapter 1] that all eigenvalues of $-\Delta$ are real and non-negative, and that the eigenvalue zero has multiplicity one. We are now ready to prove Kirchhoff's theorem, or the matrix-tree theorem [30]. This Theorem allows us to count the number of spanning trees as a determinant of (a submatrix of) the Laplacian. We will give a "Grassmannian proof" for it, which is due to [18]. A more extended version of this Theorem can be found in [1].

Theorem 5.4 (Matrix-tree theorem). *Choose an arbitrary $o \in V$. Then*

$$T_G = \det(-\Delta_{o^c,\, o^c}).$$

Proof. Let us call $-\Delta_{o^c,\, o^c} := O$. By Theorem 4.2 1. we have that

$$\int \mathcal{D}(\boldsymbol{\xi},\, \overline{\boldsymbol{\xi}})\overline{\xi}_o \xi_o \mathrm{e}^{-(\overline{\boldsymbol{\xi}},\, \Delta\boldsymbol{\xi})} = \det(O). \tag{5.1}$$

Therefore, we need to show that the left-hand side above counts the number of spanning trees of G. We expand the exponential as follows: fixing u, and taking into account the nilpotency of the generators, we have

$$\begin{aligned} \exp\left(-\sum_{v=1}^{\#V} \overline{\xi}_u \Delta(u,\, v)\xi_v\right) &= 1 + \sum_{v=1}^{\#V} \overline{\xi}_u \Delta(u,\, v)\xi_v \\ &= 1 - \sum_{v\neq u} \overline{\xi}_u \xi_v w_{uv} + \overline{\xi}_u \xi_u \left(\sum_{v\neq u} w_{uv}\right). \end{aligned}$$

By symmetry, a term like w_{uv} can appear together with a pair of variables of the same index ($\overline{\xi}_u\xi_u$ or $\overline{\xi}_v\xi_v$) or with a pair of variables with different indices ($\overline{\xi}_u\xi_v$ or $\overline{\xi}_v\xi_u$). We will now count graphically each appearance, encoding it with a different type of arrow (illustrated in Figure 1).

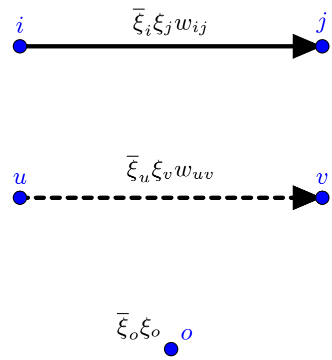

Figure 1: Arrow encoding for the proof of Theorem 5.4.

If w_{uv} appears in the first instance, we will draw a solid arrow going from u to v. If w_{uv} appears in the second instance, we will draw a dashed arrow from u to v. For $\bar{\xi}_o \xi_o$, which appears in the integrand as well, we will draw no in- or outgoing arrow from o. With this encoding, every polynomial in the left-hand side of (5.1) that has a pair of variables at the same site will have a corresponding outgoing arrow from there, while for a pair of fields with one "real" variable on one vertex and one "conjugate" variable at another vertex the arrow will be dashed (directed from the "conjugate" site to the "real" one).

By (3.7) one must have exactly one conjugate variable per site. If a site is visited by the tail of a solid arrow, like node k in Figure 2, or it is o, the pair of variables is already complete, so this vertex can only be visited by an arbitrary number of solid arrows, that bring no variable to the head. If a site is visited by the tail of a dashed arrow, like node j in Figure 2, to complete the pair of variables it must also be visited by a dashed arrow-head, and by arbitrarily many solid arrow-heads. We therefore deduce two statements:

1. dashed arrows come in closed, self-avoiding circuits (since for each outgoing arrow there is one and only one incoming arrow);

2. solid arrows create a subgraph of G whose connected components have the following property. For each connected component, there must be a "root structure" such that, for each vertex in the component, either the vertex lies in the root structure or there is a single path which connects it to the root, touching it only at the last vertex. This path is oriented towards the root structure, in other words it is a tree rooted at the "root structure".

We need know to understand what kind of root structures we can have. The possibilities are:

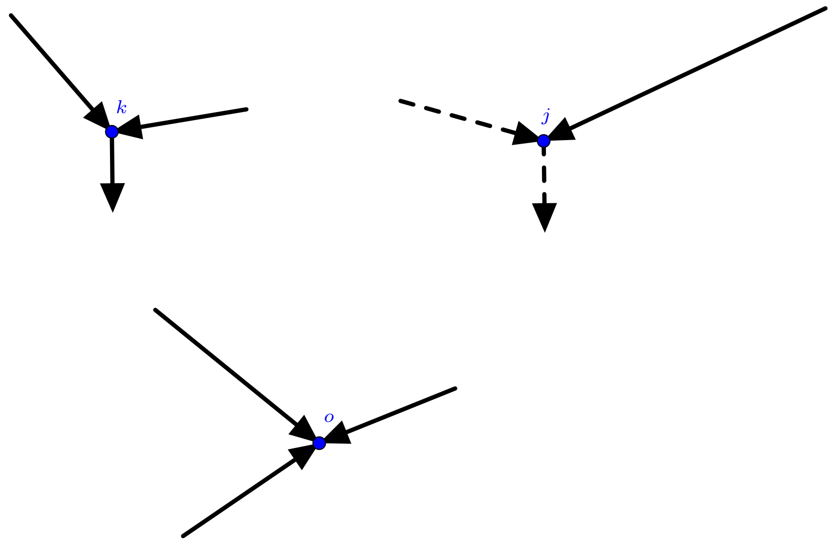

Figure 2: Skematic illustration of arrow configurations for Theorem 5.4.

(i) the vertex o;

(ii) a closed oriented circuit of dashed arrows;

(iii) a closed oriented circuit of solid arrows.

We now show that a polynomial appearing in the integral of (5.1) giving rise to a root structure of type (ii) cancels exactly the integral contribution of a polynomial with a root structure of type (iii). Indeed, if there exists an oriented cycle of dashed arrows on ℓ vertices, which without loss of generality we call $\{1, 2, \dots, \ell\}$, then the associated polynomial must be of the form

$$\begin{aligned} & \left(-\overline{\xi}_1\xi_2 w_{12}\right)\left(-\overline{\xi}_2\xi_3 w_{23}\right)\cdots\left(-\overline{\xi}_\ell\xi_1 w_{1\ell}\right) \\ \overset{(3.1)}{=} & -\overline{\xi}_1 w_{12}\left(\overline{\xi}_2\xi_2 w_{23}\right)\cdots\left(\overline{\xi}_\ell\xi_\ell w_{1\ell}\right)\xi_1 \\ = & -\left(\overline{\xi}_1\xi_1 w_{12}\right)\left(\overline{\xi}_2\xi_2 w_{23}\right)\cdots\left(\overline{\xi}_\ell\xi_\ell w_{1\ell}\right), \end{aligned} \tag{5.2}$$

where in the last equality we have used that the monomials $\overline{\xi}_i\xi_i$ are even. The proof is finished once one notices that the right-hand side of (5.2) is of type (iii). □

Thanks to Proposition 4.1, we can define the Gaussian integration on $\mathcal{G}_{\mathcal{A}}$ via its moments as

$$\langle F \rangle := (\det(O))^{-1} \int \mathcal{D}(\boldsymbol{\xi},\, \overline{\boldsymbol{\xi}}) \mathrm{e}^{(\overline{\boldsymbol{\xi}},\, O\boldsymbol{\xi})} F$$

for all $F \in \mathcal{G}_{\mathcal{A}}$ and $\{\xi_i, \overline{\xi}_i\}_{i \in V}$. In view of Theorem 5.4, this means that

$$\langle 1 \rangle = 1$$

which intuitively justifies the idea of "Gaussian probability measure" on the Grassmann algebra.

In what follows we find a deeper relation between the $\mathbf{P}$-probability to have some fixed edges in a spanning tree and the expectation $\langle \cdot \rangle$ of some fermionic polynomials. We will use throughout this Section the notation $O = -\Delta_{o^c,\, o^c}$, and use O^{-1} to denote its inverse, which exists by Theorem 5.4.

Fix an arbitrary orientation of the edges E. We will denote an oriented edge as $\vec{xy}$. Even if we have never used oriented edges before, one can prove that this choice does not matter towards the next result (and we will indeed give a "fermionic" explanation for this). We let

$$T(\vec{xy}, \vec{uv}) = O^{-1}(x,\, u) - O^{-1}(y,\, u) - O^{-1}(x,\, v) + O^{-1}(y,\, v), \tag{5.3}$$

be the transfer-impedance matrix (see [9, Section 4]). A classical result of [9, Theorem 4.2] states that the edge probabilities of the uniform spanning tree can be expressed as determinant of the transfer-impedance matrix.

Theorem 5.5 (Burton-Pemantle theorem). *For any finite collection of disjoint undirected edges $e_1, ..., e_k \in E$,*

$$\mathbf{P}(e_1, ..., e_k \in t) = \det\left(T(e_i, e_j)_{i,j=1}^{k}\right)$$

for t a spanning tree of G, where the matrix T is defined in (5.3)*.*

Define now, for any edge $e = (uv) \in E$,

$$\zeta_e := w_{uv}(\overline{\xi}_u - \overline{\xi}_v)(\xi_u - \xi_v). \tag{5.4}$$

Note that the definition of ζ_e is independent of the orientation of e.

Lemma 5.6 ([11, Lemma 4.5]). *If $\emptyset \neq F \subseteq E$, then*

$$\left\langle \prod_{f \in F} \zeta_f \right\rangle = \det\left(T(f, f')_{f, f' \in F}\right).$$

Proof. Without loss of generality let us assume that $F = \{f_1,\, \ldots,\, f_k\}$, $k \geq 1$, and each edge f_i is oriented as $\overrightarrow{u_i v_i}$. We observe that

$$\left(\overline{\xi}_{u_i} - \overline{\xi}_{v_i} \,:\, i = 1,\, \ldots,\, k\right) = \overline{\boldsymbol{\xi}}^T C, \quad \left(\overline{\xi}_{u_i} - \overline{\xi}_{v_i} \,:\, i = 1,\, \ldots,\, k\right) = B\boldsymbol{\xi},$$

where $B = C^T$ and C is a $(\#V - 1) \times k$ matrix such that the column corresponding to the i-th point is given by

$$C(\cdot, i) = (0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots, 0)^T,$$

with the -1 (resp. 1) located at the v_i-th position (resp. at the u_i-th position). Therefore,

$$\left\langle \prod_{f \in F} \zeta_f \right\rangle = (\det(O))^{-1} \int \mathcal{D}(\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) \left(\prod_{\alpha=1}^{k} \left(\bar{\boldsymbol{\xi}}^T C\right)_\alpha (B\boldsymbol{\xi})_\alpha \right) \exp\left(\bar{\boldsymbol{\xi}}, O\boldsymbol{\xi}\right). \tag{5.5}$$

The lemma now follows from item 2. of Theorem 4.2 and the computation

$$\left(BO^{-1}C\right)(f, g) = T(f, g)$$

for $f, g \in E$. □

As a comment to this lemma, note that the independence of ζ from the direction of the edges explains why the determinant of the transfer matrix, as already mentioned, is not influenced by our arbitrary choice of the edge orientation.

The following Theorem is the Grassmannian analogue of [9, Corollary 4.4], which is stated without proof in [32, Equation (3)] and proven in [42, Proposition 3.6].

Theorem 5.7 (Edge-representation with fermions). *For any* $\emptyset \neq F, F' \subseteq E$ *such that* $F \cap F' = \emptyset$,

$$\mathbf{P}(F \in t, F' \cap t = \emptyset) = \left\langle \prod_{f \in F} \zeta_f \prod_{f' \in F'} (1 - \zeta_{f'}) \right\rangle.$$

Proof. Note that, since

$$\prod_{f' \in F'} (1 - \zeta_{f'}) = \sum_{\gamma \subseteq F'} (-1)^{|\gamma|} \prod_{f' \in \gamma} \zeta_{f'},$$

we can write

$$\left\langle \prod_{f \in F} \zeta_f \prod_{f' \in F'} (1 - \zeta_{f'}) \right\rangle = \sum_{\gamma \subseteq F'} (-1)^{|\gamma|} \left\langle \prod_{f \in F} \zeta_f \prod_{f' \in \gamma} \zeta_{f'} \right\rangle. \tag{5.6}$$

By Lemma 5.6 and Theorem 5.5, we have that (5.6), for t a spanning tree, reduces to

$$\sum_{\gamma\subseteq F'}(-1)^{|\gamma|}\mathbf{P}(F\cup\gamma\in t)=\mathbf{P}(F\cup\gamma\in t)-\sum_{\substack{\#\gamma=1,\\ \gamma\subseteq F'}}\mathbf{P}(F\cup\gamma\in t)+\sum_{\substack{\#\gamma=2,\\ \gamma\subseteq F'}}\mathbf{P}(F\cup\gamma\in t)+\dots \tag{5.7}$$

Now we consider separately the two following cases:

(i) $\mathbf{P}(F\subseteq t)=0$;

(ii) $\mathbf{P}(F\subseteq t)\neq 0$.

Case (i). Note that in this case (5.7) equals 0, which proves our claim.

Case (ii). We can now divide each term of (5.7) by $\mathbf{P}(F\subseteq t)$. Thus,

$$\begin{aligned}\frac{1}{\mathbf{P}(F\subseteq t)}\left\langle\prod_{f\in F}\zeta_f\prod_{f'\in F'}(1-\zeta_{f'})\right\rangle &= 1-\mathbf{P}(\exists\ \emptyset\neq\gamma\subseteq F':\gamma\subseteq t|F\subseteq t)\\ &=\mathbf{P}\left((\exists\ \emptyset\neq\gamma\subseteq F':\gamma\subseteq t)^c|F\subseteq t\right)\\ &=\mathbf{P}(F'\cap t=\emptyset|F\subseteq t),\end{aligned}\tag{5.8}$$

where the second equality follows from the inclusion/exclusion principle. To conclude, it suffices to multiply both terms of (5.8) by $\mathbf{P}(F\subseteq t)$. □

6 Grassmannian calculus and the Abelian sandpile model

One of the intriguing applications of Grassman calculus lies in its link to combinatorial models such as the aforementioned USTs and the Abelian Sandpile Model (ASM). They are both deeply connected to self-organized criticality and the broader study of complex random systems.

The ASM is also known as the Bak-Tang-Wiesenfeld model [4, 28, 43]. It is a type of cellular automaton defined on a graph, typically a grid, where each cell (or vertex) can hold a certain number of grains of sand. In this model, height fields refer to the number of sand grains at each vertex of the graph. Each vertex i has an associated height h_i, which is an integer representing the number of grains at that vertex. The configuration of the entire system is given by the set of heights at all vertices. The model follows a dynamics in three steps. Firstly, grains of sand are added one at a time to randomly chosen vertices. Secondly, topplings may occur. If the height at any vertex exceeds a certain threshold k_{thresh}, that vertex topples, distributing one grain of sand to each of its neighboring vertices.

This can cause neighboring vertices to exceed their thresholds and topple as well, leading to a cascade of topplings, known as an avalanche. Finally, the sandpile stabilizes, meaning this process continues until all vertices are below the threshold, resulting in a stable configuration.

One of the key features of the Abelian sandpile model is its Abelian property. This means that the final stable configuration of the sandpile does not depend on the order in which the grains are added or the order in which the vertices topple. This property simplifies the analysis and allows for exact results in many cases. Despite this and its apparently simple description, studying the ASM presents several notable challenges. Firstly, the system exhibits multifractal scaling rather than simple finite-size scaling. This multifractality is a hallmark of systems exhibiting self-organized criticality. Additionally, the height fields exhibit elaborate, non-local correlations. While the height-one variables can be handled by local calculations thanks to the burning bijection [35], higher height variables involve more intricate interactions. Finally, in 2D large avalanches, though rare, dominate the statistics in the thermodynamic limit. These rare events can significantly affect the height field distributions and their correlations. These factors make the study of height fields in the Abelian sandpile model a rich and challenging area of research.

The burning bijection relates the stationary measure of the ASM on a graph to the uniform spanning tree measure of the same graph. A spanning tree of a finite connected graph G is a subgraph which has no loops and connects via its edges all points of G. Kirchhoff's theorem gives the number of spanning trees in this setup, giving rise to a uniform probability measure on all such trees, called the uniform spanning tree. This allows one to give an alternative description of many observables of the ASM. For example, for a suitable collection of vertices V the event $\{\deg_{\mathrm{UST}}(v) = 1, \forall v \in V\}$ is equivalent to $\{h_v = 1, \forall v \in V\}$, and that the UST is incident to v via a preferred edge $\eta(v)$ (see Figure 3).

This means that the height-one field is a local event, that is, an event which is measurable with respect to a finite number of edges only, even as we let the size of the UST grow. This makes it amenable to exact computations than higher heights, which in contrast are non-local [28]. The perhaps surprising fact is that this 0-1 field is closely related to a special Gaussian field: the *discrete Gaussian free field* (DGFF).

6.1 ASM and DGFF

In [22, 13] an interesting relation between the height-one field of the ASM and the discrete Gaussian free field was unveiled. In order to explain it, we first need to introduce the DGFF [49, Chapter 2]. The DGFF is a Gaussian vector indexed over a finite connected graph. In [22, 13], this

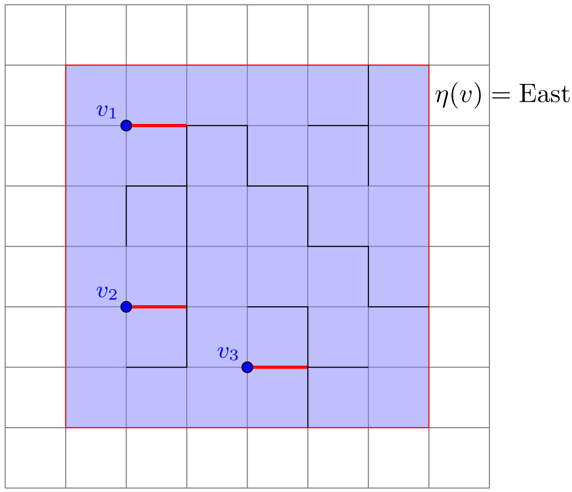

Figure 3: The event $h(v_1) = h(v_2) = h(v_3) = 1$ in ASM is equivalent to the UST incident to v_i via the east neighbor.

graph is a subset of $\mathbb{Z}^d$, although extensions to many other graphs are possible. Let us call this graph $G = (V, E)$ with its vertex set V and edge set E. On it, we define the Laplacian matrix Δ as follows: for $i, j \in V$

$$\Delta(i, j) = \begin{cases} 1 & \text{if } \|i - j\| = 1, \\ -2d & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we identify the boundary of the graph G as a subset $\emptyset \neq C \subset V$ of vertices. With these notions, we have the following definition.

Definition 6.1 (Discrete Gaussian free field). The discrete Gaussian free field (DGFF) φ on G with zero-boundary conditions on C is the mean-zero multivariate Gaussian indexed on V with density (with respect to the product Lebesgue measure on $\mathbb{R}^{V \setminus C}$) proportional to

$$\exp\left(-\frac{1}{2} \sum_{i, j \in V \setminus C} \frac{1}{2d} \varphi_i (-\Delta(i, j)) \varphi_j\right).$$

For simplicity of notation we now denote the height-one field by h. In [13] one considers under a suitable rescaling procedure a graph $G_\varepsilon = (V_\varepsilon, E_\varepsilon) \subseteq \mathbb{Z}^2$, where both the height-one h_ε and the DGFF φ_ε are defined. One studies then the ℓ-joint cumulants of first order κ of h. What one finds is that, when $x_\varepsilon^{(1)}, \ldots, x_\varepsilon^{(\ell)}$ is a set of $\ell \geq 2$ pairwise distinct points, in the limit h_ε satisfies

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon^{-2\ell} \kappa\left(h_\varepsilon\left(x_\varepsilon^{(1)}\right), \ldots, h_\varepsilon\left(x_\varepsilon^{(\ell)}\right)\right) \\ &= -4 \lim_{\varepsilon\to 0} \varepsilon^{-2\ell} \kappa\left(C\Phi_\varepsilon\left(x_\varepsilon^{(1)}\right), \ldots, C\Phi_\varepsilon\left(x_\varepsilon^{(\ell)}\right)\right) \end{aligned} \tag{6.1}$$

with C a universal explicit constant, and

$$\Phi_\varepsilon(v) := \sum_{i=1}^{2} \left(\varphi_\varepsilon(v+e_i) - \varphi_\varepsilon(v)\right)^2.$$

The field Φ_ε can be though of as the *gradient squared* of the DGFF. One of the goals of these notes is to elucidate the relation between the "squared norm" of the DGFF, and the height-one field.

Going back to (6.1), we understand now that, if we want a Gaussian field with cumulants exactly equal to those of the height-one field in the limit, we would need to take a complex version of the DGFF (and possibly sum over $2d$, rather than d, directions). However, removing the negative sign in (6.1) seems out of reach at the moment, as the next Remark discusses.

Remark 6.2. Consider two distinct random variables $\mathbf{X}, \mathbf{Y}$ defined on a common probability space on $\mathbb{R}^d$ such that $\kappa_{\mathbf{m}}(\mathbf{X}) = -\kappa_{\mathbf{m}}(\mathbf{Y})$ for any $\mathbf{m} \in \mathbb{N}^d$. Thus, formally,

$$\mathbf{E}\left[\exp\left(\mathbf{t}\cdot\mathbf{X}\right)\right] = \frac{1}{\mathbf{E}\left[\exp\left(\mathbf{t}\cdot\mathbf{Y}\right)\right]} \tag{6.2}$$

for any $\mathbf{t} \in \mathbb{R}^d$. Indeed, by means of Definition 2.3 we can formally write that

$$\begin{aligned} \mathbf{E}\left[\exp\left(\mathbf{t}\cdot\mathbf{X}\right)\right] &= \exp\left(\sum_{\mathbf{m}\in\mathbb{N}^d} \kappa_{\mathbf{m}}(\mathbf{X}) \prod_{j=1}^{d} \frac{t_j^{m_j}}{m_j!}\right) \\ &= \exp\left(-\sum_{\mathbf{m}\in\mathbb{N}^d} \kappa_{\mathbf{m}}(\mathbf{Y}) \prod_{j=1}^{d} \frac{t_j^{m_j}}{m_j!}\right) \\ &= \frac{1}{\mathbf{E}\left[\exp\left(\mathbf{t}\cdot\mathbf{Y}\right)\right]}. \end{aligned}$$

But are there any random variables that satisfy (6.2)? An immediate example are the random variables $\mathbf{X} = \mathbf{v}$ almost surely and $\mathbf{Y} = -\mathbf{v}$

almost surely, with $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^d$. The question is now whether there are more "significant" ones. If such variables exist, one can always construct a probability space in which they are independent, and in which it holds that

$$\mathbf{E}\left[\exp\left(\mathbf{t} \cdot (\mathbf{X} + \mathbf{Y})\right)\right] = 1$$

for all $\mathbf{t} \in \mathbb{R}^d$, yielding that $\mathbf{X} = -\mathbf{Y}$ almost surely. Therefore, to find a non-trivial answer, we need to consider the question in a different setup[2]. For this reason, we will now tackle the concept of *supersymmetry*, that we present in the next section.

7 Supersymmetry (SUSY)

7.1 SUSY Gaussians

For the rest of the Subsection, let $m := \#V$ and $n = 2m$. We will start working with superspins, namely vectors living in a space $\mathbb{R}^{n|n}$ which is a "hybrid" space in which both n standard, real variables and n Grassmannian generators live. More precisely, we define a *superspin*, or supervector, as the vector

$$u_i = (x_i,\, y_i,\, \xi_i,\, \overline{\xi}_i)^T, \quad i \in V$$

where $x_i,\, y_i \in \mathbb{R}$ and $\xi_i,\, \overline{\xi}_i$ are two generators of $\mathcal{G}_{\mathcal{A}}$. A smooth superfunction $F \in C^\infty(\mathbb{R}^{n|n})$ can be written as

$$F = \sum_{I \subseteq [n]} f_I(x_I, y_I)\xi_I\overline{\xi}_I$$

where $f_I \in C^\infty(\mathbb{R}^n)$ are smooth functions and $\xi_I\overline{\xi}_I$ are Grassmann monomials indexed over I. The body F_b of a superfunction F is defined as the ordinary smooth function obtained by formally setting all Grassmann variables to zero:

$$F_b = f_\emptyset(\mathbf{x}_\emptyset,\, \mathbf{y}_\emptyset).$$

The remaining part is referred to as the soul:

$$F_s = F - F_b = \sum_{\emptyset \neq I \subseteq [n]} f_I(x_I, y_I)\xi_I\overline{\xi}_I.$$

In superspaces, integration works in the following way.

---

[2] If we allow the argument to become imaginary on the right-hand side of (6.2), we get immediately the example of the independent variables $X \sim \mathcal{N}(0,\,1)$ and $Y \sim \mathrm{i}\mathcal{N}(0,1)$ in $d = 1$. In fact, such a class of random variables is well-studied, and we are grateful to Christophe Vignat who pointed us out to *van Dantzig pairs* [34]. The pair is constituted by two analytic characteristic functions $f,\, g$ such that $f(t)g(\mathrm{i}t) = 1$ for all $t \in \mathbb{R}$. See also [24] for a general approach to this problem.

7.2 Superintegration

The case of $\mathbb{R}^{0|n}$ corresponds to integration in fermionic spaces as we have already seen in Subsection 3.2. Namely,

$$\int_{\mathbb{R}^{0|n}} F = \int_{\mathbb{R}^{0|n}} \sum_{\emptyset \neq I \subseteq [n]} f_I(x_I, y_I)\xi_I\bar{\xi}_I \coloneqq \sum_{\emptyset \neq I \subseteq [n]} f_I(x_I, y_I) \int \mathcal{D}(\boldsymbol{\xi}, \bar{\boldsymbol{\xi}})\xi_I\bar{\xi}_I.$$

On $\mathbb{R}^{n|n}$, Berezin measures are written as

$$\mathrm{d}\,\mathbf{u} = \mathrm{d}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) = \sum_{I \subseteq [n]} \mathrm{d}\,\nu_I(\mathbf{x}, \mathbf{y})\xi_I\bar{\xi}_I\mathcal{D}(\boldsymbol{\xi}, \bar{\boldsymbol{\xi}})$$

where $\mathrm{d}\,\nu_I$ are measures in $\mathbb{R}^n$. Then

$$\int F\,\mathrm{d}\,\mathbf{u} \coloneqq \sum_I \int_{\mathbb{R}^n} \left(\int \mathcal{D}(\boldsymbol{\xi}, \bar{\boldsymbol{\xi}})\, F\xi_I\bar{\xi}_I \right) \mathrm{d}\,\nu_I(\mathbf{x}, \mathbf{y}).$$

For $I = \emptyset$ we take $\mathrm{d}\,\nu_\emptyset$ to be the Lebesgue measure normalized by $\pi^{n/2}$ on $\mathbb{R}^n$ and $\nu_I = 0$ otherwise, obtaining the Berezin-Lebesgue measure on $\mathbb{R}^{n|n}$. From now on, we will denote it simply as $\mathrm{d}\,\mathbf{u}$.

Example 7.1. If $n = 2$, $a \in \mathbb{R}$ and $F(u) = \exp(-a(x^2 + y^2) - a\xi\bar{\xi})$ then

$$\int F\,\mathrm{d}\,\mathbf{u} = 1.$$

Proof. Using the definition of superintegration, we obtain that

$$\int \mathrm{e}^{-a(x^2+y^2)-a\xi\bar{\xi}} = \int_{\mathbb{R}^2} \mathrm{e}^{-a(x^2+y^2)} \left(\int \mathcal{D}(\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) \left(-a\xi\bar{\xi}\right) \right) \frac{1}{\pi}\,\mathrm{d}\,x\,\mathrm{d}\,y$$
$$= \frac{a}{\pi} \left(\int_{\mathbb{R}} \mathrm{e}^{-ax^2}\,\mathrm{d}\,x \right)^2 = 1.$$

This also explains our choice of normalization of the Lebesgue measure on $\mathbb{R}^n$. □

The interesting fact is that this integral seems to be independent of a, and we would like to generalize this result, as well as using it as motivating example to the phenomenon of supersymmetry.

Let $A \in M_m(\mathbb{C})$ be an Hermitian positive-definite matrix. If $x_\alpha = \Re(Z_\alpha)$, $y_\alpha = \Im(Z_\alpha)$, we recall the identities

$$\int_{\mathbb{C}^m} \exp\left(-\left(\overline{\mathbf{Z}}, A\mathbf{Z}\right)\right) \prod_{\alpha=1}^m \frac{\mathrm{d}\,x_\alpha\,\mathrm{d}\,y_\alpha}{\pi}$$
$$= \int_{\mathbb{R}^n} \exp\left(-(\mathbf{x}, A\mathbf{x}) - (\mathbf{y}, A\mathbf{y})\right) \prod_{\alpha=1}^m \frac{\mathrm{d}\,x_\alpha\,\mathrm{d}\,y_\alpha}{\pi} = \det(A)^{-1}$$

and

$$\int \mathcal{D}(\boldsymbol{\xi},\,\overline{\boldsymbol{\xi}})\mathrm{e}^{\left(\overline{\boldsymbol{\xi}},A\boldsymbol{\xi}\right)} = \det(A).$$

Thus, by letting

$$\mathcal{D}(\mathbf{Z},\,\overline{\mathbf{Z}}) := \prod_{\alpha=1}^{m} \frac{\mathrm{d}\,x_\alpha\,\mathrm{d}\,y_\alpha}{\pi}$$

we get, as in Example 7.1,

$$\int \mathcal{D}(\boldsymbol{\xi},\,\overline{\boldsymbol{\xi}})\mathcal{D}(\mathbf{Z},\,\overline{\mathbf{Z}}) \exp\left(-\left(\overline{\mathbf{Z}},A\mathbf{Z}\right) + \left(\overline{\boldsymbol{\xi}},A\boldsymbol{\xi}\right)\right) = 1. \tag{7.1}$$

Note that the integral (7.1) does not depend on the choice of the matrix A.

Define now the *super inner product* between two superspins at i and $j \in [n]$ as

$$(u_i, u_j) := x_i x_j + y_i y_j + \frac{1}{2}\left(\xi_j \overline{\xi}_i + \xi_i \overline{\xi}_j\right). \tag{7.2}$$

Noting that

$$\left(\overline{\boldsymbol{\xi}}, A\boldsymbol{\xi}\right) \overset{(3.1)}{=} \frac{1}{2}\left(\overline{\boldsymbol{\xi}}, A\boldsymbol{\xi}\right) - \frac{1}{2}\left(\boldsymbol{\xi}, A\overline{\boldsymbol{\xi}}\right)$$

we deduce that (7.1) becomes

$$\int \mathrm{d}\mathbf{u}\ \exp\left(-\left(\mathbf{u}, A\mathbf{u}\right)\right) = 1, \tag{7.3}$$

where $\mathbf{u} = (u_1, ..., u_n)^T$ and

$$(\mathbf{u}, A\mathbf{u}) \coloneqq \sum_{i,j=1}^{n} A(i,\,j)(u_i,\,u_j).$$

The measure defined in (7.3) is called *supergaussian measure.*

7.3 Localization

Let $\mathcal{G}_{\mathcal{A}}(\mathbb{R}^n)$ be the algebra of smooth functions from $\mathbb{R}^n$ into $\mathcal{G}_{\mathcal{A}}$, which is our usual Grassmannian algebra generated by $\xi_1, ..., \xi_m, \overline{\xi}_1, ..., \overline{\xi}_m$. Consider the complex coordinates

$$z \coloneqq x + \mathrm{i}y, \qquad \bar{z} \coloneqq x - \mathrm{i}y,$$

and define

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \mathrm{i}\frac{\partial}{\partial y_i}\right), \qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \mathrm{i}\frac{\partial}{\partial y_i}\right).$$

The *supersymmetry generator* $Q : \mathcal{G}_{\mathcal{A}}(\mathbb{R}^n) \to \mathcal{G}_{\mathcal{A}}(\mathbb{R}^n)$ is then defined as

$$Q := \sum_{i=1}^{n} \left(\xi_i \frac{\partial}{\partial z_i} + \bar{\xi}_i \frac{\partial}{\partial \bar{z}_i} - 2z_i \frac{\partial}{\partial \xi_i} + 2\bar{z}_i \frac{\partial}{\partial \bar{\xi}_i} \right).$$

A function $F \in \mathcal{G}_{\mathcal{A}}(\mathbb{R}^n)$ is defined to be *supersymmetric* or Q-closed if $QF = 0$, and Q-exact if there exists $F' \in \mathcal{G}_{\mathcal{A}}(\mathbb{R}^n)$ such that $QF' = F$. If we prefer to write the supersymmetry generator in terms of the real vectors $\mathbf{x}$ and $\mathbf{y}$, we get that

$$Q = \frac{1}{\sqrt{\mathrm{i}}} \sum_{i=1}^{n} Q_i := \frac{1}{\sqrt{\mathrm{i}}} \sum_{i=1}^{n} \left(\xi_i \frac{\partial}{\partial x_i} + \bar{\xi}_i \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \bar{\xi}_i} + 2y_i \frac{\partial}{\partial \xi_i} \right).$$

The fundamental property of supersymmetry is given by the following localization theorem, whose proof can be found in [5, Theorem 11.4.5].

Theorem 7.2. *Let the element $F \in \mathcal{G}_{\mathcal{A}}(\mathbb{R}^n)$ be a smooth integrable supersymmetric form. Then*

$$\int \mathrm{d}\mathbf{u}\, F(\mathbf{u}) = F_b(\mathbf{0}),$$

where F_b is the body of F evaluated at $\mathbf{0}$.

As an example, when $F = \exp(\xi_1 + \xi_2) = 1 + \xi_1 + \xi_2$ (see Remark 3.13), one obtains $\int \mathrm{d}\mathbf{u}\, F(\mathbf{u}) = 1$.

We can prove that the super inner product is supersymmetric.

Lemma 7.3. *For all $i, j \in [n]$ one has $Q(u_i, u_j) = 0$.*

Proof. Since Q formally exchanges generators as follows:

$$Qx_i = \xi_i,\ Qy_i = \bar{\xi}_i,\ Q\bar{\xi}_i = -2x_i,\ Q\xi_i = 2y_i,$$

we have, up to a multiplicative constant factor $1/\sqrt{\mathrm{i}}$,

$$Q(u_i, u_j) = Q_i(u_i, u_j) + Q_j(u_i, u_j) = \xi_i x_j + \bar{\xi}_i y_j + \frac{1}{2}\left(-2\xi_j x_i + 2y_i \bar{\xi}_j\right)$$
$$+ x_i \xi_j + y_i \bar{\xi}_j + \frac{1}{2}\left(2y_j \bar{\xi}_i - 2\xi_i x_j\right) = 0. \square$$

We can now, for example, apply [5, Example 11.4.4] to the smooth function $\exp(-A + t\mathrm{Id}) : \mathbb{R}^{n \times n} \to \mathbb{C}$ for some $t \in \mathbb{R}^n$ to show that

$$\int \mathrm{d}\mathbf{u}\ \exp\left(-\left(\mathbf{u}, A\mathbf{u}\right)\right) \exp\left(\left(\mathbf{t}, \mathbf{u}\right)\right) = 1,$$

where formally

$$(\mathbf{t},\mathbf{u}) := \sum_{i=1}^{n} (\mathbf{t}_i,\, u_i).$$

This is formally equivalent to the cumulative generating function of $\mathbf{u}$ satisfying

$$\mathbf{E}\left[\exp\left((\mathbf{t},u)\right)\right]_{\text{SUSY G}} = 1,$$

where $\mathbf{E}\left[\cdot\right]_{\text{SUSY G}}$ means the expectation with respect to the supergaussian measure defined in (7.3). So, if $\tilde{F}$ resp. $\tilde{G}$ is the cumulant generating function of $(\boldsymbol{x},\boldsymbol{y})$ resp. $(\bar{\boldsymbol{\xi}},\boldsymbol{\xi})$ as Gaussian measures in their respective "worlds", then $\tilde{F}(\cdot) = -\tilde{G}(\cdot)$. Consequently, if F resp. G is the moment generating function of $(\boldsymbol{x},\boldsymbol{y})$ resp. $(\bar{\boldsymbol{\xi}},\boldsymbol{\xi})$ as Gaussian measures in their respective worlds, then $F(\cdot) = 1/G(\cdot)$, which accomplishes the task pointed out in Remark 6.2.

We would like to point out that supersymmetry was, in fact, not necessary to explain Remark 6.2 (see also [2]). However, we decided to introduce it here to give a glimpse into its richness and its many possible applications in probability theory.

8 Grassmannian calculus and reinforced processes

Edge reinforced random walks (ERRW) [15, 29] and vertex reinforced jump processes (VRJP) [16, 17, 44] are history-dependent stochastic processes where the particle tends to come back more often on locations it has already visited in the past. Formally, let $G = (V, E)$ be a finite undirected graph.

Definition 8.1. The ERRW with starting point $X_0 = i_0 \in V$ and initial weights $(W_{ij})_{\{i,j\}\in E}$ is the stochastic process $(X_n)_{n\geq 1}$ with transition probabilities

$$P_{i_0}\left(X_{n+1} = i \mid X_n = j, (X_m)_{m\leq n}\right) = \frac{\mathbb{1}_{\{i,j\}\in E}(W_{ij} + L_n^{ij})}{\sum_{\{i,k\}\in E}(W_{ik} + L_n^{ik})}$$

where L_n^{ik} is the number of times the edge $\{i,k\}$ has been crossed up to time n (in either direction).

Since the VRJP can be related to the ERRW [44], in what follows we only focus on the ERRW. With Grassmann calculus one is able to give an alternative characterization for the ERRW in terms of a random walk in random conductances as follows, known in the literature as *magic formula* [20, 44].

Proposition 8.2. *On the graph G, the ERRW can be described as a random walk in random conductances $c_{ij} = W_{ij} e^{u_i+u_j}$, where $\{i,j\} \in E$ and $W_{ij} > 0$ is an edge weight that may be deterministic or random. Finally, $\{u_i\}_{i\in V}$ is a family of real random variables with probability measure*

$$\mu^V_{W,i_0}(du) = \sqrt{\sum_{S\in\mathcal{S}} \prod_{\{i,j\}\in S} W_{ij} e^{u_i+u_j}} \prod_{\{i,j\}\in S} e^{-W_{ij}(\cosh(u_i-u_j)-1)} \times \prod_{i\in V\setminus\{i_0\}} \frac{e^{-u_i}}{\sqrt{2\pi}} du_i\, \delta(u_{i_0}), \tag{8.1}$$

where i_0 is the starting point of the process, and $\mathcal{S}$ is the set of spanning trees over the graph G. We denote the corresponding expectation by $\mathbb{E}^V_{W,i_0}[\cdot]$.

We refer the interested reader to [21, 20, 19, 44, 45], where key properties of the above measure were obtained using its relation to non-linear sigma models where the spin has both real and Grassmann components.

Proof of Proposition 8.2. In what follows we use Grassmann variables to derive the measure in (8.1).

Consider the Grassmann algebra $\mathcal{G}_\mathcal{A}$ generated by $\{\xi_i, \bar{\xi}_i\}_{i\in V}$ and associate with each point $i \in V$ a spin $v_i = (x_i, y_i, z_i, \xi_i, \bar{\xi}_i) \in \mathbb{R}^{3|2}$, i.e., the three components x_i, y_i, z_i are real and the two components $\xi_i, \bar{\xi}_i$ are Grassmann variables. We introduce the non-positive definite bilinear form

$$\langle v, v'\rangle := xx' + yy' - zz' + \xi\bar{\xi}' - \bar{\xi}\xi'.$$

By adding the additional constraint $\langle v, v\rangle = -1$, we further obtain that

$$z = \pm\sqrt{1 + x^2 + y^2 + 2\xi\bar{\xi}}.$$

By choosing the positive part, we obtain the *supersymmetric hyperbolic sigma model* H^2 with two additional Grassmann components, which motivates the name $H^{2|2}$. We now define the *energy* of a spin configuration $v = \{v_i\}_{i\in V}$ as

$$\begin{aligned} E_V(v) &= \sum_{\{i,j\}\in E} W_{ij}\, \langle v_i - v_j, v_i - v_j\rangle \\ &= -2 \sum_{\{i,j\}\in E} W_{ij}(1 + \langle v_i, v_j\rangle) \\ &= 2 \sum_{\{i,j\}\in E} W_{ij}(z_i z_j - (1 + x_i x_j + y_i y_j + \xi_i\bar{\xi}_j - \bar{\xi}_i\xi_j)). \end{aligned} \tag{8.2}$$

Note that $E_V(v) \in \mathcal{G}_{\mathcal{A}}$ is even and hence $\mathrm{e}^{-E_V(v)}$ is well-defined. We introduce now the analogue of the magnetic field by setting $v_{i_0} = o := (0,0,1,0,0)$. The *Gibbs measure* can be written as

$$\langle f(v)\rangle_{W,i_0}^{V} = \int_{(H^{2|2})^{V\setminus\{i_0\}}} \mathrm{d}\delta(v_{i_0}-o) \prod_{j\neq i_0} \left(\frac{\mathrm{d}x_j \mathrm{d}y_j}{2\pi} \partial_{\xi_j}\partial_{\bar{\xi}_j} \frac{1}{z_j}\right) \mathrm{e}^{-E_V(v)} f(v)$$

for regular enough functions f. The factor $1/z$ appears because we are integrating over the non-linear manifold $H^{2|2}$. This model is connected to the probability measure $\mu_{W,i_0}^{V}(u)$ introduced in (8.1) as follows:

$$\mathbb{E}_{W,i_0}^{V}[f(\mathrm{e}^u)] = \langle f(x+z)\rangle_{W,i_0}^{V}.$$

This can be seen by performing the change of coordinates $(x,y,\xi,\bar{\xi}) \longmapsto (u,s,\bar{\psi},\psi)$ (horospherical coordinates) defined via

$$x = \sinh u - \mathrm{e}^u\left(\frac{s^2}{2} + \bar{\psi}\psi\right), \qquad y = \mathrm{e}^u s, \qquad \xi = \mathrm{e}^u\bar{\psi}, \qquad \bar{\xi} = \mathrm{e}^u\psi,$$

where u and s range over the real numbers. Note that $(u,s,\bar{\psi},\psi)$ are globally defined coordinates and

$$(u,s,\bar{\psi},\psi) = (0,0,0,0) \Leftrightarrow (x,y,\xi,\bar{\xi}) = (0,0,0,0).$$

The expression for the energy (8.2) in them is

$$E_V(v) = \sum_{\{i,j\}\in E} W_{ij}(S_{ij}-1),$$

where

$$\begin{aligned} S_{ij} &= B_{ij} + (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j)\mathrm{e}^{u_i+u_j}, \\ B_{ij} &= \cosh(u_i-u_j) + \tfrac{1}{2}(s_i-s_j)^2\mathrm{e}^{u_i+u_j}. \end{aligned}$$

Conditioned on u, the variables $s, \bar{\psi}, \psi$ are Gaussian distributed with covariance $C(u)$, and therefore the corresponding integral can be performed exactly. The result is formula (8.1). For more details we refer to [21]. □

9 Grassmannian calculus and random matrix theory

This Section is based on [51, Sec. 4.4] (see also [27, Sec. 10.3.2]). Here we are going to use Grassmann calculus to prove a staple result in random matrix theory: the classical Wigner semicircular law convergence for the empirical distribution of the eigenvalues of a Gaussian unitary ensemble (GUE). The core idea is to compute the limiting spectral distribution of large random Hermitian matrices by leveraging fermionic integrals. These integrals are particularly efficient when dealing with Gaussian ensembles, where expectations over matrix entries can be performed exactly due to Wick's theorem or Gaussian integration rules (see Proposition 4.1). Thus, fermionic variables help because Grassmannian integrals encode determinants compactly and allow us to manipulate them analytically. For matrices with independent entries, the average over the ensemble can be combined with Grassmannian calculus to obtain a tractable expression for the spectral density.

We are now going to state all the necessary definitions for this result.

Definition 9.1. The GUE ensemble is the probability distribution on the set of $N \times N$ Hermitian matrices with density

$$P(\mathrm{d}A) \propto \exp\left(-\frac{N}{2}\mathrm{tr}A^2\right)\prod_{i=1}^{N}\mathrm{d}A_{ii}\prod_{1\le i<j\le N}\mathrm{d}\Re A_{ij}\mathrm{d}\Im A_{ij}.$$

Basically, we are dealing with Hermitian matrices in which the upper-triangular entries are i. i. d. with distribution $\mathcal{N}(0,1)_{\mathbb{C}}$, and the diagonal entries are i. i. d. with distribution $\mathcal{N}(0,1)_{\mathbb{R}}$. A celebrated result by Wigner states that the rescaled empirical spectral distribution (ESD)

$$\mu_N(\mathrm{d}x) := \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\delta_{\lambda_i}$$

of the eigenvalues $\lambda_1, \ldots, \lambda_N$ of a GUE matrix admits a very precise limit.

Theorem 9.2 (Wigner semicircular law convergence). *With probability one it holds that*

$$\lim_{N\to\infty}\mu_N(\mathrm{d}x) = \mu_{sc}(\mathrm{d}x)$$

where[3]

$$\mu_{sc}(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}\mathrm{d}x$$

is the Wigner semicircular law.

The method of the proof uses classical tools in random matrix theory which are then turned into Grassmann language. We will now define a few

[3] In this instance, the a.s. limit of the measures indicates that $\lim_{N\to\infty}\mu_N(-\infty,\lambda) = \mu_{sc}(-\infty,\lambda)$ a. s. for all $\lambda \in \mathbb{R}$.

objects (the interested reader can look up for example [47, 3] for in-depth monographs on the topic). For an Hermitian matrix H_N of size $N \times N$ define its resolvent as

$$R_N(z) \coloneqq (H_N - zI_N)^{-1}, \quad z \in \mathbb{C}^+,$$

where I_N is the $N \times N$ identity matrix. Moreover for a positive finite measure μ on the real line we define its Stieltjes transform as

$$S_\mu(z) \coloneqq \int \frac{\mu(\mathrm{d}\,x)}{x - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}\,. \tag{9.1}$$

The opposite $-S_\mu(\cdot)$ is often called the Cauchy transform. Finally, let us point out the difference between the notations for the trace

$$\mathrm{Tr}(H_N) \coloneqq \sum_{i=1}^{N} [H_N]_{ii}$$

and the normalized trace

$$\mathrm{tr}(H_N) = \frac{\mathrm{Tr}(H_N)}{N}$$

that we are going to use in the sequel.

The key result is that convergence of Cauchy-Stieltjes transforms implies weak convergence of the underlying probability measures. To be precise, what we are actually going to prove is that the average measure $\tilde{\mu}_N \coloneqq \mathbb{E}[\mu_N]$ converges weakly to the semicircle distribution. At the expense of not showing Theorem 9.2 in its full form, we wish to convey the gist behind the use of fermionic calculus in random matrix theory.

Proof of Theorem 9.2 (abridged version). First by complex Gaussian integration (Def. 2.4) we can write, for $1 \le i \le j \le N$,

$$[zI_N - H_N]^{-1}_{ij} = s \det\left(s\left(zI_N - H_N\right)\right) \int \mathcal{D}(\mathbf{Z}) Z_i \overline{Z_j} \mathrm{e}^{-s(\overline{\mathbf{Z}},\,(zI_N - H_N)\mathbf{Z})}, \tag{9.2}$$

where we introduce the shorthand notation

$$\mathcal{D}(\mathbf{Z}) \coloneqq \prod_{i=1}^{N} \frac{\mathrm{d}\,\Re(Z_i)\,\mathrm{d}\,\Im(Z_i)}{\pi}$$

and $s \coloneqq -\mathrm{i}\,\mathrm{sgn}(\Im z)$ is chosen so that the integral in (9.2) is convergent.

This implies that

$$\begin{aligned}&\operatorname{Tr}(zI_N - H_N)^{-1}\\ =&\sum_i (zI_N - H_N)^{-1}_{ii}\\ =&s\det\left(s\left(zI_N - H_N\right)\right)\int \mathcal{D}(\mathbf{Z})\left(\sum_{i=1}^N Z_i\overline{Z_i}\right)\mathrm{e}^{-s(\overline{\mathbf{Z}},\,(zI_N-H_N)\mathbf{Z})}.\end{aligned}$$

Thus, since $S_{\tilde{\mu}_N} = \mathbb{E}[\operatorname{tr}(R_N)]$, we get

$$-\frac{1}{N}\operatorname{Tr}(R_N(z)) = \frac{s\det\left(s\left(zI_N - H_N\right)\right)}{N}\int \mathcal{D}(\mathbf{Z})\left(\mathbf{Z},\overline{\mathbf{Z}}\right)\mathrm{e}^{-s(\overline{\mathbf{Z}},\,(zI_N-H_N)\mathbf{Z})}. \tag{9.3}$$

We use now Proposition 4.1 to develop the determinant in (9.3) obtaining

$$\begin{aligned}-S_{\tilde{\mu}_N}(z)=&\frac{s}{N}\int \mathcal{D}(\mathbf{Z})\int \mathcal{D}(\boldsymbol{\xi},\,\overline{\boldsymbol{\xi}})\left(\mathbf{Z},\overline{\mathbf{Z}}\right)\mathrm{e}^{-s(\boldsymbol{\xi},\,(zI_N-H_N)\overline{\boldsymbol{\xi}})}\mathrm{e}^{-s\left(\overline{\mathbf{Z}},\,(zI_N-H_N)\mathbf{Z}\right)}\\ =&\frac{s}{N}\int \mathcal{D}(\mathbf{Z})\int \mathcal{D}(\boldsymbol{\xi},\,\overline{\boldsymbol{\xi}})\left(\mathbf{Z},\overline{\mathbf{Z}}\right)\mathrm{e}^{-sz\left[(\overline{\mathbf{Z}},\,\mathbf{Z})+(\boldsymbol{\xi},\overline{\boldsymbol{\xi}})\right]}\mathrm{e}^{s(\overline{\mathbf{Z}},H_N\mathbf{Z})+s(\boldsymbol{\xi},H_N\overline{\boldsymbol{\xi}})}.\end{aligned} \tag{9.4}$$

To obtain (9.1) we will now compute an expectation $\mathbb{E}$ with respect to the GUE measure of one of the integrand terms, namely

$$\mathbb{E}\left[\mathrm{e}^{s(\overline{\mathbf{Z}},H_N\mathbf{Z})+(\boldsymbol{\xi},H_N\overline{\boldsymbol{\xi}})}\right] = \exp\left(-\frac{1}{2N}\left[(\overline{\mathbf{Z}},\,\mathbf{Z})^2 + 2\left(\boldsymbol{\xi},\mathbf{Z}\right)\left(\overline{\mathbf{Z}},\overline{\boldsymbol{\xi}}\right) - \left(\boldsymbol{\xi},\overline{\boldsymbol{\xi}}\right)^2\right]\right). \tag{9.5}$$

Note that in (9.4) one needs to bring the expectation with respect to the GUE measure inside the integrals. This is indeed possible because the integral has a Gaussian decay in H_N, the Grassmann integration produces only finitely many terms and lastly the bosonic integral is absolutely convergent thanks to the fact that s is chosen according to $\operatorname{sgn}(\Im z)$. This implies that the combined integrand is absolutely integrable in the H_N-variable, so that Fubini-Tonelli's theorem gives the desired result. We introduce the Hubbard-Stratonovich transformation

$$\exp\left(\frac{1}{2N}\left(\boldsymbol{\xi},\overline{\boldsymbol{\xi}}\right)^2\right) = \sqrt{\frac{N}{2\pi}}\int_{\mathbb{R}} \mathrm{d}v\exp\left(-v\left(\boldsymbol{\xi},\overline{\boldsymbol{\xi}}\right) - \frac{N}{2}v^2\right) \tag{9.6}$$

which follows from a standard Gaussian integration in $\mathbb{R}$. Plugging (9.6) and (9.5) into (9.4) we can perform first the integral in the Grassmann variables obtaining

$$\int \mathcal{D}(\boldsymbol{\xi},\,\overline{\boldsymbol{\xi}})\exp(-(\boldsymbol{\xi},K\overline{\boldsymbol{\xi}})) = \det(K)$$

with

$$K := (sz + v)I_N + \frac{1}{N}\mathbf{Z}\overline{\mathbf{Z}}^T.$$

Note that $\mathbf{Z}\overline{\mathbf{Z}}^T$ is an $N \times N$ matrix. The determinant of K is computed in [51, Eq. (4.18)] as

$$\det(K) = (sz + v)^{N-1}\left(sz + v + \frac{(\overline{\mathbf{Z}},\, \mathbf{Z})}{N}\right).$$

Therefore, (9.4) has now reduced to

$$\begin{aligned} -S_{\tilde{\mu}_N}(z) = -\frac{s}{N}\sqrt{\frac{N}{2\pi}} \int_{\mathbb{C}^N} \mathcal{D}(\mathbf{Z}) \int_{\mathbb{R}} \mathrm{d}v\, \left(\mathbf{Z}, \overline{\mathbf{Z}}\right) (sz + v)^{N-1} \\ \times \left(sz + v + \frac{(\overline{\mathbf{Z}},\, \mathbf{Z})}{N}\right) \\ \times \exp\left(-sz(\overline{\mathbf{Z}},\, \mathbf{Z}) - \frac{(\overline{\mathbf{Z}},\, \mathbf{Z})^2}{2N} - \frac{N}{2}v^2\right). \end{aligned} \tag{9.7}$$

The integral depends on the quadratic term $(\overline{\mathbf{Z}},\, \mathbf{Z})$, so introducing polar coordinates will help us calculate it. We set $\rho := (\overline{\mathbf{Z}},\, \mathbf{Z})$ and transform the right-hand side of (9.7) into

$$\frac{s}{N!}\sqrt{\frac{N}{2\pi}} \int_{\mathbb{R}} \mathrm{d}v \int_0^\infty \mathrm{d}\rho \frac{sz + v + \frac{\rho}{N}}{sz + v} h(z,\, v,\, \rho) \tag{9.8}$$

with

$$h(z,\, v,\, \rho) := \rho^N (sz + v)^N \exp\left(-sz\rho - \frac{\rho^2}{2N} - \frac{N}{2}v^2\right).$$

Since we are interested in the $N \to \infty$ limit, we will evaluate the integral through the saddle point method. The saddle points v_0, ρ_0 of $h(\cdot)$ are

$$v_0 = -\frac{sz}{2} \pm \sqrt{1 - \frac{z^2}{4}}, \quad \frac{\rho_0}{N} = -\frac{sz}{2} \pm \sqrt{1 - \frac{z^2}{4}} \tag{9.9}$$

as well as $\{v_0 = -sz,\, \rho_0 = 0\}$. However, we can exclude the latter couple of points, since the integrand vanishes at them. We also notice that if in (9.9) one takes points with square roots having opposite signs the integrand will also vanish. Therefore, we will choose ρ_0 having positive real part (since the integral in ρ runs from zero to infinity) and we obtain $h(z,\, v_0,\, \rho_0) = N^N \exp(-N)$. On the other hand, the second derivatives of $h(\cdot)$ at the saddle points are

$$\frac{\partial^2 \ln h}{\partial v^2} = -N\left(\frac{1}{(sz+v_0)^2}+1\right), \quad \frac{\partial^2 \ln h}{\partial \rho^2} = -\left(\frac{N}{\rho_0^2}+\frac{1}{N}\right)$$
$$\frac{\partial^2 \ln h}{\partial v \partial \rho} = 0, \quad \frac{\partial^2 \ln h}{\partial v^2}\frac{\partial^2 \ln h}{\partial \rho^2} = 4 - z^2.$$

From (9.8) we can conclude that

$$\begin{aligned} \lim_{N\to\infty} -S_{\tilde{\mu}_N}(z) &= \lim_{N\to\infty} \frac{N^{N+1/2}\mathrm{e}^{-N}}{N!} \frac{s}{\sqrt{2\pi}} \frac{sz+v_0+\frac{\rho_0}{N}}{sz+v_0} \frac{2\pi}{\sqrt{\frac{\partial^2 \ln h}{\partial v^2}\frac{\partial^2 \ln h}{\partial \rho^2}}} \\ &= \frac{1}{2}\left(z + s\sqrt{4-z^2}\right). \end{aligned} \tag{9.10}$$

If $z \in \mathbb{C}^+$ then $s = -\mathrm{i}$, which implies that (9.10) is equal to minus the Stieltjes transform of the semicircular law

$$\frac{z - \sqrt{z^2-4}}{2}$$

whenever we consider the square root with non-negative imaginary part (or real part if the imaginary part is zero). □

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